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ORIGINAL RESEARCH

On the Seidel spectrum of threshold graphs

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Abstract We consider a connected threshold graph G with A, S as its adjacency matrix and Seidel matrix respectively. In this paper several spectral properties of S are analysed. We compute the characteristic polynomial and determinant of S. A formula for the multiplicity of the Seidel eigenvalues ± 1 and characterisation of threshold graphs with at most five distinct Seidel eigenvalues are derived. Finally it is shown that two non isomorphic threshold graphs may be cospectral for S.

Keywords Threshold graph · Seidel matrix · Quotient matrix · Seidel cospectral

AMS Classification: 05C50

1 Introduction

Let P_n, C_n and K_n denote the path, the cycle and the complete graph on n vertices respectively. A vertex in a graph is called a dominating vertex if it is adjacent to all other vertices of the graph. A graph with no induced subgraph isomorphic to P_4, C_4 or $2K_2$ is called a threshold graph. Threshold graphs have various interesting applications [9,10] and there are various equivalent definitions for them (see [16]). One of the most interesting facts for threshold graph is that a threshold graph with n vertices has a binary representation with n digits (or every threshold graph can be represented by a binary string of length n). We can construct a threshold graph from one isolated vertex by an iterative process of two graph operations, namely, adding a new isolated vertex or adding a dominating vertex. Therefore we can represent a threshold graph by a binary string (sometimes called creation sequence of the threshold graph) $b = \alpha_1\alpha_2\alpha_3\ldots\alpha_n$. Here $\alpha_i = 0$ if the vertex v_i is added as an isolated vertex, and $\alpha_i = 1$ if v_i is added as a dominating vertex. We always consider $\alpha_1 = 0$. Every threshold graph has a unique binary string; so for each $n \geq 2$, the number of distinct connected threshold graph is exactly 2^{n-2}. For more interesting properties of threshold graphs we refer to the book by Mahadev and Peled [16].

Since last decade investigation of the spectral properties of adjacency eigenvalues gained a lot of attention. We found a lot of papers in this direction [1,2,5,7,11–14,18]. Bapat [5] determined the number of negative, zero and positive eigenvalues of a threshold graph from its binary representation. He also calculated the determinant of the adjacency matrix. Some interesting spectral properties of threshold graphs were given by Sciriha and Farrugia in [18]. Jacobs *et al.* wrote several papers (see [11–13]) with major focus on eigenvalue location, characteristic

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polynomial and energy of the adjacency matrix of threshold graphs. Lazzarin *et al.* [14] proved that no two threshold graphs are cospectral with respect to their adjacency matrices. We found articles with focus on other matrices associated to threshold graphs. In [4], Banerjee and Mehatari derived some useful results on normalized adjacency spectrum of threshold graphs, whereas the papers [3,15] are focused on distance spectra of threshold graphs.

In this paper we consider the Seidel matrix [6,8] of a connected threshold graph. Let $G = (V, E)$ be a finite, undirected, simple, connected graph and let A denote the adjacency matrix of G. Then the Seidel matrix S of the graph G is defined by

$$S = J - I - 2A.$$

Here J is all 1 matrix and I is the identity matrix. In other words, if s_{ij} is the (i, j)-th entry of S, then

$$s_{ij} = \begin{cases} -1, & \text{if } i \sim j, \\ 1, & \text{if } i \nsim j,\ i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

Let G be a threshold graph with the binary representation $b = \alpha_1\alpha_2\alpha_3 \ldots \alpha_n$. Then the adjacency matrix A of G has the form

$$A = \begin{bmatrix} 0 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_n \\ \alpha_2 & 0 & \alpha_3 & \alpha_4 & \cdots & \alpha_n \\ \alpha_3 & \alpha_3 & 0 & \alpha_4 & \cdots & \alpha_n \\ \alpha_4 & \alpha_4 & \alpha_4 & 0 & \cdots & \alpha_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_n & \alpha_n & \alpha_n & \alpha_n & \cdots & 0 \end{bmatrix}.$$

Therefore the Seidel matrix of G is given by

$$S = \begin{bmatrix} 0 & 1-2\alpha_2 & 1-2\alpha_3 & \ldots & 1-2\alpha_n \\ 1-2\alpha_2 & 0 & 1-2\alpha_3 & \ldots & 1-2\alpha_n \\ 1-2\alpha_3 & 1-2\alpha_3 & 0 & \ldots & 1-2\alpha_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1-2\alpha_n & 1-2\alpha_n & 1-2\alpha_n & \ldots & 0 \end{bmatrix}.$$

If we take $1 - 2\alpha_i = \beta_i$ for $i = 1, 2, 3, \ldots, n$, then S takes the form

$$S = \begin{bmatrix} 0 & \beta_2 & \beta_3 & \ldots & \beta_n \\ \beta_2 & 0 & \beta_3 & \ldots & \beta_n \\ \beta_3 & \beta_3 & 0 & \ldots & \beta_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_n & \beta_n & \beta_n & \ldots & 0 \end{bmatrix},$$

where $\beta_i = 1$ if $\alpha_i = 0$ and $\beta_i = -1$ if $\alpha_i = 1$. Thus if the binary representation of a threshold graph is $b = \alpha_1\alpha_2\alpha_3 \ldots \alpha_n$ then the entries of S are given by,

$$s_{ij} = \begin{cases} \beta_i, & \text{if } i > j, \\ \beta_j, & \text{if } j > i, \\ 0, & \text{otherwise.} \end{cases}$$

The paper is organized as follows: In Section 3 we give recurrence formulas for calculating the characteristic polynomial and determinant of a threshold graph. In Section 4 we prove some important properties of the Seidel quotient matrix Q_S. We prove that Q_S is diagonalizable and it has simple real eigenvalues. Later on, in that section, we derive the formula for multiplicity of the eigenvalues ± 1 of Seidel matrix. In Section 5 we derive some classes of threshold graphs with a few distinct Seidel eigenvalues. We prove that no threshold graph can have three distinct Seidel eigenvalues. In the last section we construct cospectral non-isomorphic pair of threshold graphs for the Seidel matrices.

2 Determinant and characteristic polynomial

In this section we obtain a recurrence formula for calculating the characteristic polynomial and determinant of the Seidel matrix of a threshold graph. The determinant of the Seidel matrix of a threshold graph can be found recursively using its binary representation. To obtain that, first we recall a theorem by Bapat.

Theorem 2.1 (Theorem 1, [5]) *Let $\alpha_2, \alpha_3, \ldots, \alpha_n$ be real numbers and M be the matrix of the form*

$$M = \begin{bmatrix} 0 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_n \\ \alpha_2 & 0 & \alpha_3 & \alpha_4 & \cdots & \alpha_n \\ \alpha_3 & \alpha_3 & 0 & \alpha_4 & \cdots & \alpha_n \\ \alpha_4 & \alpha_4 & \alpha_4 & 0 & \cdots & \alpha_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_n & \alpha_n & \alpha_n & \alpha_n & \cdots & 0 \end{bmatrix}.$$

Then there exists an $n \times n$ matrix P with $\det(P) = 1$ *such that*

$$PMP^T = \begin{bmatrix} -2\alpha_2 & \alpha_2 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \alpha_2 & -2\alpha_3 & \alpha_3 & 0 & \ldots & 0 & 0 & 0 \\ 0 & \alpha_3 & -2\alpha_4 & \alpha_4 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & \alpha_{n-1} & -2\alpha_n & \alpha_n \\ 0 & 0 & 0 & 0 & \ldots & 0 & \alpha_n & 0 \end{bmatrix}.$$

Therefore, using above theorem, we conclude that the determinant of S is equal to the determinant of the following tridiagonal matrix:

$$T = \begin{bmatrix} -2\beta_2 & \beta_2 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \beta_2 & -2\beta_3 & \beta_3 & 0 & \ldots & 0 & 0 & 0 \\ 0 & \beta_3 & -2\beta_4 & \beta_4 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & \beta_{n-1} & -2\beta_n & \beta_n \\ 0 & 0 & 0 & 0 & \ldots & 0 & \beta_n & 0 \end{bmatrix}.$$

By Algorithm 2.1 of [17], we know that the determinant of a tridiagonal matrix T_1, where

$$T_1 = \begin{bmatrix} b_1 & c_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \ldots & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & a_{n-1} & \beta_{n-1} & c_{n-1} \\ 0 & 0 & 0 & 0 & \ldots & 0 & a_n & b_n \end{bmatrix},$$

is given by

$$\det(T_1) = \prod_{i=1}^{n} d_i,$$

where

$$d_i = \begin{cases} b_1, & \text{if } i = 1, \\ b_i - \frac{a_i}{d_{i-1}} c_{i-1}, & \text{if } i = 2,\ 3,\ \ldots,\ n. \end{cases}$$

Now, to find the determinant of S, we apply above algorithm to T. Since $\beta_i \in \{-1, 1\}$, for $i = 1, 2, \ldots, n$, we have

$$d_i = \begin{cases} -2\beta_2, & \text{if } i = 1, \\ -2\beta_{i+1} - \frac{1}{d_{i-1}}, & \text{if } 2 \le i \le n-1, \\ -\frac{1}{d_{n-1}}, & \text{if } i = n. \end{cases}$$

Therefore the determinant of the Seidel matrix is given by

$$\det(S) = \det(T) = \prod_{i=1}^{n} d_i.$$

Example 2.1 Consider the representation $b = \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6 = 001111$ of a threshold graph G. Then $\beta_1 = \beta_2 = 1,\ \beta_3 = \beta_4 = \beta_5 = \beta_6 = -1$. The corresponding Seidel matrix is

$$S = \begin{bmatrix} 0 & 1 & -1 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 & -1 & -1 \\ -1 & -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}.$$

Here $d_1 = -2,\ d_2 = \frac{5}{2},\ d_3 = \frac{8}{5},\ d_4 = \frac{11}{8},\ d_5 = \frac{30}{11},\ d_6 = -\frac{11}{30}$.
Therefore, $\det(S) = d_1d_2d_3d_4d_5d_6 = 11$.

Theorem 2.2 *Let G be a threshold graph with the binary string $b = \alpha_1\alpha_2\ldots\alpha_n$. Let $b_r = \alpha_1\alpha_2\alpha_3\ldots\alpha_r$ and suppose $\Phi_r(x)$ denote the characteristic polynomial of Seidel matrix of the threshold graph whose binary representation is b_r, then the characteristic polynomial $\Phi_n(x)$ of the Seidel matrix is obtained by the following recurrence formula*

$$\Phi_r(x) = 2(x + \beta_{r-1})\Phi_{r-1}(x) - 2(x + \beta_{r-1})^2\Phi_{r-2}(x),$$

where $\Phi_1(x) = x$ and $\Phi_2(x) = x^2 - 1$.

Proof Let $\Phi_r(x)$ be the characteristic polynomial of the threshold graph whose binary representation is $b_r = \alpha_1\alpha_2\alpha_3\ldots\alpha_r$. Then

$$\Phi_r(x) = \begin{vmatrix} x & -\beta_2 & -\beta_3 & \ldots & -\beta_r \\ -\beta_2 & x & -\beta_3 & \ldots & -\beta_r \\ -\beta_3 & -\beta_3 & x & \ldots & -\beta_r \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -\beta_r & -\beta_r & -\beta_r & \ldots & x \end{vmatrix}.$$

We now consider the following two cases.

Case I. If $\beta_r = \beta_{r-1}$, then

$$\Phi_r(x) = \begin{vmatrix} x & -\beta_2 & -\beta_3 & \ldots & -\beta_{r-1} & -\beta_{r-1} \\ -\beta_2 & x & -\beta_3 & \ldots & -\beta_{r-1} & -\beta_{r-1} \\ -\beta_3 & -\beta_3 & x & \ldots & -\beta_{r-1} & -\beta_{r-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\beta_{r-1} & -\beta_{r-1} & -\beta_{r-1} & \ldots & x & -\beta_{r-1} \\ -\beta_{r-1} & -\beta_{r-1} & -\beta_{r-1} & \ldots & -\beta_{r-1} & x \end{vmatrix}$$

$$= \begin{vmatrix} -x & \beta_2 & \beta_3 & \ldots & \beta_{r-1} & 0 \\ \beta_2 & -x & \beta_3 & \ldots & \beta_{r-1} & 0 \\ \beta_3 & \beta_3 & -x & \ldots & \beta_{r-1} & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \beta_{r-1} & \beta_{r-1} & \beta_{r-1} & \ldots & -x & -h \\ 0 & 0 & 0 & \ldots & -h & 2h \end{vmatrix},$$

where $h = \beta_{r-1} + x$. Therefore,

$$\Phi_r(x) = 2h\Phi_{r-1}(x) + h \begin{vmatrix} x & -\beta_2 & -\beta_3 & \ldots & -\beta_{r-2} & -\beta_{r-1} \\ -\beta_2 & x & -\beta_3 & \ldots & -\beta_{r-2} & -\beta_{r-1} \\ -\beta_3 & -\beta_3 & x & \ldots & -\beta_{r-2} & -\beta_{r-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ \beta_{r-2} & \beta_{r-2} & \beta_{r-2} & \ldots & x & -\beta_{r-1} \\ 0 & 0 & 0 & \ldots & 0 & -h \end{vmatrix}$$

$$= 2h\Phi_{r-1}(x) - h^2\Phi_{r-2}(x).$$

Case II. If $\beta_{r-1} = -\beta_r$, then

$$\Phi_r(x) = \begin{vmatrix} x & -\beta_2 & -\beta_3 & \dots & -\beta_{r-1} & \beta_{r-1} \\ -\beta_2 & x & -\beta_3 & \dots & -\beta_{r-1} & \beta_{r-1} \\ -\beta_3 & \beta_3 & x & \dots & -\beta_{r-1} & \beta_{r-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\beta_{r-1} & -\beta_{r-1} & -\beta_{r-1} & \dots & x & \beta_{r-1} \\ \beta_{r-1} & \beta_{r-1} & \beta_{r-1} & \dots & \beta_{r-1} & x \end{vmatrix}$$
$$= \begin{vmatrix} x & -\beta_2 & -\beta_3 & \dots & -\beta_{r-1} & 0 \\ -\beta_2 & x & -\beta_3 & \dots & -\beta_{r-1} & 0 \\ -\beta_3 & -\beta_3 & x & \dots & -\beta_{r-1} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\beta_{r-1} & -\beta_{r-1} & -\beta_{r-1} & \dots & x & h \\ 0 & 0 & 0 & \dots & h & 2h \end{vmatrix},$$

where $h = \beta_{r-1} + x$. Therefore,

$$\Phi_r(x) = 2h\Phi_{r-1}(x) - h \begin{vmatrix} x & -\beta_2 & -\beta_3 & \dots & -\beta_{r-2} & -\beta_{r-1} \\ -\beta_2 & x & -\beta_3 & \dots & -\beta_{r-2} & -\beta_{r-1} \\ -\beta_3 & -\beta_3 & x & \dots & -\beta_{r-2} & -\beta_{r-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\beta_{r-2} & -\beta_{r-2} & -\beta_{r-2} & \dots & x & -\beta_{r-1} \\ 0 & 0 & 0 & \dots & 0 & h \end{vmatrix}$$
$$= 2h\Phi_{r-1}(x) - h^2\Phi_{r-2}(x).$$

Thus, combining Case I and Case II, we obtain the required result. □

3 Eigenvalues of threshold graphs

In this section, first we describe some properties of the quotient matrix corresponding to an equitable partition of the Seidel matrix. Using these properties we establish multiplicity of the eigenvalues ± 1.

Let G be a graph. A partition $\pi = \{V_1, V_2, \dots, V_m\}$ of the vertex set of G is called an equitable partition of G if there exists nonnegative integers b_{ij} for $1 \le i, j \le m$ such that each vertex v in V_i has exactly b_{ij} neighbours in V_j. We define a matrix $Q_S = [q_{ij}]$ of order $m \times m$, whose (i, j)-th entry is defined by

$$q_{ij} = \begin{cases} |V_j| - 2b_{ij}, & \text{if } i \ne j, \\ |V_i| - 2b_{ii} - 1, & \text{if } i = j. \end{cases}$$

The matrix Q_S is called the (Seidel) quotient matrix for the equitable partition π.

3.1 Quotient Matrix

Let us consider a threshold graph G with the binary string $b = 0^{s_1}1^{t_1}0^{s_2}\dots 0^{s_k}1^{t_k}$, where $s_i,\ t_i \ge 1$ for all i. Clearly G has $(s+t)$ vertices, where $s = \sum s_i$ and $t = \sum t_i$. Then the Seidel matrix S of G is a square matrices of size $(s+t)$, given by

$$S = \begin{bmatrix} (J-I)_{s_1} & -J_{s_1\times t_1} & J_{s_1\times s_2} & -J_{s_1\times t_2} & \dots & -J_{s_1\times t_k} \\ -J_{t_1\times s_1} & (I-J)_{t_1} & J_{t_1\times s_2} & -J_{t_1\times t_2} & \dots & -J_{t_1\times t_k} \\ J_{s_2\times s_1} & J_{s_2\times t_1} & (J-I)_{s_2} & -J_{s_2\times t_2} & \dots & -J_{s_2\times t_k} \\ -J_{t_2\times s_1} & -J_{t_2\times t_1} & -J_{t_2\times s_2} & (I-J)_{t_2} & \dots & -J_{t_2\times t_k} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -J_{t_k\times s_1} & -J_{t_k\times t_1} & -J_{t_k\times s_2} & -J_{t_k\times t_2} & \dots & (I-J)_{t_k} \end{bmatrix},$$

where $J_{m\times n}$ is all 1 block matrix of size $m \times n$. Here the diagonal blocks of S are the square matrices of size $s_1 \times s_1,\ t_1 \times t_1,\ s_2 \times s_2,\ t_2 \times t_2,\ \dots,\ t_k \times t_k$, respectively.

Let $b = 0^{s_1}1^{t_1}0^{s_2}\dots 0^{s_k}1^{t_k}$ be the binary string of a threshold graph G. For the representation $b = 0^{s_1}1^{t_1}0^{s_2}\dots 0^{s_k}1^{t_k}$, let V_{s_1} denotes the first s_1 added vertices, V_{t_1} denotes the next t_1 added vertices, V_{s_2} denotes the next s_2 added vertices and finally V_{t_k} denotes the last t_k added vertices. Then $\pi = \{V_{s_1}, V_{t_1}, V_{s_2}\dots, V_{t_k}\}$ is an equitable partition of G. Now for $1 \le i \le 2k$, we define

$$C_i = \begin{cases} |V_{s_j}| = s_j, & \text{if } i = 2j-1,\\ |V_{t_j}| = t_j, & \text{if } i = 2j.\end{cases}$$

Therefore for the vertex partition π of V, the quotient matrix Q_S is a square matrix of size $2k$, given by

$$Q_S = \begin{bmatrix} s_1-1 & -t_1 & s_2 & -t_2 & s_3 & \dots & -t_k\\ -s_1 & -(t_1-1) & s_2 & -t_2 & s_3 & \dots & -t_k\\ s_1 & t_1 & s_2-1 & -t_2 & s_3 & \dots & -t_k\\ -s_1 & -t_1 & -s_2 & -(t_2-1) & s_3 & \dots & -t_k\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ -s_1 & -t_1 & -s_2 & -t_2 & -s_3 & \dots & -(t_k-1)\end{bmatrix}.$$

We observe that all the eigenvalues of Q_S are also eigenvalues of S. Now we provide some interesting properties of Q_S. Let us consider the diagonal matrix $D = diag\{s_1, t_1, s_2, \dots, s_k, t_k\}$. We observe that the matrix $D^{\frac{1}{2}}Q_S D^{-\frac{1}{2}}$ is a symmetric matrix. Hence Q_S is diagonalizable.

Let λ be an eigenvalue of Q_S with corresponding eigenvector $X \in \mathbb{R}^{2k}$. Let $P_{n\times 2k}$ be the matrix whose i-th row is given by

$$\boldsymbol{e}_{\gamma_i+1} + \boldsymbol{e}_{\gamma_i+2} + \dots + \boldsymbol{e}_{\gamma_i+C_i},$$

where $\gamma_1 = 0$ and $\gamma_i = \sum_{j=1}^{i-1} C_j$ for $i = 2, 3, \dots, 2k$. Then it is easy to verify that $SP = PQ_S$. Thus an eigenvalue of Q_S is an eigenvalue of S also.

Theorem 3.1 *Let G be a threshold graph and $b = 0^{s_1}1^{t_1}0^{s_2}\dots 0^{s_k}1^{t_k}$ be the binary representation of G. Then*

(a) -1 *is a simple eigenvalue of Q_S if $t_k = 1$.*
(b) -1 *is not an eigenvalue of Q_S if $t_k > 1$.*

Proof Let Q_S has the eigenvalue -1. Then for the nonzero vector $X = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_{2k}\end{bmatrix}^T$, we have the matrix equation $Q_S X = -X$, which gives the following system of linear equations:

$$\begin{aligned} (s_1-1)x_1 - t_1x_2 + s_2x_3 - t_2x_4 + s_3x_5 - t_3x_6 + \dots - t_kx_{2k} &= -x_1, & (1)\\ -s_1x_1 - (t_1-1)x_2 + s_2x_3 - t_2x_4 + s_3x_5 - t_3x_6 + \dots - t_kx_{2k} &= -x_2, & (2)\\ s_1x_1 + t_1x_2 + (s_2-1)x_3 - t_2x_4 + s_3x_5 - t_3x_6 + \dots - t_kx_{2k} &= -x_3, & (3)\\ -s_1x_1 - t_1x_2 - s_2x_3 - (t_2-1)x_4 + s_3x_5 - t_3x_6 + \dots - t_kx_{2k} &= -x_4, & (4)\\ s_1x_1 + t_1x_2 + s_2x_3 + t_2x_4 + (s_3-1)x_5 - t_3x_6 + \dots - t_kx_{2k} &= -x_5, & (5)\\ \dots\ \dots\ \dots\ \dots\ \dots\quad \dots\ \dots\ \dots\ \dots & & \vdots\\ -s_1x_1 - t_1x_2 - s_2x_3 - t_2x_4 - s_3x_5 - t_3x_6 + \dots - (t_k-1)x_{2k} &= -x_{2k}. & (2k)\end{aligned}$$

Now, applying the following operations in order, we get the values of x_i, for all $i = 1, 2, 3, \dots, 2k$.
$(1) - (3)$ gives $x_2 = 0$.
Now putting $x_2 = 0$ and performing $(1) - (2)$, we get $x_1 = 0$.
Next putting $x_1 = x_2 = 0$ and performing $(1) - (5)$, we get $x_4 = 0$.
Again putting $x_1 = x_2 = x_4 = 0$ and performing $(1) - (4)$, we get $x_3 = 0$.
Proceeding in this way, and after performing $(1) - (2k-1)$ and $(1) - (2k-2)$ we obtain that any vector that satisfies eigenvalue equation corresponding to -1 must have first $2k-2$ entry equal to 0.

Finally performing $(1) - (2k)$ and $(1) + (2k)$ we get,

$$\begin{aligned} x_{2k} - s_kx_{2k-1} &= 0, & (c)\\ (1-t_k)x_{2k} &= 0. & (d)\end{aligned}$$

Now two cases arise.

Case I. If $t_k = 1$, then from (c) and (d), we get $X = \begin{bmatrix} 0 \; 0 \; 0 \cdots 0 \; 1 \; s_k \end{bmatrix}^T$ is an eigenvector corresponding to -1. In fact, in that case any nonzero eigenvector corresponding -1 is a nonzero multiple of X. Therefore -1 is a simple eigenvalue of Q_S.

Case II. If $t_k \neq 1$, then from (d) $x_{2k} = 0$. Therefore $x_{2k-1} = 0$ by (c). Therefore

$$X = \begin{bmatrix} x_1 \; x_2 \; x_3 \cdots x_{2k} \end{bmatrix}^T = \begin{bmatrix} 0 \; 0 \; 0 \cdots 0 \end{bmatrix}^T.$$

Thus -1 can not be an eigenvalue of Q_S if $t_k > 1$. □

Theorem 3.2 *Let G be a threshold graph and $b = 0^{s_1} 1^{t_1} 0^{s_2} \ldots 0^{s_k} 1^{t_k}$ be the binary representation of G. Then*

(a) 1 *is a simple eigenvalue of Q_S if $s_1 = 1$.*
(b) 1 *is not an eigenvalue of Q_S if $s_1 > 1$.*

Proof For $X \in \mathbb{R}^{2k}$ consider the matrix equation $Q_S X = X$. This gives the following system of linear equations:

$$\begin{aligned}
(s_1-1)x_1 - t_1x_2 + s_2x_3 - t_2x_4 + \cdots - t_{k-1}x_{2k-2} + s_kx_{2k-1} - t_kx_{2k} &= x_1, && (1)\\
-s_1x_1 - (t_1-1)x_2 + s_2x_3 - t_2x_4 + \cdots - t_{k-1}x_{2k-2} + s_kx_{2k-1} - t_kx_{2k} &= x_2, && (2)\\
s_1x_1 + t_1x_2 + (s_2-1)x_3 - t_2x_4 + \cdots - t_{k-1}x_{2k-2} + s_kx_{2k-1} - t_kx_{2k} &= x_3, && (3)\\
\cdots \; \cdots \; \cdots \; \cdots \; \cdots \quad \cdots \; \cdots \; \cdots \; \cdots & && \vdots\\
-s_1x_1 - t_1x_2 - s_2x_3 - t_2x_4 - \cdots - (t_{k-1}-1)x_{2k-2} + s_kx_{2k-1} \cdots - t_kx_{2k} &= x_{2k-2}, && (2k-2)\\
s_1x_1 + t_1x_2 + s_2x_3 + t_2x_4 + \cdots + t_{k-1}x_{2k-2} + (s_k-1)x_{2k-1} - t_kx_{2k} &= x_{2k-1}, && (2k-1)\\
-s_1x_1 - t_1x_2 - s_2x_3 - t_2x_4 - \cdots - t_{k-1}x_{2k-2} - s_kx_{2k-1} - (t_k-1)x_{2k} &= x_{2k}. && (2k)
\end{aligned}$$

Now, to find the x_i's, we apply the following operations:
$(2k) - (2k-2)$ gives $x_{2k-1} = 0$.
Now putting $x_{2k-1} = 0$ and performing $(2k) - (2k-1)$, we get $x_{2k} = 0$.
Putting $x_{2k} = x_{2k-1} = 0$ and performing $(2k) - (2k-4)$, we get $x_{2k-3} = 0$.
Putting $x_{2k} = x_{2k-1} = x_{2k-3} = 0$ and performing $(2k) - (2k-3)$, we get $x_{2k-2} = 0$.
Proceeding in this way, and after performing $(2k) - (2)$ and $(2k) - (3)$ we obtain $x_3 = x_4 = \cdots = x_{2k} = 0$.
Finally performing $(2k) - (1)$ and $(2k) + (1)$ we get,

$$\begin{aligned}
(s_1-1)x_1 &= 0, && (c)\\
t_1x_2 + x_1 &= 0. && (d)
\end{aligned}$$

Now two cases arise.

Case I. If $s_1 = 1$, then from (c) and (d), we get $X = \begin{bmatrix} t_1 \; -1 \; 0 \; 0 \cdots 0 \end{bmatrix}^T$ is an eigenvector corresponding to 1. In fact, in that case any nonzero eigenvector corresponding 1 is a nonzero multiple of X. Therefore 1 is a simple eigenvalue of Q_S.

Case II. If $s_1 \neq 1$, then from (c) $x_1 = 0$. Therefore $x_2 = 0$ by (d). Therefore $X = \begin{bmatrix} x_1 \; x_2 \; x_3 \cdots x_{2k} \end{bmatrix}^T = \begin{bmatrix} 0 \; 0 \; 0 \cdots 0 \end{bmatrix}^T$.
Thus 1 is not an eigenvalue of Q_S if $t_k > 1$. □

From Theorem 3.1 and Theorem 3.2 it is clear that ± 1 can be an eigenvalue of Q_S with multiplicity at most 1. In the next theorem we prove that Q_S has $2k$ distinct eigenvalues.

Theorem 3.3 *All eigenvalues of Q_S are simple.*

Proof Suppose λ is an eigenvalue of Q_S. Let $X = \begin{bmatrix} x_1 \; x_2 \; x_3 \cdots x_{2k} \end{bmatrix}^T$ be an eigenvector corresponding to λ such that $x_l \neq 0$ and $x_m = 0$ for all $m < l$, where l is minimal. Then $l = 2p - 1, 1 \leq p \leq k$. It is already proved that $\lambda = \pm 1$ is a simple eigenvalue of Q_S. So the remaining part is for $\lambda \neq \pm 1$. The matrix equation

$Q_S X = \lambda X$ gives the following system of linear equations:

$$
\begin{aligned}
(s_1-1)x_1 - t_1x_2 + s_2x_3 - t_2x_4 + \cdots - t_kx_{2k} &= \lambda x_1, && (1)\\
-s_1x_1 - (t_1-1)x_2 + s_2x_3 - t_2x_4 + \cdots - t_kx_{2k} &= \lambda x_2, && (2)\\
\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots & && \vdots\\
-s_1x_1 - t_1x_2 - s_2x_3 - (t_2-1)x_4 + \cdots - t_kx_{2k} &= \lambda x_{l-1}, && (l-1)\\
s_1x_1 + t_1x_2 + s_2x_3 + t_2x_4 + \cdots - t_kx_{2k} &= \lambda x_l, && (l)\\
-s_1x_1 - t_1x_2 - s_2x_3 - (t_2-1)x_4 + \cdots - t_kx_{2k} &= \lambda x_{l+1}, && (l+1)\\
s_1x_1 + t_1x_2 + s_2x_3 + t_2x_4 + \cdots - t_kx_{2k} &= \lambda x_{l+2}, && (l+2)\\
\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots & && \vdots\\
-s_1x_1 - t_1x_2 - s_2x_3 - t_2x_4 + \cdots - (t_k-1)x_{2k} &= -x_{2k}. && (2k)
\end{aligned}
$$

Now, applying the following operations in order, we get the values of x_i for all $i = l, l+1, l+2, \ldots, 2k$. $(l) - (l+1)$ gives

$$x_{l+1} = \frac{\left(1 + \lambda - 2s_{\frac{l+1}{2}}\right) x_l}{\lambda - 1} = c_{l+1}x_l,\ \text{(say)}.$$

Again $(l) - (l+2)$ gives,

$$x_{l+2} = \frac{1}{\lambda+1}\left[2t_{\frac{l+1}{2}} x_{l+1} + (1+\lambda)x_l\right] = c_{l+2}x_l,\ \text{(say)}$$

and so on.
Thus, proceeding in this way, we get the constants $c_{l+1}, c_{l+2}, c_{l+3}, \ldots, c_{2k}$, such that,

$$X = x_l \left[0\ 0 \cdots 0\ 1\ c_{l+1}\ c_{l+2} \cdots c_{2k}\right]^T.$$

In fact any nonzero eigenvector corresponding to λ is a constant multiple of X. Since Q_S is diagonalizable so all eigenvalues of S are simple. □

3.2 Multiplicity of the eigenvalues 1 and −1

Let us consider a threshold graph G with the binary representation $0^{s_1}1^{t_1}0^{s_2}\ldots 0^{s_k}1^{t_k}$. Let $n_{-1}(S)$ and $n_{+1}(S)$ denote the multiplicity of the eigenvalues -1 and $+1$ respectively of the matrix S. We now derive formulas for $n_{-1}(S)$ and $n_{+1}(S)$. For that first we construct eigenvectors corresponding to ± 1 which does not belong to spectrum of Q_S. Recall S has the form:

$$S = \begin{bmatrix}
(J-I)_{s_1} & -J_{s_1\times t_1} & J_{s_1\times s_2} & -J_{s_1\times t_2} & \ldots & -J_{s_1\times t_k}\\
-J_{t_1\times s_1} & (I-J)_{t_1} & J_{t_1\times s_2} & -J_{t_1\times t_2} & \ldots & -J_{t_1\times t_k}\\
J_{s_2\times s_1} & J_{s_2\times t_1} & (J-I)_{s_2} & -J_{s_2\times t_2} & \ldots & -J_{s_2\times t_k}\\
-J_{t_2\times s_1} & -J_{t_2\times t_1} & -J_{t_2\times s_2} & (I-J)_{t_2} & \ldots & -J_{t_2\times t_k}\\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\
-J_{t_k\times s_1} & -J_{t_k\times t_1} & -J_{t_k\times s_2} & -J_{t_k\times t_2} & \ldots & (I-J)_{t_k}
\end{bmatrix}.$$

For $i > 1$ define set of $(i-1)$ orthogonal row-vectors $\{X^i_j\}$ in $\mathbb{R}^i$ by

$$X^i_j = \boldsymbol{e}_1(i)^T + \boldsymbol{e}_2(i)^T + \cdots + \boldsymbol{e}_j(i)^T - j\boldsymbol{e}_{j+1}(i)^T \quad \forall 1 \le j \le i-1,$$

where $\boldsymbol{e}_j(i)$ is the j-th standard basis element of $\mathbb{R}^i$.

Now for $s_i \ge 2$, define

$$Y_{s_i}(j) = [O_{s_1}\ O_{t_1}\ \cdots\ O_{t_{i-1}}\ X^{s_i}_j\ O_{t_i}\ \cdots\ O_{t_k}]^T \quad \forall 1 \le i \le k,\quad 1 \le j \le s_i - 1,$$

where O_r denote the r-component zero row-vector. The the set $\{Y_{s_i}(1), Y_{s_i}(2), \dots, Y_{s_i}(s_i-1)\}$ contains $s_i - 1$ orthogonal eigenvectors corresponding to -1.

Again for each $t_i \geq 2$, define

$$Z_{t_i}(j) = [O_{s_1}\ O_{t_1}\ \cdots\ O_{s_i}\ X_j^{t_i}\ O_{s_{i+1}}\ \cdots\ O_{t_k}]^T \quad \forall 1 \leq i \leq k, \quad 1 \leq j \leq t_i - 1.$$

Then the set $\{Z_{t_i}(1), Z_{t_i}(2), \dots, Z_{t_i}(t_i-1)\}$ contains $t_i - 1$ orthogonal eigenvectors corresponding to 1.

Each of $Y_{s_i}(j)$'s and $Z_{t_i}(j)$'s has row sum zero in each of the vertex partition. Now let λ be an eigenvalue of Q_S with eigenvector $X \in \mathbb{R}^{2k}$. Then the eigenvector PX corresponding to the eigenvalue λ is constant in each vertex partition. Therefore PX is orthogonal to each of these $Y_{s_i}(j)$'s and $Z_{t_i}(j)$'s. Using this fact, we now calculate the multiplicity of the eigenvalues ± 1.

Theorem 3.4 *Let G be a threshold graph and $0^{s_1}1^{t_1}0^{s_2}\dots 0^{s_k}1^{t_k}$ be the binary representation of G. Then*

$$n_{-1}(S) = \begin{cases} \sum s_i - k, & \text{if } t_k > 1, \\ \sum s_i - k + 1, & \text{if } t_k = 1. \end{cases}$$

Proof We already observed that, if $s_i \geq 2$ then the set $\{Y_{s_i}(1), Y_{s_i}(2), \dots, Y_{s_i}(s_i-1)\}$ contains $s_i - 1$ orthogonal eigenvectors corresponding to -1. Now for $s_l,\ s_m \geq 2$, the vectors $Y_{s_l}(j)$ and $Y_{s_m}(k)$ are orthogonal for all $1 \leq j < m$ and $1 \leq k < m$. Therefore the set

$$\{Y_{s_i}(j) | 1 \leq i \leq k,\ 1 \leq j \leq s_i - 1 \text{ and } s_i > 1\}$$

provides a set of $\sum s_i - k$ orthogonal eigenvectors corresponding to -1. We consider two cases.

Case I. If $t_k > 1$, then Q_S does not contain the eigenvalue -1 . Therefore the multiplicity of the eigenvalue -1 in S is exactly $\sum(s_i - 1)$. Thus

$$n_{-1}(S) = \sum s_i - k.$$

Case II. If $t_k = 1$, then Q_S has eigenvalue -1 with multiplicity 1. Thus

$$n_{-1}(S) = \sum s_i - k + 1.$$

Therefore, by combining Case I and Case II, we get the required result. □

Theorem 3.5 *Let G be a threshold graph and $0^{s_1}1^{t_1}0^{s_2}\dots 0^{s_k}1^{t_k}$ be the binary representation of G. Then*

$$n_{+1}(S) = \begin{cases} \sum t_i - k, & \text{if } s_1 > 1, \\ \sum t_i - k + 1, & \text{if } s_1 = 1. \end{cases}$$

Proof By a similar argument to previous theorem, the set

$$\{Z_{t_i}(j) | 1 \leq i \leq k,\ 1 \leq j \leq t_i - 1 \text{ and } t_i > 1\}$$

provides a set of $\sum t_i - k$ orthogonal eigenvectors corresponding to 1. We now consider two cases.

Case I. If $s_1 > 1$, then spectrum of Q_S does not contain the eigenvalue 1. Therefore the multiplicity of the eigenvalue 1 in S is exactly $\sum(t_i - 1)$. Thus

$$n_{+1}(S) = \sum t_i - k.$$

Case II. If $s_1 = 1$, then 1 is a simple eigenvalue of Q_S. Thus

$$n_{+1}(S) = \sum t_i - k + 1.$$

Therefore proof of the theorem follows from Case I and Case II. □

By using Theorem 3.1, 3.2, 3.4 and 3.5 we can easily calculate eigenvectors corresponding to ± 1. For example, consider the threshold graph G with binary string $b = 01100111$. Then $n_{-1}(G) = 1$ and $n_{+1}(G) = 4$. An eigenvector corresponding to -1 is

$$Y_{s_2}(1) = \begin{bmatrix} 0\ 0\ 0\ 1\ -1\ 0\ 0\ 0 \end{bmatrix}^T.$$

Whereas eigenvectors corresponding to 1 are

$$Z_{t_1}(1) = \begin{bmatrix} 0\ 1\ -1\ 0\ 0\ 0\ 0\ 0 \end{bmatrix}^T,$$
$$Z_{t_2}(1) = \begin{bmatrix} 0\ 0\ 0\ 0\ 0\ 1\ -1\ 0 \end{bmatrix}^T,$$
$$Z_{t_2}(2) = \begin{bmatrix} 0\ 0\ 0\ 0\ 0\ 1\ 1\ -2 \end{bmatrix}^T,$$

and

$$PX = \begin{bmatrix} 2\ -1\ -1\ 0\ 0\ 0\ 0\ 0 \end{bmatrix}^T,$$

where $X = \begin{bmatrix} 2\ -1\ 0\ 0 \end{bmatrix}^T$ is an eigenvector corresponding to 1 for the quotient matrix Q_S.

4 Threshold graph with few distinct Seidel eigenvalues

Now we characterize classes of threshold graphs which have a few distinct Seidel eigenvalues. In particular we classify all such threshold graphs which have at most five distinct Seidel eigenvalues.

Theorem 4.1 *Let G be a threshold graph. Then G has two distinct Seidel eigenvalues if and only if G is either the complete graph K_n or the star graph S_n.*

Proof Suppose G has two eigenvalues then the binary string of G is of the form $0^s 1^{n-s}$. Now if $1 < s < n-1$ then ± 1 are eigenvalues of S. Again in that case Q_S has two distinct eigenvalues other than ± 1. Hence S has four distinct eigenvalues. Therefore either $s = 1$ or $s = n-1$, and in both these cases S has two distinct eigenvalues. Now, if $b = 01^{n-1}$ then $G = K_n$ and if $b = 0^{n-1}1$ then $G = S_n$. □

For a threshold graph with binary representation $b = 0^{s_1}1^{t_1}0^{s_2}\ldots 0^{s_k}1^{t_k}$, the Seidel matrix has atleast $2k$ distinct eigenvalues and atmost $2k+2$ eigenvalues. We already observed that for a threshold graph whose binary representation is $0^s 1^{n-s}$, $1 < s < n-1$, the Seidel matrix S has 4 distinct eigenvalues. Again by previous theorem if $b = 01^{n-1}$ or $b = 0^{n-1}1$, then S has exactly two distinct eigenvalues. Thus we have the following conclusion.

Theorem 4.2 *No threshold graph can have three distinct Seidel eigenvalues.*

In the next two theorems we characterize threshold graphs with exactly 4 and 5 distinct Seidel eigenvalues.

Theorem 4.3 *Let G be a threshold graph and b be the binary representation of G. Then G has 4 distinct Seidel eigenvalues if and only if $b = 0^{s_1}1^{t_1},\ s_1 > 1,\ t_1 > 1$ or $b = 01^{t_1}0^{s_2}1$.*

Proof Let $b = 0^{s_1}1^{t_1}0^{s_2}\ldots 0^{s_k}1^{t_k}$ be the binary representation of the threshold graph G. If S has 4 distinct eigenvalues then $k \le 2$. Now two cases arise.

Case I. If $k = 1$, then G has binary representation $b = 0^{s_1}1^{t_1}$. If either of s_1 or t_1 is equal to 1, then by Theorem 4.1, S has exactly two distinct eigenvalues. Whereas if $s_1 > 1, t_1 > 1$, then Q_S has two distinct eigenvalues other that ± 1. Therefore S has four distinct eigenvalues which are -1^{s_1-1}, 1^{t_1-1} and another two simple eigenvalues come from Q_S.

Case II. If $k = 2$, then $b = 0^{s_1}1^{t_1}0^{s_2}1^{t_2}$. Now if S has 4 distinct eigenvalues, then S can not have eigenvalue ± 1 outside of the spectrum of Q_S. Therefore 4 distinct eigenvalues is possible only if $s_1 = 1 = t_2$, and in that case eigenvalues of S are 1^{t_1}, -1^{s_2} and $\frac{(s_2-t_1)\pm\sqrt{(s_2-t_1)^2+4(1+t_1+s_2+2t_1s_2)}}{2}$.

Conversely if $b = 0^{s_1}1^{t_1},\ s_1 > 1,\ t_1 > 1$ or $b = 01^{t_1}0^{s_2}1$, then S has four distinct eigenvalues. □

Theorem 4.4 *Let G be a threshold graph and b be the binary representation of G. Then G has 5 distinct Seidel eigenvalues if and only if $b = 01^{t_1}0^{s_2}1^{t_2}$ with $t_2 > 1$ or $b = 0^{s_1}1^{t_1}0^{s_2}1$ with $s_1 > 1$.*

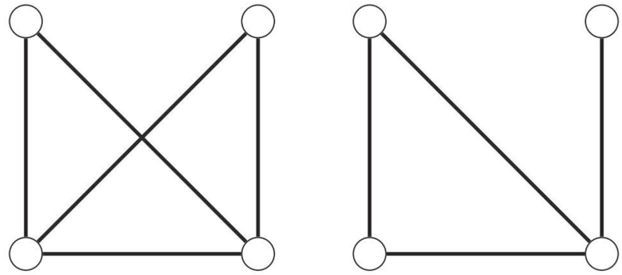

Fig. 1 Non isomorphic cospectral threshold graphs with 4 vertices

Proof Let G has the binary representation $0^{s_1}1^{t_1}0^{s_2}1^{t_2}\ldots 0^{s_k}1^{t_k}$. Now for 5 distinct eigenvalues of S, k must be equal to 2. Let $b = 0^{s_1}1^{t_1}0^{s_2}1^{t_2}$ be the binary representation of G. Then Q_S has four distinct eigenvalues. Now S will have five distinct Seidel eigenvalues if ± 1 are eigenvalues of S and exactly one of ± 1 belongs to spectrum of Q_S. Thus following two cases arise.

Case I. Let $s_1 = 1$, $t_2 > 1$. Then 1 belongs to the spectrum of Q_S, whereas -1 does not. Therefore spectrum of S is $\{-1^{s_2-1}, 1^{t_1+t_2}, \alpha_1, \beta_1, \gamma_1\}$, where α_1, β_1, and γ_1 are the distinct eigenvalues (other than 1) of Q_S. Thus S has five distinct eigenvalues.

Case II. Let $s_1 > 1$, $t_2 = 1$. Then -1 belongs to the spectrum of Q_S, whereas 1 does not. Therefore spectrum of S is $\{-1^{s_1+s_2}, 1^{t_1-1}, \alpha_2, \beta_2, \gamma_2\}$, where α_2, β_2, and γ_2 are the distinct eigenvalues (other than -1) of Q_S. Thus S has five distinct eigenvalues.

Conversely if $b = 01^{t_1}0^{s_2}1^{t_2}$ with $t_2 > 1$ or $b = 0^{s_1}1^{t_1}0^{s_2}1$ with $s_1 > 1$, then S has five distinct eigenvalues. □

5 Two threshold graphs may be Seidel cospectral

We conclude this paper by showing that two non-isomorphic threshold graphs may be cospectral with respect to their Seidel matrices. Although it is well known that two non-isomorphic threshold graphs are not cospectral with respect to their adjacency matrices and Laplacian matrices; but here we see that two threshold graphs with distinct binary strings may be Seidel cospectral. Using the following theorem we can construct nonisomorphic cospectral threshold graphs.

Theorem 5.1 *Let us consider two threshold graphs G_1 and G_2 on n vertices with the binary representation $b_1 = 0^{n-2}1^2$ and $b_2 = 010^{n-3}1$ respectively. Then G_1 and G_2 are always Seidel cospectral.*

Proof Let $f_1(x) = 0$ and $f_2(x) = 0$ be the characteristic equations of quotient matrices of G_1 and G_2 are respectively. Then

$$f_1(x) = x^2 + (4-n)x + (7-3n),$$

and

$$f_2(x) = x^4 + (4-n)x^3 + (6-3n)x^2 + (n-4)x + (3n-7).$$

It is easy to verify that

$$f_2(x) = (x^2-1)f_1(x).$$

Thus for both the strings the Seidel spectrum is same which is $\{-1^{n-3}, 1, \alpha, \beta\}$, where α, β are the zeros of $f_1(x)$.

Hence two threshold graphs G_1 and G_2 are Seidel cospectral. □

Example 5.1 If we take $n = 4$ in Theorem 5.1, we get threshold graphs G_1 and G_2 with binary string $b_1 = 0011$ and $b_2 = 0101$ respectively. In that case, G_1 and G_2 are not isomorphic (see Figure 1) but they both have Seidel eigenvalues $\pm 1, \pm\sqrt{5}$.

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