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ORIGINAL RESEARCH

Pattern polynomial graphs

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Dedicated to the memory of Professor Arbind Kumar Lal

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Abstract A graph X is said to be a *pattern polynomial graph* if its adjacency algebra is a coherent algebra. In this study we will find a necessary and sufficient condition for a graph to be a pattern polynomial graph. Some of the properties of the graphs which are polynomials in the pattern polynomial graph have been studied. We also identify known graph classes which are pattern polynomial graphs.

Keywords Adjacency Algebra of a Graph · Coherent Algebra · Automorphism of a Graph · Distance Regular Graph

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1 Introduction and preliminaries

A *directed graph* (in short, digraph), denoted $X = (V, E)$, consists of the vertex set V and the edge set $E \subset V \times V$. If $e = (u, v) \in E$, then the edge e is said to be incident with vertices u and v. An edge $e = (u, u)$ is called a *loop*. A digraph is called a *graph* if for any two elements $u, v \in V$, $(u, v) \in E$ whenever $(v, u) \in E$. We also say that the vertices u and v are adjacent. Let v be a vertex of a graph X. Then the degree of v, denoted $d(v)$, is the number of vertices adjacent to v. A graph X is said to be *k-regular* if $d(v) = k$, for each vertex v of X.

A *walk* in a graph or digraph is a finite sequence of vertices and edges, namely $v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$, in which the edge $e_i = (v_{i-1}, v_i)$, for $1 \le i \le k$. This walk is called a *path* if all the v_i's are distinct. Sometimes, we also denote the above path by $v_0 v_1 v_2 \cdots v_k$. A *directed cycle* in a digraph X is a walk $v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$ for which $v_0 = v_k$ and an added condition that the sub-walk $v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}$ is a path in X. In case X is a graph, we use the term cycle. A graph $X = (V, E)$ is said to be *connected* if for any two vertices $u, v \in V$, there exists a path from u to v.

The *distance* between two vertices u and v in X, denoted $d_X(u, v)$ (or simply $d(x, y)$, when the graph is clear from the context) is the length of a shortest path joining u to v if such a path exists, otherwise $d_X(u, v) = \infty$. A shortest u-v path is called a *geodesic* and the *diameter* of a connected graph X is the length of any longest geodesic and will be denoted by $d(X)$. Let $X = (V, E)$ be a graph. Then the *complement* of X, denoted X^c, is the graph that has V as its vertex set and two distinct vertices in X^c are adjacent if and only if they are not adjacent in X. It can be easily verified that for every graph X, either X, or X^c, is connected. For any finite set S, let $|S|$ denote the number of elements in S. Then a graph/digraph is said to be *finite*, if $|V|$ and $|E|$ are finite. A graph is called *simple* if it has no loops and no multiple edges.

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An *adjacency matrix* of a digraph $X = (V, E)$ with $|V| = n$, is an $n \times n$ matrix, denoted $A(X) = [a_{ij}]$ (or A), with $a_{ij} = 1$, if there is a directed edge from i-th vertex to the j-th vertex and 0, otherwise. If X is a graph, then $A(X)$ is a symmetric matrix. Note that the adjacency matrix is dependent on the labeling of the vertices of X, but any two adjacency matrices are similar under permutation transformation and hence, we talk of the adjacency matrix of a graph X.

Unless specified otherwise, X will denote a graph that is assumed to be finite and simple. The zero vector will be denoted by $\mathbf{0}$, the column vector of all 1's by $\mathbf{e}$, the identity matrix by I and the matrix of all 1's by $\mathbf{J} = \mathbf{e}\mathbf{e}^*$, where for any vector/matrix $\mathbf{x}$, $\mathbf{x}^*$ denotes the conjugate-transpose of the vector $\mathbf{x}$. The matrices with entries either 0 or 1 will be referred to as 0, 1-matrices. The readers are advised to see the book by Godsil and Royle [12] (Hoffman & Kunze [18]) for any definitions and results in graph theory (linear algebra) that have been used but not stated separately.

Let $\mathbb{M}_n(\mathbb{C})$ denote the set of all $n \times n$ matrices over the field of complex numbers $\mathbb{C}$. For a fixed matrix $A \in \mathbb{M}_n(\mathbb{C})$, let $\mathbb{C}[A]$ denote the set of all matrices that are polynomials in A with coefficients from $\mathbb{C}$. Then $\mathbb{C}[A]$ is an algebra over $\mathbb{C}$. Then Cayley-Hamilton Theorem gives the existence of a polynomial $f(x) \in \mathbb{C}[x]$ such that $f(A) = \mathbf{0}$. The monic polynomial of least degree satisfying $f(A) = \mathbf{0}$, is called the *minimal polynomial* of A. It is well known that the *dimension* of $\mathbb{C}[A]$, as a vector space over $\mathbb{C}$, is the degree of the minimal polynomial of A. If A is diagonalizable, then using the lemma given below, its dimension is equal to the number of distinct eigenvalues of A.

Lemma 1.1 (Hoffman & Kunze [18]) *A matrix is diagonalizable if and only if its minimal polynomial has all distinct linear factors over* $\mathbb{C}$.

Let A be the adjacency matrix of a graph X on n vertices. Then A is an $n \times n$ real symmetric matrix. Hence, A has n real (not necessarily distinct) eigenvalues, A is diagonalizable, and the eigenvectors can be chosen to form an orthonormal basis of $\mathbb{R}^n$. The eigenvalues, eigenvectors, the minimal polynomial and the characteristic polynomial of a graph X are defined to be that of its adjacency matrix. Note that in this case, $\mathbb{C}[A]$, the subalgebra of $\mathbb{M}_n(\mathbb{C})$ generated by A, is also denoted by $\mathcal{A}(X)$ and is called the *adjacency algebra* of X. Since A is the adjacency matrix of the graph X, the number $(A^\ell)_{i,j}$, for $\ell \geq 1$, gives the number of walks of length ℓ from the i-th vertex of X to the j-th vertex of X. Hence, it follows that if X is a connected graph with diameter d, then the set $\{I, A, A^2, \ldots, A^d\}$ is a linearly independent set in $\mathcal{A}(X)$. This proves the following result.

Lemma 1.2 (Biggs [5]) *Let X be a connected graph with n vertices and diameter d. Then* $d+1 \leq \dim(\mathcal{A}(X)) \leq n$.

Now, let X be a connected k-regular graph. By Perron-Frobenius theory, k is an eigenvalue of X and if λ is any other eigenvalue of X, then $-k \leq \lambda < k$. Thus, in this case, the minimal polynomial of X is of the form $(x-k)q(x)$, for some $q(x) \in \mathbb{Z}[x]$. Furthermore, $q(k) \neq 0$. Moreover, for any eigenvalue $\lambda \neq k$ of X, $q(\lambda) = 0$. Then with $q(x)$ as defined, we state the following well known result that provides a necessary and sufficient condition for a graph to be connected and regular.

Lemma 1.3 (Hoffman [17]) *A graph X is connected and regular if and only if $\mathbf{J} \in \mathcal{A}(X)$. Moreover, in this case,* $\mathbf{J} = \frac{n}{q(k)} q(A)$.

For every graph X, either X or X^c is connected. Consequently if X is a regular graph, then $\mathbf{J}$ is polynomial in either A or A^c. If we assume both X and X^c are connected and X is a regular graph, then X^c is also a regular graph. From Lemma 1.3 we have $\mathbf{J} \in \mathcal{A}(X^c)$ and hence $\mathcal{A}(X) = \mathcal{A}(X^c)$. Hence the following result.

Corollary 1.4 *Let X be a connected regular graph. Then X^c is connected if and only if* $\mathcal{A}(X) = \mathcal{A}(X^c)$.

Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then the *Hadamard product* of $A = [a_{ij}]$ and $B = [b_{ij}]$, denoted $A \odot B$, is defined as $(A \odot B)_{ij} = a_{ij} b_{ij}$, for $1 \leq i, j \leq n$. Two matrices $A, B \in \mathbb{M}_n(\mathbb{C})$ are said to be *disjoint* if $A \odot B = \mathbf{0}$.

Theorem 1.5 *[Higman [15], Brouwer, Cohen & Neumaier [8]] Let $\mathcal{M}$ be a vector subspace of the space of symmetric $n \times n$ matrices. Then $\mathcal{M}$ has a basis of mutually disjoint* 0, 1*-matrices if and only if $\mathcal{M}$ is closed under Hadamard multiplication.*

Let $\mathcal{M}$ be a subalgebra of $\mathbb{M}_n(\mathbb{C})$. Then $\mathcal{M}$ is called a *coherent algebra* if it contains the matrices I and $\mathbf{J}$ and is also closed under conjugate-transposition and the Hadamard product. For example, $\mathbb{M}_n(\mathbb{C})$ is the largest coherent algebra. It is easy to verify that the minimal polynomial of the $n \times n$ matrix $\mathbf{J}$ equals $x(x-n)$. Hence using Lemma 1.1, $\dim(\mathbb{C}[\mathbf{J}]) = 2$. Also, the set $\{I, \mathbf{J} - I\}$ is the mutually disjoint 0, 1-matrix basis for $\mathbb{C}[\mathbf{J}]$.

Thus, using Theorem 1.5, $\mathbb{C}[\mathbf{J}]$ is a coherent algebra. As any coherent algebra contains both I and $\mathbf{J}$, it is clear that $\mathbb{C}[\mathbf{J}]$ is the smallest coherent algebra. Let W_n be the adjacency matrix of a directed cycle on n vertices. Then, it can be easily checked that the i, j-th entry of W_n equals 1 whenever $j - i \equiv 1 \pmod n$. Also, recall that a matrix $A \in \mathbb{M}_n(\mathbb{F})$ is said to be a *circulant matrix* if $a_{ij} = a_{1\,j-i+1}$, whenever $2 \leq i \leq n$, $1 \leq j \leq n$ and the subscripts are read modulo n. Then it follows that W_n is a circulant matrix with $[0, 1, \dots, 0]$ as its first row. One also has the following result.

Lemma 1.6 (Davis [11]) *Let $A \in \mathbb{M}_n(\mathbb{F})$. Then A is circulant if and only if it is a polynomial in W_n.*

That is, the set of circulant matrices in $\mathbb{M}_n(\mathbb{F})$ forms a commutative algebra and its 0, 1 matrix basis is $\{I, W_n, W_n^2, \dots, W_n^{(n-1)}\}$. Moreover, it can be checked that $\mathbb{C}[W_n]$ is a coherent algebra. This example shows that a coherent algebra may be commutative but not symmetric.

Let $\mathcal{M}$ be a coherent algebra. Then a 0, 1-matrix $D \in \mathcal{M} \setminus \{\mathbf{0}\}$ is called *primitive* if for every $M \in \mathcal{M}$ there is a number $\alpha_D(M)$, depending only on M and D, such that $M \odot D = \alpha_D(M)D$. Clearly, distinct primitive 0, 1-matrices are disjoint and linearly independent. The set of primitive 0, 1-matrices will be denoted by $\mathbb{D}$. Observe that $D \odot D = D$, for all $D \in \mathbb{D}$.

The following theorem on coherent algebras is well known. We give a proof for the sake of completeness. The proof is similar to that of the proof of Theorem 1.5 given in [8].

Theorem 1.7 *[Higman [15], Brouwer, Cohen & Neumaier [8]] Every coherent algebra contains a unique basis of mutually disjoint* 0, 1*-matrices.*

Proof Let $\mathcal{M}$ be a coherent algebra. Then we claim that $\mathbb{D}$ is a basis of $\mathcal{M}$.

Let $\mathbb{D}^{\perp} = \{B \in \mathcal{M} | B \odot D = \mathbf{0}$ for all $D \in \mathbb{D}\}$. Note that $M - \sum\limits_{D \in \mathbb{D}} \alpha_D(M)D$ is an element of $\mathbb{D}^{\perp}$, for all $M \in \mathcal{M}$. Hence, we just need to show that $\mathbb{D}^{\perp} = \{\mathbf{0}\}$ to conclude $\mathbb{D}$ spans $\mathcal{M}$.

Let B' be an element of $\mathbb{D}^{\perp} \setminus \{\mathbf{0}\}$ with a minimum number of non-zero entries. Let β be one of the non-zero entries in B'. Then $B = (1/\beta)B' \in \mathbb{D}^{\perp}$ and has at least one entry equal to 1. Since $B \in \mathbb{D}^{\perp}$, $B \odot B - B \in \mathbb{D}^{\perp}$. But $B \odot B - B$ has fewer non-zero entries than B. Consequently, $B \odot B - B = \mathbf{0}$. So B must be a 0, 1-matrix.

Consider an arbitrary matrix $M \in \mathcal{M}$. Since $B \in \mathbb{D}^{\perp}$, the matrices $M \odot B$, tB and $N = M \odot B - tB$ are elements of $\mathbb{D}^{\perp}$, for any $t \in \mathbb{C}$. By choosing a suitable t, the matrix N will have fewer non-zero entries than B. Therefore, there is a choice of t, depending only on M and B, such that $N = \mathbf{0}$. Since $M \in \mathcal{M}$ was arbitrary, by definition B is a primitive element of $\mathcal{M}$. Hence $B \in \mathbb{D}$, a contradiction. This completes the proof of the claim.

Now, let us look at the proof of uniqueness. Let us assume that $\{C_1, C_2, ..., C_l\}$ and $\{D_1, D_2, ..., D_k\}$ are two bases of *disjoint* 0, 1-matrices, with the property that $C_i \odot C_j = \mathbf{0} = D_i \odot D_j$, if $i \neq j$, $C_i \odot C_i = C_i$, for all i, $1 \leq i \leq l$ and $D_j \odot D_j = D_j$, for all j, $1 \leq j \leq k$. We need to show that $l = k$ and if $D_i \odot C_j \neq \mathbf{0}$, for some $i, j \in \{1, 2, \dots, k\}$, then $D_i = C_j$.

Let $D_i \odot C_j \neq \mathbf{0}$, for some choice of i and j. As $D_i \odot C_j \in \mathcal{M}$, there exist constants α_r, such that $D_i \odot C_j = \sum\limits_r \alpha_r D_r$. But $D_r \odot D_i = \mathbf{0}$, for $r \neq i$ and $D_i \odot D_i = D_i$ implies that $D_i \odot C_j = \alpha_i D_i$. Since, D_i and C_j are 0, 1-matrices, $\alpha_i = 1$. Similarly, $D_i \odot C_j = C_j$. Hence, the result follows. □

We call this unique basis of mutually disjoint 0, 1-matrices the *standard basis*.

Corollary 1.8 *Every* 0, 1*-matrix in a coherent algebra is the sum of one or more matrices in its standard basis.*

Proof Let $\{M_1, \dots, M_t\}$ be a standard basis of the coherent algebra $\mathcal{M}$ over $\mathbb{C}$ and let $B \in \mathcal{M}$ be a 0, 1-matrix. Then $B = \sum\limits_{i=1}^{t} a_i M_i$, for some $a_i \in \mathbb{C}$, $1 \leq i \leq t$. Thus, $B = B \odot B = \sum\limits_{i=1}^{t} a_i^2 M_i$. Or equivalently, $a_i^2 = a_i$, for $1 \leq i \leq t$, and hence the required result follows. □

Corollary 1.9 *Let $\mathcal{M} \subseteq \mathbb{M}_n(\mathbb{C})$ be a commutative coherent algebra over $\mathbb{C}$. Then $\dim(\mathcal{M}) \leq n$. Moreover, every permutation matrix (including the identity matrix) which belongs to $\mathcal{M}$ is a member of the standard basis of $\mathcal{M}$.*

Proof If $\mathbf{J} \in \mathcal{M}$ and $\mathcal{M}$ is commutative, then it follows that for each matrix in $\mathcal{M}$, its row/column sums are equal. Hence, each matrix in the standard basis of $\mathcal{M}$ has at least one non-zero entry in each row and each column. Also, this non-zero entry has to be a 1, as $D \odot D = D$, for each basis element D of $\mathcal{M}$. Also; a basis of $\mathcal{M}$ contains disjoint matrices, the number of elements in the basis can be at most n. As every permutation matrix (including the identity matrix) contains exactly one 1 in each row and each column, the result follows. □

Let $A \in \mathbb{M}_n(\mathbb{C})$, then *coherent closure* of A, denoted by $\langle\langle A \rangle\rangle$ or $\mathcal{CC}(A)$, is the smallest coherent algebra containing A (see Page 59 [19]). If A is the adjacency matrix of a graph X and $\mathbb{C}[A] = \mathcal{CC}(A)$, then X will be called a *pattern polynomial* graph.

In Section 2 we will see a necessary and sufficient condition for a matrix $A \in \mathbb{M}_n(\mathbb{C})$ to have $\mathbb{C}[A] = \mathcal{CC}(A)$. In the remaining sections we consider A to be the adjacency matrix of a graph X. Relationships between pattern polynomial graphs and other graph classes are provided in Section 3. Properties of graphs which are polynomials in the pattern polynomial graph are given in Section 4.

2 $\mathbb{C}[A] = \mathcal{CC}(A)$

In this section we will see a necessary and sufficient condition for a matrix $A \in \mathbb{M}_n(\mathbb{C})$ to have $\mathbb{C}[A] = \mathcal{CC}(A)$. For that, we construct a vector space which lies in between $\mathbb{C}[A]$ and $\mathcal{CC}(A)$ as follows.

Let ℓ be the degree of the minimal polynomial of A. Then $\{I, A, \dots, A^{\ell-1}\}$ is a basis of $\mathbb{C}[A]$. Let $\mathbf{y} = (y_0, y_1, \dots, y_{\ell-1}) \in \mathbb{C}^\ell$ be a variable vector and let

$$B(\mathbf{y}) = y_0 I + y_1 A + \cdots + y_{l-1} A^{\ell-1} = \begin{bmatrix} p_{11}(\mathbf{y}) & p_{12}(\mathbf{y}) & \dots & p_{1n}(\mathbf{y}) \\ p_{21}(\mathbf{y}) & p_{22}(\mathbf{y}) & \dots & p_{2n}(\mathbf{y}) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}(\mathbf{y}) & p_{n2}(\mathbf{y}) & \dots & p_{nn}(\mathbf{y}) \end{bmatrix},$$

where $p_{ij}(\mathbf{y}) = \sum_k y_k (A^k)_{ij}$. Let us assume that $S = \{q_1(\mathbf{y}), q_2(\mathbf{y}), \dots, q_r(\mathbf{y})\}$ is the set of distinct polynomials in the matrix $B(\mathbf{y})$. We now use the set S to define r matrices, $P_1, P_2, \dots, P_r$, called the *pattern matrices* of A, by

$$(P_j)_{s,t} = \begin{cases} 1, & \text{if } B(\mathbf{y})_{s,t} = p_{st}(\mathbf{y}) = q_j(\mathbf{y}), \\ 0, & \text{otherwise.} \end{cases}$$

Then, we define $\mathcal{L}(A)$ as the linear subspace $L(P_1, P_2, \dots, P_r)$ (the linear span of $P_1, P_2, \dots, P_r$) of $\mathbb{M}_n(\mathbb{C})$. Now, one observes the following:

Lemma 2.1 *Let the pattern matrices $P_1, P_2, \dots, P_r$ be as defined above. Then*

1. *$P_i \odot P_j = \mathbf{0}$, for $1 \le i \ne j \le r$ and $P_i \odot P_i = P_i$, for $1 \le i \le r$. Also, by definition, $I \in \mathcal{L}(A)$ and since $\sum_{i=1}^{r} P_i = \mathbf{J}$, $\mathbf{J} \in \mathcal{L}(A)$.*
2. *Let $M, N \in \mathcal{L}(A)$. Then $M = \sum_{i=1}^{r} a_i P_i$ and $N = \sum_{i=1}^{r} b_i P_i$, for some $a_i, b_i \in \mathbb{C}$, $1 \le i \le r$. Therefore, by definition, $M \odot N = \sum_{i=1}^{r} a_i b_i P_i \in \mathcal{L}(A)$). Thus, $\mathcal{L}(A)$ is closed under the Hadamard product.*
3. *$\mathcal{L}(A)$ is the smallest subspace of $\mathbb{M}_n(\mathbb{C})$ that is closed under the Hadamard product and contains all powers of A. Consequently, $\mathbb{C}[A] \subseteq \mathcal{L}(A) \subseteq \mathcal{CC}(A)$ and $l \le r$.*
4. *Suppose the transpose of P^i, $P_i^T \in \{P_1, P_2, \dots, P_r\}$ for all i, $1 \le i \le r$. Then $\mathcal{L}(A)$ is also closed under conjugate transposition. In particular, if A is symmetric, then all pattern matrices are symmetric and $\mathcal{L}(A)$ is closed under conjugate transposition.*

With the above result, we are ready to state the main result of this subsection.

Theorem 2.2 *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a symmetric matrix. Then $\mathbb{C}[A] = \mathcal{CC}(A)$ if and only if $\ell = r$, where ℓ is the degree of the minimal polynomial of A and r is the number of pattern matrices of A.*

Proof If $\mathbb{C}[A] = \mathcal{CC}(A)$, then $\mathbb{C}[A] = \mathcal{L}(A)$ and hence $\ell = r$. Conversely, suppose $\ell = r$ then $\mathbb{C}[A] = \mathcal{L}(A)$. In particular, $\mathcal{L}(A)$ is closed under ordinary multiplication. Hence, using Lemma 2.1, $\mathcal{L}(A)$ is a coherent algebra. But $\mathcal{CC}(A)$ is the smallest coherent algebra containing $\mathbb{C}[A]$ and hence the required result follows. □

3 Relationships among graph classes

In this section, we provide relationships among different classes of graphs that are based on the study of adjacency algebras and coherent algebras. Throughout this paper, X is a graph on n vertices and A is its adjacency matrix and the two objects X and A will be used interchangeably. Before proceeding we need the following definitions and results.

The collection of all automorphisms of a graph X, denoted $\mathrm{Aut}(X)$, forms a group under composition. A graph $X = (V, E)$ is said to be *vertex transitive* if $\mathrm{Aut}(X)$ acts transitively on V. That is, for any two vertices $x, y \in V$, $x \neq y$ there exists $g \in \mathrm{Aut}(X)$ such that $g(x) = y$. For example, $\mathrm{Aut}(K_n) \cong \mathcal{S}_n$ and $\mathrm{Aut}(C_n) \cong \mathcal{D}_n$, where $\mathcal{S}_n$ and $\mathcal{D}_n$ denote the symmetric group on n symbols and Dihedral group of order $2n$ respectively. Hence the complete graph K_n and the cycle graph C_n are vertex transitive. Also note that $\mathrm{Aut}(X) = \mathrm{Aut}(X^c)$, and if X is a graph on n vertices then $\mathrm{Aut}(X)$ is a subgroup of $\mathcal{S}_n$. Under this correspondence, the maps in $\mathrm{Aut}(X)$ consist of $n \times n$ permutation matrices and for each $g \in \mathrm{Aut}(X)$ the corresponding permutation matrix will be denoted by P_g. With the definitions as above, we state two results, one of which gives a method to check whether a given permutation matrix is an element of $\mathrm{Aut}(X)$ or not and the other gives information about a few eigenvalues of X.

Lemma 3.1 (Biggs [5]) *Let A be the adjacency matrix of a graph X. Then $g \in \mathrm{Aut}(X)$ if and only if $P_g A = A P_g$.*

Lemma 3.2 (Peterdorf & Sachs [23]) *Let $X = (V, E)$ be a k-regular vertex transitive graph and let λ be a simple eigenvalue of X. Then λ equals k, if $|V|$ is odd and equals exactly one integer from the set $\{-k, -k+2, \ldots, k-2, k\}$, if $|V|$ is even.*

Definition 3.1 Let $v_1, v_2, \ldots, v_n$ be the vertices of a connected graph X. If d is the diameter of X then, for $0 \leq k \leq d$, the *k-th distance matrix* of X, denoted $A_k(X)$, is defined as

$$(A_k(X))_{rs} = \begin{cases} 1, & \text{if } d(v_r, v_s) = k, \\ 0, & \text{otherwise,} \end{cases}$$

where $d(u, v)$ is the distance between the vertices $u, v \in V$.

From Definition 3.1 it is clear that

- $A_0(X)$ is the identity matrix and $A_1(X)$ is the adjacency matrix of X.
- $A_0(X) + A_1(X) + \cdots + A_d(X) = \mathbf{J}$.
- the matrices $A_k(X)$, for $0 \leq k \leq d$, are symmetric.
- $A_i(X) \odot A_j(X) = 0$, for all $i \neq j$.
- the set $\{A_0(X), A_1(X), \ldots, A_d(X)\}$ is a linearly independent set in $\mathbb{M}_n(\mathbb{C})$.

For example, for C_n, the cycle graph, $\{A_0(C_n), A_1(C_n), \ldots, A_d(C_n)\}$ is a basis of $\mathcal{A}(C_n)$ and $B \in \mathbb{M}_n(\mathbb{C})$ is symmetric circulant if and only if $B \in \mathcal{A}(C_n)$ (see [21]).

Definition 3.2 Let G be a group acting on a non-empty set V. Then G also acts on $V \times V$, by $g(x, y) = (g(x), g(y))$. For each fixed element $(u, v) \in V \times V$, the set $\mathrm{Orb}(u, v) = \{g(u, v) : g \in G\}$ is called the *orbit of* (u, v), under the action of G. The distinct orbits of $V \times V$ under the action of G are called *orbitals*.

In the context of a graph $X = (V, E)$, the orbitals of X are the distinct orbits of $E \subset V \times V$ under the action of $\mathrm{Aut}(X)$. That is, the *orbitals* are the orbits of the arcs/non-arcs of the graph X. The number of orbitals is called the *rank* of X. Note that, for each fixed $(u, v) \in V \times V$, we can associate a 0, 1-matrix, say $M = [m_{ij}]$, where m_{ij} equals 1, if $(i, j) \in \mathrm{Orb}(u, v)$ and 0, otherwise. The matrices obtained by the above method are called *orbital matrices*. Also, note that in any orbital matrix, either all or none of its nonzero entries appear on the main diagonal as $g(v, v) = (g(v), g(v))$, for all $v \in V$ and $g \in \mathrm{Aut}(X)$. The orbitals containing 1's on the diagonal will be called *diagonal orbitals*.

Let G be a group acting on a set V with $|V| = n$. Then G can be represented as a subgroup of the group of $n \times n$ permutation matrices. Hence, in this paper, we may assume elements of the group G are actual permutation matrices. With this understanding of the notation, the set

$$\mathcal{V}_{\mathbb{C}}(G) = \{A \in \mathbb{M}_n(\mathbb{C}) : PA = AP, \quad \forall P \in G\}$$

forms an algebra over $\mathbb{C}$, called the *centralizer algebra* of G. It is known [for example see Page 52 of Klin, Rücker, Rücker & Tinhofer [19] or Higman [15,16]] that the set of orbitals of V, under the action of G, forms a basis for $\mathcal{V}_{\mathbb{C}}(G)$. For a graph X, its centralizer algebra, denoted $\mathcal{V}(X)$, is the centralizer algebra of $\mathrm{Aut}(X)$ acting on V, the vertex set of X. Using Lemma 3.1, it follows that for any graph X, $\mathcal{A}(X) \subseteq \mathcal{V}(X)$.

Theorem 3.3 *[Klin, Rücker, Rücker & Tinhofer [19]] Let $X = (V, E)$ be a graph and let $\mathcal{V}(X)$ be as defined above. Then $\mathcal{V}(X)$ is a coherent algebra and the orbitals of V, under the action of Aut(X), form a basis of mutually disjoint* $0, 1$*-matrices. Moreover, this basis is unique. That is, if there is a basis of $\mathcal{V}(X)$ consisting of disjoint* $0, 1$*-matrices then the basis elements must be the orbital matrices.*

It can be easily verified that $\text{Aut}(X) = \text{Aut}(X^c)$ and hence $\mathcal{V}(X) = \mathcal{V}(X^c)$. As $\mathbb{C}[A] = \mathcal{A}(X)$, using Lemma 2.1, it follows that for a graph X,

$$\mathcal{A}(X) \subseteq \mathcal{L}(X) \subseteq \mathcal{CC}(X) \subseteq \mathcal{V}(X). \tag{1}$$

Now let us define the graph classes that will be studied in this paper. Some of the definitions have already appeared before but we mention them again for the sake of clarity. We also provided a figure (Figure 1) showing their relationships. Also, recall that a square matrix A with non-negative entries is called a *doubly stochastic matrix* if its rows and columns sums are 1.

Definition 3.3 Let $X = (V, E)$ be a graph on n vertices and let A be its adjacency matrix.

1. Coherent Graph: A graph X is said to be a *coherent graph* if its adjacency matrix is a member of the standard basis of $\mathcal{CC}(X)$.
2. Compact Graph: A graph X is said to be a *compact graph* if every doubly stochastic matrix that commutes with A is a convex combination of matrices from Aut(X).
3. Distance Polynomial Graph: Let X be a connected graph with diameter d and let $A_k(X)$, for $0 \leq k \leq d$, be the k-th distance matrix of X. Then X is said to be a *distance polynomial graph* if $A_k(X) \in \mathcal{A}(X)$, for $0 \leq k \leq d$.
4. Distance Regular Graph: A connected graph X is said to be a *distance regular graph* if for any two vertices u, v of X, the number of vertices at distance i from u and distance j from v depends only on i, j and $d(u, v)$, the distance between u and v.
5. Distance Transitive Graph: A graph X is said to be a *distance transitive graph* if for any four vertices u, v, x and y of X with $d(u, v) = d(x, y)$, there exists an element $g \in \text{Aut}(X)$, such that $g(u) = x$ and $g(v) = y$.
6. Edge Regular Graph: A graph X is said to be an *edge-regular graph* if every pair of adjacent vertices of X have the same number of common neighbors.
7. Orbit Polynomial Graph: A graph X is said to be an *orbit polynomial graph* if each orbital matrix is a member of $\mathcal{A}(X)$.
8. Pattern Polynomial Graph: A graph X is said to be a *pattern polynomial graph* if $\mathcal{A}(X) = \mathcal{CC}(X)$.
9. Walk Regular Graph: A graph X is said to be a *walk-regular graph* if for each s, the number of closed walks of length s, starting at a vertex v, is independent of the choice of v.

Plenty of literature on distance regular graphs and distance transitive graphs is available. The notion of distance polynomial graph was introduced and studied by Weichsel [28]. He also made a conjecture that relates the automorphism groups and eigenvalues of a graph X with those of another graph generated from X by a polynomial. Beezer [4] defined orbit polynomial graphs and proved that Weichsel's is true, whenever X is a vertex-transitive graph with a prime number of vertices. He also provided examples in which the conjecture is false. For results related to this, the reader is advised to see the papers of Beezer [1,2,4].

Coherent graphs were introduced by Klin, Muzychuk & Ziv-Av [20]. Walk regular graphs first appeared in the paper by Godsil & McKay [13]. It can be easily observed that every walk-regular graph is an edge-regular graph as *every pair of adjacent vertices have the same number of common neighbors* corresponds to *the number of closed walks of length* 3 (a number that is independent of the vertex v in case of a walk-regular graph). The basic theory of compact graphs has been developed by Tinhofer [27]. The complete graph K_n and the cycle graph C_n, for $n \geq 3$ are both distance transitive as well as compact connected regular graphs. In this section, we mainly focus on the class of pattern polynomial graphs and their relationships with other graph classes.

We now provide relationships among some of the graph classes which we study in this paper with the help of Figure 1.

Note that in Figure 1, the sets **a, b, c, …, h** represent connected regular graphs, distance polynomial graphs, pattern polynomial graphs, …, coherent graphs, respectively. Also, for $S, T \in \{\mathbf{a,b,\ldots,h}\}$, the notation $E_{S,T}$ represents the fact that there exists an example of a graph that belongs to the set S but not the set T. In Figure 1, the bottom line/boundary for **d,e,f** is same. The top line/boundary of **f,g** is same. In the remaining part of this section, we analyze Figure 1 with respect to the relationships between different classes of graphs. Some of these relationships are already well-known but we give a proof, for the sake of completeness. We end this section with counter-examples whenever certain relationships do not hold.

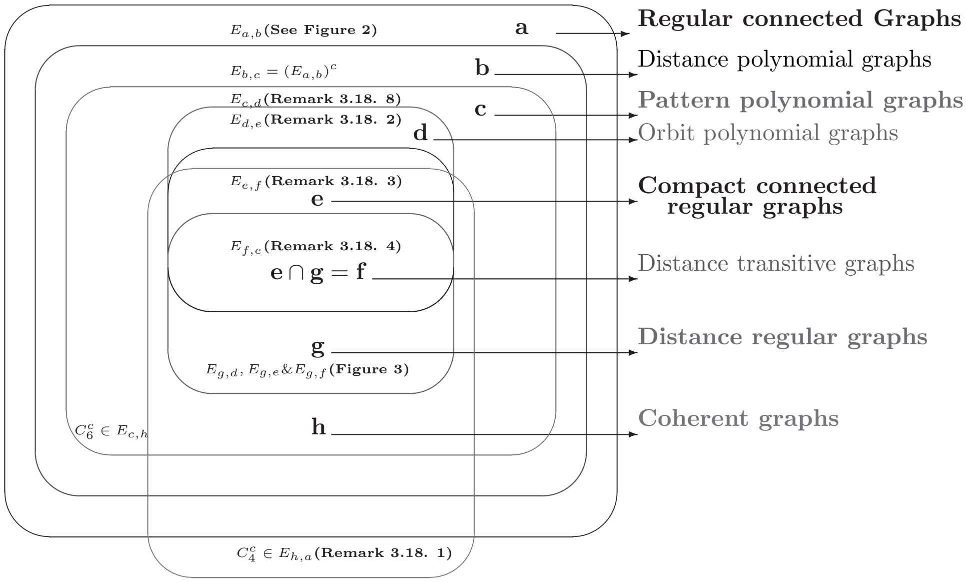

Fig. 1 Graph classes

Lemma 3.4 (Weichsel [28]) *Let X be a distance polynomial graph. Then X is a connected regular graph.*

Proof A distance polynomial graph is connected by definition. If X has diameter d, then since $A_k(X) \in \mathcal{A}(X)$, for $0 \leq k \leq d$, $\mathbf{J} = \sum_{k=0}^{d} A_k(X) \in \mathcal{A}(X)$ and the result follows from Lemma 1.3. □

To proceed further, we provide a few equivalent statements for a graph to be a pattern polynomial graph. By definition, a graph X is a pattern polynomial if $\mathcal{A}(X) = \mathcal{CC}(X)$. Consequently, $\mathcal{A}(X)$ is a coherent algebra and $\mathcal{A}(X) = \mathcal{CC}(X)$ follows from the fact that $\mathcal{A}(X) \subseteq \mathcal{CC}(X)$. Further, using Lemma 2.1, we have $\mathcal{A}(X) = \mathcal{L}(X) = \mathcal{CC}(X)$. Now, if we assume that $\mathcal{A}(X) = \mathcal{L}(X)$ then the use of Lemma 2.1 implies that $\mathbf{J} \in \mathcal{A}(X)$ and $\mathcal{A}(X)$ is closed under Hadamard product. Finally, if $\mathbf{J} \in \mathcal{A}(X)$ and $\mathcal{A}(X)$ is closed under Hadamard product, then $\mathcal{A}(X)$ is a coherent algebra as $A(X)$, the adjacency matrix of X, is a symmetric matrix and $I \in \mathcal{A}(X)$. Hence $\mathcal{A}(X) = \mathcal{CC}(X)$ and thus X is a pattern polynomial graph. Thus we have the following result.

Theorem 3.5 *The following statements are equivalent.*

- *X is a pattern polynomial graph ($\mathcal{A}(X) = \mathcal{CC}(X)$).*
- *$\mathcal{A}(X)$ is a coherent algebra.*
- *$\mathcal{A}(X) = \mathcal{L}(X)$.*
- *$\mathbf{J} \in \mathcal{A}(X)$ and $\mathcal{A}(X)$ is closed under Hadamard multiplication.*

We also need the following result of Godsil & McKay [13] that characterizes walk-regular graphs in terms of their diagonal entries.

Theorem 3.6 *(Godsil & McKay [13]) Let A be the adjacency matrix of a graph X. Then X is walk-regular if and only if the diagonal entries of A^s, for all positive integer s, are all equal.*

Thus, using Theorem 3.6 and the definition of pattern matrices, the following result is evident and hence we omit the proof.

Corollary 3.7 *A graph X is walk-regular if and only if I is a pattern matrix of $A(X)$.*

We now have the following result about pattern polynomial graphs.

Lemma 3.8 *Let X be a pattern polynomial graph. Then*

1. *X is a connected regular graph.*
2. *X is a distance polynomial graph.*
3. *X is a walk-regular graph.*

4. *X is an edge-regular graph.*

Moreover, X^c is connected if and only if X^c is a pattern polynomial graph.

Proof **Proof of Part** 1: Let X be a pattern polynomial graph with $\{P_1, P_2, \ldots, P_r\}$ as a basis of $\mathcal{L}(X)$. Then $\mathcal{A}(X) = \mathcal{CC}(X)$ and $\mathbf{J} \in \mathcal{A}(X)$. Hence from Lemma 1.3, X is a connected regular graph.

Proof of Part 2**:** Let d be the diameter of X. We need to show that the k-th distance matrix, $A_k(X)$, for $0 \le k \le d$, are elements of $\mathcal{A}(X)$. Using Theorem 3.5, it is enough to show that $A_k(X) \in \mathcal{L}(X)$, for $0 \le k \le d$. We shall do so recursively.

First note that $I = A_0(X)$, $A = A_1(X) \in \mathcal{A}(X) \subseteq \mathcal{L}(X)$. Also, from Lemma 2.1 we have $\mathbf{J} \in \mathcal{L}(X)$. Thus, $A^2 \odot (\mathbf{J} - I - A) = A^2 \odot (\mathbf{J} - A_0(X) - A_1(X)) \in \mathcal{L}(X)$ as $\mathcal{L}(X)$ is closed under the Hadamard product and $A^2 \in \mathcal{A}(X) \subseteq \mathcal{L}(X)$. Therefore, there exist integers $a_1, a_2, \ldots, a_r$ such that $A^2 \odot (\mathbf{J} - A_0(X) - A_1(X)) = \sum_{i=1}^{r} a_i P_i$. Hence, it can be easily verified that $A_2(X) = \sum_{i=1, a_i \neq 0}^{r} P_i$. Thus, $A_2(X) \in \mathcal{L}(X)$.

Now, using recursion, assume that $\mathbf{J}$, $A_0(X), A_1(X), \ldots, A_{s-1}(X) \in \mathcal{L}(X)$ and define, $M = A^s \odot (\mathbf{J} - I - A_1(X) - \cdots - A_{s-1}(X))$. Then observe that $M_{ij} \neq 0$ if and only if $(A_s(X))_{ij} = 1$. We need to show that $A_s(X) \in \mathcal{L}(X)$.

Note that $M \in \mathcal{L}(X)$ as $\mathcal{L}(X)$ is closed under the Hadamard product and $A^s \in \mathcal{A}(X) \subseteq \mathcal{L}(X)$. Moreover, if $M = \sum_{i=1}^{r} a_i P_i$, for some integers a_i, $1 \le i \le r$, then the condition $M_{ij} \neq 0$ if and only if $(A_s(X))_{ij} = 1$ implies that, $A_s(X) = \sum_{i=1, a_i \neq 0}^{r} P_i$. Hence, $A_s(X) \in \mathcal{L}(X)$ and this completes the proof of the second part.

Proof of Parts 3 **and** 4: Since $I = A_0(X) \in \mathcal{L}(X)$, I is a pattern matrix of $\mathcal{A}(X)$ and hence using Corollary 3.7, X is a walk-regular graph. Therefore, X is an edge-regular graph as well. From Corollary 1.4 we have $\mathcal{A}(X) = \mathcal{A}(X^c)$. Therefore,

$$\mathcal{A}(X^c) \subseteq \mathcal{L}(X^c) \subseteq \mathcal{CC}(X^c) = \mathcal{CC}(X) = \mathcal{A}(X) = \mathcal{A}(X^c)$$

and hence X^c is a pattern polynomial graph. Conversely, if X^c is a pattern polynomial graph then $\mathbf{J} \in \mathcal{CC}(X^c) = \mathcal{A}(X^c)$ and therefore Lemma 1.3 implies that X^c is connected. □

To state the next result, we need the following definition and a result from Weichsel [28].

Definition 3.4 (Weichsel [28]) Let d be the diameter of a graph X and let v be a vertex of X. Then the generalized degree of v is the d-tuple $(k_1, k_2, \ldots, k_d)$, where k_i is the number of vertices that are at a distance i from v. The graph X is said to be a *super-regular graph* if each vertex has the same generalized degree.

Following result shows that super-regular condition is weaker than distance-regular. If X is a connected distance-regular with diameter d, then the distance matrices $A_i(X)$ for $1 \le i \le d$ are polynomials in A.

Theorem 3.9 *(Weichsel [28]) Let X be a connected graph of diameter d. Then X is a super-regular graph if and only if $(A_i(X)A_j(X))_{rs} = (A_j(X)A_i(X))_{rs}$ for every pair of adjacent vertices r, s in X.*

Note that for a pattern polynomial graph X, Lemma 3.8 establishes that its distance matrices, $A_k(X)$, are polynomials in $A(X)$, the adjacency matrix of the graph X. Thus, $A_j(X)A_k(X) = A_k(X)A_j(X)$, for any two distinct distance matrices $A_j(X)$ and $A_k(X)$. Hence, using Theorem 3.9, we see that every pattern polynomial graph is a super regular graph. In the literature, super-regular graphs are also called *distance-degree regular* or *strongly distance-balanced graphs* (for details, see Cabello & Lukšič [9]). Thus, pattern polynomial graphs are also strongly distance-balanced graphs.

We will need the definition of a strongly regular graph and a few results related to strongly regular graphs to give an example of a pattern polynomial graph that is not a distance regular graph.

Definition 3.5 A k-regular graph X is said to be strongly regular, with parameters a and c, if for any two distinct vertices u, v of V, the number of common neighbors

1. is a, whenever u and v are adjacent, and
2. is c, whenever u and v are not adjacent.

It can be verified that X is a strongly regular graph if and only if X is a distance regular graph with diameter 2. Also, by definition, every strongly regular graph is an edge-regular graph. An important characterization of strongly regular graphs with respect to their eigenvalues, stated below for ready reference, was given by Shrikhande and Bhagwandas [25].

Theorem 3.10 *(Shrikhande and Bhagwandas [25]) Let X be a regular connected graph that is not a complete graph. Then X is strongly regular if and only if it has exactly three distinct eigenvalues.*

To prove our next result, we need to recall that if X is any graph, then either X or X^c is connected. We also need to recall Theorem 1.7 and Corollary 1.8.

Lemma 3.11 *Let X be a connected coherent graph. Then*

1. *X is a regular graph.*
2. *X is a walk-regular graph.*
3. *X is an edge-regular graph.*

Moreover, if X is not a complete graph then X and X^c are both coherent graphs if and only if X is strongly regular.

Proof **Proof of Parts** 1, 2 **and** 3: Let A be the adjacency matrix of X. On the contrary assume that X is not a regular graph. Then $A^2 \neq kI$, for any positive integer k.

Now, note that $I \odot A^2$ is a diagonal matrix. Also, $I \odot A^2 \in \mathcal{CC}(X)$ and $I \odot A^2 \neq kI$, for any positive integer k. Thus, $I \odot A^2$ and I are sums of two or more elements of the standard basis of $\mathcal{CC}(X)$.

That is, there exist matrices $B_1, B_2, \ldots, B_k$, for some $k \geq 2$, such that B_i's are elements of the standard basis of $\mathcal{CC}(X)$ and $I = B_1 + B_2 + \cdots + B_k$. This implies that $A = IA = (B_1 + B_2 + \cdots + B_k)A$. Note that A as well as the matrices B_i, for $1 \leq i \leq k$, are 0, 1-matrices. Hence, for $1 \leq i \leq k$, the matrices $B_i A$ are disjoint 0, 1-matrices. Since X is a connected graph and $I = B_1 + B_2 + \cdots + B_k$, there exist distinct i, j such that $B_i A, B_j A \neq \mathbf{0}$. Therefore, A is a sum of two or more matrices from the standard basis of $\mathcal{CC}(X)$. A contradiction, to A being an element of the standard basis of $\mathcal{CC}(X)$. Thus, we have shown that I is a pattern matrix of X. Hence, using Corollary 3.7, X must be a walk-regular graph and in particular, a regular graph, as well as an edge-regular graph. This completes the proof of the first three parts.

To prove the final statement, let X be a strongly regular graph. Then, using Theorem 3.10, it can be easily verified that $\{I, A, \mathbf{J} - I - A\}$ is a basis of the adjacency algebra $\mathcal{A}(X)$. As the adjacency matrix of X^c equals $\mathbf{J} - I - A$, $\mathcal{A}(X)$ is indeed the coherent closure of both X and X^c. Hence both X and X^c are coherent graphs.

Conversely, let us assume that both X and X^c are coherent graphs and also, without loss of generality, assume that X is a connected graph. Then, by the definition of coherent graphs, the matrices A and $\mathbf{J}-I-A$ are elements of the standard basis of $\mathcal{CC}(X)$. Also, the first part of the proof indicates that the matrix I is a pattern matrix of X. Hence, $\{I, A, \mathbf{J}-I-A\}$ is a basis of $\mathcal{CC}(X)$ and therefore $\dim(\mathcal{A}(X)) \leq \dim(\mathcal{CC}(X)) = 3$. By Lemma 3.11, it follows that X is a regular graph. Therefore, X is a regular connected graph and hence $\mathbf{J} \in \mathcal{A}(X)$. Thus, $\{I, A, \mathbf{J} - I - A\}$ is also a basis of $\mathcal{A}(X)$. That is, $\dim(\mathcal{A}(X)) = 3$ and hence X has exactly three distinct eigenvalues and by Theorem 3.10, we see that X is strongly regular. □

The second and third part of the next result was stated and proved by Beezer [2]. We give an alternate proof for the sake of completeness.

Lemma 3.12 *Let X be an orbit polynomial graph. Then*

1. *X is a pattern polynomial graph.*
2. *X is a connected regular graph.*
3. *X is a vertex-transitive graph.*

Moreover, X^c is connected if and only if X^c is an orbit polynomial graph.

Proof **Proof of Parts** 1 **and** 2: From Lemma 3.8.1 it is sufficient to prove Part 1. Since X is an orbital graph, using Equation (1) and the definition of an orbital graph,

$$\mathcal{A}(X) \subseteq \mathcal{L}(X) \subseteq \mathcal{CC}(X) \subseteq \mathcal{V}(X) = \mathcal{A}(X).$$

That is, $\mathcal{A}(X) = \mathcal{CC}(X)$ and hence X is a pattern polynomial graph.

Proof of Part 3: Let X be an orbit polynomial graph. Then $\mathcal{A}(X) = \mathcal{V}(X)$ and hence $\mathcal{V}(X)$ is a commutative algebra. Hence, using Corollary 1.9, the identity matrix is a member of the standard basis of $\mathcal{V}(X)$. Or equivalently, there is single diagonal orbital under the action of $\mathrm{Aut}(X)$ on the vertex set of X. Thus, X is vertex transitive.

For the final statement observe observe that using Part 1 and Lemma 3.8, X^c is connected if and only if X^c is a pattern polynomial graph. But $\mathrm{Aut}(X) = \mathrm{Aut}(X^c)$ implies that $\mathcal{V}(X) = \mathcal{V}(X^c)$ holds for every graph X. Thus, the required result follows. □

We now state another result of Beezer and give a prove for the sake of completeness.

Lemma 3.13 (Beezer [1]) *Let X be a distance transitive graph. Then X is an orbit polynomial graph.*

Proof Let X be a distance transitive graph with diameter d. Then by definition, the distance matrices coincide with the orbital matrices of X . Consequently, $\dim(\mathcal{V}(X)) = d + 1$, the rank (the number of orbitals) of X. But Lemma 1.2 implies that $d + 1 \leq \dim(\mathcal{A}(X))$ and Equation 1 implies that $\mathcal{A}(X) \subseteq \mathcal{V}(X)$. Thus, $\mathcal{A}(X) = \mathcal{V}(X)$ and hence the required result follows. □

Before proceeding to the next result recall that the action of a group G on a set V is said to be *generously transitive* if, given any two elements in V there is a permutation which interchanges them. The proof of the next result is a direct consequence of the following theorem of Godsil [14]. A proof is given for the sake of completeness.

Theorem 3.14 *(Godsil [14]) Let X be a compact connected regular graph with r distinct eigenvalues. Then $\mathrm{Aut}(X)$ is a generously transitive permutation group of rank r.*

Corollary 3.15 *Let X be a compact connected regular graph. Then X is an orbit polynomial graph.*

Proof If X is a compact connected graph, then by Theorem 3.14, the number of distinct eigenvalues of $A(X)$ is the same as the number of orbitals. But the number of distinct eigenvalues of $A(X)$ is also equal to the dimension of $\mathcal{A}(X)$. Thus, one has the required result (see the proof of Lemma 3.13). □

We now state a theorem of Damerell [10] for easy reference.

Theorem 3.16 *(Damerell [10]) Let X be a distance regular graph of diameter d. Then the set of distance matrices of X, $\{A_0(X), A_1(X), \ldots, A_d(X)\}$, forms a basis of the adjacency algebra $\mathcal{A}(X)$.*

We omit the proof of the next result as it is a straight-forward application of Theorem 3.16.

Lemma 3.17 *Let X be a distance regular graph. Then*

1. *X is a pattern polynomial graph.*
2. *X is a coherent graph.*

Observe that Lemma 3.17, a direct consequence of Theorem 3.16, not only establishes that every distance regular graph is a pattern polynomial graph as well as a coherent graph but also shows that every distance regular graph attains the lower bound stated in Lemma 1.2.

Before, we start a set of counter-examples, it is important to note that Godsil [14] showed that if X is a compact connected distance regular graph then X is a distance transitive graph and hence is an orbit polynomial graph as well. Similarly, if X is both an orbit polynomial graph and a distance regular graph, then the distance matrices of X coincide with its orbital matrices, hence X is a distance transitive graph.

We end this section with counter-examples that help us understand Figure 1. To do so, we first draw the figures of two important graphs, namely, the truncated tetrahedron graph and the Shrikhande graph.

Remark 3.18 The following remarks are in order.

1. Recall the cycle graph C_4 on four vertices and the matrix W_4, the companion matrix of $x^4 - 1$. Then $\{I, W_4^2, J - I - W_4^2 = W_4 + W_4^3\}$ is the standard basis of $\mathcal{CC}(C_4) = \mathcal{CC}(C_4^c)$. Hence, C_4^c is an example of a coherent graph that is not connected. Also, it can be easily checked that C_4^c is neither a distance polynomial graph nor a pattern polynomial graph. Similarly, one can verify that C_6^c is an example of a pattern polynomial graph that is not a coherent graph.
2. Let X be a connected circulant graph of prime order. Then X is an orbit polynomial graph (see Beezer [1]). But X need not be a compact graph (see Lemma 2.2 in [14]). Similarly all connected circulant graphs of prime order are not distance transitive (see Theorem 1.2 in [22]).

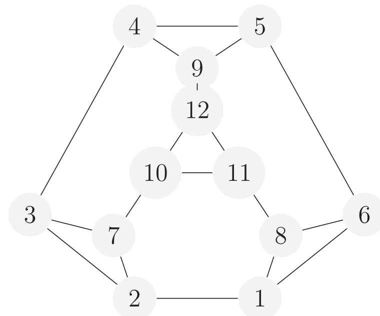

Fig. 2 Truncated Tetrahedron Graph

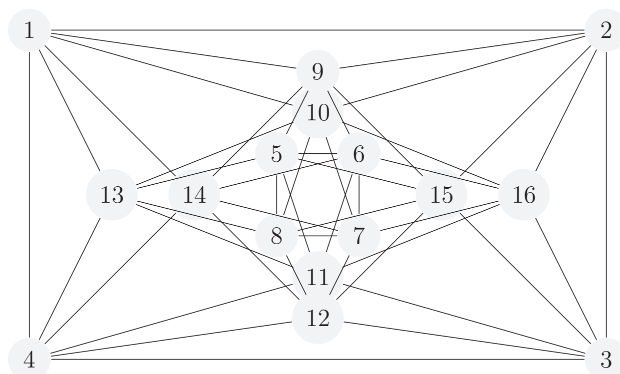

Fig. 3 Shrikhande Graph

3. Let X be a compact graph. Then, using the fact that $\mathrm{Aut}(X) = \mathrm{Aut}(X^c)$ it is easy to verify that X is compact if and only if X^c is compact. Thus, C_6^c is an example of a compact connected regular graph that is not a distance transitive graph.
4. Let X be the line graph of the complete graph K_n, for $n \geq 7$. Then X is a distance transitive graph but not a compact graph (for details, see Godsil [14]).
5. Let X be a distance transitive graph. Then, it is easy to see that X is a distance regular graph. But, the well known Shrikhande graph (see Figure 3) is a distance regular graph that is not an orbit polynomial graph. Hence, the Shrikhande graph is also not a distance transitive graph. In fact, there are many distance regular graphs whose automorphism group is trivial (see Spence [26] or Weisfeiler [30]).
6. The truncated tetrahedron graph (see Figure 2) is an example of a connected regular graph that is not a distance polynomial graph (for details, see Weichsel [28]). But, if we assume that X is a k-regular connected graph with diameter 2 then X is clearly a distance polynomial graph ($A_2(X) = \mathbf{J} - I - A$).
7. Let X be the truncated tetrahedron graph (see Figure 2). Then observe that X^c is a connected regular graph of diameter 2. Hence, X^c is an example of a distance polynomial graph, that is not a pattern polynomial graph. For if, X^c were to be a pattern polynomial graph, then using Lemma 3.8, the graph X would also be a pattern polynomial graph, giving a contradiction as X is not even a distance polynomial graph.
8. Let X be a distance regular graph of diameter greater than or equal to 3 having trivial automorphism group (for examples of such graphs, see Spence [26] or Weisfeiler [30]). Also assume that X^c is connected. Then, using Lemma 3.8, X^c is a pattern polynomial graph. But then diameter greater than or equal to 3 implies that X^c is not a coherent graph (use Lemma 3.11) and thus X^c is not a distance regular graph. Also, X^c is not an orbit polynomial graph as the automorphism group of X is trivial. Consequently, X^c is an example of a pattern polynomial graph that is neither a distance regular graph nor an orbit polynomial graph.

4 Properties of pattern polynomial graphs

Let X be a pattern polynomial graph. In this section, we show that if $X \neq K_2$ then X has at least one multiple eigenvalue. In particular, if X has an odd number of vertices, then we show that $\dim(\mathcal{A}(X)) \leq \frac{n+1}{2}$. We also find graphs that are polynomials in a given pattern polynomial graph. Recall that a graph Y is said be a polynomial in a graph X if the adjacency matrix of Y, $A(Y) \in \mathcal{A}(X)$.

The following result is a direct application of Lemma 3.2. We state it for ready reference.

Corollary 4.1 *Let $X \neq K_2$ be a vertex transitive graph. Then X has at least one multiple eigenvalue.*

The next result uses Corollary 4.1 to show that a similar result holds for pattern polynomial graphs.

Corollary 4.2 *Let X be a pattern polynomial graph on $n \geq 3$ vertices. Then X has at least one multiple eigenvalue.*

Proof On the contrary, assume that all the eigenvalues of X are simple. Then we know that $\dim(\mathcal{A}(X)) = n$. As X is a pattern polynomial, $\mathcal{A}(X) = \mathcal{L}(X) = \mathcal{CC}(X)$. Hence, using Corollary 1.9, it follows that every pattern matrix is a symmetric permutation matrix and their sum is $\mathbf{J}$. Consequently, X is a vertex transitive graph. Since $X \neq K_2$, we arrive at a contradiction to Corollary 4.1. Thus, the required result follows. □

Now, recall that $\dim(\mathcal{A}(X))$ equals the number of distinct eigenvalues of A. This helps us in extending Corollary 4.2 to arbitrary graphs in the following manner.

Corollary 4.3 *Let X be a graph on $n \geq 3$ vertices. If the corresponding coherent algebra $\mathcal{CC}(X)$ is a commutative algebra then* $\dim(\mathcal{A}(X)) \leq n - 1$.

Proof Let if possible, $\dim(\mathcal{A}(X)) = n$. If $\mathcal{CC}(X)$ is a commutative coherent algebra over $\mathbb{C}$, then Corollary 1.9 implies that $\dim(\mathcal{CC}(X)) \leq n$.

Thus, $n = \dim(\mathcal{A}(X)) \leq \dim(\mathcal{CC}(X)) \leq n$ and hence $\mathcal{CC}(X) = \mathcal{A}(X)$. Hence, X is a pattern polynomial graph on $n \geq 3$ vertices. As $\dim(\mathcal{A}(X)) = n$, X has all simple eigenvalues. This gives us a contradiction to Corollary 4.2. Hence, the required result follows. □

Remark 4.4 Recall from Page 6 that a pattern matrix is a 0, 1-matrix. Hence, given a pattern matrix A, we can get a digraph X such that A is the adjacency matrix of X. Thus, starting with a pattern polynomial graph X, we can determine the pattern matrices $I = P_1(X), P_2(X), \ldots, P_r(X)$. Then, using Lemma 2.1, we see that each $P_i(X)$ is symmetric, for $1 \leq i \leq r$. Also, the pattern matrices, different from $P_1(X)$, will give rise to graphs, say $X_2, X_3, \ldots, X_r$, such that $P_i(X)$ is the adjacency matrix of X_i, for $2 \leq i \leq r$. Since the matrices $P_i(X)$, for $2 \leq i \leq r$, commute with the matrix $\mathbf{J}$, it follows that the graphs X_i, for $2 \leq i \leq r$, are regular graphs. In general, for a graph X, the matrix $\mathbf{J}$ may not commute with the matrices $P_i(X)$ and therefore, the corresponding graphs need not be regular.

Now, let us restrict ourselves to pattern polynomial graphs that have an odd number of vertices. Then our next result improves the bound on $\dim(\mathcal{A}(X))$ stated in Corollary 4.3, by using Remark 4.4. We do so by recalling that if ℓ denotes the number of vertices of a graph X that have odd degree then ℓ is even.

Lemma 4.5 *Let X be a pattern polynomial graph have an odd number of vertices. Then* $\dim(\mathcal{A}(X)) \leq \frac{n+1}{2}$.

Proof Let X be a pattern polynomial graph. Then $\mathcal{A}(X) = \mathcal{L}(X)$ and hence $\dim(\mathcal{A}(X)) = \dim(\mathcal{L}(X))$ which in turn equals the number of pattern matrices. Thus, using Remark 4.4, we see that all pattern graphs of X are regular graphs with an odd number of vertices. Consequently, the regularity of each of these graphs must be an even integer greater than or equal to 2. Hence, X has at most $\frac{n-1}{2}$ pattern graphs. Thus, including the identity matrix, X has at most $\frac{n+1}{2}$ pattern matrices. □

We are now interested in finding graphs that are polynomials in a given pattern polynomial graph. That is, if A is the adjacency matrix of a pattern polynomial graph X, then find all graphs such that whose adjacency matrices are polynomials in A. A similar question for orbit polynomial graphs and distance regular graphs was answered by Beezer [4] and Weichsel [29], respectively. Note that both classes of graphs, namely orbit polynomial graphs and distance regular graphs, are pattern polynomial graphs. The next theorem is a direct consequence of Corollary 1.8 in which we have tried to generalize the earlier results of Beezer [4] and/or Weichsel [29] to pattern polynomial graphs. It is also important to note that if X_n denotes the path graph on n vertices, then $\mathcal{A}(X_n)$, the adjacency algebra of the path graph, is not a coherent algebra. Even then Beezer [3] found a basis for $\mathcal{A}(X_n)$ that consisted of only 0, 1-matrices. We omit the proof of our result as it directly follows from the fact that for a pattern polynomial graph X, $\mathcal{A}(X) = \mathcal{L}(X)$ and $I, \mathbf{J} \in \mathcal{L}(X)$ (see Lemma 3.8).

Theorem 4.6 *Let X be a pattern polynomial graph. If $\{I = P_1, P_2, \ldots, P_r\}$ is the standard basis of $\mathcal{A}(X)$, then a graph Y is a polynomial in X if and only if*

$$A(Y) = \sum_{i=2}^{r} a_i P_i, \quad for \;\; a_i \in \{0, 1\}, 2 \leq i \leq r,$$

where $A(Y)$ is the adjacency matrix of the graph Y.

We end this section with the following remarks that are based on Theorem 4.6.

Remark 4.7 Let X be a pattern polynomial graph with adjacency matrix A and let r be the degree of the minimal polynomial of A. Also, let Y be a graph in the adjacency algebra of X. Then

1. Then there are 2^{r-1} graphs in the adjacency algebra of X. These graphs need not be non-isomorphic.
2. $\mathcal{CC}(Y) \subseteq \mathcal{CC}(X)$.
3. $\mathcal{CC}(Y)$ is a symmetric commutative algebra. Hence,
   (a) Y is a walk-regular graph.
   (b) Y is a strongly distance-balanced graph (use Theorem 3.9).
   (c) Y has a multiple eigenvalue, whenever $Y \neq K_2$ (use Corollary 4.3).
   (d) $\dim(\mathcal{CC}(Y)) \leq n$ (use Corollary 1.9). Further if the number of vertices in Y is odd, then $\dim(\mathcal{CC}(Y)) \leq \frac{n+1}{2}$.

Now it is interesting to answer the following question: Let Y be a graph such that $\mathcal{CC}(Y)$ is a symmetric commutative algebra. Then "does there exist a pattern polynomial graph X such that Y is a polynomial in X?". For example, we know that C_n, the cycle graph on n vertices, is a pattern polynomial graph. In this case, if Y is a circulant graph on n vertices, then clearly $\mathcal{CC}(Y)$ is a symmetric commutative algebra and it is also known that Y is a polynomial in the cycle graph C_n.

Conclusion

In this paper, we looked at pattern polynomial graphs X that are defined by having $\mathcal{A}(X)$, the adjacency algebra of X, itself as a coherent algebra. Consequently, the adjacency matrices of such graphs satisfy the conditions of Theorem 2.2. Some of the properties of the graphs which are polynomials in the pattern polynomial graph have also been studied. We also identify certain known graph classes that are pattern polynomial graphs.

From the design theory point of view, a graph is a pattern polynomial graph, if its adjacency algebra is a Bose-Mesner algebra. It is also known that Bose-Mesner algebras are helpful in the construction of partially balanced incomplete block designs (see Raghavarao [24]). Thus, as a future line of research, one can also use pattern polynomial graphs for the construction of partially balanced incomplete block designs. For the definitions and results related to Bose-Mesner algebras, the reader is advised to see either the book by Brouwer, Cohen & Neumaier [8] or Bannai and Ito [6] or the original paper by Bose & Mesner [7].

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