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ORIGINAL RESEARCH

Gelfand pairs and spherical means on H-type groups

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Abstract We study the injectivity of the spherical mean operator associated to the Gelfand pairs (U, N), where N is a Heisenberg type group and U the subgroup of the group of orthogonal transformations of N that act trivially on its centre. We prove that when the dimension of the centre of N is 3, these spherical mean operator is injective on $L^p(N)$ for the optimal range $1 \leq p \leq 3$.

Keywords H-type groups · Spherical means · Gelfand pairs · Heisenberg group

Mathematics Subject Classification 22E25 · 43A90 · 44A35 · 42C10

1 Introduction and preliminaries

Finding out if a function can be reconstructed from its averages on spheres with a definite radius $r > 0$ is one of the challenges in integral geometry. By the average of the function f over the sphere of radius r centered at the point x, we mean the integral of f with respect to the normalised surface measure on the sphere $\{y \in \mathbb{R}^n : |y - x| = r\}$ in $\mathbb{R}^n$. This can be written as the convolution with the normalized surface measure ν_r on the sphere $\{x \in \mathbb{R}^n : |x| = r\}$ as

$$f * \nu_r(x) = \int_{|x|=r} f(x - y) d\nu_r(y).$$

The function can be recovered from its spherical averages only if this spherical mean operator is injective. That is,

$$f * \nu_r(x) = 0, \forall x \implies f \equiv 0.$$

But this operator is not injective in general. For $\lambda > 0$, consider the function

$$\varphi_\lambda(x) = c \, \frac{J_{\frac{n}{2}-1}(\lambda|x|)}{(\lambda|x|)^{\frac{n}{2}-1}}, \;\; x \in \mathbb{R}^n,$$

where J_α denotes the Bessel function of order α and c is a constant to normalize φ_λ in such a way that $\varphi_\lambda(0) = 1$. Then it is well known that

$$\varphi_\lambda * \nu_r(x) = \varphi_\lambda(r)\varphi_\lambda(x), \;\; x \in \mathbb{R}^n.$$

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Hence, if we choose $r > 0$ to be a zero of the function $s \to J_{\frac{n}{2}-1}(\lambda s)$, then $\varphi_\lambda * \nu_r$ vanishes identically(see [1]). Since $\varphi_\lambda \in L^p(\mathbb{R}^n)$ when $p > \frac{2n}{n-1}$, the operator $f \mapsto f * \nu_r$ is fails to be injective in $L^p(\mathbb{R}^n)$ for $\frac{2n}{n-1} < p \leq \infty$. When $1 \leq p \leq \frac{2n}{n-1}$, these operators are injective. That is, for a continuous function $f \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \frac{2n}{n-1}$, if $f * \nu_r$ is identically zero for a fixed radius $r > 0$, then f vanishes identically (see [2]).

Let H^n denote the Heisenberg group $\mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z, t)(w, s) = (z + w, t + s + \frac{1}{2}\operatorname{Im}(z \cdot \overline{w})).$$

The Lie algebra of this Lie group is $\mathfrak{h}^n = \mathbb{C}^n \times \mathbb{R}$ with the Lie bracket

$$[(z, t), (w, s)] = (0, \operatorname{Im}(z \cdot \overline{w})).$$

This makes $\mathfrak{h}^n$ a step two nilpotent Lie algebra and therefore H^n a two-step nilpotent Lie group. The spherical means of a function f on H^n can also be written in terms of the convolution as

$$f * \mu_r(z, t) = \int\limits_{|w|=r} f(z - w, t - \frac{1}{2}\operatorname{Im}(z \cdot \overline{w}))\, d\mu_r(w). \tag{1}$$

where μ_r is the normalized surface measure on the sphere $\{(z, 0) \in H^n : |z| = r\}$ of radius r in H^n. Unlike the Euclidean case, the spherical mean operators are injective on $L^p(H^n)$ for all p such that $1 \leq p < \infty$. This was proved by Thangavelu [2] using the spectral decomposition of the sublaplacian on the Heisenberg group provided by Strichartz in [3].

Notice that the unitary group $U(n)$ acts on the Heisenberg H^n by $\sigma(z, t) = (\sigma z, t)$. Hence the spherical means in (1) can be seen as the averages of the function f over $U(n)$-orbits. Since the sub-algebra of the $U(n)$ invariant functions in $L^1(H^n)$ is commutative, the pair $(H^n, U(n))$ forms a Gelfand pair. Also, the spectral decomposition studied by Strichartz [3], coincides with the expansion in terms of the spherical functions associated with this Gelfand pair. This point of view led to a general result in [4].

The Heisenberg type groups, or H-type groups, were introduced by A. Kaplan in 1980 [5] as a class of two step nilpotent groups that includes the Heisenberg groups. Let $\mathfrak{n}$ be a real two-step nilpotent Lie algebra endowed with an inner product $\langle , \rangle$. Let $\mathfrak{z}$ be the center of $\mathfrak{n}$ and $\mathfrak{v}$ be its orthogonal complement. For a unit vector $v \in \mathfrak{v}$, let $\mathfrak{k}_v$ be the kernel of the adjoint map $ad_v : \mathfrak{v} \to \mathfrak{z}$ defined by

$$ad_v(v') = [v, v'] \quad v' \in \mathfrak{v}.$$

Then the Lie algebra $\mathfrak{n}$ is said to be Heisenberg type or H-type if the adjoint map restricted to the orthogonal complement of its kernel is a surjective isometry. That is, if $ad_v : \mathfrak{v}_v \to \mathfrak{z}$ is a surjective isometry where

$$\mathfrak{v} = \mathfrak{v}_v \oplus \mathfrak{k}_v.$$

A connected and simply connected Lie group N is said to be a Heisenberg type group or H-type group if its Lie algebra is of H-type.

If $\mathfrak{n}$ is an H-type Lie algebra, for non-zero $z \in \mathfrak{z}$ we can define a skew-symmetric linear operator $J_z : \mathfrak{v} \to \mathfrak{v}$ by

$$\langle J_z(v), v' \rangle = \langle z, [v, v'] \rangle \qquad \text{for all } v, v' \in \mathfrak{v}.$$

It can be proved that $\mathfrak{n}$ is a H-type algebra if and only if $J_z^2 = -|z|^2 I$, for every nonzero $z \in \mathfrak{n}$ [5]. For $|z| = 1$, J_z defines a complex structure on $\mathfrak{v}$ and hence $\dim \mathfrak{v}$ has to be even, say $\dim \mathfrak{v} = 2n$. We will identify $\mathfrak{v}$ with $\mathbb{C}^n$ and $\mathfrak{z}$ with $\mathbb{R}^m$. Since we can identify the connected and simply connected Lie group N with its nilpotent Lie algebra $\mathfrak{n}$ via the exponential map, we will write (z, t) for points in N, where $z \in \mathbb{C}^n$ (identified with $\mathfrak{v}$) and $t \in \mathbb{R}^m$ (identified with $\mathfrak{z}$). The Haar measure on N is given by the Lebesgue measure on $\mathfrak{n}$ and will be denoted by $dzdt$. The group law is then given by

$$(z, t)(w, s) = (z + w, t + s + \frac{1}{2}[z, w]),$$

where [,] denotes the Lie bracket. For any fixed $a \in \mathfrak{z} \setminus \{0\}$ we can choose a basis with some properties (see [6, p. 294]) so that

$$\langle a, [z, w]\rangle = |a| \operatorname{Im}(z \cdot \bar{w}).$$

Let f, g be functions on N with $g(z, t) = \exp(-i\langle a, t\rangle)\varphi(z)$, then,

$$\begin{aligned} f * g(z, t) &= \int_N f((z, t)(w, s)^{-1}) g(w, s)\, dw ds \\ &= \int_{\mathbb{C}^n}\int_{\mathbb{R}^m} f(z - w, t - s - \frac{1}{2}[z, w]) g(w, s)\, dw ds \\ &= \int_{\mathbb{C}^n}\int_{\mathbb{R}^m} f(z - w, t - s - \frac{1}{2}[z, w]) \exp(-i\langle a, s\rangle)\varphi(w)\, dw ds \\ &= \int_{\mathbb{C}^n}\int_{\mathbb{R}^m} f(z - w, s) \exp(-i\langle a, t - s - \frac{1}{2}[z, w]\rangle)\varphi(w)\, dw ds \\ &= \exp(-i\langle a, t\rangle) \int_{\mathbb{C}^n} f^a(z - w)\varphi(w) \exp\left(\frac{i|a|}{2} \operatorname{Im}(z \cdot \bar{w})\right) dw \\ &= \exp\left(-i\langle a, t\rangle\right) f^a \times_{|a|} \varphi(z). \end{aligned}$$

where f^a is the Fourier transform of f in the central variable t and $\times_\lambda$ denote the twisted convolution on $\mathbb{C}^n$ of order λ, defined by,

$$F \times_\lambda G(z) = \int_{\mathbb{C}^n} F(z - w) G(w) \exp\left(\frac{i\lambda}{2} Im(z \cdot \overline{w})\right) dw.$$

The irreducible unitary representations of N that are not one dimensional are parameterized by $a \in \mathfrak{z} \setminus \{0\}$. For each $a \in \mathfrak{z}\setminus\{0\}$, we can define the Hilbert space

$$\mathcal{F}_a(\mathfrak{v}) = \left\{F : \mathfrak{v} \equiv \mathbb{C}^n \to \mathbb{C} : F \text{ is holomorphic}, \int_{\mathfrak{v}} |F(w)|^2 e^{\frac{-|a||w|^2}{2}}\, d\mathfrak{v}(w) < \infty\right\}.$$

These Hilbert spaces support the irreducible representation π_a of N, known as the Bargmann representation, defined by,

$$\pi_a(v, t)F(w) = \exp(i\langle a, t\rangle - \frac{1}{4}|a|(|v|^2 + 2\langle w, v\rangle - i\langle b, [w, v]\rangle))F(w + v)$$

for $v \in \mathfrak{v}, t \in \mathfrak{z}$, where $b = \frac{a}{|a|}$. Moreover, any infinite dimensional unitary representation π of N such that $\pi|_{\mathfrak{z}} = e^{i<a,t>}$ Id is equivalent to π_a [7, p. 420].

Let $A(N)$ be the group of orthogonal transformations of $N = \mathfrak{v} \oplus \mathfrak{z}$, which are automorphisms of N. Let U be the subgroup of $A(N)$ that act trivially on $\mathfrak{z}$. That is,

$$U = \{k \in A(N) : k(z) = z, \text{ for all } z \in \mathfrak{z}\}.$$

For $z \in \mathfrak{z}$, the map $J_z : \mathfrak{v} \to \mathfrak{v}$ extends to N as an automorphism by defining

$$J_z(z) = z \text{ and } J_z(z') = -z' \text{ if } z' \perp z.$$

We denote by $\operatorname{Pin}(m)$ the subgroup of $A(N)$ generated by $\{J_z : z \in \mathfrak{z}\}$. Then U and $\operatorname{Pin}(m)$ commute and their intersection contains at most four elements [8]. Also $A(N) = U \cdot \operatorname{Pin}(m)$ unless $m \equiv 1 \pmod 4$ and in that case $A(N)/(U \cdot \operatorname{Pin}(m))$ has two elements [7].

We recall that for any Lie group N and any compact subgroup K of its automorphism group, the pair (K, N) is said to be a Gelfand pair if the set $L^1_K(N)$ of integrable K-invariant functions on N forms a commutative algebra under convolution. For the particular case of Heisenberg type groups we have the following classification theorem[9, p. 266].

Theorem 1.1 *The groups N of H-type for which $L^1_{A(N)}(N)$ is commutative, that is (A(N),N) is a Gelfand pair, are those for which*

$$\dim(\mathfrak{z}) = m = \begin{cases} 1, 2 \text{ or } 3 \\ 5, 6 \text{ or } 7 \text{ and } \mathfrak{v} \text{ is irreducible} \\ 7, \mathfrak{v} \text{ is isotypic and } \dim(\mathfrak{v}) = 16. \end{cases}$$

Here irreducibility of $\mathfrak{v}$ means irreducible under the action of the associated Clifford algebra. See [9] for more details. From the proof of the above theorem we obtain the following corollary[9, p. 268],

Corollary 1.1 *Let U be the subgroup of $A(N)$ that act trivially on the center $\mathfrak{z}$. Then (U, N) is a Gelfand pair if and only if* $\dim \mathfrak{z} = 1, 2,$ *or* 3.

We consider the above cases in some detail. First we notice that when $m = 1$, N is the Heisenberg group $\mathbb{C}^k \oplus \mathbb{R}$ and $U = U(k)$ is the unitary group. The U-averages give rise to the spherical means on N and the injectivity result follow from [2, Theorem 5.1]. See also [4, Theorem 5.2].

When $m = 2$, $N \cong \mathbb{H}^k \oplus \mathbb{R}^2$ (See [9, p. 268]) where $\mathbb{H}$ is the space of quaternions and $U = Sp(k)$, the compact symplectic group. In this case U acts transitively on the spheres centered at origin in $\mathbb{H}^k (\cong \mathbb{C}^{2k})$. Therefore the averages over U-orbits coincide with the following spherical means

$$f * \mu_r(z, t) = \int_{|w|=r} f(z - w, t - \frac{1}{2}[z, w])\, d\mu_r(w)$$

defined in [10] in terms of the normalised surface measure μ_r on the sphere $\{(z, 0) \in N : |z| = r\}$ for a continuous function f on N. This is one of the three spherical mean operators which were shown to be injective on $L^p(N)$ for $1 \le p \le 2m/(m-1)$, where $m = \dim \mathfrak{z}$ (See [10, Theorem 1.1]).

When $m = 3$, $\mathfrak{v} \cong \mathbb{H}^k \oplus \mathbb{H}^l$ and $U = Sp(k) \times Sp(l)$ (see [9, p. 268]). The orbit of U in $\mathfrak{v}$ is the product of spheres in $\mathbb{H}^k$ and $\mathbb{H}^l$. So the U- averages give rise to new type of spherical means, not considered in [10].

We consider the above case $m = 3$ and prove injectivity result for averages over U-orbit. Fix $r_2, r_2 > 0$. Let $S^k_{r_1}$ and $S^l_{r_2}$ be the spheres of radii r_1 and r_2 centered at the origin in $\mathbb{H}^k$ and $\mathbb{H}^l$, respectively. Let $\mu^k_{r_1}$ and $\mu^l_{r_2}$ be the normalized surface measures on $S^k_{r_1}$ and $S^l_{r_2}$ respectively and let $\nu_{r_1,r_2} = \mu^k_{r_1} \times \mu^l_{r_2}$ realised as a measure on $N = \mathbb{H}^k \oplus \mathbb{H}^l \oplus \mathbb{R}^3$. Then the U-spherical means can be defined as

$$f * \nu_{r_1,r_2}(z, w, t) = \int_{S^k_{r_1} \times S^l_{r_2}} f(z - u, w - v, t - \frac{1}{2}[(z, w), (u, v)])\, d\mu^k_{r_1}(u) d\mu^l_{r_2}(v)$$

Our result is the following:

Theorem 1.2 *If $f \in L^p(N)$ for $1 \le p \le 3$ and $f * \nu_{r_1,r_2} \equiv 0$ then $f \equiv 0$.*

For the proof of the above, we closely follow the arguments in [2] and [10]. First we compute the spherical functions for the Gelfand pair $(Sp(k) \times Sp(l), N)$ (see 2). Then we obtain an expansion of L^2-functions in term of the spherical functions and establish the Abel summability of this expansion in L^p (see Theorem 3.4). Then the proof of the injectivity will follow as in [10].

2 Spherical functions for the case $m = 3$

Let (K, N) be a Gelfand pair and π be an irreducible unitary representation of N on a Hilbert space $\mathcal{H}_\pi$. Define,

$$K_\pi = \{k \in K : \pi \circ k \text{ unitarily equivalent to } \pi.\}$$

Let $\mathcal{H} = \oplus_\alpha P_\alpha$ be the decomposition into the K_π-irreducible subspaces. The following theorem was proved in [11, p. 415].

Theorem 2.1 *If ϕ is a bounded K-spherical function on N, then there exist a unique (up to unitary equivalence) irreducible representation π and a subspace $\mathcal{P}_\alpha$ in the decomposition of the representation space $\mathcal{H} = \oplus_\alpha \mathcal{P}_\alpha$ into K_π-irreducible subspaces, such that,*

$$\phi(x) = \phi_{\pi,v}(x) = \int_K \langle \pi(k \cdot x)v, v\rangle \, dk,$$

for any unit vector $v \in \mathcal{P}_\alpha$ and $x \in N$. In particular, if $K = K_\pi$ and $\{v_1, v_2, \ldots, v_l\}$ is any orthonormal basis for $\mathcal{P}_\alpha$, then

$$\phi_{\pi,\alpha}(x) = \frac{1}{l}\sum_{j=1}^{l} \langle \pi(x)v_j, v_j\rangle.$$

When $\mathfrak{v} = \mathbb{H}^k \oplus \mathbb{H}^l$, the action of $U = Sp(k) \times Sp(l)$ on the space $\mathcal{P}(\mathfrak{v})$ of holomorphic polynomials on $\mathfrak{v}$, decomposes as

$$\mathcal{P}(\mathfrak{v}) = \bigoplus_{p=0,q=0}^{\infty} \mathcal{P}^p(\mathbb{H}^k) \otimes \mathcal{P}^q(\mathbb{H}^l)$$

where $\mathcal{P}^p(\mathbb{H}^k) = \mathcal{P}^p(\mathbb{C}^{2k})$ is the space of homogeneous polynomials of degree p and $\mathcal{P}^q(\mathbb{H}^l) = \mathcal{P}^q(\mathbb{C}^{2l})$ is the space of homogeneous polynomial of degree l [9, p. 268].

The U action on $\mathfrak{n}$ is via the $Sp(k)$ action on $\mathcal{P}^p(\mathbb{H}^k)$ and the $Sp(l)$ action on $\mathcal{P}^q(\mathbb{H}^l)$ and so is trivial on the centre $\mathfrak{z}$. Hence for every $k \in U$, $(\pi_a \circ k)|_{\mathfrak{z}} = \pi_a|_{\mathfrak{z}}$. That is,

$$U_{\pi_a} = \{k \in U : \pi_a \circ k \equiv \pi_a\} = U.$$

Hence, by Theorem 2.1 every spherical function is of the form $\phi_{\pi_a,v}$, for some $a \in \mathfrak{z} \setminus \{0\}$ and a unit vector $v = (p, q) \in \mathcal{P}^p(\mathbb{H}^k) \otimes \mathcal{P}^q(\mathbb{H}^l)$. Hence the spherical functions are parameterised by (p, q) as

$$e^a_{p,q}(z, w, t) = \phi_{\pi_a,v}(z, w, t)$$

where $(z, w) \in \mathbb{H}^k \times \mathbb{H}^l = \mathbb{C}^{2k} \times \mathbb{C}^{2l}$.

To obtain the explicit expression for $e^a_{p,q}$, consider an orthonormal basis $\{u_\alpha(\xi) = \xi^\alpha, \xi \in \mathbb{C}^{2k} : \alpha \in \mathbb{N}^{2k}, |\alpha| = p\}$ for $\mathcal{P}^p(\mathbb{C}^{2k})$ and an orthonormal basis $\{v_\beta(\eta) = \eta^\beta, \eta \in \mathbb{C}^{2l} : \eta \in \mathbb{N}^{2l}, |\beta| = q\}$ for $\mathcal{P}^q(\mathbb{C}^{2l})$. Then $\{u_\alpha \otimes v_\beta : \alpha \in \mathbb{N}^{2k}, \beta \in \mathbb{N}^{2l}, |\alpha| = p, |\beta| = q\}$ is an orthonormal basis for $\mathcal{P}^p(\mathbb{C}^k) \otimes \mathcal{P}^q(\mathbb{C}^l)$. Let $d_p = \dim \mathcal{P}^p(\mathbb{C}^{2k})$ and $d_p = \dim \mathcal{P}^q(\mathbb{C}^{2l})$, then

$$\begin{aligned} e^a_{p,q}(z, w, t) &= \frac{1}{d_p d_q} \sum_{\substack{|\alpha|=p \\ |\beta|=q}} < \pi_a(z, w, t)u_\alpha \otimes v_\beta, u_\alpha \otimes v_\beta > \\ &= e^{i<a,t>} \left(\frac{1}{d_p} \sum_{|\alpha|=p} \langle \pi_a(z, 0, 0)u_\alpha, u_\alpha\rangle \right) \\ &\quad \times \left(\frac{1}{d_q} \sum_{|\beta|=q} \left\langle \pi_a(0, w, 0)v_\beta, v_\beta \right\rangle \right) \end{aligned}$$

Since the action of $\pi_a(z, 0, 0)$ on $\mathcal{P}(\mathbb{C}^{2k})$ is same as the action of the representation $\pi_{|a|}(z, 0)$ of the Heisenberg group $H^{2k} = \mathbb{C}^{2k} \times \mathbb{R}$ on $\mathcal{P}(\mathbb{C}^{2k})$, we have

$$\frac{1}{d_p} \sum_{|\alpha|=p} \langle \pi_a(z, 0, 0)u_\alpha, u_\alpha\rangle = \frac{1}{d_p} \sum_{|\alpha|=p} \langle \pi_{|a|}(z, 0)u_\alpha, u_\alpha\rangle.$$

Then by [11, Proposition 6.2],

$$\frac{1}{d_p}\sum_{|\alpha|=p}\langle \pi_a(z,0,0)u_\alpha, u_\alpha\rangle = L_p^{2k-1}\left(\frac{|a|}{2}|z|^2\right)e^{-\frac{|a|}{4}|z|^2},$$

where L_r^δ is the r-th Laguerre polynomial of type $\delta > -1$. Similarly,

$$\frac{1}{d_q}\sum_{|\beta|=q}\langle \pi_a(0,w,0)v_\beta, v_\beta\rangle = L_q^{2l-1}\left(\frac{|a|}{2}|w|^2\right)e^{-\frac{|a|}{4}|w|^2}.$$

Therefore, we have,

$$\begin{aligned} e^a_{p,q}(z,w,t) &= e^{i\langle a,t\rangle}L_p^{2k-1}\left(\frac{1}{2}|a||z|^2\right)L_q^{2l-1}\left(\frac{1}{2}|a||w|^2\right)e^{-\frac{1}{4}|a|\left(|z|^2+|w|^2\right)} \\ &= e^{i\langle a,t\rangle}\varphi^a_{p,q}(z,w) \end{aligned}$$

where,

$$\varphi^a_{p,q}(z,w) = L_p^{2k-1}\left(\frac{1}{2}|a||z|^2\right)L_q^{2l-1}\left(\frac{1}{2}|a||w|^2\right)e^{-\frac{1}{4}|a|\left(|z|^2+|w|^2\right)}$$

Therefore,

$$e^a_{p,q}(z,w,t) = e^{i\langle a,t\rangle}\varphi_p^{|a|,2k}(z)\varphi_q^{|a|,2l}(w) \tag{2}$$

where $\varphi_j^{\lambda,n}(z) = L_j^{n-1}\left(\frac{1}{2}\lambda|z|^2\right)e^{-\frac{1}{4}\lambda|z|^2}$, $\lambda > 0$ is the scaled Laguerre function on $\mathbb{C}^n$.

3 Spectral decomposition and Abel summability

An important step in obtaining the injectivity results for spherical means in [10] is the spectral decomposition. For $f \in L^2(N)$ in the H-type group $N \cong \mathbb{C}^n \times \mathbb{R}^m$,

$$f(z,t) = \frac{1}{(2\pi)^{n+m}}\sum_{r=0}^{\infty}\int_{\mathbb{R}^n} f * e^a_r(z,t)\ |a|^n\ da$$

and the fact that the eigenfunctions e^a_r satisfy

$$e^a_r * \mu_k(z,t) = c_r e^a_r(k,0)e^a_r(z,t) \qquad \text{for } (z,t)\in N$$

where μ_k is the surface measure on the sphere of radius k and c_r an appropriate constant. When $m = 3$, elements in N can be written as (z,w,t) with $z \in \mathbb{H}^k \equiv \mathbb{C}^{2k}$, $w \in \mathbb{H}^l \equiv \mathbb{C}^{2l}$, $2k+2l = n$ and the above decomposition becomes

$$f(z,w,t) = \frac{1}{(2\pi)^{2k+2l+3}}\int_{\mathbb{R}^3\setminus\{0\}}\sum_{r=0}^{\infty} f * e^a_r(z,w,t)|a|^{2k+2l}\, da. \tag{3}$$

In order to obtain a similar expansion for $f \in L^2(N)$ in terms of the U-spherical functions, we first prove the following Lemma.

Lemma 3.1

$$\begin{aligned} &\sum_{p+q=k} L_p^{2k-1}\left(\frac{1}{2}|a||z|^2\right)L_q^{2l-1}\left(\frac{1}{2}|a||w|^2\right)e^{-\frac{1}{4}|a|(|z|^2+|w|^2)} \\ &\quad = L_j^{n-1}\left(\frac{1}{2}|a|\left(|z|^2+|w|^2\right)\right)e^{-\frac{1}{4}|a|\left(|z|^2+|w|^2\right)} \qquad j = 0,1,2,\ldots \end{aligned}$$

where $n = 2k+2l$.

Proof We use the generating function of the Laguerre polynomials of type $\alpha > -1$,

$$\sum_{k=0}^{\infty} L_k^{\alpha}(x) r^k = (1-r)^{-\alpha-1} e^{-\frac{r}{1-r}x} \qquad |r| < 1.$$

Hence for $x \geq 0$,

$$\sum_{k=0}^{\infty} L_k^{\alpha}(x) e^{-\frac{x}{2}} r^k = (1-r)^{-\alpha-1} e^{-\frac{1}{2}\left(\frac{1+r}{1-r}\right)x} \qquad |r| < 1,$$

since $2k + 2l = 2n$, for $x, y \geq 0$,

$$\begin{aligned}\left(\sum_{p=0}^{\infty} L_p^{2k-1}(x) e^{-\frac{x}{2}} r^p\right)\left(\sum_{q=0}^{\infty} L_q^{2l-1}(y) e^{-\frac{y}{2}} r^q\right) &= (1-r)^{-2k-2l} e^{-\frac{1}{2}\left(\frac{1+r}{1-r}\right)(x+y)} \\ &= (1-r)^{-n} e^{-\frac{1}{2}\left(\frac{1+r}{1-r}\right)(x+y)} \\ &= \sum_{j=0}^{\infty} L_j^{n-1}(x+y) e^{-\frac{x+y}{2}} r^j\end{aligned}$$

Since the power series expansion is unique, by comparing the coefficients we get,

$$\sum_{p+q=j} L_p^{2k-1}(x) L_q^{2l-1}(y) e^{-\frac{x+y}{2}} = L_j^{n-1}(x+y) e^{-\frac{x+y}{2}}.$$

The lemma will follow by taking $x = \frac{1}{2}|a||z|^2$ and $y = \frac{1}{2}|a||w|^2$. □

Since

$$e_r^a(z, w, t) = e^{i\langle a, t\rangle} L_r^{n-1}\left(\frac{1}{2}|a|(|z|^2 + |w|^2)\right) e^{-\frac{1}{4}|a|(|z|^2+|w|^2)}$$

using the Lemma 3.1 we can write,

$$e_r^a(z, w, t) = \sum_{p+q=r} e_{p,q}^a(z, w, t) \qquad \text{for } r = 0, 1, 2, \ldots \tag{4}$$

Hence from (3) we get the following

Proposition 3.1 *If $f \in L^2(N)$ we have*

$$f(z, w, t) = \int_{\mathbb{R}^3\setminus\{0\}} \sum_{p,q} f * e_{p,q}^a(z, w, t)|a|^n \, da$$

where the above expansion converges in $L^2(N)$.

When f is a Schwartz class function on N,

$$f * e_{p,q}^a(z, w, t) = e^{i\langle a, t\rangle} f^a \times_{|a|} \varphi_{p,q}^{|a|}(z, w).$$

The functions $\varphi_{p,q}^{|a|}$ satisfy the orthogonality relation

$$\begin{aligned}\varphi_{p,q}^{|a|} \times_{|a|} \varphi_{r,s}^{|a|}(z, w) &= \int_{\mathbb{C}^{2k}\times\mathbb{C}^{2l}} \varphi_{p,q}^{|a|}(z-u, w-v)\varphi_{r,s}^{|a|}(u, v) e^{\frac{i|a|}{2} Im(z\cdot\overline{u} + w\cdot\overline{v})} \, du\, dv \\ &= \int_{\mathbb{C}^{2k}\times\mathbb{C}^{2l}} \Big(\varphi_p^{|a|,2k}(z-u)\varphi_q^{|a|,2l}(w-v)\varphi_r^{|a|,2k}(u)\varphi_s^{|a|,2l}(v)\end{aligned}$$

$$
\begin{aligned}
& e^{\frac{i|a|}{2} Im(z\cdot\overline{u}+w\cdot\overline{v})}\Big) \, du \, dv \\
&= \varphi_p^{|a|,2k} \times_{|a|} \varphi_r^{|a|,2k}(z) \, \varphi_q^{|a|,2l} \times_{|a|} \varphi_s^{|a|,2l}(w) \\
&= \frac{(2\pi)^n}{|a|^n} \, \delta_{p,r} \, \delta_{q,s} \, \varphi_{p,q}^{|a|}(z,w)
\end{aligned}
$$

which follows from the the orthogonality property of the Lagurre functions

$$
\varphi_i^{\lambda,d} \times_\lambda \varphi_j^{\lambda,d} = \frac{(2\pi)^d}{\lambda^d} \delta_{ij} \varphi_i^{\lambda,d}
$$

As a consequence of the above orthogonality property of $\varphi_{p,q}^{|a|}$, we see that the operator

$$
\mathcal{P}_{p,q} : f \mapsto \int\limits_{\mathbb{R}^3\setminus\{0\}} f * e_{p,q}^a(z,w,t)|a|^n \, da
$$

are projection operators.

Our aim is to write the spectral projection operator $\mathcal{P}_{p,q}$ as a convolution operator and prove its L^p boundedness. To write this operator as a convolution operator, we define the kernel,

$$
\begin{aligned}
P_{p,q}(z,w,t) &= \int\limits_{\mathbb{R}^3\setminus\{0\}} e_{p,q}^a(z,w,t) \, |a|^n \, da \\
&= \int\limits_{\mathbb{R}^3\setminus\{0\}} e^{-i\langle a,t\rangle} \, \varphi_{p,q}^{|a|}(z,w)|a|^n \, da.
\end{aligned}
$$

Since $\varphi_{p,q}^{|a|}(z,w) = L_p^{2k-1}\left(\frac{|a||z|^2}{2}\right) L_q^{2l-1}\left(\frac{|a||w|^2}{2}\right) e^{-\frac{|a|}{4}(|z|^2+|w|^2)}$, the kernel $P_{p,q}(z,w,t)$ is a linear combination of functions of the form

$$
P_{p,q}^{i,j}(z,w,t) = |z|^{2i}|w|^{2j} \int\limits_{\mathbb{R}^3\setminus\{0\}} e^{-i\langle a,t\rangle} \, e^{-\frac{|a|}{4}(|z|^2+|w|^2)} \, |a|^{n+i+j} \, da,
$$

$i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$. A simple change of variables shows that

$$
P_{p,q}^{i.j}(sz, sw, , s^2t) = s^{-(2n+6)} P_{p,q}^{j}(z,w,t),
$$

which is the required homogeneity for singular integral operators on $N = \mathbb{C}^n \oplus \mathbb{R}^3$

Since $P_{p,q}^{i,j}(z,w,t)$ is radial in z and w, we can write

$$
P_{p,q}^{i,j}(z,w,t) = c|z|^{2i}|w|^{2j} \int_0^\infty \frac{J_{\frac{1}{2}}(\lambda|t|)}{(\lambda|t|)^{\frac{1}{2}}} e^{-\frac{\lambda}{4}(|z|^2+|w|^2)} \lambda^{n+i+j+2} \, d\lambda,
$$

where c is a constant. We prove that $P_{p,q}(z,w,t)$ is a Calderón-Zygmund kernel by showing that each $P_{p,q}^{j}(z,w,t)$ is. Since $P_{p,q}^{j}(z,w,t)$ is homogeneous of degree $-Q = -2n-6$ and belongs to $C^\infty(N\setminus\{0\})$, by the Lemma 2.2 in [10], the required cancellation condition will be obtained from the following lemma.

Lemma 3.2 *For $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$,*

$$
\int\limits_{\mathbb{C}^{2k}} \int\limits_{\mathbb{C}^{2l}} P_{p,q}^{i,j}(z,w,1) \, dz dw = 0.
$$

Proof We start with the integral

$$I(\tau) = \int_0^\infty \frac{J_{\frac{1}{2}}(\lambda)}{\lambda^{\frac{1}{2}}} \, e^{-\tau\lambda} \, \lambda^2 \, d\lambda, \; \tau > 0. \tag{5}$$

Then for any $t \in \mathbb{R}^3$ such that $|t| = 1$, it is easy to see that (up to a constant)

$$I(\tau) = \int\limits_{\mathbb{R}^3} e^{-i\langle x,t\rangle} \, e^{-\tau|x|} \, dx.$$

The above equals the Poisson kernel,

$$c \frac{\tau}{(1+\tau^2)^2}$$

for some constant c.

Now,

$$\int_0^\infty \frac{J_{\frac{1}{2}}(\lambda)}{\lambda^{\frac{1}{2}}} \, e^{-\tau\lambda} \, \lambda^{n+i+j+2} \, d\lambda = \frac{d^{n+i+j}}{d\tau^{n+i+j}} \left(I(\tau)\right) \\ = I^{(n+i+j)}(\tau).$$

Hence, to prove the lemma, we need to show that

$$\int\limits_{\mathbb{C}^{2k}} \int\limits_{\mathbb{C}^{2l}} |z|^{2i} |w|^{2j} \, I^{(n+i+j)} \left(\frac{|z|^2 + |w|^2}{2} \right) \, dz \, dw = 0, \;\; j = 0, 1, 2, \ldots, k.$$

Since the integrand is radial in z and w, this reduces to showing that

$$\int\limits_0^\infty \int\limits_0^\infty I^{(n+i+j)} \left(\frac{r^2 + s^2}{4} \right) \, r^{4k+2i-1} s^{4l+2j-1} \, dr \, ds = 0.$$

By taking $r = \rho\cos\theta, \; s = \rho\sin\theta, \rho > 0$ and $0 \le \theta \le \frac{\pi}{2}$, we obtain,

$$\begin{aligned} &\int\limits_0^\infty \int\limits_0^\infty I^{(n+i+j)} \left(\frac{r^2 + s^2}{4} \right) r^{4k+2i-1} s^{4l+2j-1} \, dr \, ds \\ &\qquad = \left(\int\limits_0^{\frac{\pi}{2}} \cos^{4k+2i-1}(\theta) \sin^{4l+2j-1}(\theta) \, d\theta \right) \\ &\qquad\quad \times \left(\int\limits_0^\infty I^{(n+i+j)} \left(\frac{\rho^2}{4} \right) \rho^{4k+4l+2i+2j-2} d\rho \right) \\ &\qquad = 2^{4k+4l+2i+2j-2} \left(\int\limits_0^{\frac{\pi}{2}} \cos^{4k+2i-1}(\theta) \sin^{4l+2j-1}(\theta) \, d\theta \right) \\ &\qquad\quad \times \left(\int\limits_0^\infty I^{(n+i+j)}(\rho) \; \rho^{n+i+j-1} d\rho \right) \end{aligned}$$

Now, writing

$$\Psi(\rho) = \frac{1}{(1+\rho^2)^2},$$

we get,

$$I^{(n+i+j)}(\rho) = \rho\Psi^{(n+i+j)}(\rho) + (n+i+j)\Psi^{(n+i+j-1)}(\rho).$$

Hence

$$\begin{aligned}\int_0^\infty I^{(n+i+j)}(\rho)\, \rho^{n+i+j-1}\, d\rho &= \int_0^\infty \Psi^{(n+i+j)}(\rho)\, \rho^{n+i+j}\, d\rho \\ &\quad + (n+i+j)\int_0^\infty \Psi^{(n+i+j-1)}(\rho)\, \rho^{n+i+j-1}\, d\rho \\ &= \lim_{\rho\to\infty} \rho^{n+i+j}\Psi^{(n+j-1)}(\rho)\end{aligned}$$

which is easily verified to be zero. This proves the lemma. □

Since the kernel is radial in t, it follows from the Lemma 3.2, that

$$\int_{\mathbb{C}^k}\int_{\mathbb{C}^l}\int_{S^2} P^{i,j}_{p,q}(z,w,t)\, dz\, dw\, d\sigma(t) = 0,\ \ j = 0, 1, 2, \ldots, p+q.$$

where σ is the normalised surface measure on the unit sphere in $\mathbb{R}^3$. We need the following well-known theorem.

Theorem 3.1 *Let N be a connected, simply connected H-type group. Let $K \in C^\infty(G \setminus \{0\})$ be a kernel which is homogeneous of degree $-Q$. Assume that K satisfies the cancellation condition*

$$\int_{a<|(z,t)|<b} K(z,t)\, dzdt = 0, \forall\ 0 < a < b < \infty.$$

Then the singular integral operator

$$f \mapsto f * K$$

is bounded on $L^2(N)$.

Proof This is a special case of Theorem 1 in [12, p. 494]. □

The next theorem says that for the above operators, the L^2-boundedness imply the L^p-boundedness.

Theorem 3.2 *Let N be an H-type group and $K \in C^\infty(N \setminus \{0\})$ be a kernel that satisfy the cancellation condition and is homogeneous of degree $-Q$. If the operator $f \mapsto f * K$ is bounded on $L^2(N)$, then it is bounded on $L^p(N)$ for $1 < p < \infty$.*

Proof Follows from Theorem 5.1 of [13]. □

From Theorem 3.1 and Theorem 3.2, we obtain the following result.

Theorem 3.3 *For each $(p.q)$ the spectral projection operator*

$$\mathcal{P}_{p,q} : f \mapsto \int_{\mathbb{R}^3\setminus\{0\}} f * e^a_{p,q}(z,w,t)|a|^n\, da$$

is a bounded operator on $L^r(N)$ for $1 < r < \infty$.

Next we show the Abel summability of the spectral decomposition for $f \in L^p(N)$

Theorem 3.4 *For $2 \leq p < \infty$ we have the Abel summability*

$$\lim_{s\to 1} \sum_{d=0}^{\infty} s^d \sum_{p+q=d} \int_{\mathbb{R}^3\setminus\{0\}} f * e^a_{p,q}(z,w,t)|a|^n \, da = f(z,w,t)$$

Proof From Theorem 3.2 in [10] we have, for $2 \leq p < \infty$ and $f \in L^p(N)$,

$$\lim_{s\to 1} \sum_{d=0}^{\infty} s^d \int_{\mathbb{R}^m} f * e^a_k(z,t) \, |a|^n \, da = f(z,t)$$

in the L^p norm. Then the result follows from (4). □

4 Spherical means and injectivity

Recall that $U = Sp(k) \times Sp(l)$. An orbit of U is of the form $S_r \times S_s$, where S_r is the sphere of radius r in $\mathbb{C}^{2k}$ and S_s is the sphere of radius s in $\mathbb{C}^{2l}$. Let $\mu_{r,s}$ be the normalized surface measure on the product of S_r and S_s.

If f is of the form $f(z,w,t) = e^{i<a,t>} g(z)h(w)$, for $(z,w,t) \in N$ then

$$\begin{aligned}
f * \mu_{r,s}(z,w,t) &= \int_{\mathbb{C}^k}\int_{\mathbb{C}^l} f\left((z,w,t)(\xi,\eta,0)^{-1}\right) d\mu_{r,s}(\xi,\eta) \\
&= \int_{\mathbb{C}^k}\int_{\mathbb{C}^l} f\left(z-\xi, w-\eta, t-\frac{1}{2}[z,\xi]-\frac{1}{2}[w,\eta]\right) d\mu_{r,s}(\xi,\eta) \\
&= c\int_{\mathbb{C}^k}\int_{\mathbb{C}^l} g(z-\xi)h(w-\eta)e^{i\langle a,t-\frac{1}{2}[z,\xi]-\frac{1}{2}[w,\eta]\rangle} \, d\mu_r(\xi)\, d\mu_s(\eta) \\
&= c\, e^{i\langle a,t\rangle} \int_{\mathbb{C}^k} g(z-\xi)e^{-i\langle a,\frac{1}{2}[z,\xi]\rangle} \, d\mu_r(\xi) \\
&\quad \times \int_{\mathbb{C}^l} h(w-\eta)e^{-i\langle a,\frac{1}{2}[w,\eta]\rangle} \, d\mu_s(\eta) \\
&= c\, e^{i\langle a,t\rangle} \left(g \times_{|a|} \mu_r\right)(z) \left(h \times_{|a|} \mu_s\right)(w)
\end{aligned}$$

Since (see [2, Proposition 5.1])

$$\varphi_k^{|a|} \times_{|a|} \mu_r(z) = \frac{k!(n-1)!}{(k+n-1)!} \varphi_k^{|a|}(r)\varphi_k^{|a|}(z)$$

we obtain,

$$e^a_{p,q} * \mu_{r,s}(z,w,t) = e^a_{p,q}(r,s,0)e^a_{p,q}(z,w,t) \tag{6}$$

We need the following result.

Theorem 4.1 *Let $f \in L^p(\mathbb{R}^m)$ and support of $\widehat{f}$ (distributional Fourier transform of f) is contained in a C^1-manifold of dimension d, $0 < d < m$. Then f vanishes identically provided $1 \leq p \leq \frac{2m}{d}$. If $d = 0$, f vanishes identically provided $1 \leq p < \infty$.*

Proof When the support is a sphere, this follows from [2] (see Lemma 2.2 and Theorem 2.2 there). For the general case see [14] (Theorem 1). □

Theorem 4.2 *Let $f \in L^p(N)$, $1 \leq p \leq 3$. If $f * \mu_{r,s} = 0$ then $f \equiv 0$.*

Proof Let $f \in L^p(N)$, $1 \leq p \leq 3$ and assume that $f * \mu_{r,s}$ vanishes identically. Convolving f with a smooth approximate identity, we may assume that $f \in L^p$ for $2 \leq p \leq 3$. From (6), the spectral decomposition of $f * \mu_{r,s}$ is given by

$$f * \mu_{r,s}(z, w, t) = \sum_{p,q=0}^{\infty} \int_{\mathbb{R}^3 \setminus \{0\}} c\, e^a_{p,q}(r, s, 0)\ f * e^a_{p,q}(z, w, t)\ |a|^n\, da.$$

If $f * \mu_{r,s}(z, w, t) = 0$ for all (z, w, t), by Theorem 3.4,

$$\lim_{u \to 1^-} \sum_{d=0}^{\infty} u^d \sum_{p+q=d} \int_{\mathbb{R}^3 - \{0\}} e^a_{p,q}(r, s, 0)\ f * e^a_{p,q}(z, w, t)\ |a|^n\, da = 0$$

where the convergence is in $L^p(N)$. Applying the (p, q)-th spectral projection operator $\mathcal{P}_{p,q}$ and using Theorem 3.3 we obtain that, for all $(z, w, t) \in N$ and for all $p, q = 1, 2, \ldots$,

$$\int_{\mathbb{R}^3 \setminus \{0\}} \varphi^{|a|}_p(r) \varphi^{|a|}_q(s) f^a \times_{|a|} \varphi^{|a|}_{p,q}(z, w)\ e^{-i\langle a,t\rangle} |a|^n\, da = 0.$$

Arguing as in [2, p.276] (also see [4, pp.257-258]), we obtain that, for almost all $(z, w) \in \mathbb{C}^k \times \mathbb{C}^l$, the support of $f^a \times_{|a|} \varphi^{|a|}_{p,q}(z, w)|a|^n$, the distributional Fourier transform of $\mathcal{P}_{p,q} f(z, w, \cdot)$, is contained in the zero set of $a \mapsto L^{2k-1}_p(\frac{1}{2}|a|r^2) L^{2l-1}_p(\frac{1}{2}|a|s^2)$, which is a finite union of spheres in $\mathbb{R}^m$. But this implies, by Theorem 4.1, that $\mathcal{P}_{p,q} f(z, w, t)$ is zero as $\mathcal{P}_{p,q} f \in L^p$ for $1 \leq p \leq 3$. This completes the proof. $\square$

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Declarations

Conflict of interest The author states that there is no conflict of interest.

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