Indian J Pure Appl Math (2024) 55:1329–1349
https://doi.org/10.1007/s13226-023-00442-x

ORIGINAL RESEARCH

Numerical solution of a time dependent singularly perturbed delay differential equation on an exponentially graded mesh

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Received: 4 October 2022 / Accepted: 1 June 2023 / Published online: 13 June 2023

Abstract In this article, we consider a singularly perturbed time-dependent reaction-diffusion problem with a large delay in space direction. The problem possess two boundary layers and also an interior layer. To find the numerical solution, we proposed a method which is comprised of an implicit Euler method in the time direction and a finite difference scheme on the exponentially graded mesh. The term containing the delay is treated using the theory of interpolation. The numerical method is shown to be first order parameter uniform convergent in space and time directions in the maximum norm. Theoretical estimates support the obtained numerical results.

Keywords Singular perturbation problem · Partial differential equations · Delay differential equations · Finite difference method · Exponentially graded mesh · Parameter uniform convergence

1 Introduction

Singular perturbation problem is defined as the differential equation in which the highest order derivative is multiplied with a perturbation parameter $\varepsilon \to 0$. For such problems difficulty arise in obtaining the solution due to the presence of boundary layers in the solution. In the study of bistable devices, human pupil-light reflex, variational problems related to control theory are few areas in which singularly perturbed delay differential equations (SPDDE) arise. Stein [1] studied a mathematical model related to neuronal variability which turns out to be a singularly perturbed differential-difference equation (SP-DDE). Later the model was generalized by Musila and Lánský [2]. Lange and Miura [3–5] studied various singularly perturbed boundary value problems in differential-difference equations and observed that small shift terms produce large amplitude and multiphase behaviour over some parts of the domain. These studies attracted many researcher to work on SP-DDE.

As far as the large delay problems are concerned very little literature is available. Amiraliyev and Erdogan [6] studied the first order SPDDE with a fixed delay. Later Amiraliyeva et al. [7] approximated the solution of quasilinear SPDDE on a uniform mesh. Nicasi and Xenophontos [8] considered a second-order singularly perturbed reaction-diffusion problem with constant delay at $x = 1$ and approximated the solution by using hp finite element method. Subburayan and Ramanujam [9] used an initial value technique on Shishkin mesh for the numerical solution of a second-order SPDDE with discontinuous convection-diffusion coefficients. Zarin [10] developed a non-symmetric discontinuous Galerkin FEM with interior penalties on singularly perturbed

Communicated by G.D. Veerappa Gowda.

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reaction-diffusion problem with constant delay on layer adapted mesh. Rai and Sharma [11] studied SPDDE with a turning point. In [12,13], the authors discussed numerical methods for SP-DDE in which shift terms are of any order from $o(\varepsilon)$ to $O(\varepsilon)$. To the best of our knowledge, very few papers are on time dependent SPDDE of reaction-diffusion type with large delay in space. Bansal and Sharma [14] proposed a finite difference scheme on Shishkin mesh. Kumar and Kumari [15] constructed a finite difference scheme for time-dependent singularly perturbed reaction-diffusion problem on Shishkin mesh where the solution is shown to be first order in space and second-order in the time direction.

We proposed a finite difference scheme for 1D-singularly perturbed parabolic reaction-diffusion problem with delay in space. The method comprises of Euler method in time and a finite difference scheme in space direction on an exponentially graded mesh and the delay term is treated using interpolation. The exponentially graded mesh is constructed in such a way that it adapts the boundary layers and the interior layer. The error estimates are obtained and shown to be the first order of uniform convergence in space and time directions.

The paper is arranged as follows: In Section 2, we stated the problem under consideration and is semi-discretized by using an implicit Euler method on an equidistant mesh in the time direction. Also, exponentially graded mesh is generated in space direction and a finite difference method is constructed on this mesh. In Section 3, theoretical estimates are obtained for the proposed method. Numerical results which shows a parameter uniform convergence of first order in space and time directions are presented for two test problems in Sect. 4. Conclusions are given at the end of last section.

In this paper C is a generic constant which is positive and is independent of the perturbation parameter and the mesh parameters $N_x,\ N_t$. The maximum norm for a function f in a domain $\mathfrak{D} \in \mathbb{R}^2$ is defined as $\|f\| = \max\limits_{(x,t)\in\mathfrak{D}} |f(x,t)|$.

2 Statement of the problem

For our study, we consider the following singularly perturbed parabolic delay partial differential equation of reaction-diffusion type on a rectangular domain. For fixed T on $\mathfrak{D} = \Omega \times \Lambda$ where $\Omega = (0, 2),\ \Lambda = (0, T)$ and $\partial\mathfrak{D} = \bar{\mathfrak{D}} - \mathfrak{D} = \{(0, t) \cup (x, 0) \cup (2, t) \cup (x, T) : 0 \le x \le 2,\ 0 \le t \le T\}$,

$$\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + \alpha(x)u(x,t) + \beta(x)u(x-1,t) = g(x,t), \quad (x,t) \in \mathfrak{D}, \tag{1}$$

with the initial and boundary conditions

$$u(x,0) = u_0(x), \qquad \forall\, x \in \bar{\Omega} = [0,2], \tag{2}$$

$$u(x,t) = \varphi(x,t), \qquad \forall\, (x,t) \in \mathfrak{D}_l = \{-1 \le x \le 0,\ t \in (0,T]\}, \tag{3}$$

$$u(2,t) = \vartheta(t), \qquad \forall\, (2,t) \in \mathfrak{D}_r = \{(2,t) : t \in [0,T]\} \tag{4}$$

where, $0 < \varepsilon \ll 1$. The functions $\alpha(x),\ \beta(x),\ g(x,t),\ u_0(x),\ \varphi(x,t),\ \vartheta(t)$ assumed to be sufficiently smooth, bounded and independent of perturbation parameter ε. Further, it is assumed that

$$\alpha(x) + \beta(x) \ge 2\kappa > 0, \quad \forall x \in \bar{\Omega} \tag{5}$$

and $\beta(x) < 0$, where κ is positive constant. The solution has boundary layer of boundary width of $O(\sqrt{\varepsilon})$ is expected in the right neighbourhood of $x = 0$ and also in the right neighbourhood of $x = 1$ and left neighbourhood of $x = 2$ and also in the left neighbourhood of $x = 1$.
If we let $\Omega_- = (0, 1]$, $\Omega_+ = (1, 2)$, we have $\mathfrak{D}_1 = \Omega_- \times \Lambda$ and $\mathfrak{D}_2 = \Omega_+ \times \Lambda$. Since $u(x,t)$ is known in $[-1, 0] \times \Lambda$, the problem (1)–(4) can be rewritten as

$$\mathfrak{L}u(x,t) = G(x,t) \tag{6}$$

where

$$\mathfrak{L}u(x,t) = \begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon \dfrac{\partial^2 u}{\partial x^2} + \alpha(x)u(x,t), & \forall (x,t) \in \mathfrak{D}_1 \\ \dfrac{\partial u}{\partial t} - \varepsilon \dfrac{\partial^2 u}{\partial x^2} + \alpha(x)u(x,t) + \beta(x)u(x-1,t), & \forall (x,t) \in \mathfrak{D}_2 \end{cases} \tag{7}$$

and

$$G(x,t) = \begin{cases} g(x,t) - \beta(x)\varphi(x-1,t), & \forall (x,t) \in \mathfrak{D}_1 \\ g(x,t), & \forall (x,t) \in \mathfrak{D}_2 \end{cases} \tag{8}$$

with

$$u(x,0) = u_0(x), \quad \forall x \in \bar{\Omega},$$

$$u(0,t) = \varphi(0,t), \quad u(2,t) = \vartheta(t).$$

For the existence and uniqueness of the solution (1)–(4), the given data is assumed to be Hölder continuous and satisfy the required compatibility conditions

$$\varphi(0,0) = u_0(0), \quad \vartheta(0) = u_0(2), \tag{9}$$

$$\frac{\partial \varphi(0,0)}{\partial t} - \varepsilon \frac{\partial^2 u_0(0)}{\partial x^2} + \alpha(0)u_0(0) + \beta(0)\varphi(-1,0) = g(0,0), \tag{10}$$

$$\frac{\partial \vartheta(0)}{\partial t} - \varepsilon \frac{\partial^2 u_0(2)}{\partial x^2} + \alpha(2)u_0(2) + \beta(2)u(1,0) = g(2,0). \tag{11}$$

Using the compatibility conditions (9)–(11), one can prove the estimate [c.f. [16]]

$$|u(x,t) - u(x,0)| = |u(x,t) - u_0(x)| \le Ct, \quad \forall (x,t) \in \bar{\mathfrak{D}}. \tag{12}$$

Lemma 2.1 [14] *The solution $u(x,t)$ of the problem* (1)–(4) *is bounded*

$$|u(x,t)| \le C, \quad \forall (x,t) \in \bar{\mathfrak{D}}.$$

Lemma 2.2 *Let $\xi(x,t) \in C^{2,1}(\bar{\mathfrak{D}})$ such that $\xi(x,t) \ge 0$, $\forall (x,t) \in \partial\mathfrak{D}$ and $\mathfrak{L}\xi(x,t) \ge 0$, $\forall (x,t) \in \mathfrak{D}$ it implies $\xi(x,t) \ge 0$, $\forall (x,t) \in \bar{\mathfrak{D}}$.*

Proof We can prove this by using the method of contradiction principle. If there is a point say $\xi(x^*,t^*) = \min_{(x,t)\in\mathfrak{D}} \xi(x,t) < 0$ for some (x^*,t^*), then we have $\xi_{xx}(x^*,t^*) > 0$ and $\xi_t(x^*,t^*) = \xi_x(x^*,t^*) = 0$. It is clear that the point (x^*,t^*) does not lie on the boundary $\partial\mathfrak{D}$. We have the following two cases:
Case I: For $(x,t) \in \mathfrak{D}_1$

$$\mathfrak{L}\xi(x^*,t^*) = \frac{\partial \xi}{\partial t} - \varepsilon \frac{\partial^2 \xi}{\partial x^2} + \alpha(x^*)\xi(x^*,t^*) = -\varepsilon \frac{\partial^2 \xi}{\partial x^2} + \alpha(x^*)\xi(x^*,t^*) < 0$$

Case II: For $(x,t) \in \mathfrak{D}_2$

$$\mathfrak{L}\xi(x^*,t^*) = \frac{\partial \xi}{\partial t} - \varepsilon \frac{\partial^2 \xi}{\partial x^2} + \alpha(x^*)\xi(x^*,t^*) + \beta(x^*)\xi(x^*-1,t^*) \le -\varepsilon \frac{\partial^2 \xi}{\partial x^2} + (\alpha(x^*) + \beta(x^*))\xi(x^*,t^*) < 0$$

we obtain the contradiction from both the cases. Thus $\xi(x^*,t^*) \ge 0$, which implies $\xi(x,t) \ge 0 \; \forall (x,t) \in \bar{\mathfrak{D}}$. □

Lemma 2.3 *Under the assumption of $u_0(x) = \varphi(x,t) = \vartheta(t) = 0$, the derivative bound of the solution $u(x,t)$ with respect to t given by*

$$\left\| \frac{\partial^i u}{\partial t^i} \right\| \le C, \quad i = 0, 1, 2.$$

Proof The proof can be followed from [15]. □

2.1 Time discretization

The considered problem (1)–(4) is semi-discretized on the time domain $[0, T]$ using backward Euler method with uniform mesh in time such that $\Lambda^{N_t} = \{t_n = n\Delta t,\ n = 0, 1, \dots, N_t\}$, where N_t is the number of mesh elements considered in t-direction. The semi-discrete problem at $(n+1)$-th time level given by

$$\mathfrak{L}^{N_t} u^{n+1}(x) = G(x, t_{n+1}) \tag{13}$$

where

$$\mathfrak{L}^{N_t} u^{n+1}(x) = \begin{cases} -\varepsilon\Delta t \dfrac{d^2u^{n+1}}{dx^2} + \widehat{\alpha}(x)u^{n+1}(x), & x \in (0, 1] \\ -\varepsilon\Delta t \dfrac{d^2u^{n+1}}{dx^2} + \widehat{\alpha}(x)u^{n+1}(x) + \beta(x)\Delta t u^{n+1}(x-1), & x \in (1, 2) \end{cases} \tag{14}$$

and

$$G(x, t_{n+1}) = \begin{cases} \Delta t g(x, t_{n+1}) - \beta(x)\Delta t \varphi(x-1, t_{n+1}) + u^n(x), & x \in (0, 1] \\ \Delta t g(x, t_{n+1}) + u^n(x), & x \in (1, 2) \end{cases} \tag{15}$$

$$u^0(x) = u_0(x), \qquad x \in \bar{\Omega} = [0, 2], \tag{16}$$

$$u^{n+1}(x) = \varphi(x, t_{n+1}), \qquad -1 \le x \le 0,\ 1 \le n+1 \le N_t, \tag{17}$$

$$u^{n+1}(2) = \vartheta(t_{n+1}), \qquad 1 \le n+1 \le N_t, \tag{18}$$

where, $\widehat{\alpha}(x) = 1 + \Delta t\alpha(x)$ and $u^n(x)$ is the semi-discrete solution at n-th time level. It is clearly seen that the operator $\mathfrak{L}^{N_t}$ satisfies the maximum principle and so from [17], we have

$$\|(\mathfrak{L}^{N_t})^{-1}\|_\infty \le \frac{1}{1 + 2\kappa\Delta t},$$

which ensures the method (13)–(18) is stable.

Lemma 2.4 *Let $\xi^{n+1}(x)$ be a smooth function such that $\xi^{n+1}(0) \ge 0$ and $\xi^{n+1}(2) \ge 0$ then $\mathfrak{L}^{N_t}\xi^{n+1}(x) \ge 0\ \forall\, x \in \Omega$ it implies $\xi^{n+1}(x) \ge 0\ \forall\, x \in \bar{\Omega}$.*

Proof By using the similar approach followed in Lemma 2.2, the above result can be proved easily. □

Lemma 2.5 [14] *Let $u^{n+1}(x)$ be the solution of the semi-discretized problem* (13)–(18)*, then the solution $u^{n+1}(x)$ satisfies*

$$|u^{n+1}(x)| \le \max\left\{|u^{n+1}(0)|, \frac{1}{1 + 2\kappa\Delta t}\|G\|, |u^{n+1}(2)|\right\}, \quad \forall x \in \bar{\Omega}.$$

The error estimates for local truncation error(LTE) defined by $e_{n+1} = u(x, t_{n+1}) - u^{n+1}(x)$ and the global error E_n for the time semi-discretization (13)–(18) is presented in following lemma.

Lemma 2.6 [18] *In the solution of* (13)–(18) *the LTE estimate is given by*

$$\|e_{n+1}\|_\infty \le C(\Delta t)^2,$$

and the global error satisfies

$$\|E_n\|_\infty \le C\Delta t, \quad n \le T/\Delta t.$$

From Lemma 2.6, it is clear that the semi-discretization process (13)–(18) is first-order convergent in time.

The solution $u^{n+1}(x)$ of the semi-discretized problem (13)–(18) can be decomposed in its regular and singular components as

$$u^{n+1}(x) = v^{n+1}(x) + w^{n+1}(x),$$

where $v^{n+1}(x)$ be regular component further decompose as

$$v^{n+1}(x) = \begin{cases} v_1^{n+1}(x), & 0 < x < 1, \\ v_2^{n+1}(x), & 1 < x < 2, \end{cases} \tag{19}$$

for $x \in (0, 1^-)$, $v_1^{n+1}(x)$ satisfies the following differential equation

$$-\varepsilon \Delta t \frac{d^2 v_1^{n+1}}{dx^2} + \widehat{\alpha}(x) v_1^{n+1}(x) = \Delta t g(x, t_{n+1}) - \beta(x) \Delta t \varphi(x-1, t_{n+1}) + u^n(x),$$

$$v_1^{n+1}(0) = v_0(0, t_{n+1}), \quad v_1^{n+1}(1^-) = \frac{1}{\widehat{\alpha}(1)} (\Delta t g(1, t_{n+1}) - \beta(1) \Delta t \varphi(0, t_{n+1}) + u^n(1)),$$

where $v_0(x, t_{n+1})$ be the solution of reduced problem.

For $x \in (1^+, 2)$, $v_2^{n+1}(x)$ satisfies the following differential equation

$$-\varepsilon \Delta t \frac{d^2 v_2^{n+1}}{dx^2} + \widehat{\alpha}(x) v_2^{n+1}(x) + \beta(x) \Delta t v_2^{n+1}(x-1) = \Delta t g(x, t_{n+1}) + u^n(x),$$

$$v_2^{n+1}(1^+) = \frac{1}{\widehat{\alpha}(1)} (\Delta t g(1, t_{n+1}) - \beta(1) \Delta t u_0(0) + u^n(1)), \quad v_2^{n+1}(2) = v_0(2, t_{n+1}),$$

where $w^{n+1}(x)$ be singular component further decompose as

$$w^{n+1}(x) = \begin{cases} w_1^{n+1}(x), & 0 < x < 1, \\ w_2^{n+1}(x), & 1 < x < 2, \end{cases} \tag{20}$$

$$-\varepsilon \Delta t \frac{d^2 w_1^{n+1}}{dx^2} + \widehat{\alpha}(x) w_1^{n+1}(x) = 0, \quad \text{for } x \in (0, 1^-)$$

$$-\varepsilon \Delta t \frac{d^2 w_2^{n+1}}{dx^2} + \widehat{\alpha}(x) w_2^{n+1}(x) + \beta(x) \Delta t w_2^{n+1}(x-1) = 0, \quad \text{for } x \in (1^+, 2)$$

$$w_1^{n+1}(0) = u^{n+1}(0) - v_1^{n+1}(0),$$

$$w_2^{n+1}(2) = u^{n+1}(2) - v_2^{n+1}(2),$$

$$v_1^{n+1}(1^-) + w_1^{n+1}(1^-) = v_2^{n+1}(1^+) + w_2^{n+1}(1^+),$$

$$(v_1^{n+1}(1^-))_x + (w_1^{n+1}(1^-))_x = (v_2^{n+1}(1^+))_x + (w_2^{n+1}(1^+))_x.$$

Further, we decompose it as

$$w_1^{n+1}(x) = w_{1,L}^{n+1}(x) + w_{1,R}^{n+1}(x),$$
$$w_2^{n+1}(x) = w_{2,L}^{n+1}(x) + w_{2,R}^{n+1}(x),$$

$$-\varepsilon\Delta t\frac{d^2w_{1,L}^{n+1}}{dx^2} + \widehat{\alpha}(x)w_{1,L}^{n+1}(x) = 0, \quad \text{for } x \in (0, 1^-)$$

$$w_{1,L}^0(x) = 0, \quad w_{1,L}^{n+1}(0) = w_1^{n+1}(0), \quad w_{1,L}^{n+1}(1) = 0,$$

$$-\varepsilon\Delta t\frac{d^2w_{1,R}^{n+1}}{dx^2} + \widehat{\alpha}(x)w_{1,R}^{n+1}(x) = 0, \quad \text{for } x \in (0, 1^-)$$

$$w_{1,R}^0(x) = 0, \quad w_{1,R}^{n+1}(0) = 0, \quad w_{1,R}^{n+1}(1) = M_1,$$

$$-\varepsilon\Delta t\frac{d^2w_{2,L}^{n+1}}{dx^2} + \widehat{\alpha}(x)w_{2,L}^{n+1}(x) + \beta(x)\Delta t w_{2,L}^{n+1}(x-1) = 0, \quad \text{for } x \in (1^+, 2)$$

$$w_{2,L}^0(x) = 0, \quad w_{2,L}^{n+1}(1) = M_2, \quad w_{2,L}^{n+1}(2) = 0,$$

$$-\varepsilon\Delta t\frac{d^2w_{2,R}^{n+1}}{dx^2} + \widehat{\alpha}(x)w_{2,R}^{n+1}(x) + \beta(x)\Delta t w_{2,R}^{n+1}(x-1) = 0, \quad \text{for } x \in (1^+, 2)$$

$$w_{2,R}^0(x) = 0, \quad w_{2,R}^{n+1}(1) = 0, \quad w_{2,R}^{n+1}(2) = w_2^{n+1}(2).$$

Here, M_1 and M_2 are chosen constants and satisfy the jump condition at $(1, t_{n+1})$, $\forall\ 1 \le n+1 \le N_t + 1$.

Now we give the derivative bounds of the solution of (13)–(18) and also for its regular and singular components.

Lemma 2.7 [19,20] *The solution $u^{n+1}(x)$ of the semi-discretized problem* (13)–(18) *satisfies for $i = 0, 1, 2, 3$*

$$\left|\frac{d^i u^{n+1}(x)}{dx^i}\right| \le C\left(1 + \varepsilon^{-i/2}e_1(x,\kappa)\right), \quad \forall x \in \Omega_-$$

$$\left|\frac{d^i u^{n+1}(x)}{dx^i}\right| \le C\left(1 + \varepsilon^{-i/2}e_2(x,\kappa)\right), \quad \forall x \in \Omega_+$$

where,

$$e_1(x,\kappa) = \exp\left(-\sqrt{\frac{\kappa}{\varepsilon}}x\right) + \exp\left(-\sqrt{\frac{\kappa}{\varepsilon}}(1-x)\right),$$

$$e_2(x,\kappa) = \exp\left(-\sqrt{\frac{\kappa}{\varepsilon}}(x-1)\right) + \exp\left(-\sqrt{\frac{\kappa}{\varepsilon}}(2-x)\right).$$

Lemma 2.8 [19,20] *The derivative bounds of the regular component $v^{n+1}(x)$ and singular component $w^{n+1}(x)$ satisfy the following estimates for $i = 0, 1, 2, 3$*

$$\left|\frac{d^i v^{n+1}(x)}{dx^i}\right| \le C\left(1 + \varepsilon^{-(i-2)/2} e_1(x,\kappa)\right), \quad \forall x \in \Omega_-$$

$$\left|\frac{d^i v^{n+1}(x)}{dx^i}\right| \le C\left(1 + \varepsilon^{-(i-2)/2} e_2(x,\kappa)\right), \quad \forall x \in \Omega_+$$

and

$$\left|\frac{d^i w^{n+1}(x)}{dx^i}\right| \le C\varepsilon^{-i/2} e_1(x,\kappa), \quad \forall x \in \Omega_-$$

$$\left|\frac{d^i w^{n+1}(x)}{dx^i}\right| \le C\varepsilon^{-i/2} e_2(x,\kappa), \quad \forall x \in \Omega_+.$$

2.2 Exponentially graded mesh construction

Let $N_x \ge 8$ be even and divided the interval $I = [0, 2]$ into N_x sub-intervals using the mesh $\triangle^{N_x} = \{x_i\}_{i=0}^{N_x}$. The mesh generating function $\Phi(\wp) = -\ln(1 - 4\theta\wp)$ generate the non-uniform layer adapted mesh which depends on the perturbation parameter, the function θ defined as [21]

$$\theta = 1 - \exp\left(\frac{-\sqrt{\kappa}}{\tau\varepsilon^{1/2}}\right) \in \mathbb{R}^+$$

where, $\tau \ge 1$ user chosen parameter and $\kappa \in (0, 1)$. First we divide the interval $[0, 2]$ into two intervals $[0, 1]$ and $(1, 2]$, where the mesh points $\{x_i\}_{i=0}^{N_x}$ are defined as,

for $[0, 1]$ such that $x_{\frac{N_x}{2}} = 1$.

$$x_i = \begin{cases} \dfrac{\tau\varepsilon^{1/2}}{4\sqrt{\kappa}}\Phi(\wp_i), & i = 0, 1, \ldots, \frac{N_x}{8} - 1, \\ x_{\frac{N_x}{8}-1} + \dfrac{x_{\frac{3N_x}{8}+1} - x_{\frac{N_x}{8}-1}}{N_x/4+2}\left(i - \dfrac{N_x}{8} + 1\right), & i = \frac{N_x}{8}, \frac{N_x}{8}+1, \ldots, \frac{3N_x}{8}, \\ 1 - \dfrac{\tau\varepsilon^{1/2}}{4\sqrt{\kappa}}\Phi(1-\wp_i), & i = \frac{3N_x}{8}+1, \frac{3N_x}{8}+2, \ldots, \frac{N_x}{2}. \end{cases} \tag{21}$$

We divide the interval $[0, 1]$ into three subintervals as

$$[0,1] = \left[0, x_{\frac{N_x}{8}-1}\right] \cup \left[x_{\frac{N_x}{8}-1}, x_{\frac{3N_x}{8}+1}\right] \cup \left[x_{\frac{3N_x}{8}+1}, 1\right],$$

for interval $(1, 2]$ the mesh points are defined as

$$x_i = \begin{cases} 1 + \dfrac{\tau\varepsilon^{1/2}}{4\sqrt{\kappa}}\Phi(\wp_i - 1), & i = \frac{N_x}{2}+1, \frac{N_x}{2}+2, \ldots, \frac{5N_x}{8} - 1, \\ x_{\frac{N_{5x}}{8}-1} + \dfrac{x_{\frac{7N_x}{8}+1} - x_{\frac{5N_x}{8}-1}}{N_x/4+2}\left(i - \dfrac{5N_x}{8} + 1\right), & i = \frac{5N_x}{8}, \frac{5N_x}{8}+1, \ldots, \frac{7N_x}{8}, \\ 2 - \dfrac{\tau\varepsilon^{1/2}}{4\sqrt{\kappa}}\Phi(2-\wp_i), & i = \frac{7N_x}{8}+1, \frac{7N_x}{8}+2, \ldots, N_x. \end{cases} \tag{22}$$

We divide the interval $(1, 2]$ into three subintervals as

$$(1,2] = \left[x_{\frac{N_x}{2}+1}, x_{\frac{5N_x}{8}}\right] \cup \left[x_{\frac{5N_x}{8}-1}, x_{\frac{7N_x}{8}+1}\right] \cup \left[x_{\frac{7N_x}{8}+1}, 2\right],$$

where $\wp_i = \dfrac{i}{N_x}$, for $i = 0, 1, \ldots, N_x$. The mesh points in $\left[0, x_{\frac{N_x}{8}-1}\right]$, $\left[x_{\frac{3N_x}{8}+1}, 1\right]$, $\left[x_{\frac{N_x}{2}+1}, x_{\frac{5N_x}{8}-1}\right]$, and $\left[x_{\frac{7N_x}{8}+1}, 2\right]$ are gradually distributed, whereas the mesh points in $\left[x_{\frac{N_x}{8}-1}, x_{\frac{3N_x}{8}+1}\right]$ and $\left[x_{\frac{5N_x}{8}-1}, x_{\frac{7N_x}{8}+1}\right]$ are equidistant.

Follows from [22]

$$\max\{e^{-\sqrt{\frac{\kappa}{\varepsilon}}x_{\frac{N_x}{8}-1}}, e^{-\sqrt{\frac{\kappa}{\varepsilon}}(1-x_{\frac{3N_x}{8}+1})}, e^{-\sqrt{\frac{\kappa}{\varepsilon}}(1+x_{\frac{5N_x}{8}-1})}, e^{-\sqrt{\frac{\kappa}{\varepsilon}}(2-x_{\frac{7N_x}{8}+1})}\} \le CN_x^{\frac{-\tau}{4}} \tag{23}$$

and

$$\max|\Phi'| \le CN_x.$$

For the subset of indices

$$\begin{aligned} S_1 &= \{1, 2, \ldots, \frac{N_x}{8} - 1\}, & S_4 &= \{\frac{N_x}{2} + 1, \frac{N_x}{2} + 2, \ldots, \frac{5N_x}{8} - 1\}, \\ S_2 &= \{\frac{N_x}{8}, \frac{N_x}{8} + 1, \ldots, \frac{3N_x}{8}\}, & S_5 &= \{\frac{5N_x}{8}, \frac{5N_x}{8} + 1, \ldots, \frac{7N_x}{8}\}, \\ S_3 &= \{\frac{3N_x}{8} + 1, \frac{3N_x}{8} + 2, \ldots, \frac{N_x}{2}\}, & S_6 &= \{\frac{7N_x}{8} + 1, \frac{7N_x}{8} + 2, \ldots, N_x\}. \end{aligned}$$

The mesh step size $h_i = x_i - x_{i-1}$, for $i = 1, 2, \ldots, N_x$,

$$h_i \le \begin{cases} \dfrac{\tau\varepsilon^{1/2}}{4\kappa}\left[\Phi(\wp_i) - \Phi(\wp_{i-1})\right] \le C\varepsilon^{1/2}N_x^{-1}\max|\Phi'| \le C\varepsilon^{1/2}, & i \in S_j,\ j = 1, 3, 4, 6, \\ CN_x^{-1}, & i \in S_j,\ j = 2, 5. \end{cases} \tag{24}$$

Introducing the mesh characterizing functions $\Psi = \exp(-\Phi)$ and it is observed that $\max|\Psi'| \le 4\theta < 4$. We derive the other properties of mesh width of exponentially graded mesh in the layer region given as

$$h_i \le \frac{\tau\varepsilon^{1/2}}{4\kappa}\max_{\rho\in[\rho_{i-1},\rho_i]}(-\Psi'(\rho)e^{\Phi(\rho)}) \le C\varepsilon^{1/2}N_x^{-1}\max_{\rho\in[\rho_{i-1},\rho_i]}|\Psi'|e^{\Phi(\rho)} \le C\varepsilon^{1/2}N_x^{-1}\exp\left(\frac{4\sqrt{\kappa}x_i}{\tau\varepsilon^{1/2}}\right),\ i \in S_1, \tag{25}$$

similarly, following the above approach we get

$$h_i \le C\varepsilon^{1/2}N_x^{-1}\exp\left(\frac{4\sqrt{\kappa}(1 - x_{i-1})}{\tau\varepsilon^{1/2}}\right), \quad i \in S_3, \tag{26}$$

$$h_i \le C\varepsilon^{1/2}N_x^{-1}\exp\left(\frac{4\sqrt{\kappa}(x_i - 1)}{\tau\varepsilon^{1/2}}\right), \quad i \in S_4, \tag{27}$$

$$h_i \le C\varepsilon^{1/2}N_x^{-1}\exp\left(\frac{4\sqrt{\kappa}(2 - x_{i-1})}{\tau\varepsilon^{1/2}}\right), \quad i \in S_6, \tag{28}$$

these estimates are used to prove the convergence analysis.

3 Numerical scheme

We denote $\Omega^{N_x} = \{x_0, x_1, \ldots, x_{N_x}\} \setminus \{x_0, x_{\frac{N_x}{2}}, x_{N_x}\}$ and we define $\Omega_-^{N_x} = \{x_i\}_{i=0}^{\frac{N_x}{2}-1}$ and $\Omega_+^{N_x} = \{x_i\}_{i=\frac{N_x}{2}+1}^{N_x-1}$ where the layer adapted mesh is constructed in such a way that the more points lies in layer region. The proposed scheme is given by

$$\mathfrak{L}_1^{N_x,N_t}U^{n+1}(x_i) = G_1^{n+1}(x_i), \qquad x_i \in \Omega_-^{N_x}, \tag{29}$$

$$\mathfrak{L}_2^{N_x,N_t}U^{n+1}(x_i) = G_2^{n+1}(x_i), \qquad x_i \in \Omega_+^{N_x}, \tag{30}$$

where,

$$\mathfrak{L}_1^{N_x,N_t}U^{n+1}(x_i) = -\varepsilon\Delta t D^+D^-U^{n+1}(x_i) + \widehat{\alpha}(x_i)U^{n+1}(x_i), \qquad x_i \in \Omega_-^{N_x},$$
$$\mathfrak{L}_2^{N_x,N_t}U^{n+1}(x_i) = -\varepsilon\Delta t D^+D^-U^{n+1}(x_i) + \widehat{\alpha}(x_i)U^{n+1}(x_i) + \beta^{n+1}(x_i)\Delta t U_*^{n+1}(x_i), \qquad x_i \in \Omega_+^{N_x},$$

and

$$G_1^{n+1}(x_i) = \Delta t g(x_i, t_{n+1}) - \beta(x_i)\Delta t\varphi(x_i - 1, t_{n+1}) + U^n(x_i), \qquad x_i \in \Omega_-^{N_x},$$
$$G_2^{n+1}(x_i) = \Delta t g(x_i, t_{n+1}) + U^n(x_i), \qquad x_i \in \Omega_+^{N_x},$$

also we have

$$D^-U_{\frac{N_x}{2}}^{n+1} = D^+U_{\frac{N_x}{2}}^{n+1}, \tag{31}$$

with the initial condition

$$U^0(x_i) = u_0(x_i), \quad \text{for } i = 1, 2, \ldots, N_x - 1, \tag{32}$$

and boundary condition for $i = -\frac{N_x}{2}, -\frac{N_x}{2}+1, \ldots, 0,$ and $n = 0, 1, \ldots, N_t - 1$

$$U^{n+1}(x_i) = \varphi(x_i, t_{n+1}), \qquad U^{n+1}(x_{N_x}) = \vartheta(t_{n+1}), \tag{33}$$

where, we define

$$D^-D^+U^{n+1}(x_i) = \frac{2}{h_i + h_{i+1}}\Big(D^+U^{n+1}(x_i) - D^-U^{n+1}(x_i)\Big),$$

$$D^+U^{n+1}(x_i) = \frac{U^{n+1}(x_{i+1}) - U^{n+1}(x_i)}{h_{i+1}}, \quad D^-U^{n+1}(x_i) = \frac{U^{n+1}(x_i) - U^{n+1}(x_{i-1})}{h_i},$$

$$U_*^{n+1} = U^{n+1}(x_j)\left[\frac{x_{j+1} - (x_i - 1)}{x_{j+1} - x_j}\right] + U^{n+1}(x_{j+1})\left[\frac{(x_i - 1) - x_j}{x_{j+1} - x_j}\right], \ x_j \le x_i - 1 \le x_{j+1}, \ \ x_i \in \Omega_+^{N_x}.$$

Lemma 3.1 *Assume $\xi^{n+1}(x_0) \ge 0$ and $\xi^{n+1}(x_{N_x}) \ge 0$, $n = 0, 1, \ldots, N_t - 1$ such that $\mathfrak{L}_1^{N_x,N_t}\xi^{n+1}(x_i) \ge 0$ for $x_i \in \Omega_-^{N_x}$, $\mathfrak{L}_2^{N_x,N_t}\xi^{n+1}(x_i) \ge 0$ for $x_i \in \Omega_+^{N_x}$ and $D^+\xi^{n+1}(x_{\frac{N_x}{2}}) - D^-\xi^{n+1}(x_{\frac{N_x}{2}}) \le 0$. Then $\xi^{n+1}(x_i) \ge 0$, $\forall\, i = 0, 1, \ldots N_x$.*

Proof Let i^* be such that $\xi(x_{i^*}) = \min_i \xi(x_i)$ then $\xi(x_{i^*}) < 0$. It is clear that $i^* \notin \{0, N_x\}$.

Case I: For $i^* = 1, 2, \ldots, \frac{N_x}{2} - 1$

$$\mathfrak{L}_1^{N_x,N_t}\xi_{i^*}^{n+1} = -\varepsilon\Delta t D^+D^-\xi^{n+1}(x_{i^*}) + \widehat{\alpha}(x_{i^*})\xi^{n+1}(x_{i^*}) < 0$$

Case II: For $i^* = \frac{N_x}{2} + 1, \ldots, N_x$

$$\begin{aligned}
\mathfrak{L}_2^{N_x,N_t}\xi_{i^*}^{n+1} &= -\varepsilon\Delta t D^+D^-\xi^{n+1}(x_{i^*}) + (1 + \alpha(x_{i^*})\Delta t)\xi^{n+1}(x_{i^*}) + \beta(x_{i^*})\Delta t\xi_*^{n+1}(x_{i^*}) \\
&\le -\varepsilon\Delta t D^+D^-\xi^{n+1}(x_{i^*}) + (1 + \alpha(x_{i^*})\Delta t)\xi^{n+1}(x_{i^*}) + \beta(x_{i^*})\Delta t\xi^{n+1}(x_{i^*}) \\
&\le -\varepsilon\Delta t D^+D^-\xi^{n+1}(x_{i^*}) + (\alpha(x_{i^*}) + \beta(x_{i^*}))\Delta t\xi^{n+1}(x_{i^*}) + \Delta t\xi^{n+1}(x_{i^*}) \\
&< 0.
\end{aligned}$$

Since, if $i^* = \frac{N_x}{2}$, then $\xi^{n+1}(x_{i^*}) - \xi^{n+1}(x_{i^*-1}) \le 0$ and $\xi^{n+1}(x_{i^*+1}) - \xi^{n+1}(x_{i^*}) \ge 0$, gives

$$D^+\xi^{n+1}(x_{i^*}) - D^-\xi^{n+1}(x_{i^*}) \ge 0,$$

to avoid contradiction, we assume $D^+\xi^{n+1}(x_i) - D^-\xi^{n+1}(x_i) = 0$ and so $\xi^{n+1}(x_{\frac{N_x}{2}-1}) = \xi^{n+1}(x_{\frac{N_x}{2}}) = \xi^{n+1}(x_{\frac{N_x}{2}+1}) < 0$. Hence, we proof the lemma. □

Lemma 3.2 (Stability estimate) *Let* ξ_i^{n+1} *be any function, for* $0 \le i \le N_x$ *it satisfies*

$$|\xi_i^{n+1}| \le \max\left\{|\xi_0^{n+1}|, |\xi_{N_x}^{n+1}|, \frac{\|\mathfrak{L}_1^{N_x,N_t}\xi_i^{n+1}\|}{(1+2\kappa\Delta t)}, \frac{\|\mathfrak{L}_2^{N_x,N_t}\xi_i^{n+1}\|}{(1+2\kappa\Delta t)}\right\}.$$

Proof For $0 \le i \le N_x$, let us consider two barrier function

$$(\Pi_i^{n+1})^{\pm} = \max\left\{|\xi_0^{n+1}|, |\xi_{N_x}^{n+1}|, \frac{\|\mathfrak{L}_1^{N_x,N_t}\xi_i^{n+1}\|}{(1+2\kappa\Delta t)}, \frac{\|\mathfrak{L}_2^{N_x,N_t}\xi_i^{n+1}\|}{(1+2\kappa\Delta t)}\right\} \pm \xi_i^{n+1},$$

from above equation we have, $(\Pi_0^{n+1})^{\pm} \ge 0$ and $(\Pi_{N_x}^{n+1})^{\pm} \ge 0$,

$$\mathfrak{L}_1^{N_x,N_t}(\Pi_i^{n+1})^{\pm} \ge 0, \quad \text{for } i = 1, 2, \ldots, \frac{N_x}{2} - 1,$$

$$\mathfrak{L}_2^{N_x,N_t}(\Pi_i^{n+1})^{\pm} \ge 0, \quad \text{for } i = \frac{N_x}{2} + 1, \ldots, N_x,$$

$$(D^+ - D^-)(\Pi_i^{n+1})^{\pm} = 0, \quad \text{for } i = \frac{N_x}{2}.$$

Hence using Lemma 3.1, we have $(\Pi_i^{n+1})^{\pm} \ge 0$, for $0 \le i \le N_x$ which leads to required estimate. □

4 Convergence analysis

To derive the parameter uniform convergence of the fully discretize scheme (29)–(33), we obtain the consistency and stability estimates of (13)–(18) and then generalize it for the fully discretize scheme (29)–(33). To obtain the error estimates, consider a finite difference scheme

$$\mathfrak{L}_1^{N_x,N_t}\tilde{U}^{n+1}(x_i) = G_1^{n+1}(x_i), \qquad x_i \in \Omega_-^{N_x}, \tag{34}$$

$$\mathfrak{L}_2^{N_x,N_t}\tilde{U}^{n+1}(x_i) = G_2^{n+1}(x_i), \qquad x_i \in \Omega_+^{N_x}, \tag{35}$$

with

$$D^-\tilde{U}_{\frac{N_x}{2}}^{n+1} = D^+\tilde{U}_{\frac{N_x}{2}}^{n+1}, \tag{36}$$

with the initial condition

$$\tilde{U}^0(x_i) = u_0(x_i), \quad i = 1, 2, \ldots, N_x - 1, \tag{37}$$

and boundary condition for $i = -\frac{N_x}{2}, -\frac{N_x}{2} + 1, \ldots, 0$, and $n = 0, 1, \ldots, N_t - 1$

$$\tilde{U}^{n+1}(x_i) = \varphi(x_i, t_{n+1}), \qquad \tilde{U}^{n+1}(x_{N_x}) = \vartheta(t_{n+1}), \tag{38}$$

where the $\mathfrak{L}_1^{N_x,N_t}$, $\mathfrak{L}_2^{N_x,N_t}$, $G_1^{n+1}(x_i)$ and $G_2^{n+1}(x_i)$ defined as in (29)–(33).

The discrete solution $\tilde{U}^{n+1}(x_i)$ can be decomposed into regular and singular component as

$$\tilde{U}^{n+1}(x_i) = \tilde{V}^{n+1}(x_i) + \tilde{W}^{n+1}(x_i), \tag{39}$$

where the regular component $\tilde{V}^{n+1}(x_i)$ is further decomposed into

$$\tilde{V}^{n+1}(x_i) = \begin{cases} \tilde{V}_1^{n+1}(x_i), & 0 \le i \le \frac{N_x}{2} - 1, \\ \tilde{V}_2^{n+1}(x_i), & \frac{N_x}{2} + 1 \le i \le N_x, \end{cases} \tag{40}$$

where, for $i = 1, 2, \ldots \frac{N_x}{2} - 1$,

$$\mathfrak{L}_1^{N_x,N_t} \tilde{V}_1^{n+1}(x_i) = G_1^{n+1}(x_i),$$

$$\tilde{V}_1^{n+1}(x_0) = v_1^{n+1}(0), \quad \tilde{V}_1^{n+1}(x_{\frac{N_x}{2}-1}) = v_1^{n+1}(1^-),$$

for $i = \frac{N_x}{2} + 1, \ldots, N_x$,

$$\mathfrak{L}_2^{N_x,N_t} \tilde{V}_2^{n+1}(x_i) = G_2^{n+1}(x_i),$$

$$\tilde{V}_2^{n+1}(x_{\frac{N_x}{2}+1}) = v_2^{n+1}(1^+), \quad \tilde{V}_2^{n+1}(x_{N_x}) = v_2^{n+1}(2),$$

and singular component $W^{n+1}(x_i)$ is further decomposed into

$$\tilde{W}^{n+1}(x_i) = \begin{cases} \tilde{W}_1^{n+1}(x_i), & 0 \le i \le \frac{N_x}{2} - 1, \\ \tilde{W}_2^{n+1}(x_i), & \frac{N_x}{2} + 1 \le i \le N_x, \end{cases} \tag{41}$$

for $i = 1, 2, \ldots \frac{N_x}{2} - 1$,

$$\mathfrak{L}_1^{N_x,N_t} \tilde{W}_1^{n+1}(x_i) = 0,$$

$$\tilde{W}_1^{n+1}(x_0) = w_1^{n+1}(0),$$

for $i = \frac{N_x}{2} + 1, \ldots, N_x$,

$$\mathfrak{L}_2^{N_x,N_t} \tilde{W}_2^{n+1}(x_i) = 0,$$

$$\tilde{W}_2^{n+1}(x_{N_x}) = w_2^{n+1}(2),$$

and

$$\tilde{V}_1^{n+1}(x_{\frac{N_x}{2}-1}) + \tilde{W}_1^{n+1}(x_{\frac{N_x}{2}-1}) = \tilde{V}_2^{n+1}(x_{\frac{N_x}{2}+1}) + \tilde{W}_2^{n+1}(x_{\frac{N_x}{2}+1}),$$

$$D^- \tilde{V}_1^{n+1}(x_{\frac{N_x}{2}}) + D^- \tilde{W}_1^{n+1}(x_{\frac{N_x}{2}}) = D^+ \tilde{V}_2^{n}(x_{\frac{N_x}{2}+1}) + D^+ \tilde{W}_2^{n+1}(x_{\frac{N_x}{2}}).$$

Lemma 4.1 *If the smooth component* $\tilde{V}^{n+1}(x_i)$ *be the solution of problem* (34)–(38) *and* $v^{n+1}(x_i)$ *be the solution of semi-discrete scheme* (13)–(18)*, then*

$$\begin{aligned} |\mathfrak{L}_1^{N_x,N_t}(\tilde{V}^{n+1}(x_i) - v^{n+1}(x_i))| &\le CN_x^{-1}, \quad x_i = \Omega_-^{N_x}, \ n = 1, 2, \ldots, N_t - 1 \\ |\mathfrak{L}_2^{N_x,N_t}(\tilde{V}^{n+1}(x_i) - v^{n+1}(x_i))| &\le CN_x^{-1}, \quad x_i = \Omega_+^{N_x}, \ n = 1, 2, \ldots, N_t - 1 \end{aligned}$$

Proof For $i = 1, 2, \dots, \frac{N_x}{2} - 1$, using Lemma 2.8 and $h_i \le CN_x^{-1}$,

$$\begin{aligned}\left|\mathfrak{L}_1^{N_x,N_t}(\tilde{V}_1^{n+1}(x_i) - v_1^{n+1}(x_i)\right| &\le C\varepsilon\Delta t \left|\left(D^+D^- - \frac{d^2}{dx^2}\right)v_1^{n+1}(x_i)\right| \\ &\le C\varepsilon\Delta t(h_{i+1}+h_i)\left|\frac{d^3v^{n+1}(x_i)}{dx^3}\right| \\ &\le CN_x^{-1}\end{aligned}$$

Similarly, for $i = \frac{N_x}{2} + 1, \dots, N_x$, using Lemma 2.8 and $h_i \le CN_x^{-1}$,

$$\begin{aligned}\left|\mathfrak{L}_2^{N_x,N_t}(\tilde{V}_2^{n+1}(x_i) - v_2^{n+1}(x_i))\right| &\le C\varepsilon\Delta t \left|\left(D^+D^- - \frac{d^2}{dx^2}\right)v_2^{n+1}(x_i) + \Delta t\left(\tilde{V}_*^{n+1}(x_i) - v^{n+1}(x_i-1)\right)\right| \\ &\le C\varepsilon\Delta t(h_{i+1}+h_i)\left|\frac{d^3v^{n+1}(x_i)}{dx^3}\right| + \Delta t\left(\tilde{V}_*^{n+1}(x_i) - v^{n+1}(x_i-1)\right) \\ &\le CN_x^{-1},\end{aligned}$$

$|\tilde{V}_*^{n+1}(x_i) - v^{n+1}(x_i-1)| \le CN_x^{-2}$ (as the linear interpolation is second order accurate). □

Now $\tilde{W}_1^{n+1}(x_i)$ and $\tilde{W}_2^{n+1}(x_i)$ further decompose as follows

$$\tilde{W}_1^{n+1}(x_i) = \tilde{W}_{1,L}^{n+1}(x_i) + \tilde{W}_{1,R}^{n+1}(x_i) \quad \text{and} \quad \tilde{W}_2^{n+1}(x_i) = \tilde{W}_{2,L}^{n+1}(x_i) + \tilde{W}_{2,R}^{n+1}(x_i) \tag{42}$$

where $\tilde{W}_{1,L}^{n+1},\ \tilde{W}_{1,R}^{n+1},\ \tilde{W}_{2,L}^{n+1},\ \tilde{W}_{2,R}^{n+1}$ are defined by

$$\mathfrak{L}_1^{N_x,N_t}\tilde{W}_{1,L}^{n+1} = 0, \quad \text{for } i = 1, 2, \dots, \frac{N_x}{2} - 1,$$

$$\tilde{W}_{1,L}^0(x) = 0, \quad \tilde{W}_{1,L}^{n+1}(x_0) = w_{1,L}^{n+1}(0), \quad \tilde{W}_{1,L}^{n+1}(x_{\frac{N_x}{2}}) = w_{1,L}^{n+1}(1),$$

$$\mathfrak{L}_2^{N_x,N_t}\tilde{W}_{2,L}^{n+1} = 0, \quad \text{for } i = \frac{N_x}{2} + 1, \dots, N_x,$$

$$\tilde{W}_{2,L}^0(x) = 0, \quad \tilde{W}_{2,L}^{n+1}(x_{\frac{N_x}{2}}) = w_{2,L}^{n+1}(1), \quad \tilde{W}_{2,L}^{n+1}(x_{N_x}) = w_{2,L}^{n+1}(2),$$

$$\mathfrak{L}_1^{N_x,N_t}\tilde{W}_{1,R}^{n+1} = 0, \quad \text{for } i = 1, 2, \dots, \frac{N_x}{2} - 1,$$

$$\tilde{W}_{1,R}^0(x) = 0, \quad \tilde{W}_{1,R}^{n+1}(x_0) = w_{1,R}^{n+1}(0), \quad \tilde{W}_{1,L}^{n+1}(x_{\frac{N_x}{2}}) = w_{1,R}^{n+1}(1),$$

$$\mathfrak{L}_2^{N_x,N_t}\tilde{W}_{2,R}^{n+1} = 0, \quad \text{for } i = \frac{N_x}{2} + 1, \dots, N_x,$$

$$\tilde{W}_{2,R}^0(x) = 0, \quad \tilde{W}_{2,R}^{n+1}(x_{\frac{N_x}{2}}) = w_{2,R}^{n+1}(1), \quad \tilde{W}_{2,R}^{n+1}(x_{N_x}) = w_{2,R}^{n+1}(2).$$

Lemma 4.2 *The error of the singular component satisfies*

$$|\mathfrak{L}_1^{N_x,N_t}(\tilde{W}_{1,L}^{n+1}(x_i) - w_{1,L}^{n+1}(x_i))| \le CN_x^{-1} \quad for\ i = 0, 1, \dots, \frac{N_x}{2} - 1, \quad n = 1, 2, \dots, N_t - 1.$$

Proof The local truncation error of the singular component, as $h_{i+1} > h_i$ for $x_i \in [0, x_{\frac{N_x}{8}-1}]$

$$\begin{aligned}
|\mathfrak{L}_1^{N_x,N_t}(\tilde{W}_{1,L}^{n+1}(x_i) - w_{1,L}^{n+1}(x_i))| &\le C(h_{i+1}+h_i)\varepsilon\left|\frac{d^3 w_{1,L}^{n+1}}{dx^3}\right| \\
&\le Ch_{i+1}\varepsilon\left|\frac{d^3 w_{1,L}^{n+1}}{dx^3}\right| \\
&\le CN_x^{-1}\exp\left(\sqrt{\frac{\kappa}{\varepsilon}}\frac{4}{\tau}x_{i+1}\right)\exp\left(\sqrt{-\frac{\kappa}{\varepsilon}}x_i\right) \\
&\le CN_x^{-1}\exp\left(\sqrt{\frac{\kappa}{\varepsilon}}\frac{4}{\tau}(x_i+h_i)\right)\exp\left(\sqrt{-\frac{\kappa}{\varepsilon}}x_i\right) \\
&\le CN_x^{-1}\exp\left(-\sqrt{\frac{\kappa}{\varepsilon}}x_i\left(-\frac{4}{\tau}+1\right)\right)\exp\left(\sqrt{\frac{\kappa}{\varepsilon}}\frac{4}{\tau}h_i\right) \\
&\le CN_x^{-1}, \quad \text{provided } \tau \ge 4
\end{aligned}$$

by using the derivative bounds from lemma 2.8 and the bound of h_i in (25).

Following above approach, for $x_i \in [x_{\frac{3N_x}{8}+1}, x_{\frac{N_x}{2}-1}]$ similar estimates can be obtained.

On the other hand, for $x_i \in [x_{\frac{N_x}{8}-1}, x_{\frac{3N_x}{8}+1}]$, the local truncation error of the left layer boundary, we have

$$\left|\mathfrak{L}_1^{N_x,N_t}\left(\tilde{W}_{1,L}^{n+1}(x_i) - w_{1,L}^{n+1}(x_i)\right)\right| \le C\varepsilon\left(D^+D^- - \frac{d^2}{dx^2}\right)w_{1,L}^{n+1},$$

but,

$$|D^+D^-w_{1,L}^{n+1}(x_i)| \le C\max_{x\in\{x_{i-1},x_{i+1}\}}\left|\frac{d^2 w_{1,L}^{n+1}}{dx^2}\right|.$$

Hence

$$\left|\frac{d^2 w_{1,L}^{n+1}}{dx^2}\right| \le C\varepsilon^{-1}\exp\left(\sqrt{-\frac{\kappa}{\varepsilon}}x_i\right),$$

for all $x_i \ge x_{\frac{N_x}{8}-1}$

$$\left|\mathfrak{L}_1^{N_x,N_t}\left(\tilde{W}_{1,L}^{n+1}(x_i) - w_{1,L}^{n+1}(x_i)\right)\right| \le C\exp\left(\sqrt{-\frac{\kappa}{\varepsilon}}x_{\frac{N_x}{8}-1}\right) \le CN_x^{-1}.$$

Here we used lemma 2.8, (23) and τ value from the last result. □

Similarly, one can obtain bounds for

$$\left|\mathfrak{L}_1^{N_x,N_t}\left(\tilde{W}_{1,R}^{n+1}(x_i) - w_{1,R}^{n+1}(x_i)\right)\right| \le CN_x^{-1}, \quad \text{for } i = 1, 2, \ldots, \frac{N_x}{2} - 1,$$

Consider a point at $(x_{\frac{N_x}{2}}, t_n)$, let $h_{\frac{N_x}{2}} = h^* = h_{\frac{N_x}{2}+1}$, as $(D^- - D^+)\tilde{U}^{n+1}(x_i) = 0$ so the error estimates, we have

$$\begin{aligned}
\left|(D^+ - D^-)(\tilde{U}^{n+1}(x_{\frac{N_x}{2}}) - u^{n+1}(x_{\frac{N_x}{2}}))\right| &= \left|\left(\left((D^+ - \frac{d}{dx}\right) - \left(D^- - \frac{d}{dx}\right)\right)u^{n+1}(x_{\frac{N_x}{2}})\right| \\
&\le \left|\left((D^+ - \frac{d}{dx}\right)u^{n+1}(x_{\frac{N_x}{2}})\right| + \left|\left(D^- - \frac{d}{dx}\right)u^{n+1}(x_{\frac{N_x}{2}})\right| \\
&\le \frac{h_{\frac{N_x}{2}+1}}{2}\max_{\varsigma\in(1,2)}\left|\frac{d^2u^{n+1}(\varsigma)}{dx^2}\right| + \frac{h_{\frac{N_x}{2}}}{2}\max_{\varsigma\in(0,1)}\left|\frac{d^2u^{n+1}(\varsigma)}{dx^2}\right|
\end{aligned}$$

$$\leq h^* \max_{x\in(0,1)\cup(1,2)} \left|\frac{d^2u^{n+1}(x)}{dx^2}\right|$$
$$\leq C\frac{h^*}{\varepsilon}.$$

Define a set of discrete barrier function given by, for $i = 0, 1, 2, \ldots, N_x$,

$$\omega_i^{n+1} = \begin{cases} \dfrac{\prod_{k=1}^{i}\left(1+\sqrt{\frac{\kappa}{\varepsilon}}h_k\right)}{\prod_{k=1}^{\frac{N_x}{2}}\left(1+\sqrt{\frac{\kappa}{\varepsilon}}h_k\right)}, & 0 \leq i \leq \dfrac{N_x}{2}, \\[2ex] \dfrac{\prod_{k=i}^{N_x-1}\left(1+\sqrt{\frac{\kappa}{\varepsilon}}h_{k+1}\right)}{\prod_{k=\frac{N_x}{2}}^{N_x-1}\left(1+\sqrt{\frac{\kappa}{\varepsilon}}h_{k+1}\right)}, & \dfrac{N_x}{2} \leq i \leq N_x. \end{cases} \tag{43}$$

We can observe that

$$\omega_0^{n+1} = 0,\ \omega_{\frac{N_x}{2}}^{n+1} = 1,\ \omega_{N_x}^{n+1} = 0 \tag{44}$$

and

$$0 \leq \omega_i^{n+1} \leq 1, \quad \text{for } 0 \leq i \leq N_x. \tag{45}$$

For $i = 1, 2, \ldots, \dfrac{N_x}{2}$,

$$D^+\omega_i^{n+1} = \frac{\omega_{i+1}^{n+1} - \omega_i^{n+1}}{h_{i+1}} = \sqrt{\frac{\kappa}{\varepsilon}}w_i^{n+1}, \tag{46}$$

$$D^-\omega_i^{n+1} = \frac{\omega_i^{n+1} - \omega_{i-1}^{n+1}}{h_i} = \sqrt{\frac{\kappa}{\varepsilon}}\frac{1}{\left(1+\sqrt{\frac{\kappa}{\varepsilon}}h_i\right)}\omega_i^{n+1}, \tag{47}$$

$$D^+D^-\omega_i^{n+1} = \frac{2(D^+ - D^-)\omega_i^{n+1}}{h_i + h_{i+1}} \leq \frac{2\kappa}{\varepsilon}\omega_i^{n+1}. \tag{48}$$

Similarly, for $i = \dfrac{N_x}{2} \ldots, N_x$,

$$D^+\omega_i^{n+1} = \frac{\omega_i^{n+1} - \omega_{i-1}^{n+1}}{h_i} = -\sqrt{\frac{\kappa}{\varepsilon}}\frac{1}{\left(1+\sqrt{\frac{\kappa}{\varepsilon}}h_{i+1}\right)}\omega_i^{n+1}, \tag{49}$$

$$D^-\omega_i^{n+1} = \frac{\omega_{i+1}^{n+1} - \omega_i^{n+1}}{h_{i+1}} = -\sqrt{\frac{\kappa}{\varepsilon}}\omega_i^{n+1}, \tag{50}$$

$$D^+D^-\omega_i^{n+1} = \frac{2(D^+ - D^-)w_i^{n+1}}{h_i + h_{i+1}} \leq \frac{2\kappa}{\varepsilon}\omega_i^{n+1}. \tag{51}$$

In particular, at $i = \frac{N_x}{2}$ using (44), (47) and (49),

$$(D^+ - D^-)\omega_i^{n+1} \leq -\frac{C}{\sqrt{\varepsilon}}. \tag{52}$$

Considering,

$$\mathfrak{L}_1^{N_x,N_t}\omega_i^{n+1} = -\varepsilon\Delta t D^+D^-\omega_i^{n+1} + \widehat{\alpha}(x_i)\omega_i^{n+1} \tag{53}$$
$$\geq (-2\kappa\Delta t + \widehat{\alpha}(x_i))\,\omega_i^{n+1}. \tag{54}$$

Similarly, we get

$$\mathfrak{L}_2^{N_x,N_t}\omega_i^{n+1} = -\varepsilon\Delta t D^+D^-\omega^{n+1}(x_i) + \widehat{\alpha}(x_i)\omega_i^{n+1} + \beta^{n+1}(x_i)\Delta t\omega_{*,i}^{n+1} \tag{55}$$
$$\geq (-2\kappa\Delta t + \widehat{\alpha}(x_i))\,\omega_i^{n+1} + \beta^{n+1}(x_i)\Delta t\omega_{*,i}^{n+1}. \tag{56}$$

Following above approach, we can also prove the bounds given below

$$\left|\mathfrak{L}_2^{N_x,N_t}\left(\tilde{W}_{2,L}^{n+1}(x_i) - w_{2,L}^{n+1}(x_i)\right)\right| \leq CN_x^{-1}, \quad \text{for } i = \frac{N_x}{2}+1,\ldots,N_x,\ n=1,2,\ldots,N_t-1.$$
$$\left|\mathfrak{L}_2^{N_x,N_t}\left(\tilde{W}_{2,R}^{n+1}(x_i) - w_{2,R}^{n+1}(x_i)\right)\right| \leq CN_x^{-1}, \quad \text{for } i = \frac{N_x}{2}+1,\ldots,N_x,\ n=1,2,\ldots,N_t-1.$$

Lemma 4.3 *Let $u^{n+1}(x_i)$ be the solution of problem* (13)–(18) *and $\tilde{U}^{n+1}(x_i)$ be the solution of problem* (34)–(38)*, then for $0 \leq i \leq N_x$,*

$$|u^{n+1}(x_i) - \tilde{U}^{n+1}(x_i)| \leq CN_x^{-1}$$

Proof Consider two mesh functions given by

$$(\Upsilon_i^{\pm})^{n+1} = C_1N_x^{-1} + C_2\sqrt{\frac{\kappa}{\varepsilon}}h^*\omega_i^{n+1} \pm e_i^{n+1}, \quad 0 \leq i \leq N_x$$

where C_1 and C_2 are constants independent of perturbation parameter and mesh parameters, then

$$\mathfrak{L}_1^{N_x,N_t}(\Upsilon_i^{\pm})^{n+1} = C_1\widehat{\alpha}(x_i)N_x^{-1} + C_2\sqrt{\frac{\kappa}{\varepsilon}}h^*\mathfrak{L}_1^{N_x,N_t}\omega_i^{n+1} \pm \mathfrak{L}_1^{N_x,N_t}e_i^{n+1},$$

using (54), we obtain

$$\mathfrak{L}_1^{N_x,N_t}(\Upsilon_i^{\pm})^{n+1} \geq C_1\widehat{\alpha}(x_i)N_x^{-1} + C_2\sqrt{\frac{\kappa}{\varepsilon}}h^*\,(-2\kappa\Delta t + \widehat{\alpha}(x_i))\,\omega_i^{n+1} \pm CN_x^{-1}.$$

Similarly,

$$\mathfrak{L}_2^{N_x,N_t}(\Upsilon_i^{\pm})^{n+1} = C_1\widehat{\alpha}(x_i)N_x^{-1} + C_2\sqrt{\frac{\kappa}{\varepsilon}}h^*\mathfrak{L}_2^{N_x,N_t}\omega_i^{n+1} \pm \mathfrak{L}_2^{N_x,N_t}e_i^{n+1},$$

using (56), we obtain

$$\mathfrak{L}_2^{N_x,N_t}(\Upsilon_i^{\pm})^{n+1} \geq C_1(\widehat{\alpha}(x_i) + \Delta t\beta(x_i))N_x^{-1} + C_2\sqrt{\frac{\kappa}{\varepsilon}}h^*\,(-2\kappa\Delta t + \widehat{\alpha}(x_i))\,\omega_i^{n+1} + \beta(x_i)\Delta t\omega_*^{n+1}(x_i) \pm CN_x^{-1}.$$

Further

$$(D^+ - D^-)(\Upsilon_{\frac{N_x}{2}}^{\pm})^{n+1} \leq -C_2\frac{Ch^*}{\varepsilon} \pm \frac{Ch^*}{\varepsilon},$$
$$\leq 0, \text{ for a suitable choice of } C_2.$$

We have $(\Upsilon_0^{\pm})^{n+1} \geq 0$ and $(\Upsilon_{N_x}^{\pm})^{n+1} \geq 0$. Therefore using Lemma 3.1, it follows that $(\Upsilon_i^{\pm})^{n+1} \geq 0$ for $0 \leq i \leq N_x$, we obtain the required result. □

Corollary 4.4 *If we take $N_x^{\kappa} \leq C\Delta t$ with $0 < \kappa < 1$ then we have*

$$|u^{n+1}(x_i) - \tilde{U}^{n+1}(x_i)| \leq C\Delta tN_x^{-1+\kappa},\ i = 1,2,\ldots,N_x.$$

Theorem 4.5 *Let $u(x, t)$ be the exact solution of* (1)–(4) *and $U^n(x_i)$ be the approximate solution of the fully discrete scheme* (29)–*(33) at the n^{th} −time level. Assume that $N_x^{-\kappa} \le C\Delta t$ with $0 < \kappa < 1$. Then the corresponding error is given by*

$$\|u(x_i, t_n) - U^n(x_i)\| \le C\left(\Delta t + N_x^{-1+\kappa}\right).$$

Proof Let the error at the time level t_n by $\chi_i^n = u(x_i, t_n) - U^n(x_i)$. Splitting the error χ_i^n and using the triangle inequality, we have

$$\|\chi_i^n\| \le \|u(x_i, t_n) - u^n(x_i)\| + \|u^n(x_i) - \tilde{U}^n(x_i)\| + \|\tilde{U}^n(x_i) - U^n(x_i)\|.$$

Using Lemma 2.6 and Corollary 4.4, we have

$$\|\chi_i^n\| \le C\Delta t\left(\Delta t + N_x^{-1+\kappa}\right) + \|\tilde{U}^n(x_i) - U^n(x_i)\|.$$

Using Lemma 3.2 and stability estimates of the fully discretize scheme, we obtain

$$\|\tilde{U}^n(x_i) - U^n(x_i)\| \le \|u(x_i, t_{n-1}) - U^{n-1}(x_i)\|.$$

From above equations, we obtain

$$\|\chi_i^n\| \le C\Delta t\left(\Delta t + N_x^{-1+\kappa}\right) + \|\chi_i^{n-1}\|,$$

which proves the required result. □

5 Numerical experiments

Two test problems whose solutions exhibit interior and boundary layers are encountered to verify the theoretical results estimated in the previous section.

Example 5.1 In this problem, we take $\alpha(x) = 3$, $\beta(x) = -1$, $g(x, t) = 1$, $u_0(x) = 0$, $\varphi(x, t) = 0$, and $\vartheta(t) = 0$.

Example 5.2 In this problem, we take $\alpha(x) = x + 6$, $\beta(x) = -(x^2 + 1)$, $g(x, t) = 3$, $u_0(x) = 0$, $\varphi(x, t) = 0$, and $\vartheta(t) = 0$.

Since the analytical/exact solution of considered problems are not available, for a fixed ε the absolute maximum pointwise error of the proposed method is calculated using the double mesh principle. Let $\mathfrak{D}^{N_x,\Delta t} = \triangle^{N_x} \times \Lambda^{N_t}$ be the discretized domain, by bisecting the original mesh $\mathfrak{D}^{N_x,\Delta t}$ in x and t direction gives $\mathfrak{D}^{2N_x,\Delta t/2}$. If $U(x_i, t_n)$ and $\widehat{U}(x_i, t_n)$ are the approximate solutions on the mesh $\mathfrak{D}^{N_x,\Delta t}$ and $\mathfrak{D}^{2N_x,\Delta t/2}$, respectively. Then for each ε the maximum point-wise error $E_\varepsilon^{N_x,\Delta t}$ over all time levels is calculated as

$$E_\varepsilon^{N_x,\Delta t} = \max_{1\le i,n\le N_x,N_t} |U(x_i, t_n) - \widehat{U}(x_{2i}, t_n)|$$

and the corresponding order of convergence $R_\varepsilon^{N_x,\Delta t}$ is calculated as

$$R_\varepsilon^{N_x,\Delta t} = \log_2\left(\frac{E_\varepsilon^{N_x,\Delta t}}{E_\varepsilon^{2N_x,\Delta t/2}}\right).$$

The maximum-pointwise errors and corresponding order of convergence of Example's 5.1 and 5.2 is presented in Table 1-2 for different values of N_x, N_t as $\varepsilon \to 0$. Comparison of present method is shown in Table 3 with method present in [15]. From the table we can observe that the maximum-pointwise error decreases as N_x increases. Numerical solution of proposed method are shown to be parameter uniform convergence of first order. Furthermore, Figure 1-2 represent the surface plot for $N = 64$, $\Delta t = 1/64$ of Example's 5.1 and 5.2 respectively, from the graph it can be observed that there is a weak layer at $x = 1$ due to presence of large delay at $x = 1$ and boundary layer at $x = 0$ and $x = 2$. Figure's 3–4 represent the numerical solution of of Example's 5.1 and 5.2 for different value of ε and $N_x = 64$ at time level $N_t = 64$.

Table 1 $E_\varepsilon^{N_x,\Delta t}$ and $R_\varepsilon^{N_x,\Delta t}$ of Example 5.1 for different values of ε

	$N_x = 16$ $\Delta t = 1/16$	$N_x = 32$ $\Delta t = 1/32$	$N_x = 64$ $\Delta t = 1/64$	$N_x = 128$ $\Delta t = 1/128$	$N_x = 256$ $\Delta t = 1/256$
$E_{\varepsilon=2^{-8}}^{N_x,\Delta t}$	7.8263e-03	3.6485e-03	1.6365e-03	8.0081e-04	3.9883e-04
$R_{\varepsilon=2^{-8}}^{N_x,\Delta t}$	1.1010	1.1567	1.0311	1.0057	
$E_{\varepsilon=2^{-10}}^{N_x,\Delta t}$	6.8457e-03	3.4772e-03	1.6491e-03	8.0353e-04	3.9957e-04
$R_{\varepsilon=2^{-10}}^{N_x,\Delta t}$	0.97727	1.0762	1.0373	1.0079	
$E_{\varepsilon=2^{-12}}^{N_x,\Delta t}$	6.0211e-03	3.4767e-03	1.6509e-03	8.0400e-04	3.9960e-04
$R_{\varepsilon=2^{-12}}^{N_x,\Delta t}$	0.79231	1.0745	1.0380	1.0087	
$E_{\varepsilon=2^{-14}}^{N_x,\Delta t}$	5.6725e-03	3.4730e-03	1.6509e-03	8.0401e-04	3.9960e-04
$R_{\varepsilon=2^{-14}}^{N_x,\Delta t}$	0.70779	1.0729	1.0380	1.0087	
$E_{\varepsilon=2^{-16}}^{N_x,\Delta t}$	5.6042e-03	3.4710e-03	1.6509e-03	8.0401e-04	3.9960e-04
$R_{\varepsilon=2^{-16}}^{N_x,\Delta t}$	0.69116	1.0721	1.0380	1.0087	
$E_{\varepsilon=2^{-18}}^{N_x,\Delta t}$	5.6040e-03	3.4699e-03	1.6509e-03	8.0401e-04	3.9960e-04
$R_{\varepsilon=2^{-18}}^{N_x,\Delta t}$	0.69155	1.0716	1.0380	1.0087	
$E_{\varepsilon=2^{-20}}^{N_x,\Delta t}$	5.6040e-03	3.4694e-03	1.6509e-03	8.0401e-04	3.9960e-04
$R_{\varepsilon=2^{-20}}^{N_x,\Delta t}$	0.69177	1.0714	1.0380	1.0087	

Table 2 $E_\varepsilon^{N_x,\Delta t}$ and $R_\varepsilon^{N_x,\Delta t}$ of Example 5.2 for different values of ε

	$N_x = 16$ $\Delta t = 1/16$	$N_x = 32$ $\Delta t = 1/32$	$N_x = 64$ $\Delta t = 1/64$	$N_x = 128$ $\Delta t = 1/128$	$N_x = 256$ $\Delta t = 1/256$
$E_{\varepsilon=2^{-8}}^{N_x,\Delta t}$	2.2569e-02	1.0986e-02	5.3067e-03	2.6077e-03	1.2978e-03
$R_{\varepsilon=2^{-8}}^{N_x,\Delta t}$	1.0386	1.0498	1.0250	1.0067	
$E_{\varepsilon=2^{-10}}^{N_x,\Delta t}$	2.0297e-02	1.1074e-02	5.3781e-03	2.6405e-03	1.3100e-03
$R_{\varepsilon=2^{-10}}^{N_x,\Delta t}$	0.87408	1.0420	1.0263	1.0113	
$E_{\varepsilon=2^{-12}}^{N_x,\Delta t}$	1.8675e-02	1.1095e-02	5.3939e-03	2.6493e-03	1.3147e-03
$R_{\varepsilon=2^{-12}}^{N_x,\Delta t}$	0.75130	1.0405	1.0257	1.0108	
$E_{\varepsilon=2^{-14}}^{N_x,\Delta t}$	1.7829e-02	1.1105e-02	5.4013e-03	2.6531e-03	1.3170e-03
$R_{\varepsilon=2^{-14}}^{N_x,\Delta t}$	0.68301	1.0398	1.0256	1.0104	
$E_{\varepsilon=2^{-16}}^{N_x,\Delta t}$	1.7402e-02	1.1110e-02	5.4053e-03	2.6550e-03	1.3182e-03
$R_{\varepsilon=2^{-16}}^{N_x,\Delta t}$	0.64733	1.0394	1.0256	1.0102	
$E_{\varepsilon=2^{-18}}^{N_x,\Delta t}$	1.7187e-02	1.1113e-02	5.4073e-03	2.6560e-03	1.3187e-03
$R_{\varepsilon=2^{-18}}^{N_x,\Delta t}$	0.62909	1.0393	1.0257	1.0101	
$E_{\varepsilon=2^{-20}}^{N_x,\Delta t}$	1.7079e-02	1.1114e-02	5.4083e-03	2.6565e-03	1.3190e-03
$R_{\varepsilon=2^{-20}}^{N_x,\Delta t}$	0.61986	1.0392	1.0256	1.0101	

Table 3 Comparison of $E_\varepsilon^{N_x,\Delta t}$ of Example 5.1 for $\varepsilon = 2^{-30}$, $\Delta t = \frac{1}{N_x}$

Method in [15]		Present Method	
N_x	$E^{N_x,\Delta t}$	N_x	$E^{N_x,\Delta t}$
18	1.12E-02	16	5.6040e-03
36	6.98E-03	32	3.4689e-03
72	2.96E-03	64	1.6509e-03
144	1.14E-03	128	8.0401e-04
288	3.89E-04	256	3.9960e-04

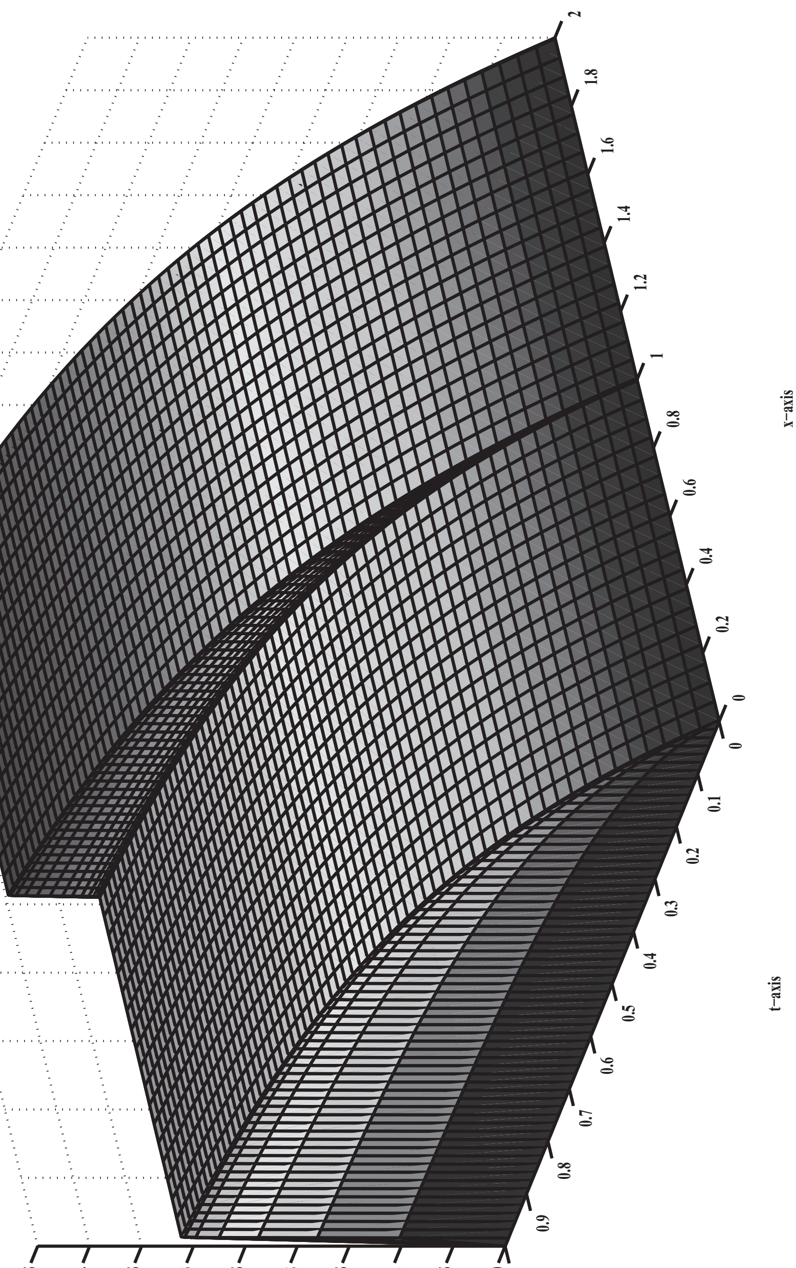

Fig. 1 Surface plot of Example 5.1 for $\varepsilon = 2^{-20}$ and $N_x = N_t = 64$

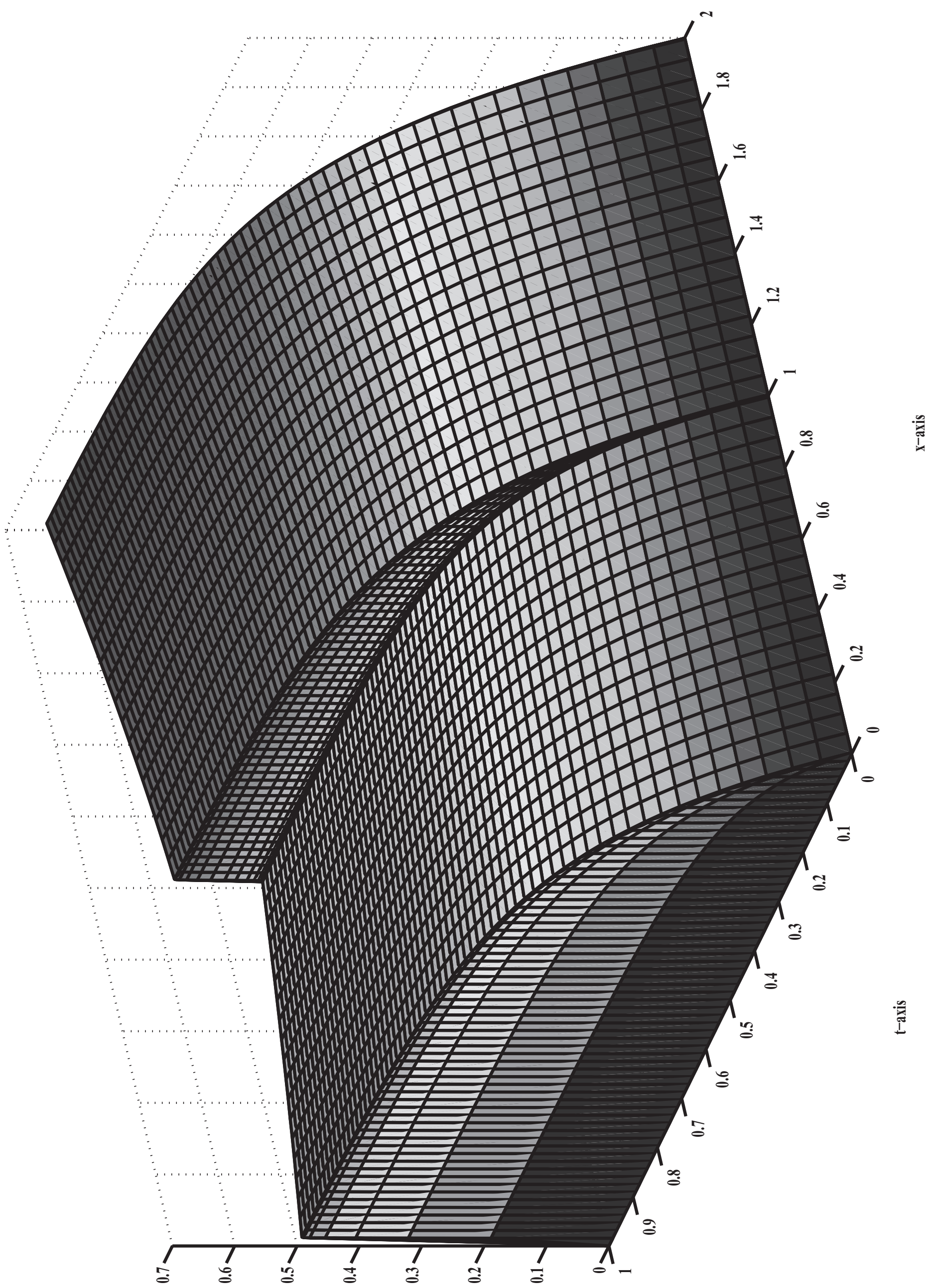

Fig. 2 Surface plot of Example 5.2 for $\varepsilon = 2^{-20}$ and $N_x = N_t = 64$

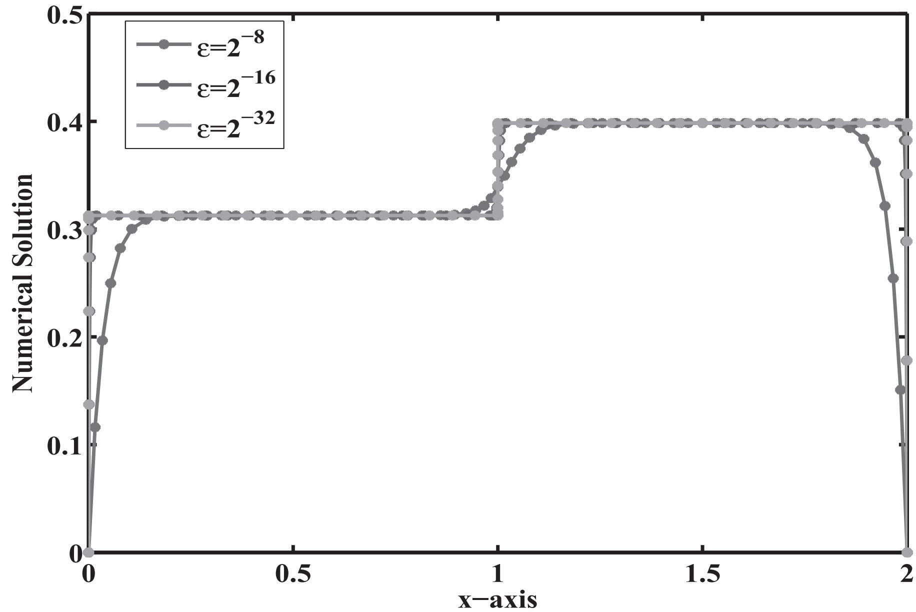

Fig. 3 Numerical solution of Example 5.1for $\varepsilon = 2^{-20}$ at time level $N_t = 64$ $N_x = 64$

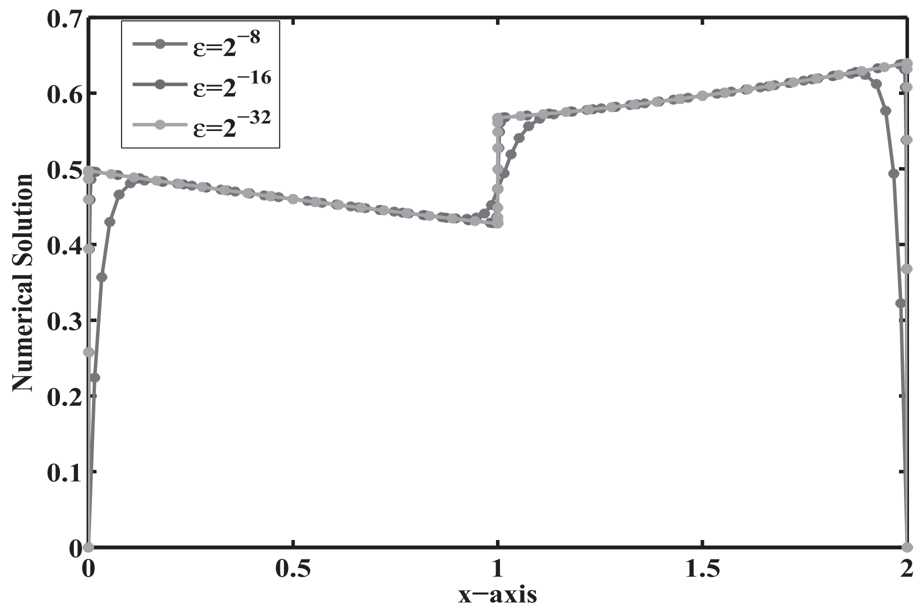

Fig. 4 Numerical solution of Example 5.2 for $\varepsilon = 2^{-20}$ at time level $N_t = 64$ $N_x = 64$

6 Conclusions

In this article, we proposed a parameter uniform numerical method comprise of backward Euler method in $t-$direction and central difference scheme on exponentially graded mesh in $x-$ direction for singularly perturbed parabolic delay differential equation exhibiting boundary layers at $x = 0$, $x = 2$ and an interior layer at $x = 1$. The method is shown to first order of convergence in x and t directions and theoretical estimates were verified by considering two test examples numerically. The proposed method has better accuracy as compared to the previous existing results.

Acknowledgements The authors wish to thank National Board for Higher Mathematics, Department of Atomic Energy, Government of India, for their financial support under project No. 2/48(33)/2016 NBHM(R.P.)/R&D II/4533 dated 31^{st} March 2017.

Declarations

Conflicts of interest The authors declare that there is no conflict of interests regarding the publication of this paper.

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