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ORIGINAL RESEARCH

An efficient numerical approximation for mixed singularly perturbed parabolic problems involving large time-lag

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Abstract The objective of this work is to provide an efficient numerical scheme for solving singularly perturbed parabolic convection-diffusion problems with a large time lag having Robin-type boundary conditions. The implicit Euler scheme is used in the temporal direction on a uniform mesh. To handle the layer behavior caused due to the presence of perturbation parameter, the problem is solved using the upwind scheme on two layer-resolving meshes in the spatial direction. As the Shishkin mesh makes the order of convergence to one up to a logarithmic factor, in the spatial direction, the presence of this effect is prevented by the use of the Bakhvalov-Shishkin mesh. Further, the robustness of the scheme is tested over semi-linear time-lagged initial boundary value problems. Numerical outputs are presented in the form of tables and graphs to prove the parameter-uniform nature and robustness of the proposed scheme.

Keywords Singular perturbation · Convection-diffusion problem · Time lag · Robin-type boundary condition · Semi-linear parabolic problem

Mathematics Subject Classification 65M06 · 65M12 · 35K20 · 35K58

1 Introduction

Singularly perturbed problems (SPPs) are the major tools for describing various types of problems in many branches of applied mathematics. The peculiar character of these type of problems is an arbitrarily small parameter multiplied to the highest-order derivative term. The presence of this parameter makes the solution vary rapidly in some parts of the domain. This unique feature makes the treatment of these problems quite challenging. In a quest to address this challenge, numerous amount of numerical experiments is conducted which provide parameter uniform solutions for these problems. One can refer [6,7,11,17,21,25,26] and the references therein, which provide the solutions of SPPs for both ordinary and partial differential equations.

The topic of time delay/lag is rather an old topic of research dating back to the works of Euler-Bernoulli in XVIII century. The time lag is not just a mathematical exercise but is deeply rooted in many natural and man-made systems such as biology, process industries, and mechatronic motions. The

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lag can capture a wide variety of possible gestures of a simple system. In engineering, time lag usually describes the propagation phenomenon, energy or material transfer in inter-command systems, and data transmission in communication systems [13,29]. Usually, these lags are the major reasons for inducing oscillations, poor control performance, and instability. In biology, it can represent a bunch of biological occurrences, accounting for only the time required by them to complete. It sometimes can be related to the duration of the certain hidden process like stages of the life cycle, the time between the infection of a cell and the production of new viruses, the gestation period, the incubation period, the duration of the infectious period, the time required by the immune system to respond to any treatment, or can simply be the delays that happen due to the transport [2,24]. Any system certainly can contain a time lag if it has a feedback control or if it requires prior information about itself to act further. In easier words, the evolution of a system at a particular time depends on past history/memory. If the delay would be small enough then some analytical approximations could work [26]. But, as the time-lag in SPPDDEs are large (*i.e.,* of $o(1)$), they require more mathematical maturity than their without-delay counter parts.

The major motivation behind this work is to construct, analyze and apply a robust and efficient ε-uniform numerical algorthim for a special class of parameter (both perturbation and time delay) dependent differential equation, as its solution is highly impacted by some minute changes in the parameters. Several numerical experiments are conducted in this regard. In [9] and [10], Gowrisankar and Natesan have used the implicit Euler scheme in time on a uniform mesh and the upwind scheme in spatial direction upon the non-uniform mesh constructed via the equidistribution of a monitor function, for singularly perturbed parabolic delayed differential equations (SPPDDEs) of reaction-diffusion and convection-diffusion types, respectively. Das and Natesan [4] used the hybrid scheme on the piecewise uniform Shishkin mesh (S-mesh) for spatial direction and the implicit Euler scheme on a uniform mesh for the temporal direction, Govindrao *et.al.* [8] used the Richardson extrapolation technique to solve the similar problem in reaction-diffusion case. The use of a finite difference scheme and the adaptive grid is investigated by Kaushik and Sharma [12], and Podila and Kumar [20] used the NSFD scheme for the same. For problems with a time lag and two small parameters, Govindrao *et.al.* [6] proposed the implicit Euler scheme on a uniform mesh and the upwind scheme on S-mesh for the temporal and spatial direction, respectively. Kumar and Kumar [14] used the monotone hybrid finite difference scheme and Sahu and Mohapatra [25] experimented with a similar scheme as in [4]. Recently in [22], the authors have used the Richardson extrapolation technique to give more accurate results for a similar class of problems. For semilinear SPPDEs having time lag, Priyadarshana *et.al.,* in [21] and [23] have used the Crank-Nicholson scheme for temporal direction along with the upwind scheme and the Monotone hybrid scheme in spatial direction, respctively, after resolving the semilinearity with the quasilinearization process.

In literature, the convection-diffusion SPPs in ordinary differential equations with Robin boundary condition are explored by a few of the researchers [3,17]. Again, all the time-lagged SPPDEs discussed above are problems with Dirichlet boundary conditions. As per our knowledge, the only work discussing the time-lagged SPPDEs with Robin boundary condition is by Selvi and Ramanujam [27]. But, it is worthy enough to mention that the problem considered in [27] is of reaction-diffusion type. It clearly indicates the gap of numerical discussions for such kind of model problems with a convective term. Hence, the motive of this work is to provide a robust finite difference scheme for convection-diffusion type SPPDEs with a time lag and Robin boundary conditions. The scheme involves the use of the implicit Euler scheme in the time direction and the upwind scheme in the spatial direction on two layer-resolving meshes namely, S-mesh and Bakhvalov-Shishin mesh (B-S-mesh), which provides a first-order global accurate result. The accuracy of the presented scheme has again experimented on semi-linear time-lagged convection-diffusion problems having Robin-type boundary conditions.

The rest of the manuscript is structured as the statement of the problem along with its bounds are discussed in Section 2. The complete numerical scheme is discussed in Section 3. The convergence of the numerical scheme is derived in Section 4. Section 5 describes the existence and the approximation to handle a similar kind of model problem with semi-linearity associated with it. Section 6 provides the numerical outputs followed by the conclusion in Section 7.

Throughout the manuscript, $C > 0$ is considered as a generic constant, independent of ε, mesh points, and mesh sizes. For any $\zeta(s,t)$ defined on $\mathcal{D}$, the standard supremum norm is defined as $\|\zeta\|_\infty = \sup_{(s,t)\in\mathcal{D}} \zeta(s,t)$.

2 Statement of the problem

The problem of consideration with the initial-Robin boundary conditions is of the following type:

$$\begin{cases} L_\varepsilon z(s,t) = (\varepsilon z_{ss} + a z_s - bz - z_t)(s,t) = -z(s,t-\tau) + f(s,t), \quad (s,t) \in \mathcal{D} \\ z\big|_{\Upsilon_d} = \Phi_d(s,t), \quad (s,t) \in \Upsilon_d = [0,1] \times [-\tau, 0], \\ B_{l,\varepsilon} z(0,t) = (z - \varepsilon z_s)(0,t) = \Phi_l(t), \quad (s,t) \in \Upsilon_l = 0 \times (0,T], \\ B_{r,\varepsilon} z(1,t) = (z + \varepsilon z_s)(1,t) = \Phi_r(t), \quad (s,t) \in \Upsilon_r = 1 \times (0,T]. \end{cases} \tag{1}$$

Here, $\overline{\mathcal{D}} = \mathcal{D} \cup \partial\mathcal{D}$ with $\mathcal{D} = (0,1) \times (0,T]$ and $\partial\mathcal{D} = \Upsilon_d \cup \Upsilon_l \cup \Upsilon_r$. $\tau > 0$ is the temporal lag. $T = i\tau$ for i in $\mathbb{N}$. We assume $a(s)$, $b(s)$, $f(s,t)$, $\Phi_d(s,t)$, $\Phi_l(t)$, $\Phi_r(t)$ are sufficient smooth and bounded functions that satisfy $a(s) \geq \alpha > 0$ and $b(s) \geq \beta > 0$ for $(s,t) \in \overline{\mathcal{D}}$. The temporal and spatial domains are denoted by $\mathcal{D}_t = [-\tau, T]$ and $\mathcal{D}_s = [0,1]$, respectively. The reduced problem (with ε tends to zero) exhibits a layer in the neighbourhood of Υ_l *i.e.,* on the left side of the lateral boundary with a width of order $O(\varepsilon)$. The considered problem (1) satisfies the following minimum principle [27].

Lemma 1 *Assume that $\Psi(s,t)$ be a sufficient smooth function such that it satisfies $L_\varepsilon \Psi(s,t) \leq 0$, $(s,t) \in \mathcal{D}$, $\Psi(s,t) \geq 0$ for $(s,t) \in \Upsilon_0$, $B_{l,\varepsilon}\Psi(0,t) \geq 0$ and $B_{r,\varepsilon}\Psi(1,t) \geq 0$, where $\Upsilon_0 = \{(s,0) : s \in [0,1]\}$. Then, $\Psi(s,t) \geq 0$, $(s,t) \in \overline{\mathcal{D}}$.*

Proof Assume that $\Psi(s,t)$ attains its minimum value at (s^*,t^*) *i.e.,*

$$\Psi(s^*,t^*) = \min_{(s,t) \in \overline{\mathcal{D}}} \Psi(s,t) < 0.$$

Clearly, $(s^*,t^*) \notin \Upsilon_0$. $\dfrac{\partial\Psi}{\partial s}(s^*,t^*) = 0$, $\dfrac{\partial\Psi}{\partial t}(s^*,t^*) = 0$ and $\dfrac{\partial^2\Psi}{\partial s^2}(s^*,t^*) \geq 0$. Then, the following cases can happen. For Case:1, if $(s^*,t^*) \in \Upsilon_l$ then, $\Psi(s^*,t^*) - \varepsilon\dfrac{\partial\Psi}{\partial s}(s^*,t^*) < 0$ and for Case:2, if $(s^*,t^*) \in \Upsilon_r$ then, $\Psi(s^*,t^*) + \varepsilon\dfrac{\partial\Psi}{\partial s}(s^*,t^*) < 0$. Again, in Case:3, if $(s^*,t^*) \in \mathcal{D}$, $L_\varepsilon\Psi(s^*,t^*) = \varepsilon\dfrac{\partial^2\Psi}{\partial s^2}(s^*,t^*) + a\dfrac{\partial\Psi}{\partial s}(s^*,t^*) - b\Psi(s^*,t^*) - \dfrac{\partial\Psi}{\partial t}(s^*,t^*) > 0$. In all the above cases, we arrive at a contradiction, which proves the statement. □

Lemma 2 *The solution $z(s,t)$ of (1) satisfies,*

$$\|z(s,t)\|_{\overline{\mathcal{D}}} \leq C \max\left\{\|B_{l,\varepsilon}z\|, \|B_{r,\varepsilon}z\|, \|z\|_{\Upsilon_d}\right\} + \alpha^{-1}\|L_\varepsilon z\|_{\mathcal{D}}.$$

Proof This lemma can be proved by using Lemma 1 and the barrier function

$$\Theta^{\pm}(s,t) = M \pm z(s,t), \quad (s,t) \in \overline{\mathcal{D}},$$

where,

$$M = \max\left\{\|B_{l,\varepsilon}z\|, \|B_{r,\varepsilon}z\|, \|z\|_{\Upsilon_d}\right\} + \alpha^{-1}\|L_\varepsilon z\|_{\mathcal{D}}.$$

For the details, one can refer to [5]. The stability and ε-uniform bound for the solution of (1) in the maximum norm follows immediately from the statement. □

For the solution of problem (1), where the lag values of $(t-\tau)$ are bounded away from t by a positive constant, the existence of the solution of (1) is verified by method of steps. As, the solution is known on Υ_d *i.e.,* $z(s,t-\tau)$ is given, so (1) becomes a classical singularly perturbed parabolic problem, which can be solved and treated by some known existing theories [5,16]. Again, $z(s,t-\tau)$ will be known while solving for $(s,t) \in [0,1] \times [\tau, 2\tau]$ and so on. The existence and uniqueness of the solution $z(s,t)$ of (1) can be proved in a similar approach as in [27] *i.e.,* by assuming all the datas to be Hölder continuous and also satisfy appropriate compatible conditions at the corner points $(0,0)$, $(1,0)$, $(0,-\tau)$, $(1,-\tau)$. $\Phi_l, \Phi_r \in C^k([0,T])$, $\Phi_d \in C^{(1,k)}(\Upsilon_d)$ are said to have the k^{th}-order compatibility conditions if

$$\frac{\partial^k}{\partial t^k}\left[\Phi_d - \varepsilon\frac{\partial\Phi_d}{\partial s}\right](0,0) = \frac{d^k}{dt^k}\Phi_l(0) \quad \text{and} \quad \frac{\partial^k}{\partial t^k}\left[\Phi_d + \varepsilon\frac{\partial\Phi_d}{\partial s}\right](1,0) = \frac{d^k}{dt^k}\Phi_r(0).$$

Theorem 3 *Let coefficients and the source function* $a, b, f \in C^{(4+\gamma)}(\overline{\mathcal{D}})$, $\Phi_l(t)$, $\Phi_r(t)$ *and* $\Phi_d(s,t) \in C^{(5+\gamma)}(\overline{\mathcal{D}})$, $\gamma \in (0,1)$. *Assuming that the compatibility conditions are satisfied for* $k = 0, 1$, *the problem (1) has a unique solution* $z \in C^{(2+\gamma)}(\overline{\mathcal{D}})$.

Proof The proof can be followed from [15]. □

2.1 Bounds of the solution and its derivatives

The following theorem provides the bounds on the solution of (1) and its derivatives.

Theorem 4 *For* $0 \le i + j \le 4$, *under sufficient compatibility conditions, the solution of (1) and its partial derivatives satisfy,*

$$\left|\frac{\partial^{i+j} z(s,t)}{\partial s^i \partial t^j}\right|_{\overline{\mathcal{D}}} \le C(1 + \varepsilon^{-i} e^{-\alpha s/\varepsilon}) \;\; for \;\; (s,t) \in \overline{\mathcal{D}}.$$

Proof For the proof one can refer Lemma 2.3 - Lemma 2.8 of [16]. □

The classical bounds on $z(s,t)$ and its derivatives are insufficient for the derivation of ε-uniform error estimates, hence $z(s,t)$ needs to be decomposed into its regular $(z_r(s,t))$ and boundary layer components $(z_b(s,t))$ as $z(s,t) = z_r(s,t) + z_b(s,t)$. Again, the regular component has to be decomposed into $z_r(s,t) = z_{r,0}(s,t) + \varepsilon z_{r,1}(s,t) + \varepsilon^2 z_{r,2}(s,t)$, where $z_{r,0}$ is the solution of the following inital boundary value problem

$$\begin{cases} [a(z_{r,0})_s - bz_{r,0} - (z_{r,0})_t](s,t) = -z_{r,0}(s,t-\tau) + f(s,t), & (s,t) \in \mathcal{D} \\ z_{r,0}\big|_{\Upsilon_d}(s,t) = \Phi_d(s,t), & (s,t) \in \Upsilon_d \\ B_{r,\varepsilon} z_{r,0}(1,t) = \Phi_r(t), & 0 \le t \le T. \end{cases} \tag{2}$$

$z_{r,1}$ and $z_{r,2}$ are the solutions of

$$\begin{cases} [a(z_{r,1})_s - bz_{r,1} - (z_{r,1})_t](s,t) = -z_{r,1}(s,t-\tau) + (z_{r,0})_{ss}, & (s,t) \in \mathcal{D} \\ z_{r,1}\big|_{\Upsilon_d}(s,t) = 0, & (s,t) \in \Upsilon_d \\ B_{r,\varepsilon} z_{r,1}(1,t) = 0, & 0 \le t \le T, \end{cases} \tag{3}$$

and

$$\begin{cases} L_\varepsilon z_{r,2}(s,t) = -z_{r,2}(s,t-\tau) + (z_{r,1})_{ss}, & (s,t) \in \mathcal{D} \\ z_{r,2}\big|_{\Upsilon_d}(s,t) = 0, & (s,t) \in \Upsilon_d \\ B_{l,\varepsilon} z_{r,2}(0,t) = 0 = B_{r,\varepsilon} z_{r,2}(1,t), & 0 \le t \le T, \end{cases} \tag{4}$$

respectively. Hence, z_r satisfies,

$$\begin{cases} L_\varepsilon z_r(s,t) = -z_r(s,t-\tau) + f(s,t), & (s,t) \in \mathcal{D} \\ z_r\big|_{\Upsilon_d}(s,t) = \Phi_d(s,t), & (s,t) \in \Upsilon_d \\ B_{l,\varepsilon} z_r(0,t) = \sum_{i=0}^{2} \varepsilon^i B_{l,\varepsilon} z_{r,i}(0,t), & 0 \le t \le T \\ B_{r,\varepsilon} z_r(1,t) = \Phi_r(t), & 0 \le t \le T. \end{cases} \tag{5}$$

Now, the boundary layer component satisfies,

$$\begin{cases} L_\varepsilon z_b(s,t) = -z_b(s,t-\tau), & (s,t) \in \mathcal{D} \\ z_b\big|_{\Upsilon_d} = 0, & (s,t) \in \Upsilon_d \\ B_{l,\varepsilon} z_b(s,t) = B_{l,\varepsilon} z(0,t) - B_{l,\varepsilon} z_r(0,t), & 0 \le t \le T \\ B_{r,\varepsilon} z_b(1,t) = 0, & 0 \le t \le T. \end{cases} \tag{6}$$

The following theorem provides the bounds for the smooth and layer components of $z(s,t)$.

Theorem 5 *Under certain smoothness and compatibility conditions on $a(s)$, $b(s)$ and $f(s,t)$, the smooth and boundary layer components satisfy the following bounds*

$$\left|\frac{\partial^{i+j} z_r(s,t)}{\partial s^i \partial t^j}\right| \le C(1+\varepsilon^{2-i}), \textit{for } i+j \le 3,$$
$$\left|\frac{\partial^{i+j} z_b(s,t)}{\partial s^i \partial t^j}\right| \le C\varepsilon^{-i} e^{-\alpha s/\varepsilon} \textit{ for } i+2j \le 4.$$

Proof Since, both $z_{r,0}$ and $z_{r,1}$ are the solutions of (2) and (3) that are independent of ε, therefore the bounds can be rightfully obtained as

$$\left|\frac{\partial^{i+j} z_{r,k}(s,t)}{\partial s^i \partial t^j}\right| \le C_k, \quad k = 0, 1. \tag{7}$$

Again, $z_{r,2}$ is the solution of (4), which is in the form of (1), hence it will satisfy the bounds as mentioned in Theorem 4. So,

$$\left|\frac{\partial^{i+j} z_{r,2}(s,t)}{\partial s^i \partial t^j}\right| \le C_2(1+\varepsilon^{-i} e^{-\alpha s/\varepsilon}), \quad \text{for} \quad (s,t) \in \overline{\mathcal{D}}. \tag{8}$$

Thus, by the use of (7) and (8), the bounds for the solution of (5) can be obtained as

$$\begin{aligned}\left|\frac{\partial^{i+j} z_r(s,t)}{\partial s^i \partial t^j}\right| &\le \left|\frac{\partial^{i+j} z_{r,0}(s,t)}{\partial s^i \partial t^j}\right| + \varepsilon\left|\frac{\partial^{i+j} z_{r,1}(s,t)}{\partial s^i \partial t^j}\right| + \varepsilon^2\left|\frac{\partial^{i+j} z_{r,2}(s,t)}{\partial s^i \partial t^j}\right| \\ &\le C_0 + C_1\varepsilon + C_2\varepsilon^2[1+\varepsilon^{-i} e^{-\alpha s/\varepsilon}].\end{aligned}$$

By choosing sufficiently large C, the required bounds can be obtained. Before prooving the bounds for $z_b(s,t)$, we must show

$$|z_b(s,t)| \le Ce^{-\alpha s/\varepsilon}.$$

Consider the barrier functions as

$$\Theta^{\pm}(s,t) = Ce^{-\alpha s/\varepsilon} e^t \pm z_b(s,t).$$

If $(s,t) \in \Upsilon_l$, then choosing sufficiently large C, the following can be concluded

$$\Theta^{\pm}(0,t) - \varepsilon\Theta_s^{\pm}(0,t) = Ce^t \pm z_b(0,t) + C\alpha e^t \mp \varepsilon(z_b)_s(0,t) \ge 0.$$

Similarly, for $(s,t) \in \Upsilon_r$ and $(s,t) \in \mathcal{D}$, $\Theta^{\pm}(1,t) + \varepsilon\Theta_s^{\pm}(1,t) \ge 0$ and $L_\varepsilon\Theta^{\pm}(s,t) < 0$, respectively can be obtained. $\Theta^{\pm}(s,t)$ satisfies the conditions for Lemma 1 hence, $\Theta^{\pm}(s,t) \ge 0$ and $z_b(s,t) \le Ce^{-\alpha s/\varepsilon} e^t \le Ce^{-\alpha s/\varepsilon}$ for $0 \le t \le T$. Now, the bounds for the derivatives of $z_b(s,t)$ can be obtained by using the approach as done in [18]. □

3 Numerical scheme

The complete numerical scheme is discussed in this section, comprising the implicit Euler scheme in the time direction and the upwind scheme in the spatial direction.

3.1 Temporal discretization

With constant step length Λt, the time direction is handled using an uniform mesh. Let $\overline{\mathcal{D}_t} = [-\tau, 2\tau = T]$ be divided as,

$$\mathcal{D}_t^{M_1} = \{t_m = m\Lambda t, m = 0, 1, \ldots, M_1, t_{M_1} = \tau, \Lambda t = \tau/M_1\},$$
$$\mathcal{D}_t^{M_2} = \{t_m = m\Lambda t, m = 0, 1, \ldots, M_2, t_{M_2} = T, \Lambda t = T/M_2\}.$$

Let, $M = (M_1 + M_2)$, M_1 and M_2 are the number of mesh intervals in $[-\tau, T]$, $[-\tau, 0]$ and $[0, T]$, respectively.

3.2 Spatial discretization

With $\varrho > 0$ as the transition parameter, let $N > 4$ be an even integer considered as the total number of mesh intervals in space. With $\varrho_0 \geq 2/\alpha$, ϱ is defined as:

$$\varrho = \min\left(\frac{1}{2}, \varrho_0 \varepsilon \ln N\right).$$

Due to the presence of ε, a layer of $O(\varepsilon)$ appears near $s = 0$. $\overline{\mathcal{D}}_s$ is split into two equal sub-domains namely, $[0, \varrho]$ and $[\varrho, 1]$. $\overline{\mathcal{D}}_s^N$ with $s_0 = 0$, $s_{N/2} = \varrho$ and $s_N = 1$ is

$$\overline{\mathcal{D}}_s = \{s_0 = 0, s_1, s_2, \ldots, s_{N/2} = \varrho, \ldots, s_N = 1\}.$$

Let $\overline{\mathcal{D}}^L = [0, \varrho] \times [0, T]$, $\overline{\mathcal{D}}^R = [\varrho, 1] \times [0, T]$, $h_n = s_n - s_{n-1}$ and $\hbar_n$ is denoted as: $\hbar_n = h_n + h_{n+1}$.. S-mesh is denoted as:

$$s_n = \begin{cases} n\dfrac{2\varrho}{N}, & \text{if } n = 1, 2, \ldots, \dfrac{N}{2}, \\ \varrho + \dfrac{2(1-\varrho)}{N}(n - N/2), & \text{if } n = \dfrac{N}{2} + 1, \ldots, N - 1. \end{cases}$$

The B-S-mesh is:

$$s_n = \begin{cases} -\varrho_0 \varepsilon \ln\left[1 - \left(2(1 - 1/N)(n/N)\right)\right], & \text{if } n = 1, 2, \ldots, \dfrac{N}{2}, \\ \varrho + \dfrac{2(1-\varrho)}{N}(n - N/2), & \text{if } n = \dfrac{N}{2} + 1, \ldots, N - 1. \end{cases}$$

3.3 Fully discretized scheme

With N number of mesh intervals in spatial direction and Λt as the uniform mesh width in temporal direction, let $\mathcal{D}^{N,\Lambda t}$ is the fully discrete version of $\mathcal{D}$. The similar argument works for $L_\varepsilon^{N,\Lambda t}$,$B_{l,\varepsilon}^{N,\Lambda t}$,$B_{r,\varepsilon}^{N,\Lambda t}$, and $\Upsilon_d^{N,\Lambda t}$. The discrete form of (1) on $\mathcal{D}^{N,\Lambda t}$ is

$$\begin{cases} L_\varepsilon^{N,\Lambda t} Z_n^{m+1} \cong \\ (\varepsilon\delta_s^2 + a_n\delta_s^+ - b_n^{m+1} - \delta_t^-)Z_n^{m+1} = -Z_n^{m+1-M_1} + f_n^{m+1}, \\ Z_n^{m+1-M_1} = \Phi_d(s_n, t_{m+1-M_1}) \quad \text{for} \quad m = M_1, \ldots, M \quad \text{and} \quad n = 0, \ldots, N-1, \\ B_{l,\varepsilon} Z(0, t_{m+1}) = Z(0, t_{m+1}) - \varepsilon\delta_s^+ Z(0, t_{m+1}) = \Phi_l(t_{m+1}), \\ B_{r,\varepsilon} Z(1, t_{m+1}) = Z(1, t_{m+1}) + \varepsilon\delta_s^- Z(1, t_{m+1}) = \Phi_r(t_{m+1}), \quad \forall (s,t) \in \mathcal{D}^{N,\Lambda t}, \end{cases} \tag{9}$$

where,

$$\delta_s^2 Z_n^{m+1} = \frac{2}{\hbar_n}\left(\delta_s^+ Z_n^{m+1} - \delta_s^- Z_n^{m+1}\right), \quad \delta_s^+ Z_n^{m+1} = \frac{Z_{n+1}^{m+1} - Z_n^{m+1}}{h_{n+1}}$$

$$\delta_s^- Z_n^{m+1} = \frac{Z_n^{m+1} - Z_{n-1}^{m+1}}{h_n} \quad \text{and} \quad \delta_t^- Z_n^{m+1} = \frac{Z_n^{m+1} - Z_n^m}{\Lambda t}.$$

After rearranging the terms, we have

$$\begin{cases} A_n^- Z_{n-1}^{m+1} + A_n^c Z_n^{m+1} + A_n^+ Z_{n+1}^{m+1} = g_n^{m+1}, \quad (s_n, t_{m+1}) \in \mathcal{D}^{N,\Lambda t}, \\ Z_n^{m+1-M_1} = \Phi_d(s_n, t_{m+1-M_1}), \quad 0 \leq n \leq N, \quad m = M_1 \ldots, M, \\ B_{l,\varepsilon} Z(0, t_{m+1}) = Z(0, t_{m+1}) - \varepsilon\delta_s^+ Z(0, t_{m+1}) = \Phi_l(t_{m+1}), \\ B_{r,\varepsilon} Z(1, t_{m+1}) = Z(1, t_{m+1}) + \varepsilon\delta_s^- Z(1, t_{m+1}) = \Phi_r(t_{m+1}) \quad m = M_1, \ldots, M. \end{cases}$$

The coefficients are given by,

$$\begin{cases} A_n^+ = \dfrac{2\varepsilon}{\hbar_n h_{n+1}} + \dfrac{a_n}{h_{n+1}}, \; A_n^c = -\dfrac{2\varepsilon}{h_n h_{n+1}} - \dfrac{a_n}{h_{n+1}} - b_n - \dfrac{1}{\Lambda t}, \\ A_n^- = \dfrac{2\varepsilon}{\hbar_n h_n}, \qquad g_n^{m+1} = -\dfrac{Z_n^m}{\Lambda t} - Z_n^{m+1-M_1} + f_n^{m+1}. \end{cases}$$

for $n = 1, 2, \ldots, (N-1)$ and $m = M_1, (M_1+1), \ldots, M$. $A_0^+ = -\dfrac{\varepsilon}{h_1}$, $A_0^c = 1 + \dfrac{\varepsilon}{h_1}$, $A_N^- = -\dfrac{\varepsilon}{h_N}$ and $A_N^c = 1 + \dfrac{\varepsilon}{h_N}$. The coefficient matrix of the discrete problem in (9) along with the discrete boundary conditions will create an $(N+1)\times(N+1)$ system of linear equations with $(N+1)$ number of unknowns *i.e.,* $Z_0, Z_1, \ldots, Z_N$. The tridiagonal matrix obtained for each time level can thus be proved to be an M-matrix. This demonstrates that the discrete scheme applied is stable and oscillation free. The solution of (9) will satisfy the discrete minimum principle which is analogous to Lemma 1 satisfied by the solution of (1). The following lemma will provide the ε- uniform stability for the discrete operator $L_\varepsilon^{N,\Lambda t}$.

Lemma 6 *Assume that $Z(s_n, t_{m+1})$ be a sufficient smooth function such that it satisfies*

$$\begin{cases} L_\varepsilon^{N,\Lambda t} Z(s_n, t_{m+1}) \leq 0, \quad for \quad (s_n, t_{m+1}) \in \overline{\mathcal{D}}^{N,\Lambda t}, \\ Z(s_n, t_{m+1}) \geq 0, for (s_n, t_{m+1}) \in \Upsilon_0^{N,\Lambda t}, \\ B_{l,\varepsilon}^{N,\Lambda t} Z(s_0, t_{m+1}) \geq 0, \\ B_{r,\varepsilon}^{N,\Lambda t} Z(s_N, t_{m+1}) \geq 0, \quad where \quad \Upsilon_0^{N,\Lambda t} = \{(s_n, 0) : n = 0, 1, \ldots, N\}. \end{cases}$$

Then, $Z(s_n, t_{m+1}) \geq 0$, $(s_n, t_{m+1}) \in \overline{\mathcal{D}}^{N,\Lambda t}$.

Lemma 7 *Solution $Z(s_n, t_{m+1})$ of the discrete problem (9) satisfies the following bounds*

$$\begin{aligned} \left\| Z(s_n, t_{m+1}) \right\|_{\overline{\mathcal{D}}^{N,\Lambda t}} \leq C \max \Big\{ \left\| B_{l,\varepsilon}^{N,\Lambda t} Z(s_0, t_{m+1}) \right\|, \left\| B_{r,\varepsilon}^{N,\Lambda t} Z(s_N, t_{m+1}) \right\|, \\ \left\| Z(s_n, t_{m+1}) \right\|_{\Upsilon_d^{N,\Lambda t}} \Big\} + \alpha^{-1} \left\| L_\varepsilon^{N,\Lambda t} Z(s_n, t_{m+1}) \right\|_{\mathcal{D}^{N,\Lambda t}}. \end{aligned}$$

Proof Analogous to Lemma 2 for the continuous problem, this lemma can be proved by using Lemma 6 and the appropriate barrier function. □

4 Convergence of the numerical scheme

This section explains the order of convergence of the proposed numerical scheme. The analysis will be divided into two steps for two different time levels. Let $[0,1] \times [0,\tau] = \mathcal{D}_1$ and $[0,1] \times [\tau, 2\tau] = \mathcal{D}_2$. Before the proof of converegnce in the entire domain, the following results are needed.

Lemma 8 *For $n = 0, 1, \ldots, N$ and for fixed time at $(m+1)^{th}$ level, define a mesh function as,* $\psi(s_n, t_{m+1}) = \psi_n^{m+1} = \prod_{k=1}^{n} \left[1 + \dfrac{\alpha h_k}{\varepsilon}\right]^{-1}$ *with $\psi_0^{m+1} = 1$. Then, for $n = 1, \ldots, N$, we have* $L_\varepsilon^{N,\Lambda t} \psi_n^{m+1} \geq \dfrac{C}{\max\{\varepsilon, h_{n+1}\}} \psi_n^{m+1}$.

Lemma 9 *For $n = 1, \ldots, N-1$ we have,* $\exp\big(-\alpha s_n/\varepsilon\big) \leq \prod_{k=1}^{n} \left[1 + \dfrac{\alpha h_k}{\varepsilon}\right]^{-1}$.

Proof For any n, $\left[1 + \dfrac{\alpha h_n}{\varepsilon}\right]^{-1} \geq \exp\big(-\alpha h_n/\varepsilon\big)$. So, $\prod_{k=1}^{n} \left[1 + \dfrac{\alpha h_k}{\varepsilon}\right]^{-1} \geq \prod_{k=1}^{n} \exp\big(-\alpha h_k/\varepsilon\big) = \exp\big(-\alpha s_n/\varepsilon\big)$.
□

Theorem 10 *Let $z(s,t)$ and $Z(s_n,t_{m+1})$ be respectively the solutions of continuous problem (1) and the fully discrete problem (9) in $\mathcal{D}_1$, satisfying the compatibility conditions at the corners. Then, the following error bound is obtained for $(s_n,t_{m+1}) \in \overline{\mathcal{D}}_1^{N,\Lambda t}$*

$$\max_{(s_n,t_{m+1})}\left|Z(s_n,t_{m+1})-z(s_n,t_{m+1})\right| \le C(N^{-1}\ln N+\Lambda t), \quad \textit{on S-mesh and}$$

$$\max_{(s_n,t_{m+1})}\left|Z(s_n,t_{m+1})-z(s_n,t_{m+1})\right| \le C(N^{-1}+\Lambda t), \quad \textit{on B-S-mesh.}$$

Proof While finding the error bound for $\mathcal{D}_1$, the term $Z(s,t-\tau)$ in the problem will have the values from the domain $[0,1]\times[-\tau,0]$, which is the given initial condition and hence is independent of ε. So, the error bounds for the domain $\mathcal{D}_1$ can be done in a similar approach as for any usual SPPDE with Robin boundary conditions [5]. □

Theorem 11 *Let $z(s,t)$ and $Z(s_n,t_{m+1})$ be respectively the solutions of continuous problem (1) and the fully discrete problem (9) in $\mathcal{D}_2$, satisfying the compatibility conditions at the corners. Then, for $(s_n,t_{m+1}) \in \overline{\mathcal{D}}_2^{N,\Lambda t}$, the corresponding error bounds on S-mesh and B-S-mesh are obtained as*

$$\max_{(s_n,t_{m+1})}\left|Z(s_n,t_{m+1})-z(s_n,t_{m+1})\right| \le C(N^{-1}\ln N+\Lambda t),$$

and

$$\max_{(s_n,t_{m+1})}\left|Z(s_n,t_{m+1})-z(s_n,t_{m+1})\right| \le C(N^{-1}+\Lambda t).$$

Proof On $\overline{\mathcal{D}}_2$, the following time-lagged SPPDE with Robin boundary condition is considered.

$$\begin{cases} L_\varepsilon z(s,t) = (\varepsilon z_{ss}+az_s-bz-z_t)(s,t) = -z(s,t-\tau)+f(s,t), \quad (s,t)\in\overline{\mathcal{D}}_2 \\ z|_{\Upsilon_d} = z_d(s,t), \quad (s,t)\in\overline{\mathcal{D}}_1=[0,1]\times[0,\tau], \\ B_{l,\varepsilon}z(0,t)=(z-\varepsilon z_s)(0,t)=\Phi_l(t), \quad \tau\le t\le 2\tau \\ B_{r,\varepsilon}z(1,t)=(z+\varepsilon z_s)(1,t)=\Phi_r(t), \quad \tau\le t\le 2\tau, \end{cases} \tag{10}$$

where, $z_d(s,t)$ is the solution obtained in $\overline{\mathcal{D}}_1$. The solution of (10) is split into its regular and boundary layer components as $z(s,t)=z_r(s,t)+z_b(s,t)$. Since, $\overline{\mathcal{D}}_2\subset\overline{\mathcal{D}}$, the estimates of Theorem 5 can also be used for both z_r and z_b. Again, the smooth component will be decomposed into $z_r(s,t)=z_{r,0}(s,t)+\varepsilon z_{r,1}(s,t)+\varepsilon^2 z_{r,2}(s,t)$, where $z_{r,0}$ is the solution of the following problem

$$\begin{cases} [a(z_{r,0})_s-bz_{r,0}-(z_{r,0})_t](s,t) = -z_{r,0}(s,t-\tau)+f(s,t), \quad (s,t)\in\overline{\mathcal{D}}_2 \\ z_{r,0}(s,t)=z_d(s,t), \quad (s,t)\in\overline{\mathcal{D}}_1 \\ B_{r,\varepsilon}z_{r,0}(1,t)=\Phi_r(t), \quad \tau\le t\le 2\tau. \end{cases} \tag{11}$$

$z_{r,1}$ and $z_{r,2}$ are the solutions of

$$\begin{cases} [a(z_{r,1})_s-bz_{r,1}-(z_{r,1})_t](s,t) = -z_{r,1}(s,t-\tau)+(z_{r,0})_{ss}, \quad (s,t)\in\overline{\mathcal{D}}_2 \\ z_{r,1}(s,t)=0, \quad (s,t)\in\overline{\mathcal{D}}_1 \\ B_{r,\varepsilon}z_{r,1}(1,t)=0, \quad \tau\le t\le 2\tau, \end{cases} \tag{12}$$

and

$$\begin{cases} L_\varepsilon z_{r,2}(s,t) = -z_{r,2}(s,t-\tau)+(z_{r,1})_{ss}, \quad (s,t)\in\overline{\mathcal{D}}_2 \\ z_{r,2}(s,t)=0, \quad (s,t)\in\overline{\mathcal{D}}_1 \\ B_{l,\varepsilon}z_{r,2}(0,t)=0=B_{r,\varepsilon}z_{r,2}(1,t), \quad \tau\le t\le 2\tau, \end{cases} \tag{13}$$

respectively. Hence, the smooth components satisfy,

$$\begin{cases} L_\varepsilon z_r(s,t) = -z_r(s,t-\tau) + f(s,t), \quad (s,t) \in \overline{\mathcal{D}}_2 \\ z_r(s,t) = z_d(s,t), \quad (s,t) \in \overline{\mathcal{D}}_1 \\ B_{l,\varepsilon} z_r(0,t) = \sum_{i=0}^{2} \varepsilon^i B_{l,\varepsilon} z_{r,i}(0,t), \quad B_{r,\varepsilon} z_r(s,t) = \Phi_r(t), \quad \tau \le t \le 2\tau. \end{cases} \tag{14}$$

Now, the boundary layer components satisfy,

$$\begin{cases} L_\varepsilon z_b(s,t) = -z_b(s,t-\tau), \quad (s,t) \in \overline{\mathcal{D}}_2 \\ z_b(s,t) = 0, \quad (s,t) \in \overline{\mathcal{D}}_1 \\ B_{l,\varepsilon} z_b(s,t) = B_{l,\varepsilon} z(0,t) - B_{l,\varepsilon} z_r(0,t), \\ B_{r,\varepsilon} z_b(s,t) = B_{r,\varepsilon} z(1,t) - B_{r,\varepsilon} z_r(1,t), \quad \tau \le t \le 2\tau. \end{cases} \tag{15}$$

The fully discrete form of (10) in $(s_n, t_{m+1}) \in \mathcal{D}_2^{N,\Delta t}$ can be written as

$$\begin{cases} L_\varepsilon^{N,\Delta t} Z_n^{m+1} \cong (\varepsilon\delta_s^2 + a_n\delta_s^+ - b_n - \delta_t^-) Z_n^{m+1} = -Z_n^{m+1-M_1} + f_n^{m+1}, \\ Z_n^{m+1-M_1} = Z_d(s_n, t_{m+1-M_1}), \quad 0 \le n \le N, \quad m = (M_1 + (M_2/2)), \ldots, M, \\ B_{l,\varepsilon}^{N,\Delta t} Z(0, t_{m+1}) = Z(0, t_{m+1}) - \varepsilon\delta_s^+ Z(0, t_{m+1}), \quad m = (M_1 + (M_2/2)), \ldots, M, \\ B_{r,\varepsilon}^{N,\Delta t} Z(1, t_{m+1}) = Z(1, t_{m+1}) + \varepsilon\delta_s^- Z(1, t_{m+1}), \quad m = (M_1 + (M_2/2)), \ldots, M. \end{cases} \tag{16}$$

Again, the solution of (16) can be decomposed into the regular and boundary layer components as $Z(s_n, t_{m+1}) = Z_r(s_n, t_{m+1}) + Z_b(s_n, t_{m+1})$, where $Z_r(s_n, t_{m+1})$ and $Z_b(s_n, t_{m+1})$ are the solutions of following discrete problems

$$\begin{cases} L_\varepsilon^{N,\Delta t} Z_{r,n}^{m+1} = -Z_{r,n}^{m+1-M_1} + f_n^{m+1}, \quad (s_n, t_{m+1}) \in \mathcal{D}_2^{N,\Delta t} \\ Z_{r,n}^{m+1-M_1} = Z_d(s_n, t_{m+1-M_1}), \quad 0 \le n \le N, \quad m = (M_1 + (M_2/2)), \ldots, M, \\ B_{l,\varepsilon}^{N,\Delta t} Z_r(0, t_{m+1}) = B_{l,\varepsilon}^{N,\Delta t} z_r(0, t_{m+1}), \quad m = (M_1 + (M_2/2)), \ldots, M, \\ B_{r,\varepsilon}^{N,\Delta t} Z_r(1, t_{m+1}) = B_{r,\varepsilon}^{N,\Delta t} z_r(1, t_{m+1}), \quad m = (M_1 + (M_2/2)), \ldots, M. \end{cases} \tag{17}$$

and

$$\begin{cases} L_\varepsilon^{N,\Delta t} Z_{b,n}^{m+1} = -Z_{b,n}^{m+1-M_1}, \quad (s_n, t_{m+1}) \in \mathcal{D}_2^{N,\Delta t} \\ Z_{b,n}^{m+1-M_1} = 0, \quad 0 \le n \le N, \quad m = (M_1 + (M_2/2)), \ldots, M, \\ B_{l,\varepsilon}^{N,\Delta t} Z_b(0, t_{m+1}) = \Phi_l(t_{m+1}) - B_{l,\varepsilon}^{N,\Delta t} Z_r(0, t_{m+1}), \quad m = (M_1 + (M_2/2)), \ldots, M, \\ B_{r,\varepsilon}^{N,\Delta t} Z_b(1, t_{m+1}) = \Phi_r(t_{m+1}) - B_{r,\varepsilon}^{N,\Delta t} Z_r(1, t_{m+1}), \quad m = (M_1 + (M_2/2)), \ldots, M. \end{cases} \tag{18}$$

$z_d(s,t)$ and $Z_d(s_n, t_{m+1})$ are the solutions of (1) and (9) in $\mathcal{D}_1$ and $\mathcal{D}_1^{N,\Delta t}$, respectively. At any node (s_n, t_{m+1}), the error can be written as

$$(Z - z)(s_n, t_{m+1}) = (Z_r - z_r)(s_n, t_{m+1}) + (Z_b - z_b)(s_n, t_{m+1})$$

which implies

$$\left|(Z - z)(s_n, t_{m+1})\right| \le \left|(Z_r - z_r)(s_n, t_{m+1})\right| + \left|(Z_b - z_b)(s_n, t_{m+1})\right|.$$

Hence, the errors have to be estimated separately for Z_r and Z_b.

Error in smooth component:

For the left-end boundary conditions, the following can be obtained

$$B_{l,\varepsilon}^{N,\Lambda t}(Z_r - z_r)(s_0, t_{m+1}) = B_{l,\varepsilon}^{N,\Lambda t} Z_r(s_0, t_{m+1}) - B_{l,\varepsilon}^{N,\Lambda t} z_r(s_0, t_{m+1})$$
$$= \Big[z_r(s_0, t_{m+1}) - \varepsilon (z_r)_s(s_0, t_{m+1}) \Big] - \Big[Z_r(s_0, t_{m+1}) - \varepsilon \delta_s^+ Z_r(s_0, t_{m+1}) \Big]$$
$$= \varepsilon [\delta_s^+ - \frac{\partial}{\partial s}] z_r(s_0, t_{m+1})$$

Hence,

$$\Big| B_{l,\varepsilon}^{N,\Lambda t}(Z_r - z_r)(s_0, t_{m+1}) \Big| \leq C\varepsilon (s_1 - s_0)(z_r)_{ss}(s_0, t_{m+1}).$$

Using the bounds on regular components from Theorem 5 and the fact $\varepsilon \leq CN^{-1}$, we have

$$\Big| B_{l,\varepsilon}^{N,\Lambda t}(Z_r - z_r)(s_0, t_{m+1}) \Big| \leq CN^{-1}, \quad M_1 + 1 \leq m \leq M. \tag{19}$$

Similarly, for the right-end boundary, one can get

$$\Big| B_{r,\varepsilon}^{N,\Lambda t}(Z_r - z_r)(s_N, t_{m+1}) \Big| \leq CN^{-1}, \quad M_1 + 1 \leq m \leq M. \tag{20}$$

Now, for $1 \leq n \leq N-1$, using Theorem 10, (14) and (17),

$$L_\varepsilon^{N,\Lambda t}(Z_r - z_r)(s_n, t_{m+1})$$
$$= z_r(s_n, t_{m+1-M_1}) - Z_r(s_n, t_{m+1-M_1}) + (L_\varepsilon - L_\varepsilon^{N,\Lambda t}) z_r(s_n, t_{m+1}),$$
$$= z_d(s_n, t_{m+1-M_1}) - Z_d(s_n, t_{m+1-M_1}) + (L_\varepsilon - L_\varepsilon^{N,\Lambda t}) z_r(s_n, t_{m+1})$$
$$= C(N^{-1} + \Lambda t) + \left[\varepsilon \left(\frac{\partial^2}{\partial s^2} - \delta_s^2 \right) + a(s_n) \left(\frac{\partial}{\partial s} - \delta_s^+ \right) - \left(\frac{\partial}{\partial t} - \delta_t^- \right) \right] z_r(s_n, t_{m+1})$$

Hence,

$$\Big| L_\varepsilon^{N,\Lambda t}(Z_r - z_r)(s_n, t_{m+1}) \Big| \leq C(N^{-1} + \Lambda t) + \frac{\varepsilon}{3}(s_{n+1} - s_{n-1}) \left\| \frac{\partial^3 z_r}{\partial s^3} \right\|_{\overline{\mathcal{D}}_2^{N,\Lambda t}}$$
$$+ \frac{\alpha}{2}(s_{n+1} - s_n) \left\| \frac{\partial^2 z_r}{\partial s^2} \right\|_{\overline{\mathcal{D}}_2^{N,\Lambda t}} + \frac{\Lambda t}{2} \left\| \frac{\partial^2 z_r}{\partial t^2} \right\|_{\overline{\mathcal{D}}_2^{N,\Lambda t}}$$

Choosing sufficient large C and by the use of Lemma 1 and Theorem 5, the required bounds can be obtained as

$$\Big| Z_r(s_n, t_{m+1}) - z_r(s_n, t_{m+1}) \Big| \leq C(N^{-1} + \Lambda t), \quad \forall (s_n, t_{m+1}) \in \overline{\mathcal{D}}_2^{N,\Lambda t} \tag{21}$$

Error in layer component:

As the error estimate for the layer component depends upon the transition parameter ϱ, so the proof of this section is divided into two cases according to the choice of ϱ.

Case:1 If $\varrho = 1/2$, then the mesh formed inside the layer region will be uniform in nature and $\varepsilon^{-1} \leq C \ln N$. Using Theorem 10, $s_{n+1} - s_{n-1} = 2N^{-1}$, $s_n - s_{n-1} = N^{-1}$, we have

$$B_{l,\varepsilon}^{N,\Lambda t}(Z_b - z_b)(s_0, t_{m+1}) = B_{l,\varepsilon}^{N,\Lambda t} Z_b(s_0, t_{m+1}) - B_{l,\varepsilon}^{N,\Lambda t} z_b(s_0, t_{m+1})$$
$$= \Big[z_b(s_0, t_{m+1}) - \varepsilon (z_b)_s(s_0, t_{m+1}) \Big] - \Big[Z_b(s_0, t_{m+1}) - \varepsilon \delta_s^+ Z_b(s_0, t_{m+1}) \Big]$$
$$= \varepsilon [\delta_s^+ - \frac{\partial}{\partial s}] z_b(s_0, t_{m+1}). \tag{22}$$

Hence,

$$\left|B_{l,\varepsilon}^{N,\Lambda t}(Z_b - z_b)(s_0, t_{m+1})\right| \leq C\varepsilon(s_1 - s_0)(z_b)_{ss}(s_0, t_{m+1}) \leq CN^{-1}\ln N.$$

Similarly, for the right end boundary, one can derive

$$\left|B_{r,\varepsilon}^{N,\Lambda t}(Z_b - z_b)(s_N, t_{m+1})\right| \leq CN^{-1}\ln N.$$

Again, using the initial conditions in $\overline{\mathcal{D}}_1$ from (18) and the bounds in Theorem 10;

$$\begin{aligned}
&L_{\varepsilon}^{N,\Lambda t}(Z_b - z_b)(s_n, t_{m+1})\\
&= z_b(s_n, t_{m+1-M_1}) - Z_b(s_n, t_{m+1-M_1}) + (L_{\varepsilon} - L_{\varepsilon}^{N,\Lambda t})z_b(s_n, t_{m+1}),\\
&= (L_{\varepsilon} - L_{\varepsilon}^{N,\Lambda t})z_b(s_n, t_{m+1})\\
&= \left[\varepsilon\left(\frac{\partial^2}{\partial s^2} - \delta_s^2\right) + a(s_n)\left(\frac{\partial}{\partial s} - \delta_s^+\right) - \left(\frac{\partial}{\partial t} - \delta_t^-\right)\right]z_b(s_n, t_{m+1}).
\end{aligned} \tag{23}$$

Hence,

$$\begin{aligned}
\left|L_{\varepsilon}^{N,\Lambda t}(Z_b - z_b)(s_n, t_{m+1})\right| \leq \frac{\varepsilon}{3}(s_{n+1} - s_{n-1})\left\|\frac{\partial^3 z_b}{\partial s^3}\right\|_{\overline{\mathcal{D}}_2^{N,\Lambda t}} + \frac{\alpha}{2}(s_{n+1} - s_n)\left\|\frac{\partial^2 z_b}{\partial s^2}\right\|_{\overline{\mathcal{D}}_2^{N,\Lambda t}}\\
+\frac{\Lambda t}{2}\left\|\frac{\partial^2 z_b}{\partial t^2}\right\|_{\overline{\mathcal{D}}_2^{N,\Lambda t}} \leq C(N^{-1}\ln N + \Lambda t).
\end{aligned}$$

Case:2 If $\varrho = \varrho_0\varepsilon\ln N$, for the two boundaries it is easy to derive

$$\left|B_{l,\varepsilon}^{N,\Lambda t}(Z_b - z_b)(s_0, t_{m+1})\right| \leq CN^{-1}\ln N$$

and

$$\left|B_{r,\varepsilon}^{N,\Lambda t}(Z_b - z_b)(s_N, t_{m+1})\right| \leq CN^{-1}\ln N.$$

Now, the proof for the smooth and the coarse mesh will be done separately. If $(s_n, t_{m+1}) \in (\varrho, 1] \times [\tau, 2\tau]$, using triangle inequality,

$$\left|(Z_b - z_b)(s_n, t_{m+1})\right| \leq \left|Z_b(s_n, t_{m+1})\right| + \left|z_b(s_n, t_{m+1})\right|.$$

From the bounds of the coarse components in Theorem 5, the following can be achieved for $n = \frac{N}{2}, \frac{N}{2} + 1, \ldots, N$,

$$\left|z_b(s_n, t_{m+1})\right| \leq Ce^{\frac{-\alpha s_n}{\varepsilon}} \leq Ce^{\frac{-\alpha\varrho}{\varepsilon}} \leq CN^{-1}.$$

In a similar fashion, $\left|Z_b(s_n, t_{m+1})\right| \leq CN^{-1}$ can also be obtained (using Lemma 9). Therefore for all $(s_n, t_{m+1}) \in (\varrho, 1] \times [\tau, 2\tau]$, we have

$$\left|(Z_b - z_b)(s_n, t_{m+1})\right| \leq C(N^{-1} + \Lambda t).$$

If $(s_n, t_{m+1}) \in (0, \varrho) \times [\tau, 2\tau]$,

$$\left| L_\varepsilon^{N,\Lambda t}(z_b - Z_b)(s_n, t_{m+1}) \right| \leq C\left[\Lambda t + \int_{s_{n-1}}^{s_{n+1}} \varepsilon |z_b^{'''}(\lambda, t_{m+1})| d\lambda + \int_{s_{n-1}}^{s_{n+1}} |z_b^{''}(\lambda, t_{m+1})| d\lambda \right].$$

Using the bounds on the singular component, we have

$$\begin{aligned} \left| L_\varepsilon^{N,\Lambda t}(z_b - Z_b)(s_n, t_{m+1}) \right| &\leq C\left[\Lambda t + \varepsilon^{-1} e^{-\alpha s_n/\varepsilon} (e^{\alpha h/\varepsilon} - e^{-\alpha h/\varepsilon}) \right] \\ &\leq C\left[\Lambda t + \varepsilon^{-1} e^{-\alpha s_n/\varepsilon} \sinh(\alpha h/\varepsilon) \right]. \end{aligned}$$

Since, for any $0 \leq \theta \leq 2$, $\sinh\theta \leq C\theta$. So, $\sinh(\alpha h/\varepsilon) \leq C\alpha h/\varepsilon$. Now, using Lemma 9, the required bounds can be found. The similar argument can follow to prove the bounds for B-S-mesh along with the idea followed by [1]. □

Theorem 12 *Let $z(s,t)$ and $Z(s_n, t_{m+1})$ be respectively the solutions of continuous problem (1) and the fully discrete problem (9), satisfying the compatibility conditions at the corners. Then, the following error bounds are obtained with S-mesh and B-S-mesh, respectively for $(s_n, t_{m+1}) \in \overline{\mathcal{D}}^{N,\Lambda t}$*

$$\max_{(s_n,t_{m+1})} \left| Z(s_n, t_{m+1}) - z(s_n, t_{m+1}) \right| \leq C(N^{-1} \ln N + \Lambda t),$$

$$\max_{(s_n,t_{m+1})} \left| Z(s_n, t_{m+1}) - z(s_n, t_{m+1}) \right| \leq C(N^{-1} + \Lambda t).$$

Proof The result follows from Theorem 10 and Theorem 11. □

5 Semilinear singularly perturbed parabolic problem

In this section, the time-lagged semilinear singularly perturbed parabolic problem of the following type is considered:

$$\begin{cases} L_\varepsilon z(s,t) = (\varepsilon z_{ss} + a z_s - bz - z_t)(s,t) = -z(s, t-\tau) + f(s,t,z), & (s,t) \in \mathcal{D} \\ z\big|_{\Upsilon_d} = \Phi_d(s,t), \quad (s,t) \in \Upsilon_d = [0,1] \times [-\tau, 0], \\ B_{l,\varepsilon} z(0,t) = (z - \varepsilon z_s)(0,t) = \Phi_l(t), \quad (s,t) \in \Upsilon_l = 0 \times (0,T] \\ B_{r,\varepsilon} z(1,t) = (z + \varepsilon z_s)(1,t) = \Phi_r(t), \quad (s,t) \in \Upsilon_r = 1 \times (0,T]. \end{cases} \tag{24}$$

For deducing some major estimates for the solution of (24), following the concept of the upper and lower solutions (see Definition 3.1 in [19]), we can write

$$\big(-\check{z}_t + \varepsilon \check{z}_{ss} + a(s)\check{z}_s - b(s)\check{z} \big)(s,t) \geq \big(-\hat{z}_t + \varepsilon \hat{z}_{ss} + a(s)\hat{z}_s - b(s)\hat{z} \big)(s,t).$$

Here, $\check{z}$ and $\hat{z}$ are denoted as the upper and lower solutions of (24). Now, for any $z \in \langle \check{z}, \hat{z} \rangle$, by the use of weaker assumption $f_z \geq 0$ in $\mathcal{D}$, problem (24) have a unique solution [15,19].

5.1 Quasilinearization

The quasilinearization process is used to solve (24). The function $z(s,t)$ is approximated with a suitable initial guess *i.e.,* $z^{(0)}(s,t)$ in $f(s,t,z)$. From the conditions given in (24), we can introduce $z^{(0)}(s,t) = z^{(0)}$. Now for all $k > 0$, expansion of $f(s,t,z^{(k+1)})$ around $z^{(k)}$ gives,

$$f(s,t,z^{(k+1)}(s,t)) \approx f(s,t,z^{(k)}) + \big[z^{(k+1)} - z^{(k)} \big] \frac{\partial f}{\partial z(s,t)}\bigg|_{(s,t,z^{(k)})} + \ldots \tag{25}$$

Applying (25) in (24), we get

$$\begin{aligned} L_\varepsilon z &\cong \left[-z_t^{(k+1)} + \varepsilon z_{ss}^{(k+1)} + a(s) z_s^{(k+1)} - b(s) z^{(k+1)} - \tfrac{\partial f}{\partial z}\big|_{(s,t,z^{(k)})} z^{(k+1)} \right](s,t) \\ &= -z^{(k+1)}(s, t-\tau) + f(s,t,z^{(k)}) - z^{(k)} \tfrac{\partial f}{\partial z}\big|_{(s,t,z^{(k)})}. \end{aligned}$$

Table 1 Numerical results for Example 1

ε	Mesh	N/ Λt				
		$32/\frac{1}{32}$	$64/\frac{1}{64}$	$128/\frac{1}{128}$	$256/\frac{1}{256}$	$512/\frac{1}{512}$
10^{-3}	S-mesh	8.3558e-1	5.0029e-1	2.9266e-1	1.6785e-1	9.4731e-2
		0.7400	0.7735	0.8020	0.8252	
	B-S-mesh	3.6065e-1	1.7527e-1	8.5608e-2	4.2181e-2	2.0927e-2
		1.0410	1.0337	1.0212	1.0112	
10^{-5}	S-mesh	8.3465e-1	4.9949e-1	2.9215e-1	1.6756e-1	9.4573e-2
		0.7407	0.7737	0.8020	0.8251	
	B-S-mesh	3.6167e-1	1.7545e-1	8.5509e-2	4.2026e-2	2.0794e-2
		1.0436	1.0369	1.0248	1.0151	
10^{-7}	S-mesh	8.3464e-1	4.9948e-1	2.9215e-1	1.6755e-1	9.4572e-2
		0.7407	0.7737	0.8020	0.8251	
	B-S-mesh	3.6168e-1	1.7545e-1	8.5509e-2	4.2025e-2	2.0793e-2
		1.0436	1.0369	1.0248	1.0152	

Table 2 Numerical results on S-mesh for Example 2

N	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-5}$	
	$\overline{\mathcal{D}}^L$	$\overline{\mathcal{D}}^R$	$\overline{\mathcal{D}}^L$	$\overline{\mathcal{D}}^R$
16	3.9169e-1	9.4077e-3	3.9184e-1	9.2666e-3
	0.6038	0.8728	0.6041	0.8649
32	2.5773e-1	5.1372e-3	2.5778e-1	5.0881e-3
	0.6735	0.9210	0.6736	0.9141
64	1.6159e-1	2.7130e-3	1.6160e-1	2.7001e-3
	0.7332	0.9477	0.7333	0.9427
128	9.7200e-2	1.4065e-3	9.7206e-2	1.4047e-3
	0.7795	0.9646	0.7795	0.9608
256	5.6625e-2	7.2071e-4	5.6628e-2	7.2170e-4

Table 3 Numerical results on B-S-mesh for Example 2

N	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-5}$	
	$\overline{\mathcal{D}}^L$	$\overline{\mathcal{D}}^R$	$\overline{\mathcal{D}}^L$	$\overline{\mathcal{D}}^R$
16	2.4629e-1	9.5311e-3	2.4675e-1	9.2678e-3
	0.9483	0.8633	0.9472	0.8648
32	1.2763e-1	5.2389e-3	1.2797e-1	5.0891e-3
	0.9831	0.9157	0.9829	0.9140
64	6.4568e-2	2.7770e-3	6.4748e-2	2.7007e-3
	0.9971	0.9465	0.9971	0.9426
128	3.2348e-2	1.4409e-3	3.2439e-2	1.4051e-3
	1.0012	0.9670	1.0015	0.9608
256	1.6161e-2	7.3711e-4	1.6203e-2	7.2191e-4

On further arrangement we have,

$$\begin{aligned} L_\varepsilon z &\cong -z_t^{(k+1)}(s,t) + \varepsilon z_{ss}^{(k+1)}(s,t) + a(s) z_s^{(k+1)}(s,t) - B(s,t) z^{(k+1)}(s,t) \\ &= -z^{(k+1)}(s,t-\tau) + F(s,t). \end{aligned} \tag{26}$$

$B(s,t) = b(s) + \frac{\partial f}{\partial z}\Big|_{(s,t,z^{(k)})}$ and $F(s,t) = f(s,t,z^{(k)}) - z^{(k)} \frac{\partial f}{\partial z}\Big|_{(s,t,z^{(k)})}$. Afterwards, each iteration is solved numerically with a stopping criterion

$$\max_{(s_n,\ t_m)\ \in\ \mathcal{D}} \left| Z^{k+1}(s_n,\ t_m) - Z^k(s_n,\ t_m) \right| \leq TOL.$$

For computation, the TOL is chosen to be 10^{-4}. The numerical approximation and the analysis for convergence of (26) can be done in a similar approach as discussed for (1).

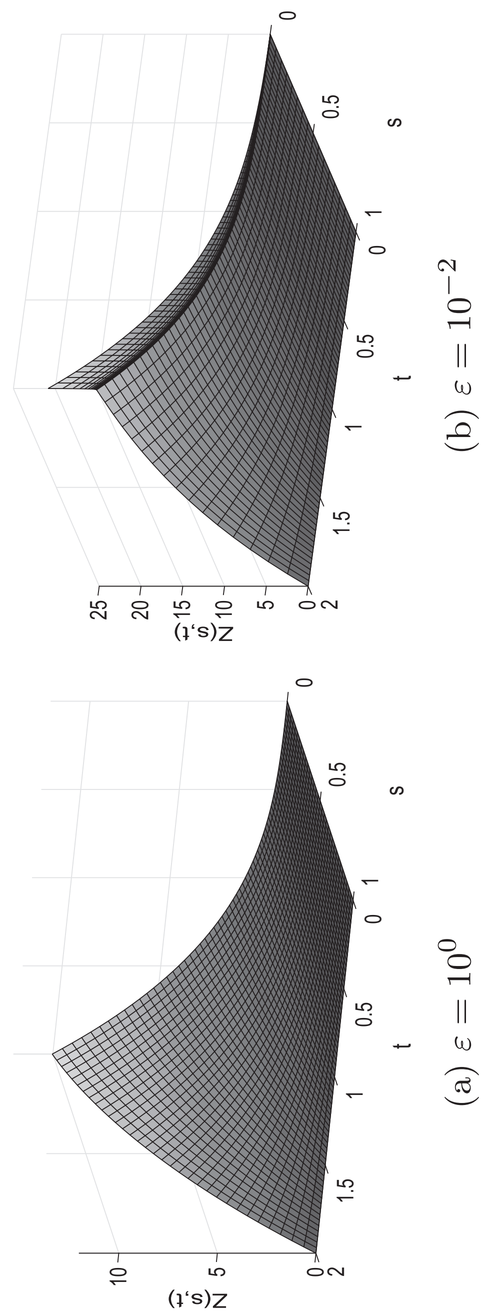


Fig. 1 Surface plots for Example 1

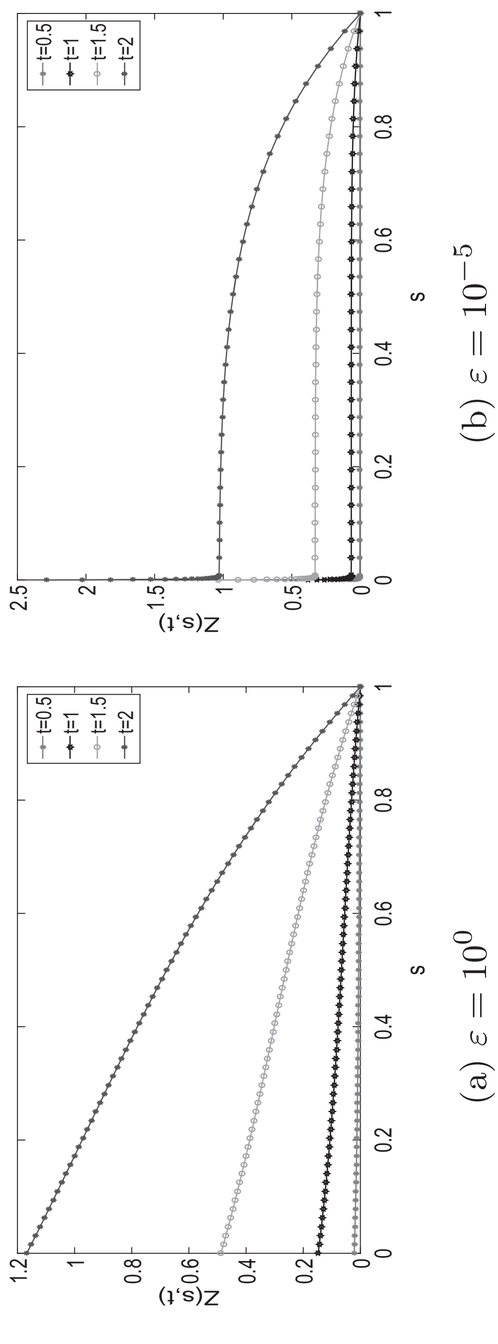


Fig. 2 Solution plots at different time levels for Example 1

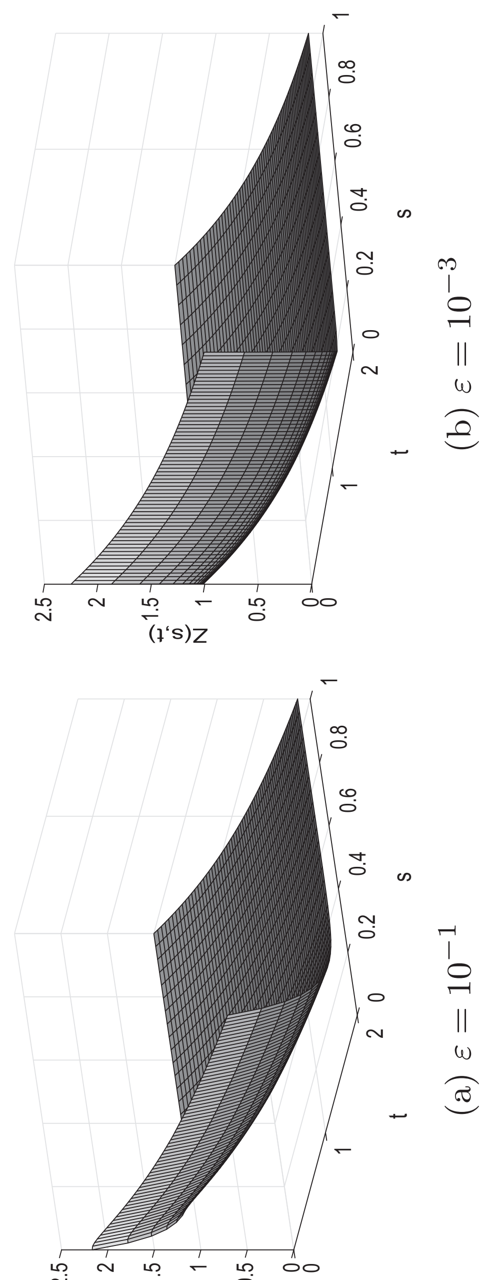
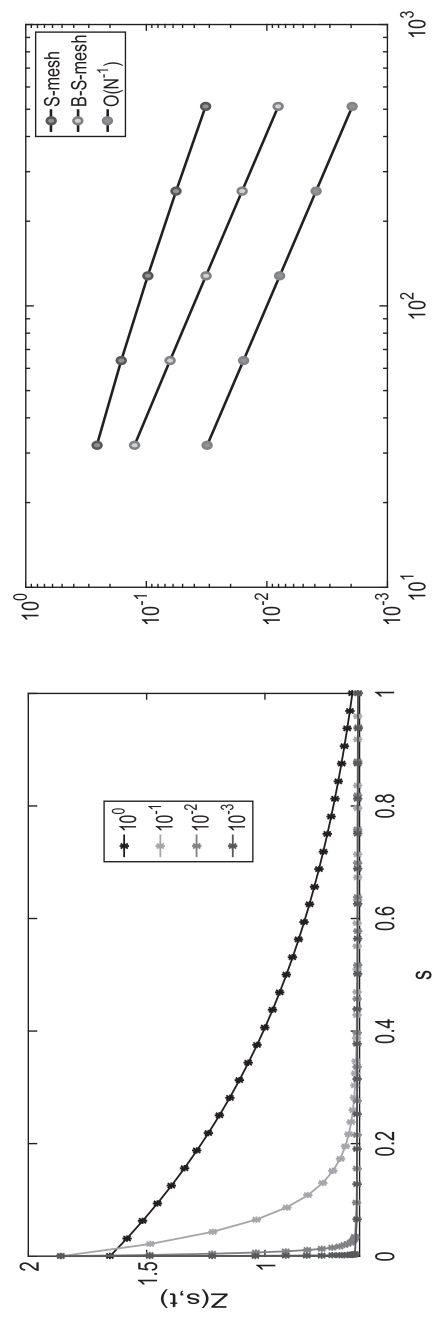


Fig. 3 Surface plots for Example 2



(a) layer plot for different ε

(b) log-log plot for $\varepsilon = 10^{-2}$

Fig. 4 Solution plots for Example 2

Table 4 Numerical results for Example 3

ε	Mesh	N/ Λt				
		$32/\frac{1}{32}$	$64/\frac{1}{64}$	$128/\frac{1}{128}$	$256/\frac{1}{256}$	$512/\frac{1}{512}$
10^{-3}	S-mesh	7.5393e-1	4.4912e-1	2.6089e-1	1.4881e-1	8.3652e-2
		0.7473	0.7836	0.8099	0.8310	
	B-S-mesh	2.6548e-1	1.3938e-1	7.1010e-2	3.5725e-2	1.7904e-2
		0.9295	0.9729	0.9910	0.9967	
10^{-5}	S-mesh	8.5859e-1	5.1819e-1	3.0266e-1	1.7303e-1	9.7362e-2
		0.7284	0.7757	0.8067	0.8295	
	B-S-mesh	3.5592e-1	1.8740e-1	9.5030e-2	4.7531e-2	2.3709e-2
		0.9254	0.9796	0.9995	1.0034	
10^{-7}	S-mesh	8.6131e-1	5.1978e-1	3.0355e-1	1.7352e-1	9.7632e-2
		0.7286	0.7759	0.8068	0.8297	
	B-S-mesh	3.5807e-1	1.8842e-1	9.5518e-2	4.7765e-2	2.3823e-2
		0.9262	0.9801	0.9998	1.0036	

6 Numerical outputs

Three exemplifying problems are presented in this section in order to validate the theoretical findings. The errors ($\mathbf{E}_\varepsilon^{N,\Lambda t}$) and convergence rates ($\mathbf{R}_\varepsilon^{N,\Lambda t}$) are computed by,

$$\mathbf{E}_\varepsilon^{N,\Lambda t} = \max_{(s_n,\ t_m)\ \in\ \mathcal{D}} \left| Z^{N,M}(s_n,\ t_m) - \mathcal{Z}^{N,M}(s_n,\ t_m) \right|, \quad \mathbf{R}_\varepsilon^{N,\Lambda t} = \log_2\left(\frac{\mathbf{E}_\varepsilon^{N,\Lambda t}}{\mathbf{E}_\varepsilon^{2N,\Lambda t/2}}\right).$$

As the exact solution is not known, so the double mesh principle is applied to calculate $\mathbf{E}_\varepsilon^{N,\Lambda t}$ and $\mathbf{R}_\varepsilon^{N,\Lambda t}$ for Example 1 and Example 3. So, $\mathcal{Z}^{N,M}(s_n,\ t_m) = Z^{2N,2M}(s_n,\ t_m)$ for Example 1, Example 3 and $\mathcal{Z}^{N,M}(s_n,\ t_m) = z(s_n,\ t_m)$ for Example 2.

Example 1 Consider the following SPPDE having time lag of the form (1), with $(s, t) \in (0, 1) \times (0, 2]$.

$$\begin{cases} (\varepsilon z_{ss} + z_s - z_t)(s, t) = -z(s, t-1) - 4t^3 \\ z\big|_{\Upsilon_d}(s, t) = 0, \quad \Upsilon_d = (0, 1) \times [-1, 0] \\ -\varepsilon z_s(0, t) = t^2, \quad z(1, t) = 0. \end{cases}$$

Example 2 Next, we consider the following test problem with $(s, t) \in (0, 1) \times (0, 2]$.

$$\begin{cases} (\varepsilon z_{ss} + 2z_s - z_t)(s, t) = -z(s, t-1) + f(s, t) \\ z\big|_{\Upsilon_d}(s, t) = \Phi_d(s, t), \quad \Upsilon_d = (0, 1) \times [-1, 0] \\ (z - \varepsilon z_s)(0, t) = \Phi_l(t), \quad (z + \varepsilon z_s)(1, t) = \Phi_r(t). \end{cases}$$

The source term and the initial boundary conditions are made to satisfy

$$z(s, t) = \frac{\exp(\frac{-2s}{\varepsilon}) - \exp(\frac{-2}{\varepsilon})}{1 - \exp(\frac{-2}{\varepsilon})} + \exp(-t).$$

Example 3 Consider the following semi-linear SPPDE having time lag of the form (1), with $(s, t) \in (0, 1) \times (0, 2]$.

$$\begin{cases} (\varepsilon z_{ss} + z_s - z_t)(s, t) - \exp(z) = -z(s, t-1) - 4t^3 \\ z\big|_{\Upsilon_d}(s, t) = 0, \quad \Upsilon_d = (0, 1) \times [-1, 0] \\ -\varepsilon z_s(0, t) = t^2, \quad z(1, t) = 0. \end{cases}$$

The numerical solutions on both S-mesh and B-S-mesh for Example 1 and Example 3 are tabulated in Table 1 and Table 4, respectively. Both the tables conclude the efficacy of B-S-mesh over S-mesh. Table 2 and Table 3 provide a comparison of ($\mathbf{E}_\varepsilon^{N,\Lambda t}$) and ($\mathbf{R}_\varepsilon^{N,\Lambda t}$) between inside and outside layer region for Example 2. It is evident from Table 2 and Table 3 that in case of the outer layer region both S-mesh and B-S-mesh provide almost similar kind of results whereas inside the layer region, being graded in nature, B-S-mesh gives better result. The surface plots for Example 1 with $\varepsilon = 10^0$ and $\varepsilon = 10^{-2}$ are presented in Figure 1. The layer behavior in Example 1 is shown clearly in Figure 2 by providing the solution profiles at different time levels with $\varepsilon = 10^0$ and $\varepsilon = 10^{-5}$. The layer behavior in Example 2 can be identified from the surface plots in Figure 3 and the solution plots for different ε in Figure 4a. Figure 4b confirms the global first-order accuracy through the log-log plot in both the S-mesh and B-S-mesh for Example 2. The claim of our scheme to be parameter uniform is confirmed through the numerical outputs as the error for each example can be seen to decrease after certain values of ε and N.

7 Conclusion

The singularly perturbed parabolic convection-diffusion problem with Robin-type boundary conditions having large time lag is solved using an efficient numerical approach. The temporal direction is handled with the implicit Euler scheme along with the upwind scheme on both the Shishkin mesh and the Bakhvalov-Shishkin mesh in the spatial direction. Along with the linear case, the scheme is also tested over the semi-linear convection-diffusion problem and proved to be efficient for this case, as well. Numerical experiments and analysis prove the global first-order accuracy and robustness of the proposed scheme. From the numerical experiments it is evident that B-S-mesh provides better accuracy than the S-mesh.

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Data Availibility Statement The data sets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflicts of interest The authors declare that there are no conflicts of interest.

References

1. Arslan, D.: A uniformly convergent numerical study on Bakhvalov-Shishkin mesh for singularly perturbed problem. Commun. Math. Appl. **11(1)**, 161-171 (2020)
2. Britton, N.: Spatial structures and periodic travelling waves in an integro-differential reaction-diffusion population model. SIAM J. Appl. Math. **50 (6)**, 1663-1688 (1990)
3. Cai, X., Liu, F.: Uniform convergence difference schemes for singularly perturbed mixed boundary problems. J. Comput. Appl. Math. **166**, 31-54 (2004)
4. Das, A., Natesan, S.: Uniformly convergent hybrid numerical scheme for singularly perturbed delay parabolic convection-diffusion problems on Shishkin mesh. Appl. Math. Comput. **271**, 168-186 (2015)
5. Gelu, F.W., Duressa, G.F.: A novel numerical approach for singularly perturbed parabolic convection-diffusion problems on layer-adapted meshes. Res. Math. **9(1)**, (2022). https://doi.org/10.1080/27658449.2021.2020400
6. Govindarao, L., Sahu, S.R., Mohapatra, J.: Uniformly convergent numerical method for singularly perturbed time delay parabolic problem with two small parameters. Iran. J. Sci. Technol. Trans. A: Sci. **43(5)**, 2373-2383 (2019)
7. Govindarao, L., Mohapatra, J.: Numerical analysis and simulation of delay parabolic partial differential equation involving a small parameter. Eng. Comput. **37(1)**, 289-312 (2019)
8. Govindarao, L., Mohapatra, J., Das, A.: A fourth-order numerical scheme for singularly perturbed delay parabolic problem arising in population dynamics. J. Appl. Math. Comput. **63(1)**, 171-195 (2020)
9. Gowrisankar, S., Natesan, S.: A robust numerical scheme for singularly perturbed delay parabolic initial-boundary-value problems on equidistributed grids. Electron. Trans. Numer. Anal. **41**, 376-395 (2014)
10. Gowrisankar, S., Natesan, S.: ε-Uniformly convergent numerical scheme for singularly perturbed delay parabolic partial differential equations. Int. J. Comput. Math. **94(5)**, 902-921 (2017). https://doi.org/10.1080/00207160.2016.1154948
11. Gowrisankar, S., Natesan, S.: An efficient robust numerical method for singularly perturbed Burgers' equation. Appl. Math. Comput. **346**, 385-394 (2019). https://doi.org/10.1016/j.amc.2018.10.049
12. Kaushik, A., Sharma, M.: A robust numerical approach for singularly perturbed time delayed parabolic partial differential equations. Comput. Math. Model. **23(1)**, 96-106 (2012)
13. Kyrycho, Y. N. and Hogan, S. J.: On the use of delay equations in engineering applications. J. Vib. Control. **16(7-8)**, 943-960 (2010)
14. Kumar, S., Kumar, M.: A robust numerical method for a two-parameter singularly perturbed time delay parabolic problem. Comput. Appl. Math. **39(3)**, 1-25 (2020)
15. Ladyženskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N.: Linear and Quasi-linear Equations of Parabolic Type. American Mathematical Soc., **23**, (1968)
16. Mbroh, N.A., Noutchie, S.C.O., Massoukou, R.Y.M.: A uniformly convergent finite difference scheme for Robin-type singularly perturbed parabolic convection diffusion problem. Math. Comput. Simulation. **174**, 218-232 (2020)

17. Mohapatra, J., Natesan, S.: Parameter-uniform numerical methods for singularly perturbed mixed boundary value problems using grid equidistribution. J. Appl. Math. Comput. **37(1)**, 247-265 (2011)
18. O Riordan, E., Pickett, M., Shishkin, G.: Parameter–uniform finite difference schemes for singularly perturbed parabolic diffusion–convection–reaction problems. Math. Comp. **75(255)**, 1135-1154 (2006)
19. Pao, C.V.: Nonlinear Parabolic and Elliptic Equations. Plenum Press, New York (1992). https://doi.org/10.1007/978-1-4615-3034-3
20. Podila, P.C., Kumar, K.: A new stable finite difference scheme and its convergence for time-delayed singularly perturbed parabolic PDEs. Comput. Appl. Math. **39**, 1-16 (2020)
21. Priyadarshana, S., Mohapatra, J., Govindrao, L.: An efficient uniformly convergent numerical scheme for singularly perturbed semilinear parabolic problems with large delay in time. J. Appl. Math. Comput. **12(1)**, 55-72 (2021)
22. Priyadarshana, S., Mohapatra, J., Pattanaik, S.R.: Parameter uniform optimal order numerical approximations for time-delayed parabolic convection diffusion problems involving two small parameters. Comput. Appl. Math. **41 (233)** (2022). https://doi.org/10.1007/s40314-022-01928-w
23. Priyadarshana, S., Mohapatra, J.: Weighted variable based numerical scheme for time-lagged semilinear parabolic problems including small parameter, J. Appl. Math. Comput., (2023), https://doi.org/10.1007/s12190-023-01841-3
24. Rihan, F. A.: Delay Differential Equations and Applications to Biology. Springer Singapore. (2021). https://doi.org/10.1007/978-981-16-0626-7
25. Sahu, S.R., Mohapatra, J.: Numerical investigation of time delay parabolic differential equation involving two small parameters. Eng. Comput. **38(6)**, 2882-2899 (2021)
26. Sahu, S.R., Mohapatra, J.: Numerical study of time delay singularly perturbed parabolic differential equations involving both small positive and negative space shift. J. Appl. Anal. (2021). https://doi.org/10.1515/jaa-2021-2064
27. Selvi, P.A., Ramanujam, N.: A parameter uniform difference scheme for singularly perturbed parabolic delay differential equation with Robin-type boundary condition. Appl. Math. Comput. **296**, 101-115 (2017)
28. Shakti, D., Mohapatra, J.: A second order numerical method for a class of parameterized singular perturbation problems on adaptive grid. Nonlinear Eng. **6(3)**, 221-228 (2017)
29. Wang, P. K. C.: Asymptotic stability of a time-delayed diffusion system. J. Appl. Mech. **30(4)**, 500–504 (1963)