Indian J Pure Appl Math (2024) 55:1429–1439
https://doi.org/10.1007/s13226-023-00448-5

ORIGINAL RESEARCH

A note on the distinct eigenvalues of quotient matrices

Bilal Ahmad Rather

Received: 8 November 2022 / Accepted: 9 June 2023 / Published online: 18 June 2023

Abstract For a square matrix M of order n partitioned into m^2 blocks B_{ij}, the quotient matrix Q is a square matrix of order k ($m \leq k \leq n-1$) with (i, j)-th entries as the average of the row (columns) sums of B_{ij} of M. The partition is said to be equitable if each block B_{ij} of M has some constant row (column) sum, in such case Q is known as the equitable quotient matrix of M and each eigenvalue of Q is the eigenvalue of M. Atik [(2020), Linear Multilinear Algebra] asked the problem: *What is the necessary and sufficient condition on Q to contain all the distinct eigenvalues of M*? In this paper, we consider this problem: we give some classes of matrices such that Q contains all the distinct eigenvalues of M, classes of matrices such that Q does not contains all the distinct eigenvalues of M with respect to some partition and contains all the distinct eigenvalues of M with respect to another partitions. The significant observation is that the problem of the distinct eigenvalue of Q is about the smallest possible equitable partition. We restate the problem in terms of the smallest possible equitable partition.

Keywords Block matrices, Distinct Eigenvalues · Quotient Matrices · Equitable Partitions

Mathematics Subject Classification 15A18 · 15A42 · 05C50

1 Introduction

We denote set of all square matrices over a field $\mathbb{F}$ by $\mathbb{M}_n(\mathbb{F})$. The set of all eigenvalues of $M \in \mathbb{M}_n(\mathbb{F})$ is known as the *spectrum* of M. The largest eigenvalue of M is the *spectral radius* of M. A matrix is said to be non-negative if all its entries are greater than or equal to zero and matrix is non-positive if it contains arbitrary entries (both positive and negative). If an eigenvalue λ of M occurs with algebraic multiplicity k, then we write it as $\lambda^{[k]}$. For $k = 1$, we say λ is the simple eigenvalue of M.

Consider a square matrix M written in block form as given below:

$$M = \begin{pmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,m-1} & B_{1,m} \\ B_{2,1} & B_{2,2} & \cdots & B_{2,m-1} & B_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{m-1,1} & B_{m-1,2} & \cdots & B_{m-1,m-1} & B_{m-1,m} \\ B_{m,1} & B_{m,2} & \cdots & B_{m,m-1} & B_{m,m} \end{pmatrix},$$

Communicated by S Sivaramakrishnan.

B. A. Rather
Mathematical Sciences Department, College of Science, United Arab Emirates University, Al Ain 15551, Abu Dhabi, United Arab Emirates
E-mail: bilalahmadrr@gmail.com

such that the rows and the columns of M are partitioned according to a partition

$$P = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{m-1}, \mathcal{P}_m\} \text{ of the index set } \mathcal{I} = \{1, 2, \ldots, n\}.$$

If q_{ij} denotes the average row (column) sum of the block B_{ij} of M, then the matrix $Q = (q_{ij})_{m\times m}$ is called the *quotient matrix* (see [3,4,6]). The partition P is said to be *regular* (equitable), if every block $B_{i,j}$ of M has some constant row (column) sum and in this case Q is known as the *equitable quotient matrix*. More about matrix theory terminology can be seen in [5,7].

For, two sequences of real numbers $\{s_1, s_2, \ldots, s_n\}$ and $\{s'_1, s'_2, \ldots, s'_m\}$ with $n > m$. The latter is said to interlace the former, whenever

$$s_i \geq s'_i \geq s_{n-m+i}, \quad \text{for } i = 1, 2, \ldots, m.$$

The interlacing is said to be tight if there exists a positive integer $k \in [0, m]$, such that

$$s_i = s'_i \quad \text{for } i = 1, 2, \ldots, k \text{ and } s_{n-m+i} = s'_i \quad \text{for } k+1 \leq i \leq m.$$

If $m = n - 1$, then the interlacing becomes

$$s_1 \geq s'_1 \geq s_2 \geq s'_2 \geq \cdots \geq s_{n-1} \geq s'_{n-1} \geq s_n.$$

Next result give the relation between the eigenvalues of M and the eigenvalues of Q.

Lemma 1.1 *([3,4]) Let M be a real symmetric matrix of order n and Q be its quotient matrix. Then the following hold.*

(i) If the partition P of $\mathcal{I}$ of matrix M is not equitable, then eigenvalues of Q interlace the eigenvalues of M.
(ii) If the partition P of $\mathcal{I}$ of matrix M is equitable, then the interlacing is tight and each eigenvalue of Q is the eigenvalue of M.

Quotient matrices are very interesting and have many applications in linear algebra especially spectral theory of graphs. By the interlacing property of Q, we can find bounds for the eigenvalues of the original matrix M. Quotient matrix plays rich role in the spectral theory of graphs defined on algebraic structures like zero-divisor graphs of commutative rings and Von Neumann regular rings, commuting graphs of finite groups, power graphs of finite groups and many other algebraic graphs, see [2,8–12].

Since, the eigenvalues of Q interlace the eigenvalues of M. So question is: What about the spectral radii of M and Q. In case of non-negative matrices, we have the following result.

Lemma 1.2 *([2,12]) The spectral radius of a non-negative matrix M and the spectral radius of an equitable quotient matrix Q coincide.*

The above result is not true for an arbitrary matrix, one such infinite class of matrix is the matrix given by (2.1).

By (ii) of Lemma 1.1, it follows that distinct eigenvalues of Q are contained in the spectrum of M. What about other way round, that is, if $\lambda_1, \lambda_2, \ldots, \lambda_t$, $(t \geq 2)$ are the distinct eigenvalues of M, then is it true that $\lambda_1, \lambda_2, \ldots, \lambda_t$ are always the eigenvalues of Q? In this regard, Atik [1] asked the following problem about the distinct eigenvalues of the equitable quotient matrix of M.

Problem 1 (Atik, *[1]) What is the necessary and sufficient condition on the equitable quotient matrix Q to contain all the distinct eigenvalues of M?*

In the next section, we consider Problem 1, we provide several important and interesting classes of matrices such that Q may contain all the distinct eigenvalues of M with some equitable partition and may not contain all the distinct eigenvalues of M with other equitable partition. We also give classes of matrices such that Q contains all the distinct eigenvalues of M. The purpose of presenting these particular matrices is that a general conclusion about Problem 1 can be obtained. The most significant observation is that there seems no necessary and sufficient conditions on equitable quotient matrix to contain all the distinct eigenvalues of the original matrix. The problem is about the smallest possible equitable partition of the quotient matrix Q to contain all the distinct eigenvalues of M. We end up the article with the restatement of Problem 1 in terms of the smallest possible equitable partitions.

2 Equitable quotient matrices and the distinct eigenvalues of original matrix

In this section, we consider several classes of matrices and study their eigenvalue relations with the parent matrix for different equitable partitions.

We begin with a result, which gives the spectrum of the classes of non-negative matrices.

Theorem 2.1 *Let M be a real symmetric matrix of order $n = 2l_1 + 1,\ l_1 \geq 2$ and is given in Equation (2.1). Then the spectrum of M is*

$$\left\{0, \left(\frac{1}{2}\left(\alpha + \Phi \pm \sqrt{(\alpha - \Phi)^2 + 4\gamma^2}\right)\right)^{[l_1-1]}, \frac{1}{2}\left(-2\gamma + 2\delta l_1 + l_1\xi + l_1\sigma + \xi + \sigma \pm \sqrt{D}\right)\right\},$$

where $D = 4\gamma^2 - 8\gamma\delta l_1 + 4\gamma l_1\xi + 4\gamma l_1\sigma + 4\delta^2 l_1^2 - 4\delta l_1^2\xi - 4\delta l_1^2\sigma + l_1^2\xi^2 + 2l_1\xi^2 + 2l_1^2\xi\sigma - 4l_1\xi\sigma + l_1^2\sigma^2 + 2l_1\sigma^2 + \xi^2 - 2\xi\sigma + \sigma^2$, $\alpha = l_1(\beta + \delta) + \xi - \gamma$, $\Phi = l_1(\Psi + \delta) + \sigma - \gamma$, $\mu = l_1(\xi + \sigma)$ and $\beta, \gamma, \delta, \xi$ and σ are real numbers.

Proof Let l_1 be a positive integer such that $n = 2l_1 + 1$ and $M_{n\times n}$ be the matrix given below

$$M = \begin{pmatrix} \alpha I_{l_1} - \beta J_{l_1} & \gamma I_{l_1} - \delta J_{l_1} & -\xi J_{l_1\times 1} \\ \gamma I_{l_1} - \delta J_{l_1} & \Phi I_{l_1} - \Psi J_{l_1} & -\sigma J_{l_1\times 1} \\ -\xi J_{1\times l_1} & -\sigma J_{1\times l_1} & \mu \end{pmatrix}, \tag{2.1}$$

where $\alpha, \beta, \gamma, \delta, \xi, \sigma, \Phi, \mu$ and Ψ are real numbers. We will find the eigenvalues of the matrix M with the help of eigenvector analysis.

For $i = 2, 3, \ldots, l_1$, consider a sequence of vectors over $\mathbb{R}$ as follows:

$$X_{i-1} = \big(-\beta_1, x_{i2}, x_{i3}, \ldots, x_{il_1-1}, x_{il_1}, -1, x^*_{i2}, x^*_{i3}, \ldots, x^*_{il_1-1}, x^*_{il_1}, 0\big) \in \mathbb{R}^n(\mathbb{R}),$$

where

$$\beta_1 = \frac{1}{a\gamma}\big(\alpha - \Phi - \sqrt{(\alpha - \Phi)^2 + 4\gamma^2}\big),\ x_{ij} = \begin{cases} \beta_1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases},\ \text{and}$$

$$x^*_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

For $i = 2, 3, \ldots, l_1$, it is easy to verify that X_{i-1}, are linearly independent vectors. Also, for the vector X_1 and the definition of eigenequation, we have

$$MX_1 = \begin{pmatrix} -\alpha\beta_1 - \gamma \\ \alpha\beta_1 + \gamma \\ 0 \\ \vdots \\ 0 \\ -\beta_1\gamma - \Phi \\ \beta_1\gamma + \Phi \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{\alpha}{2\gamma}\left(\alpha - \Phi - \sqrt{(\alpha - \Phi)^2 + 4\gamma^2}\right) - \gamma \\ \frac{\alpha}{2\gamma}\left(\alpha - \Phi - \sqrt{(\alpha - \Phi)^2 + 4\gamma^2}\right) + \gamma \\ 0 \\ \vdots \\ 0 \\ -\frac{1}{2}\left(\alpha + \Phi - \sqrt{(\alpha - \Phi)^2 + 4\gamma^2}\right) \\ \frac{1}{2}\left(\alpha + \Phi - \sqrt{(\alpha - \Phi)^2 + 4\gamma^2}\right) \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

$$
= \begin{pmatrix}
-\frac{\left(\alpha+\Phi-\sqrt{\alpha^2-2\alpha\Phi+4\gamma^2+\Phi^2}\right)\left(\alpha+\Phi-\sqrt{\alpha^2-2\alpha\Phi+4\gamma^2+\Phi^2}\right)}{4\gamma} \\
\frac{\left(\alpha+\Phi-\sqrt{\alpha^2-2\alpha\Phi+4\gamma^2+\Phi^2}\right)\left(\alpha+\Phi-\sqrt{\alpha^2-2\alpha\Phi+4\gamma^2+\Phi^2}\right)}{4\gamma} \\
0 \\ \vdots \\ 0 \\
-\frac{1}{2}\left(\alpha+\Phi-\sqrt{(\alpha-\Phi)^2+4\gamma^2}\right) \\
\frac{1}{2}\left(\alpha+\Phi-\sqrt{(\alpha-\Phi)^2+4\gamma^2}\right) \\
0 \\ \vdots \\ 0 \\ 0
\end{pmatrix}
= \lambda \begin{pmatrix} -\beta_1 \\ \beta_1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \lambda X_1,
$$

where $\lambda = \frac{1}{2}\left(\alpha+\Phi-\sqrt{(\alpha-\Phi)^2+4\gamma^2}\right)$. Thus, it follows that $\frac{1}{2}\left(\alpha+\Phi-\sqrt{(\alpha-\Phi)^2+4\gamma^2}\right)$ is the eigenvalue of M with the corresponding eigenvector X_1. Proceeding in a similar manner, we can show that $X_2, X_3, \dots, X_{l_1-1}$ are the eigenvectors corresponding to the eigenvalue $\frac{1}{2}(\alpha+\Phi-\sqrt{(\alpha-\Phi)^2+4\gamma^2})$. In this manner, we obtain l_1-1 eigenvalues of M.

Again, For $i = 2, 3, \dots, l_1$, let

$$Y_{i-1} = \big(-\beta_2, y_{i2}, y_{i3}, \dots, y_{il_1-1}, y_{il_1}, -1, y^*_{i2}, y^*_{i3}, \dots, y^*_{il_1-1}, y^*_{il_1}, 0\big) \in \mathbb{R}^n(\mathbb{R}),$$

where

$$\beta_2 = \frac{1}{2}\left(\alpha+\Phi+\sqrt{(\alpha-\Phi)^2+4\gamma^2}\right), \; y_{ij} = \begin{cases} \beta & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}, \text{ and}$$

$$y^*_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}.$$

Clearly, Y_{i-1}, with $i = 2, 3, \dots, l_1$ are linearly independent vectors. Further, as above we can verify that $Y_1, Y_2, \dots, Y_{l_1-1}$ are the eigenvectors corresponding to the eigenvalue $\frac{1}{2}(\alpha+\Phi+\sqrt{(\alpha-\Phi)^2+4\gamma^2})$. So, we have obtained $2l_1-2$ eigenvalues of M. The remaining three eigenvalues of the block matrix M are the eigenvalues of the following equitable quotient matrix

$$Q = \begin{pmatrix} \alpha - l_1\beta & \gamma - l_1\delta & -\xi \\ -\gamma - l_1\delta & \Phi - l_1\Psi & -\sigma \\ -l_1\xi & -l_1\sigma & \mu \end{pmatrix}. \tag{2.2}$$

Since the eigenvalues of Q are not easy to locate, without loss of generality, we assume that $\alpha = l_1(\beta+\delta)+\xi-\gamma$, $\Phi = l_1(\Psi+\delta)+\sigma-\gamma$ and $\mu = l_1(\xi+\sigma)$, so that the eigenvalues of Q are easy to calculate. With such a choice, each row sum of Q is zero, it follows that 0 is the eigenvalue of Q with eigenvector $(1,\ 1,\ 1)^T$. Now, it is easy to see that the other two eigenvalues of matrix (2.2) are

$$\left\{\frac{1}{2}\left(-2\gamma+2\delta l_1+l_1\xi+l_1\sigma+\xi+\sigma\pm\sqrt{D}\right)\right\},$$

where $D = 4\gamma^2-8\gamma\delta l_1+4\gamma l_1\xi+4\gamma l_1\sigma+4\delta^2 l_1^2-4\delta l_1^2\xi-4\delta l_1^2\sigma+l_1^2\xi^2+2l_1\xi^2+2l_1^2\xi\sigma-4l_1\xi\sigma+l_1^2\sigma^2+2l_1\sigma^2+\xi^2-2\xi\sigma+\sigma^2$. That gives the spectrum of M completely. □

From Theorem 2.1, we note that the matrix M has five distinct eigenvalues and its associated equitable quotient matrix contains only three distinct eigenvalues. So, the equitable quotient Q of the block matrix M does not contains all the distinct eigenvalues M. Next, we try to partition the matrix M of (2.1) in such a way that its corresponding equitable quotient matrix contains all the distinct eigenvalues of M. In this regard, we have the following result.

Proposition 2.2 *Let M be a matrix as given in (2.1) and consider its new block representation as in (2.3). Then the resulting equitable quotient matrix contains all the distinct eigenvalues of M.*

Proof First by brute force, we see that there does not exist equitable quotient matrix of order 4 and certainly that does not contains all the distinct eigenvalues of M, since there are five distinct eigenvalues of M. Let $\{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5\}$, be the partition of the index set $\{1, 2, \ldots, l_1, l_1+1, \ldots, 2l_1, 2l_1+1\}$, where $\mathcal{P}_1 = \{1, 2, \ldots, l_1 - 1\}$, $\mathcal{P}_2 = \{l_1\}$, $\mathcal{P}_3 = \{l_1+1, l_1+2, \ldots, 2l_1-1\}$, $\mathcal{P}_4 = \{2l_1\}$ and $\mathcal{P}_5 = \{2l_1+1\}$ and the block matrix with this partition is shown in (2.3)

$$\begin{pmatrix} \alpha I_{l_1-1} - \beta J_{l_1-1} & -\beta J_{(l_1-1)\times 1} & \gamma J_{l_1-1} - \delta J_{l_1-1} & -\delta J_{(l_1-1)\times 1} & -\xi J_{(l_1-1)\times 1} \\ -\beta J_{1\times(l_1-1)} & \alpha - \beta & -\delta J_{1\times(l_1-1)} & \gamma - \delta & -\xi \\ \gamma I_{l_1-1} - \delta J_{l_1-1} & -\delta J_{(l_1-1)\times 1} & \Phi J_{l_1-1} - \Psi J_{l_1-1} & -\Psi J_{(l_1-1)\times 1} & -\delta J_{(l_1-1)\times 1} \\ -\delta J_{1\times(l_1-1)} & \gamma - \delta & -\Psi J_{1\times(l_1-1)} & \Phi - \Psi & -\sigma \\ -\xi J_{1\times(l_1-1)} & -\xi & -\sigma J_{1\times(l_1-1)} & -\sigma & \mu \end{pmatrix}. \tag{2.3}$$

It is clear that (2.3) has constant row sum for each partition cell (block), so the equitable quotient matrix of (2.3) is

$$Q' = \begin{pmatrix} \alpha - \beta(l_1-1) & -\beta & \gamma - \delta(l_1-1) & -\delta & -\xi \\ \beta(-(l_1-1)) & \alpha - \beta & \delta(-(l_1-1)) & \gamma - \delta & -\xi \\ \gamma - \delta(l_1-1) & -\delta & \Phi - (l_1-1)\Psi & -\Psi & -\sigma \\ \delta(-(l_1-1)) & \gamma - \delta & -(l_1-1)\Psi & \Phi - \Psi & -\sigma \\ -(l_1-1)\xi & -\xi & -(l_1-1)\sigma & -\sigma & \mu \end{pmatrix}, \tag{2.4}$$

where the parameters $\alpha, \beta, \gamma, \delta, \sigma, \mu, \xi, \Phi$ and Ψ are as in Theorem 2.1. By using symbolic computation (Wolfram Mathematica) on a computer, the eigenvalues of Q' are

$$\left\{0, \frac{1}{2}\left(\alpha + \Phi \pm \sqrt{(\alpha - \Phi)^2 + 4\gamma^2}\right), \frac{1}{2}\left(-2\gamma + 2\delta l_1 + l_1\xi + l_1\sigma + \xi + \sigma \pm \sqrt{D}\right)\right\},$$

where $D = 4\gamma^2 - 8\gamma\delta l_1 + 4\gamma l_1\xi + 4\gamma l_1\sigma + 4\delta^2 l_1^2 - 4\delta l_1^2\xi - 4\delta l_1^2\sigma + l_1^2\xi^2 + 2l_1\xi^2 + 2l_1^2\xi\sigma - 4l_1\xi\sigma + l_1^2\sigma^2 + 2l_1\sigma^2 + \xi^2 - 2\xi\sigma + \sigma^2$. Thus, Matrix (2.4) contains all the distinct eigenvalues of Matrix (2.1). □

We have the following observation from Theorem 2.1 and Proposition 2.2.

Remark 2.3 From Theorem 2.1, we see that Matrix (2.2) is the smallest possible equitable quotient matrix and it does not contain all the distinct eigenvalues of the original Matrix (2.1). Besides, it is clear that there exists no equitable quotient matrix of order 4, which contains all the distinct eigenvalues (or even 4 distinct eigenvalues) of Matrix (2.1). By Proposition 2.2, the smallest quotient matrix corresponding to the equitable partition of minimum size is 5 and it contains all the distinct eigenvalues of (2.1) and that solves the problem of spectral radius of matrix with negative entries and Problem 1 for these infinite class of matrices. But the problem is not done yet, it was an experiment for a class of matrices to gain some insight about Problem 1.

Next, we have an another type of result.

Theorem 2.4 *Let M be the matrix of order $n \geq 6$ as given in (2.5). Then the spectrum of M consists of the eigenvalue $2n - 4$ with multiplicity $n - 4$ and the zeros of the following polynomial*

$$x^4 - (10n - 20)x^3\left(35n^2 - 148n + 160\right)x^2 + \left(-50n^3 + 318n^2 - 666n + 448\right)x + 24n^4 \\ - 192n^3 + 528n^2 - 520n + 64.$$

Proof Consider the matrix M, given below

$$M = \begin{pmatrix} 2nI_{n-3} + 2e_{n-3} & J_{(n-3)\times 1} & 2J_{(n-3)\times 1} & 3J_{(n-3)\times 1} \\ J_{1\times(n-3)} & n & 1 & 1 \\ 2J_{1\times(n-3)} & 1 & 2n-4 & 1 \\ 3J_{1\times(n-3)} & 2 & 1 & 3n-6 \end{pmatrix}, \tag{2.5}$$

where $e_{n-3} = -2I_{n-3} + J_{n-3}$.

Choosing the vector $X_{i-1} = (1, x_{i2}, x_{i3}, \ldots, x_{i(n-4)}, 0, 0, 0)^T$, where $x_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases}$ for $i = 2, 3, \ldots, n-3$. We can easily verify that X_{i-1}, $i = 2, 3, \ldots, n-3$ are the $n-4$ eigenvectors corresponding to the eigenvalue $2n-4$ of M. The other four eigenvalues of the block matrix M are the eigenvalues of the following equitable quotient matrix (with partition cells $\{1, 2, \ldots, n-3\}$, $\{n-2\}$, $\{n-1\}$, $\{n\}$)

$$\begin{pmatrix} 4n-10 & 1 & 2 & 3 \\ n-3 & n & 1 & 2 \\ 2(n-3) & 1 & 2n-4 & 1 \\ 3(n-3) & 2 & 1 & 3n-6 \end{pmatrix}. \tag{2.6}$$

By computer calculations, the characteristic polynomial of the above matrix is

$$\begin{aligned} x^4 - &(10n-20)x^3 + \left(35n^2 - 148n + 160\right)x^2 + \big(-50n^3 + 318n^2 \\ &- 666n + 448\big)x + 24n^4 - 192n^3 + 528n^2 - 520n + 64. \end{aligned} \tag{2.7}$$

□

The zeros of the polynomial in (2.7) cannot be found explicitly, but by using intermediate value theorem and computer calculations, it is easy to verify that it has four distinct zeros and they lie in four mutually disjoint intervals $(n-1, n)$, $(2n-8, 2n-5)$, $(3n-15, 3n-14)$ and $(4n, 4n+2)$. Thus, the matrix given in (2.5) has five distinct eigenvalues and we note that the equitable quotient matrix (2.6) does not contain all the distinct eigenvalues of M. However, if we choose the partition $\{\{1, 2, \ldots, n-4\}, \{n-3\}, \{n-2\}, \{n-1\}, \{n\}\}$ of n, then the corresponding equitable quotient matrix contains all the distinct eigenvalues of M as can be seen below:

$$M' = \begin{pmatrix} 2nI_{n-4} + 2e_{n-4} & 2J_{(n-4)} & J_{(n-4)\times 1} & 2J_{(n-4)\times 1} & 3J_{(n-4)\times 1} \\ 2J_{1\times(n-4)} & 2n-2 & 1 & 2 & 3 \\ J_{1\times(n-4)} & 1 & n & 1 & 1 \\ 2J_{1\times(n-4)} & 2 & 1 & 2n-4 & 1 \\ 3J_{1\times(n-4)} & 3 & 2 & 1 & 3n-6 \end{pmatrix},$$

and its equitable quotient matrix is

$$Q' = \begin{pmatrix} 4n-12 & 2 & 1 & 2 & 3 \\ 2(n-4) & 2n-2 & 1 & 2 & 3 \\ n-4 & 1 & n & 1 & 2 \\ 2(n-4) & 2 & 1 & 2n-4 & 1 \\ 3(n-4) & 3 & 2 & 1 & 3n-6 \end{pmatrix}.$$

The characteristic polynomial of Q' is

$$\begin{aligned} (x-2n+4)(x^4 - (10n-20)x^3 &\left(35n^2 - 148n + 160\right)x^2 + \big(-50n^3 + 318n^2 \\ &- 666n + 448\big)x + 24n^4 - 192n^3 + 528n^2 - 520n + 64). \end{aligned}$$

Therefore, Q' contains all the distinct eigenvalues of the matrix M given in (2.5).

From Theorems 2.1 and 2.4, it follows that the eigenvalue with algebraic multiplicity greater than 1 are not the eigenvalues of the quotient matrix corresponding to the smallest possible equitable partition. Next, we have a class of matrices whose all eigenvalues with multiplicities more than one are also the eigenvalues of the equitable quotient matrix except one eigenvalue.

Theorem 2.5 *Let M be defined as in (2.8) with $n = p^4 - 1$, where p is prime. Then the following holds*

(i) The spectrum of M is

$$\begin{aligned} \Big\{0, (p^4-1)^{[p-1]}, (2p^4-p-1)^{[p^2-p]}, (2p^4-p^2-1)^{[p^3-p^2-1]}, \\ (2p^4-p^3-1)^{[p^4-p^3]}\Big\}. \end{aligned}$$

(ii) The equitable quotient Matrix (2.10) contains all the distinct eigenvalues of M.

Proof Consider the following block M matrix of order $n = p^4 - 1$, for prime p

$$M = \left(\begin{array}{c|c|c|c} d_1 I_{p-1} - J_{p-1} & -J_{(p-1)\times(p^2-p)} & -J_{(p-1)\times(p^3-p^2)} & -J_{(p-1)\times(p^4-p^3)} \\ \hline -J_{(p^2-p)\times(p-1)} & d_2 I_{p^2-p} - J_{p^2-p} & -J_{(p^2-p)\times(p^3-p^2)} & -2J_{(p^2-p)\times(p^4-p^3)} \\ \hline -J_{(p^3-p^2)\times(p-1)} & -J_{(p^3-p^2)\times(p^2-p)} & d_3 I_{p^3-p^2} - 2J_{p^3-p^2} & -2J_{(p^3-p^2)\times(p^4-p^3)} \\ \hline -J_{(p^4-p^3)\times(p-1)} & -2J_{(p^4-p^3)\times(p^2-p)} & -2J_{(p^4-p^3)\times(p^3-p^2)} & d_4 I_{p^4-p^3} - 2J_{p^4-p^3} \end{array}\right), \tag{2.8}$$

where $d_1 = p^4 - 1,\ d_2 = 2p^4 - p^3 - 1,\ d_3 = 2p^4 - p^2 - 1$ and $d_4 = 2p^4 - p - 1$. Consider the sequence of non-vectors of $\mathbb{R}^n(\mathbb{R})$ defined as:

$$\begin{aligned} X_{i-1} =& (1, x_{12}, \ldots, x_{1(p-1)}, 0, 0, \ldots, 0, 0, 0, \ldots, 0, 0, 0, \ldots, 0)^T, \\ Y_{i-1} =& (0, 0, \ldots, 0, 1, y_{12}, \ldots, y_{1(p^2-p)}, 0, 0, \ldots, 0, 0, 0, \ldots, 0)^T, \\ Z_{i-1} =& (0, 0, \ldots, 0, 0, 0, \ldots, 0, 1, z_{12}, \ldots, z_{1(p^3-p)}, 0, 0, \ldots, 0)^T, \\ T_{i-1} =& (0, 0, \ldots, 0, 0, 0, \ldots, 0, 0, 0, \ldots, 0, 1, t_{12}, \ldots, t_{1(p^4-p^3)})^T, \end{aligned}$$

where, for $i = 2, 3, \ldots, p-1$, $x_{ij} = \begin{cases} -1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$,

for $i = 2, 3, \ldots, p^2 - p$, $y_{ij} = \begin{cases} -1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$,

for $i = 2, 3, \ldots, p-1$, $z_{ij} = \begin{cases} -1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$,

and for $i = 2, 3, \ldots, p-1$, $t_{ij} = \begin{cases} -1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$.

Now, it is verify that $X_1, \ldots, X_{p-2}$ are the linearly independent eigenvectors corresponding to the eigenvalues p^4-1, $Y_1, \ldots, Y_{p^2-p-1}$ are the linearly independent eigenvectors corresponding to the eigenvalues $2p^4-p^3-1$, $Z_1, \ldots, Z_{p^3-p^2-1}$ are the linearly independent eigenvectors corresponding to the eigenvalues $2p^4 - p^2 - 1$ and $T_1, \ldots, T_{p^4-p^3-1}$ are the linearly independent eigenvectors corresponding to the eigenvalues $2p^4 - p - 1$. Since the block matrix given in (2.8) with partition $\{\{1, 2, \ldots, p-1\}, \{1, 2, \ldots, p^2 - p\}, \{1, 2, \ldots, p^3 - p\}, \{1, 2, \ldots, p^4 - p^3\}\}$ is equitable, so by Lemma 1.1, the remaining 4 eigenvalues of the block matrix M are the eigenvalues of the following equitable quotient matrix

$$\begin{pmatrix} p^4 - p & -\left(p^2 - p\right) & -\left(p^3 - p^2\right) & -\left(p^4 - p^3\right) \\ -(p-1) & 2p^4 - p^3 - p^2 + p - 1 & -\left(p^3 - p^2\right) & -2\left(p^4 - p^3\right) \\ -(p-1) & -\left(p^2 - p\right) & 2p^4 - 2p^3 + p^2 - 1 & -2\left(p^4 - p^3\right) \\ -(p-1) & -2\left(p^2 - p\right) & -2\left(p^3 - p^2\right) & 2p^3 - p - 1 \end{pmatrix}. \tag{2.9}$$

The eigenvalues of the above matrix (2.9) are

$$\left\{0, p^4 - 1, 2p^4 - p - 1, 2p^4 - p^3 - 1\right\}.$$

That proves (i).
Since matrix (2.9) does not contain all the distinct eigenvalues of M, so we modify the partition as $\{\{1, 2, \ldots, p-1\}, \{1, 2, \ldots, p^2 - p\}, \{1\}, \{2, \ldots, p^3 - p\}, \{1, 2, \ldots, p^4 - p^3\}\}$ and the new equitable quotient matrix is

$$\begin{pmatrix} p^4 - p & -\left(p^2 - p\right) & -1 & -\left(p^3 - p^2 - 1\right) & -\left(p^4 - p^3\right) \\ -(p-1) & 2p^4 - p^3 - p^2 + p - 1 & -1 & -\left(p^3 - p^2 - 1\right) & -2\left(p^4 - p^3\right) \\ -(p-1) & -\left(p^2 - p\right) & 2p^4 - p^2 - 3 & -2\left(p^3 - p^2 - 1\right) & -2\left(p^4 - p^3\right) \\ -(p-1) & -\left(p^2 - p\right) & -2 & 2p^4 - 2p^3 + p^2 + 1 & -2\left(p^4 - p^3\right) \\ -(p-1) & -2\left(p^2 - p\right) & -2 & -2\left(p^3 - p^2 - 1\right) & 2p^3 - p - 1 \end{pmatrix} \tag{2.10}$$

and its eigenvalue are

$$\left\{0, p^4 - 1, 2p^4 - p - 1, 2p^4 - p^2 - 1, 2p^4 - p^3 - 1\right\}.$$

Thus, it is clear that the above matrix contains all the distinct eigenvalues of the matrix given in (2.8). □

Theorem 2.5 can be generalized in the following result.

Theorem 2.6 *Consider the following matrix of order $n = p^{2t} - 1$, where $p \geq 2$ is a real and $t \geq 2$ is a positive integer.*

$$M = \begin{pmatrix}
a_1 & l_{1,2} & l_{1,3} & \cdots & l_{1,t} & l_{1,t+1} & l_{1,t+2} & \cdots & l_{1,2t-2} & l_{1,2t-1} & l_{1,2t} \\
l_{2,1} & a_2 & l_{2,3} & \cdots & l_{2,t} & l_{2,t+1} & l_{2,t+2} & \cdots & l_{2,2t-2} & l_{2,2t-1} & 2l_{2,2t} \\
l_{3,1} & l_{3,2} & a_3 & \cdots & l_{3,t} & l_{3,t+1} & l_{3,t+2} & \cdots & l_{3,2t-2} & 2l_{3,2t-1} & 2l_{3,2t} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
l_{t,1} & l_{t,2} & l_{t,3} & \cdots & a_t & l_{t,t+1} & l_{t,t+2} & \cdots & 2l_{t,2t-2} & 2l_{t,2t-1} & 2l_{t,2t} \\
l_{t+1,1} & l_{t+1,2} & l_{t+1,3} & \cdots & l_{t+1,t} & a_{t+1} & 2l_{t+1,t+2} & \cdots & 2l_{t+1,2t-2} & 2l_{t+1,2t-1} & 2l_{t+1,2t} \\
l_{t+2,1} & l_{t+2,2} & l_{t+2,3} & \cdots & l_{t+2,t} & 2l_{t+2,t+1} & a_{t+2} & \cdots & 2l_{t+2,2t-2} & 2l_{t+2,2t-1} & 2l_{t+2,2t} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
l_{2t-2,1} & l_{2t-2,2} & l_{2t-2,3} & \cdots & 2l_{2t-2,t} & 2l_{2t-2,t+1} & 2l_{2t-2,t+2} & \cdots & a_{2t-2} & 2l_{2t-2,2t-1} & 2l_{2t-2,2t} \\
l_{2t-1,1} & l_{2t-1,2} & 2l_{2t-1,3} & \cdots & 2l_{2t-1,t} & 2l_{2t-1,t+1} & 2l_{2t-1,t+2} & \cdots & 2l_{2t-1,2t-2} & a_{2t-1} & 2l_{2t-1,2t} \\
l_{2t,1} & 2l_{2t,2} & 2l_{2t,3} & \cdots & 2l_{2t,t} & 2l_{2t,t+1} & 2l_{2t,t+2} & \cdots & 2l_{2t,2t-2} & 2l_{2t,2t-1} & a_{2t}
\end{pmatrix},$$

where $a_i = d_i I_{p^i - p^{i-1}} - J_{p^i - p^{i-1}}$, for $i = 1, 2, \ldots, t-2, t-1$, $a_i = d_i I_{p^i - p^{i-1}} - 2J_{p^i - p^{i-1}}$, for $i = t, t+1, \ldots, 2t-2, 2t-1, 2t$ and $l_{i,j} = J_{(p^i - p^{i-1}) \times (p^j - p^{j-1})}$. Also, $d_i = 2p^{2t} - p^{2t+1-1} - 1$, for $i = 1, 2, \ldots, t-1, t, t+1, 2t-2, 2t-1, 2t$.

As in Theorem 2.5 (here we avoid calculations), the eigenvalues of M are the simple eigenvalue 0 and the eigenvalues

$$2p^{2t} - p^{2t+1-i} - 1, \text{ with multiplicity } p^i - p^{i-1} - 1,$$

where $i = 1, 2, \ldots, t-1, t, t+1, \ldots, 2t-2, 2t-1, 2t$. Thus, M has $2t+1$ distinct eigenvalues.

$$Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \text{ where}$$

$$A = \begin{pmatrix}
d_1 - (p-1) & -(p^2-p) & -(p^3-p^2) & \ldots & -(p^{t-1}-p^{t-2}) & -(p^t-p^{t-1}) \\
-(p-1) & d_2 - (p^2-p) & -(p^3-p^2) & \ldots & -(p^{t-1}-p^{t-2}) & -(p^t-p^{t-1}) \\
-(p-1) & -(p^2-p) & d_3 - (p^3-p^2) & \ldots & -(p^{t-1}-p^{t-2}) & -(p^t-p^{t-1}) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-(p-1) & -(p^2-p) & -(p^3-p^2) & \ldots & d_{t-1} - (p^{t-1}-p^{t-2}) & -(p^t-p^{t-1}) \\
-(p-1) & -(p^2-p) & -(p^3-p^2) & \ldots & -(p^{t-1}-p^{t-2}) & d_t - (p^t-p^{t-1})
\end{pmatrix},$$

$$B = \begin{pmatrix}
-(p^{t+1}-p^t) & -(p^{t+2}-p^{t+1}) & \ldots & -(p^{2t-2}-p^{2t-3}) & -(p^{2t-1}-p^{2t-2}) & -(p^{2t}-p^{2t-1}) \\
-(p^{t+1}-p^t) & -(p^{t+2}-p^{t+1}) & \ldots & -(p^{2t-2}-p^{2t-3}) & -(p^{2t-1}-p^{2t-2}) & -2(p^{2t}-p^{2t-1}) \\
-(p^{t+1}-p^t) & -(p^{t+2}-p^{t+1}) & \ldots & -(p^{2t-2}-p^{2t-3}) & -2(p^{2t-1}-p^{2t-2}) & -2(p^{2t}-p^{2t-1}) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-(p^{t+1}-p^t) & -(p^{t+2}-p^{t+1}) & \ldots & -2(p^{2t-2}-p^{2t-3}) & -2(p^{2t-1}-p^{2t-2}) & -2(p^{2t}-p^{2t-1}) \\
-(p^{t+1}-p^t) & -2(p^{t+2}-p^{t+1}) & \ldots & -2(p^{2t-2}-p^{2t-3}) & -2(p^{2t-1}-p^{2t-2}) & -2(p^{2t}-p^{2t-1})
\end{pmatrix},$$

$$C = \begin{pmatrix}
-(p-1) & -(p^2-p) & -(p^3-p^2) & \ldots & -(p^{t-1}-p^{t-2}) & -(p^t-p^{t-1}) \\
-(p-1) & -(p^2-p) & -(p^3-p^2) & \ldots & -(p^{t-1}-p^{t-2}) & -2(p^t-p^{t-1}) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-(p-1) & -(p^2-p) & -(p^3-p^2) & \ldots & -2(p^{t-1}-p^{t-2}) & -2(p^t-p^{t-1}) \\
-(p-1) & -(p^2-p) & -2(p^3-p^2) & \ldots & -2(p^{t-1}-p^{t-2}) & -2(p^t-p^{t-1}) \\
-(p-1) & -2(p^2-p) & -2(p^3-p^2) & \ldots & -2(p^{t-1}-p^{t-2}) & -2(p^t-p^{t-1})
\end{pmatrix}$$

$$D = \begin{pmatrix}
d_{t+1} - 2(p^{t+1}-p^t) & -2(p^{t+2}-p^{t+1}) & \ldots & -2(p^{2t-1}-p^{2t-2}) & -2(p^{2t}-p^{2t-1}) \\
-2(p^{t+1}-p^t) & d_{t+2} - 2(p^{t+2}-p^{t+1}) & \ldots & -2(p^{2t-1}-p^{2t-2}) & -2(p^{2t}-p^{2t-1}) \\
-2(p^{t+1}-p^t) & -2(p^{t+2}-p^{t+1}) & \ldots & -2(p^{2t-1}-p^{2t-2}) & -2(p^{2t}-p^{2t-1}) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-2(p^{t+1}-p^t) & -2(p^{t+2}-p^{t+1}) & \ldots & -2(p^{2t-1}-p^{2t-2}) & -2(p^{2t}-p^{2t-1}) \\
-2(p^{t+1}-p^t) & -2(p^{t+2}-p^{t+1}) & \ldots & d_{2t-1} - 2(p^{2t-1}-p^{2t-2}) & -2(p^{2t}-p^{2t-1}) \\
-2(p^{t+1}-p^t) & -2(p^{t+2}-p^{t+1}) & \ldots & -2(p^{2t-1}-p^{2t-2}) & d_{2t} - 2(p^{2t}-p^{2t-1})
\end{pmatrix}.$$

one way to find the eigenvalues of Q is to verify that $2p^{2t} - p^{2t+1-i} - 1$, $i = 1, 2, \ldots, m-1, m, m+2, m+3, \ldots, 2t-2, 2t-1, 2t$ are the eigenvalues of Q. We note that $2p^{2t} - p^m - 1$ is only remaining eigenvalue of M

which is not the eigenvalue of Q. However dividing $(t+1)$-th column (row, respectively) in two columns (rows) as in the matrix given in (2.10), so the new equitable quotient matrix (say Q^*) contains all the eigenvalues of Q and the eigenvalue $2p^{2t} - p^m - 1$. Thus, Q^* contains all the distinct eigenvalues of M.

By gaining the information from Theorems 2.1, 2.4 and 2.5, it certainly follows that there exists an equitable partition such that its corresponding quotient matrix contains all the distinct eigenvalues of M. In this direction, we have the following result.

Theorem 2.7 *Let M be a matrix of order n and Q be its equitable quotient matrix. Then the following hold.*

(i) There exists an equitable partition such that its quotient matrix Q contains the spectral radius of M.
(ii) There exists an equitable partition such that its quotient matrix Q contains all the distinct eigenvalues of M.

Proof Let $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_j, \ldots, \mathcal{P}_m\}$ be the equitable partition of $\{1, 2, \ldots, n\}$, $Q_{n\times n}$ be the corresponding equitable quotient matrix and λ be the spectral radius of M (other than non-negative matrix). Let λ be the eigenvalue of M with multiplicity greater than or equal to 1. If λ is contained in the spectrum of Q, then the result follows. If λ is not the eigenvalue of Q, then there is some partition cell $\mathcal{P}_j = \{1, 2, \ldots, z\}, 2 \leq z < n$, which when modified, the resulting quotient matrix contains the spectral radius. That is, we can write $\mathcal{P}_j = \{1, 2, \ldots, z-1\} \cup \{z\} = \mathcal{P}_1' \cup \mathcal{P}_2''$, so that $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{j-1}, \mathcal{P}_1', \mathcal{P}_2'', \mathcal{P}_{j+1}, \ldots, \mathcal{P}_m\}$ is the new equitable partition of $\{1, 2, \ldots, n\}$ and its corresponding quotient matrix Q' of order $m+1$ contains the eigenvalue λ. Similarly, if $\eta^{[k]}$ with multiplicity $k \geq 1$ is the any other eigenvalue of M, then η may or may not be the eigenvalue of Q. If η is not in the spectrum of Q, then as above we modify the previous partition as $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{j-1}, \mathcal{P}_1', \mathcal{P}_2'', \mathcal{P}_{j+1}, \ldots, \mathcal{P}_{t-1}, \mathcal{P}_1^*, \mathcal{P}_2^{**}, \mathcal{P}_{t+1}, \ldots, \mathcal{P}_m\}$, so that the resulting quotient matrix of order $m+2$ is equitable and its spectrum contains the eigenvalue η. Repeating the same process of modifying the partitions of the set $\{1, 2, \ldots, n\}$ for the remaining eigenvalues not contained in the spectrum of the quotient matrix. Thus there exists a partition of $\{1, 2, \ldots, n\}$, so that the spectrum of the resulting equitable quotient matrix contains all the distinct eigenvalues of M. □

We illustrate Theorem 2.7 with the help of the following example: Consider a matrix

$$M = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 1 \end{pmatrix}.$$

The spectrum of M is $\{7.54138, 1.45862, 1^{[2]}, (-1)^{[2]}\}$. Clearly, M have four distinct eigenvalues and its quotient matrix with smallest possible equitable partition

$$Q = \begin{pmatrix} 4 & 3 \\ 3 & 5 \end{pmatrix},$$

and its eigenvalues are 7.54138 and 1.45862. Since, M has four distinct eigenvalues, so the smallest order of the equitable quotient matrix must be 4. Consider a new partition $\pi = \{\{1, 2\}, \{3\}, \{4, 5\}, \{6\}\}$, we note that such a partition is not unique. Thus, the quotient matrix Q' with partition π is

$$Q' = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 1 & 3 & 2 \\ 2 & 1 & 4 & 1 \end{pmatrix},$$

and it is easy to see that the spectrum of Q' is $\{7.54138, 1.45862, -1, 1\}$. Therefore, Q' with the enlarged partition π contains all the distinct eigenvalues of M.

Theorem 2.7 gives an insight that there exists an equitable partition such that equitable quotient matrix with such partition contains all the distinct eigenvalues of M. How do we know which partition cell of the equitable partition of $\{1, 2, \ldots, n\}$ need to be written into two disjoint cells so that the resulting quotient matrix is equitable and its spectrum contains the required eigenvalue. We need to be very smart in picking up such a partition cell, sometimes the missing eigenvalue and the symmetric pattern of some matrices may help but in general it is an inspection method and all cells need to be checked. Further, we note that for matrices with symmetric eigenvalue, (that is, if λ is eigenvalue of a matrix M, then $-\lambda$ is also eigenvalue of M), we simultaneously need to modify their corresponding partitions as we did in Proposition 2.2 for the matrix given in (2.1). Also an interesting question is: How small is the equitable partition such that Q contains all the eigenvalues of M? For suppose, next we have a very rare type of matrix such that Q with the smallest equitable partition contains all the distinct eigenvalues of M.

Proposition 2.8 *Let M be the matrix as defined in (2.11). Then it's equitable quotient matrix Q given in (2.12) contains all the distinct eigenvalues of M.*

Proof Consider the block matrix of order $n+2$ given below

$$M = \left(\begin{array}{c|c|cccccc} 4n-2 & -(2n-2) & -2 & -2 & \dots & -2 & -2 \\ \hline -1 & 4n+1 & -4 & -4 & \dots & -4 & -4 \\ \hline -1 & -2(2n-2) & 8n-7 & -4 & \dots & -4 & -4 \\ -1 & -2(2n-2) & -4 & 8n-7 & \dots & -4 & -4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -2(2n-2) & -4 & -4 & \dots & 8n-7 & -4 \\ -1 & -2(2n-2) & -4 & -4 & \dots & -4 & 8n-7 \end{array}\right). \tag{2.11}$$

Now, it can be easily verified that $8n-3$ is the eigenvalue of M with corresponding eigenvectors

$$X_1 = (0, 0, 1, -1, 0, \dots, 0, 0),\ X_2 = (0, 0, 1, 0, -1, 0, \dots, 0, 0),\ X_3 = (0, 0, 1, 0, -1, 0, \dots, 0, 0),$$
$$\vdots$$
$$X_{n-3} = (0, 0, 1, 0, 0, \dots, -1, 0, 0),\ X_{n-2} = (0, 0, 1, 0, \dots, 0, -1, 0),\ X_{n-1} = (0, 0, 1, 0, \dots, 0, 0, -1).$$

The remaining three eigenvalues of M are the eigenvalues of the following equitable quotient matrix

$$Q = \begin{pmatrix} 4n-2 & -(2n-2) & -2n \\ -1 & 4n+1 & -4n \\ -1 & -2(2n-2) & 4n-3 \end{pmatrix}. \tag{2.12}$$

Clearly, 0, $4n-1$ and $8n-3$ are the eigenvalues of (2.12). Therefore, all the distinct eigenvalues of M are also the eigenvalues of its equitable quotient matrix Q. □

Remark 2.9 Proposition 2.8 is a special case of the following general matrix

$$M = \begin{pmatrix} \alpha & -\beta & -\gamma J_{1\times n} \\ -\delta & \Phi & -\Psi J_{1\times n} \\ -\eta J_{n\times 1} & -\xi J_{n\times 1} & b_1 I_n - b_2 J_n \end{pmatrix}, \tag{2.13}$$

where $\alpha, \beta, \gamma, \delta, \Phi, \eta, \xi, b_1$ and b_2 are real numbers.

The equitable quotient matrix of Matrix (2.13) is

$$Q = \begin{pmatrix} \alpha & -\beta & -n\gamma \\ -\delta & \Phi & -n\Psi \\ -\eta & -\xi & b_1 - nb_2. \end{pmatrix}. \tag{2.14}$$

Now, with $\alpha = \beta + n\gamma$, $\Phi = \delta + n\Psi$, $b_1 = \eta + \xi + nb_2$ and $\beta = 2n-2$, $\gamma = 2$, $\delta = 1$, $\Psi = 4$, $\eta = 1$, $\xi = 2(2n-2)$, it is easy to see that Proposition 2.8 follows. We note that in general the equitable quotient matrix in (2.14) may not contain all the distinct eigenvalues of Matrix (2.13) for arbitrary values of parameters included there.

In view of the above results, we observe that the equitable quotient matrix may or may not contain all the distinct eigenvalues of M. As we increase the order of partition, we get an equitable quotient matrix which contains all the distinct eigenvalues (including the spectral radius in case of non-positive matrices) of M. Therefore, problem is about the partitions of matrix M and there seems no conditions on matrix itself, so that the distinct eigenvalues of M are also the eigenvalues of Q. Thus, it seems very non-trivial to identify those classes of matrices whose quotient matrices contain all the distinct eigenvalues (including spectral radius in case of matrices other than non-negative) with respect to the smallest equitable partition. Therefore, we ask Problem 1 and the problem of the spectral radius of non-positive matrices in the following way.

Problem 2 *Characterize those non-positive matrices, whose quotient matrix with the smallest equitable partition contains the spectral radius?*

Problem $1'$ *Characterize those matrices such that the spectrum of the quotient matrix contains all the distinct eigenvalues of parent matrix with respect to the smallest equitable partition?*

Acknowledgements The authors are very thankful to the referees for their valuable comments and input to improve the paper.

Data availibility There is no data associated with this article.

References

1. F. Atik, On equitable partition of matrices and its applications, *Linear Multilinear Algebra* **68**(11) (2020) 2143–2156.
2. F. Atik and P. Panighari, On the distance and distance signless Laplacian eigenvalues of graphs and the smallest Grešgorin disc, *Electronic J. Linear Algebra* **34** (2018) 191–204.
3. A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Springer, New York, 2010.
4. D. M. Cvetković, P. Rowlison and S. Simić, *An Introduction to Theory of Graph Spectra*, London Math. Society Student Text, 75, Cambridge University Press, UK, 2010.
5. S. R. Garcia and R. A. Horn, *A Second Course in Linear Algebra*, Cambridge University Press, UK, 2017.
6. W. H. Haemers, Interlacing eigenvalues and graphs, *Linear Algebra Appl.* **226-288** (1995) 593–616.
7. R. Horn and C. Johnson, *Matrix Analysis*, Second Edition, Cambridge University Press, New York, (2013).
8. Z. Mehranian, A. Gholami and A. R. Ashrafi, The spectra of power graphs of certain finite groups, Linear Multilinear Algebra **65**(5) (2016) 1003–1010.
9. S. Pirzada, Bilal A. Rather, M. Aijaz and T. A. Chishti, Distance signless Laplacian spectrum of graphs and spectrum of zero divisor graphs of $\mathbb{Z}_n$, *Linear MultiLinear Algebra* **70**(17) (2022) 3354–3369.
10. B. A. Rather, S. Pirzada and Z. Goufei, On distance Laplacian spectra of power graphs of certain finite groups, *Acta Math. Sinica, English Series* **39** (2023) 603–617.
11. D. Stevanović, Large sets of long distance equienergetic graphs, *Ars Math. Contemp.* **2**(1) (2009) 35–40.
12. L. You, M. Yang, W. So and W. Xi, On the spectrum of an equitable quotient matrix and its applications, *Linear Algebra Appl.* **577** (2019) 21–40.