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ORIGINAL RESEARCH

Distribution of monomial-prime numbers and Mertens sum evaluations

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Abstract In this paper, we mainly study the monomial-prime numbers, which are of the form pn^k for primes p and integers $k \geq 2$. First, we give an asymptotic estimate on the number of numbers of a general form $pf(n)$ for arithmetic functions f satisfying certain growth conditions, which generalizes Bhat's recent result on the Square-Prime Numbers. Then, we prove three Mertens-type theorems related to numbers of a more general form, partially extending the recent work of Qi-Hu, Popa and Tenenbaum on the Mertens sum evaluations. At the end, we evaluate the average and variance of the number of distinct monomial-prime factors of positive integers by applying our Mertens-type theorems.

Keywords Monomial-Prime Number · Mertens Theorem · Prime Number Theorem · Zeta Function

Mathematics Subject Classification 11N25 · 11N37 · 11N80

1 Introduction and statement of main results

The distribution of integers with certain constraints is a fundamental topic in analytic number theory. Recently, Bhat [1] studied the numbers of the form pn^2, which are named Square-Prime (SP) Numbers, for integers $n > 1$ and primes p. He showed that the number of SP Numbers up to x is asymptotic to $(\zeta(2) - 1)x/\log x$, where $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ $(s > 1)$ is the Riemann zeta function. In this paper, we study the numbers of a general form $pf(n)$ for certain primes p and arithmetic functions $f : \mathbb{N} \to \mathbb{R}_{\geq 1}$. If $f(n) = n^s$ is a monomial for some positive real number $s > 1$, we call pn^s a *monomial-prime number* of power s. Our first result gives an asymptotic formula for the number of monomial-prime numbers.

Theorem 1.1 *Let S be a set of primes satisfying the following asymptotic estimate*

$$\sum_{p \leq x, p \in S} 1 = \frac{cx^{\gamma}}{\log^{\alpha}(x)} \left(1 + O\left(\frac{1}{\log^{\beta}(x)}\right)\right)$$

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for some positive constants c, α, β, and $0 < \gamma \leq 1$. *Let* $f : \mathbb{N} \to \mathbb{R}_{\geq 1}$ *be an increasing positive sequence satisfying* $\sum_{f(n)\geq x} 1/f^{\gamma}(n) \ll x^{-\delta}$ *for some* $\delta > 0$. *Then as* $x \to \infty$ *we have*

$$\#\{(p,n) : pf(n) \leq x,\, p \in S\} \sim \left(\sum_{n=1}^{\infty} \frac{1}{f^{\gamma}(n)}\right) \cdot \frac{cx^{\gamma}}{\log^{\alpha}(x)}.$$

In particular, for any real number $s > 1$ *we have*

$$\#\{(p,n) : pn^{s} \leq x,\, p \text{ prime}\} \sim \zeta(s) \cdot \frac{x}{\log x}. \tag{1}$$

Remark 1 For $s = 1$ in (1), Bănescu and Popa proved in [2, Proposition 5(ii)] that

$$\#\{(p,n) : pn \leq x,\, p \text{ prime}\} \sim x \log\log x.$$

When $s = 2$, see [3,4] for more properties on SP Numbers, such as their bounded gaps, SP Numbers of the form $x^2 + 1$, and existence of SP Numbers between x^2 and $(x+2)^2$, etc.. When $s = k \geq 2$ is an integer, the numbers of the form pn^k with $n > 1$ is called by Bhat [4] KP_k Numbers. And he also gave an asymptotic estimate for the distribution of KP_k Numbers, and its main term can be deduced from (1).

Theorem 1.1 builds a connection between the density of sets of primes and the asymptotic behavior on numbers of the form $pf(n)$. We see that the asymptotic behavior of monomial-prime numbers differs from that of prime numbers by the special value $\zeta(s)$ of the Riemann zeta function as a factor. Several common examples of S will be given in section 2.1 including primes in arithmetic progressions, in the Chebatorev density theorem, in the Beatty sequences, in the Piatetski-Shapiro sequences, and with preassigned digits. If we take S to be the set of all primes and $f(n) = n^2$, then Bhat's asymptotic formula on the number of SP Numbers is recovered from Theorem 1.1 by the prime number theorem. Moreover, analogous to the SP Numbers, we call pn^3 a Cube-Prime (CP) Number[1] for $n > 1$. Taking $f(n) = n^3$, we get the following asymptotic formula on the number of CP Numbers.

Corollary 1 *The number of CP Numbers smaller than x is asymptotic to* $(\zeta(3) - 1)x/\log x$.

In [1, Theorem 6.1] Bhat also gave an asymptotic estimate on the number of SP Numbers ending in 1. As another application of Theorem 1.1, we get the following asymptotic estimate on the number of CP Numbers ending in 1 as well.

Corollary 2 *The number of CP Numbers ending in 1 is asymptotic to*

$$\frac{x}{4000\log x}\left(\zeta\left(3, \frac{1}{10}\right) + \zeta\left(3, \frac{3}{10}\right) + \zeta\left(3, \frac{7}{10}\right) + \zeta\left(3, \frac{9}{10}\right) - 1000\right), \tag{2}$$

where $\zeta(s,t) = \sum_{n=0}^{\infty} 1/(n+t)^s$ *is the Hurwitz zeta function.*

Remark 2 The asymptotic estimates of CP Numbers ending in 3, 7, and 9 respectively are the same as that ending in 1. Similarly, one can count the number of CP Numbers ending in 2, 4, 5, 6, and 8 in the same way as the case ending in 1, but the asymptotic estimates will be slightly different.

Next, we evaluate some sums and products of Mertens type for monomial-prime numbers. Recall that Mertens' theorems are three results related to the reciprocals of primes (e.g., see [5, Chapter I.1]), which are stated as follows:

$$\sum_{p\leq x} \frac{\log p}{p} = \log x + O(1) \qquad \text{(Mertens' first theorem)}, \tag{3}$$

$$\sum_{p\leq x} \frac{1}{p} = \log\log x + M + O\left(\frac{1}{\log x}\right) \qquad \text{(Mertens' second theorem)}, \tag{4}$$

$$\prod_{p\leq x}\left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right) \qquad \text{(Mertens' third theorem)}, \tag{5}$$

[1] Here we require $n > 1$ for CP Numbers to agree with the SP Numbers defined by Bhat. But $n = 1$ is allowed for monomial-prime numbers defined in this paper.

where M is the Mertens' constant and γ is the Euler's constant.

Generalizations of Mertens' theorems have been widely studied in many literatures. With respect to the Mertens' first theorem, Qi and Hu [6] evaluated the following sum

$$\sum_{p_1\cdots p_k\le x}\frac{\log^s(p_1\cdots p_k)}{p_1\cdots p_k}$$

for positive integers k and s. By their result, for $s=1$ there is a polynomial $F(X)$ of degree $k-1$ such that

$$\sum_{p_1\cdots p_k\le x}\frac{\log(p_1\cdots p_k)}{p_1\cdots p_k}=F(\log\log x)\log x+O\left((\log\log x)^k\right). \tag{6}$$

See Lemma 2.1 in Section 2.2 for the explicit description of $F(X)$. In particular, when $k=2$, one may take $F(X)=2X+2M-2$, see [7, Theorem 3.3] as well.

With respect to the Mertens' second theorem, if we let

$$S_k(x):=\sum_{p_1\cdots p_k\le x}\frac{1}{p_1\cdots p_k},$$

where p_j denotes a prime number, then Mertens' second theorem evaluates $S_1(x)$. In [8,9], Popa proved asymptotic estimates for $S_2(x)$ and $S_3(x)$ respectively. In 2017, Tenenbaum [10,11] showed the following asymptotic formula for general $S_k(x)$ using the Selberg-Delange method: for $k\ge 1$, we have

$$S_k(x)=P_k(\log\log x)+O\left(\frac{(\log\log x)^{k-1}}{\log x}\right)\qquad(x\ge 3), \tag{7}$$

where $P_k(X):=\sum_{0\le j\le k}\lambda_{j,k}X^j$ and

$$\lambda_{j,k}:=\sum_{0\le m\le k-j}\binom{k}{m,\,j,\,k-m-j}(M-\gamma)^{k-m-j}\left(\frac{1}{\Gamma}\right)^{(m)}(1)\qquad(0\le j\le k).$$

Here M is the Mertens' constant and γ is the Euler's constant as in Mertens' third theorem, $\Gamma(x)$ is the Gamma function, and $(1/\Gamma)^{(m)}$ is the m-th derivative of $1/\Gamma$. We remark that recently Qi and Hu [6] proved another formula for (7) and Bayless et al. [12] showed these two formulas ere equivalent to each other.

Our second main result is the following theorem of Mertens type for the numbers of more general form $p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)$ for some arithmetic functions f_i and integers $k, r\ge 1$, $1\le i\le r$.

Theorem 1.2 *Let k and r be two positive integers. Let $F(X)$ and $P_k(X)$ be the same polynomials as in (6) and (7) respectively. For $1\le i\le r$, let $f_i:\mathbb{N}\to\mathbb{R}_{\ge 1}$ be an increasing positive function satisfying $\sum_{f_i(n)\ge x}1/f_i(n)\ll x^{-\delta_i}$ for some $\delta_i>0$. Then we have*

$$\text{(I)}:\quad \sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x}\frac{\log(p_1\cdots p_k f_1(n_1)\cdots f_r(n_r))}{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)}$$
$$=\left(\prod_{i=1}^r\sum_{n=1}^\infty\frac{1}{f_i(n)}\right)F(\log\log x)\log x+O\left((\log\log x)^k\right); \tag{8}$$

$$\text{(II)}:\quad \sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x}\frac{1}{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)}$$
$$=\left(\prod_{i=1}^r\sum_{n=1}^\infty\frac{1}{f_i(n)}\right)P_k(\log\log x)+O\left(\frac{(\log\log x)^k}{\log x}\right); \tag{9}$$

$$\text{(III)}:\quad \prod_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x}\left(1-\frac{1}{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)}\right)$$
$$=e^{-\left(\prod_{i=1}^r\sum_{n=1}^\infty\frac{1}{f_i(n)}\right)P_k(\log\log x)+c(k,f_1,\cdots,f_r)}\left(1+O\left(\frac{(\log\log x)^{k-1}}{\log x}\right)\right). \tag{10}$$

Here the constant $c(k,f_1,\cdots,f_r)$ in (10) depends on k and the functions f_i, $1\le i\le r$, only.

Taking $k = r$ and $f_i(n) = n^{s_i}$ for any positive real number $s_i > 1$, $1 \le i \le k$, we get the following result of Mertens type for the products of k monomial-prime numbers.

Corollary 3 *Let $s_1 > 1, \ldots, s_k > 1$ be k positive real numbers. Let $F(X)$ and $P_k(X)$ be as in Theorem 1.2. Then we have*

$$\text{(I)}': \quad \sum_{p_1 n_1^{s_1} \cdots p_k n_k^{s_k} \le x} \frac{\log(p_1 n_1^{s_1} \cdots p_k n_k^{s_k})}{p_1 n_1^{s_1} \cdots p_k n_k^{s_k}}$$

$$= \left(\prod_{i=1}^{k} \zeta(s_i) \right) F(\log\log x) \log x + O\left((\log\log x)^k \right); \tag{11}$$

$$\text{(II)}': \quad \sum_{p_1 n_1^{s_1} \cdots p_k n_k^{s_k} \le x} \frac{1}{p_1 n_1^{s_1} \cdots p_k n_k^{s_k}} = \left(\prod_{i=1}^{k} \zeta(s_i) \right) P_k(\log\log x) + O\left(\frac{(\log\log x)^k}{\log x} \right); \tag{12}$$

$$\text{(III)}': \quad \prod_{p_1 n_1^{s_1} \cdots p_k n_k^{s_k} \le x} \left(1 - \frac{1}{p_1 n_1^{s_1} \cdots p_k n_k^{s_k}} \right)$$

$$= e^{-(\prod_{i=1}^{r} \zeta(s_i)) P_k(\log\log x) + c(k, s_1, \cdots, s_k)} \left(1 + O\left(\frac{(\log\log x)^k}{\log x} \right) \right). \tag{13}$$

Here the constant $c(k, s_1, \cdots, s_k)$ in (13) depends on the constants k, $s_1, \cdots$, and s_k.

Remark 3 On Mertens' first theorem, notice that in the case $k = 1$ the error term in (3) is better than that in (6). Due to this observation, the following result gives a more precise estimate than (11) for $k = 1$: let $s > 1$ be a positive number, then

$$\sum_{pn^s \le x} \frac{\log(pn^s)}{pn^s} = \zeta(s) \log x - s\zeta'(s) \log\log x + O(1). \tag{14}$$

Remark 4 There are other generalizations of Mertens type theorems too. For example, Garcia and Lee [13] recently obtained unconditional and effective number-field analogues of the three Mertens? theorems. In [14], Lichtman proved an asymptotic formula for the dissecting sum of reciprocals of numbers with exactly k prime divisors, $k \ge 1$. We leave the monomial-prime analogues of these results to the interested readers.

Finally, we apply (12) in Corollary 3 to compute the average and variance of the number of distinct monomial-prime factors of integers. Let $k \ge 2$ be an integer. Let $\omega_k(n)$ be the number of distinct monomial-prime factors of n of power k. That is, $\omega_k(n) = \sum_{pm^k | n} 1$.

Theorem 1.3 *We have*

$$\sum_{n \le x} \omega_k(n) = \zeta(k) x \log\log x + \zeta(k) M x + O\left(\frac{x \log\log x}{\log x} \right), \tag{15}$$

and

$$\sum_{n \le x} (\omega_k(n) - \zeta(k) \log\log x)^2 = \frac{\zeta^3(k) - \zeta^2(k)\zeta(2k)}{\zeta(2k)} x (\log\log x)^2 + O(x \log\log x), \tag{16}$$

where M is the Mertens' constant.

Remark 5 Recall $\omega(n)$ is the arithmetic function that counts the number of distinct prime factors of n, then one may view $\omega_k(n)$ as an analogue of $\omega(n)$. Although $\omega_k(n)$ is not a multiplicative function as $\omega(n)$ is, some properties of $\omega(n)$ still hold with respect to $\omega_k(n)$. For example, let $k = 2$, if there are infinitely many natural numbers n such that $\omega(n) = \omega(n+2) = 1$ or $\omega_2(n) = \omega_2(n+2) = 1$, then both of them would imply the famous twin prime conjecture is true, which is still far from reach now. We do have some results on $\omega(n)$ and $\omega(n+2)$, as well as $\omega_2(n)$ and $\omega_2(n+2)$, taking the same values. For example, it follows by [15, Theorem 5] that there are infinitely many integers n such that $\omega(n) = \omega(n+2) = 5$ and $\omega_2(n) = \omega_2(n+2) = 9$. Also,

one may think of $\omega(n)$ as the limit of $\omega_k(n)$ as $k \to \infty$, then letting $k \to \infty$ in Theorem 1.3, we recover the following estimates on $\omega(n)$ (e.g., see [16, Theorems 3.1.1 and 3.1.2]):

$$\sum_{n\le x}\omega(n) = x\log\log x + Mx + O\left(\frac{x\log\log x}{\log x}\right)$$

and

$$\sum_{n\le x}(\omega(n) - \log\log x)^2 = O(x\log\log x).$$

This paper is organized as follows. In Section 2 we introduce some interesting examples of sets of prime numbers and some technical theorems and lemmas that will be applied later. In Section 3 we first use the key technical tool, Lemma 2.2, to prove Theorem 1.1, and then we apply Theorem 1.1 to prove Corollary 2. In Section 4 we prove Theorem 1.2 and Equation (14) by Lemma 2.2. In Section 5 we compute the average and variance of $\omega_k(n)$ in Theorem 1.3 by applying (12) in Corollary 3 and Lemma 2.5.

Notation. The letters $p, q, p_1, \ldots,$ and p_k always denote primes. We write $f(x) = O(g(x))$ or $f(x) \ll g(x)$ if there exists some constant $C > 0$ such that $|f(x)| \le C|g(x)|$ for all x. The implied constant C may depend on some parameters, say k, m, or ε. We write $f(x) \sim g(x)$ if $\lim_{x\to\infty} f(x)/g(x) = 1$. As usual $[a, b]$ is the least common multiple of a and b, (a, b) is the greatest common divisor of a and b, and $\lfloor x\rfloor$ is the floor function.

2 Nuts and bolts

In this section, we list some examples of interesting sets of prime numbers, state a theorem on Mertens sums, and prove some lemmas that will be used in the proofs of main results. In particular, Lemma 2.2 is the key technical tool to be frequently used in the following sections.

2.1 Examples of subsets of primes

Let $\mathcal{P}$ be the set of all primes. Let S be a subset of primes and let $\pi_S(x) = \{p \in S : p \le x\}$ be the number of primes in S up to x. The following list gives several common interesting examples of S in the literature. The asymptotic estimates on $\pi_S(x)$ satisfy the assumptions in Theorem 1.1.

1. Arithmetic progressions. Let $q \ge 2$ and $1 \le a < q$ be two integers with $(a, q) = 1$. Let $S = \{p \in \mathcal{P} : p \equiv a \pmod q\}$, then by the prime number theorem in arithmetic progressions (e.g., see [5, Theorem II .4.1]), we have
$$\pi_S(x) = \frac{1}{\varphi(q)}\frac{x}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right).$$
2. Chebatorev density theorem. Let $K/\mathbb{Q}$ be a finite Galois extension with Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$. For any conjugacy class $C \subset G$, let
$$S_C = \left\{p \in \mathcal{P} : p \text{ unramified}, \left[\frac{K/\mathbb{Q}}{p}\right] = C\right\},$$
where $\left[\frac{K/\mathbb{Q}}{p}\right]$ is the conjugacy class of Artin symbols with respect to an unramified prime p. Then by effective versions of the Chebotarev density theorem from Lagarias and Odlyzko [17, Theorems 1.3 and 1.4], we have
$$\pi_{S_C}(x) = \frac{|C|}{|G|}\frac{x}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right).$$
3. Beatty sequences. Let α be positive and irrational of finite type. Let
$$S_\alpha = \{p \in \mathcal{P} : p = \lfloor \alpha n\rfloor \text{ for some } n \in \mathbb{N}\}.$$
Then by the prime number theorem for Beatty sequences [18, Corollary 5.5], we have
$$\pi_{S_\alpha}(x) = \frac{x}{\alpha\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

4. Piatetski-Shapiro primes. Let $c \geq 1$ be positive and set

$$S_c := \left\{ p \in \mathcal{P} : p = \lfloor n^c \rfloor \text{ for some } n \in \mathbb{N} \right\}.$$

Then by Piatetski-Shapiro's work [19], we have

$$\pi_{S_c}(x) = \frac{x^{1/c}}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right)$$

for $c \in [1, 12/11)$.

5. Primes with preassigned digits. Let $q \geq 2$ be an integer and $A_q = \{0, 1, 2, \ldots, q-1\}$. For integers $n \geq 0$ and $j \geq 0$, let $a_j(n) \in A_q$ be defined by $n = \sum_{j=0}^{\infty} a_j(n) q^j$. Let $b \geq 1$ be an integer with q-ary expansion $b = \sum_{j=0}^{r} b_j q^j$ with $b_0, b_1, \ldots, b_r \in A_q$ and $(b_0, q) = 1$. For a sequence of indexes $1 \leq l_1 < \cdots < l_r$, we take

$$S_b := \left\{ p \in \mathcal{P} : a_0(p) = b_0, a_{l_j}(p) = b_j, \forall\, 1 \leq j \leq r \right\}.$$

Then by [20, Theorem 1], for $q^N < x < q^{N+1}$, $N \geq 1$, $0 \leq r < \sqrt{N}$, $1 \leq l_1 < \cdots < l_r \leq N$, we have

$$\pi_{S_b}(x) = \frac{1}{\varphi(q) q^r} \frac{x}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

2.2 Multiple Mertens evaluations

Suppose $\{a_n\}$ is a sequence related to the Riemann zeta function, that is,

$$a_2 = -\zeta(2), a_3 = 2\zeta(3), a_4 = 3\zeta(2)^2 - 6\zeta(4),$$
$$a_k = \sum_{i=1}^{k-3} (-1)^i C_{k-1}^i i! \zeta(i+1) a_{k-1-i} + (-1)^{k-1}(k-1)!\zeta(k) \quad (k > 4),$$

and $C_k^l = \binom{k}{l}$. Then we set a series of polynomials $\{Q_i(y) : i \geq 0\}$:

$$Q_0(y) = 1, Q_1(y) = y + M,$$
$$Q_k(y) = (y+M)^k + \sum_{m=2}^{k} C_k^m a_m (y+M)^{k-m} \quad (k \geq 2),$$

where M is Mertens' constant.

Now, we state a theorem about multiple Mertens evaluations by Qi and Hu.

Lemma 2.1 *([[6], Theorem 1.1]) For any positive integers k and s, the following evaluation holds*

$$\sum_{p_1 \cdots p_k \leq x} \frac{\log^s(p_1 \cdots p_k)}{p_1 \cdots p_k} = \sum_{l=0}^{k-1} (-1)^l \frac{A_k^{l+1}}{s^{l+1}} Q_{k-1-l}(\log\log x) \cdot \log^s(x) + f(2)\log^{s-1}(2) + O\left(\log^{s-1}(x) \cdot (\log\log x)^k\right),$$

where

$$f(x) = \sum_{l=0}^{k-1} (-1)^l A_k^{l+1} Q_{k-1-l}(\log\log x) \cdot \log x,$$

and the combinatorial number $A_k^l = \binom{k}{l} \cdot l!$.

In (6), one may take $F(X) = \sum_{i=0}^{k-1} (-1)^i A_k^{i+1} Q_{k-1-i}(X)$.

2.3 Some lemmas

Now we establish some lemmas that will be used in the proofs of our main results. In particular, in Lemma 2.2, A and B are two expressions on some variables. For example, we may take $A = p_1 \cdots p_k$, then $p_1, \ldots, p_k$ are variables in this expression. If we take $B = [a^k, b^k]$ or $f_1(n_1) \cdots f_r(n_r)$, then a and b or $n_1, \ldots, n_r$ are variables in B. In our applications, the variables of A and B are easy to see from their explicit expressions related to the summations.

Lemma 2.2 *Let A and B be two expressions. Let $g(A)$ be a nonnegative function on A and $h(B)$ a nonnegative function on B. Suppose that $g(A)$ has the following asymptotic estimate:*

$$\sum_{A\le x} g(A) = x^\gamma P(\log\log x, \log x)\left(1 + O\left(\frac{(\log\log x)^\beta}{\log^\alpha(x)}\right)\right)$$

for some $\alpha > 0$, $\beta \in \mathbb{R}$, and $\gamma \ge 0$. Here $P(x, y) = \sum_{u,v\in\mathbb{R}} c_{u,v} x^u y^v$ is a finite sum of some monomials of two variables. If $h(B)$ satisfies $\sum_B h(B)/B^\gamma < \infty$ with the following decaying rate

$$\sum_{B\ge x} \frac{h(B)}{B^\gamma} \ll x^{-\delta}$$

for some $\delta > 0$, then we have

$$\sum_{AB\le x} g(A)h(B) = \left(\sum_B \frac{h(B)}{B^\gamma}\right) x^\gamma P(\log\log x, \log x)\left(1 + O\left(\frac{\log\log x}{\log x}\right) + O\left(\frac{(\log\log x)^\beta}{\log^\alpha(x)}\right)\right). \quad (17)$$

If $P(\log\log x, \log x)$ has no $\log x$ term, then the error term $O\left(\frac{\log\log x}{\log x}\right)$ in (17) can be replaced by $O\left(\frac{1}{\log x}\right)$.

Proof Let $\ell \ge \max\left\{\delta^{-1}, \delta^{-1}\alpha\right\}$. We break the double summations up into two parts:

$$\begin{aligned}\sum_{AB\le x} g(A)h(B) &= \sum_{B\le\log^\ell(x)} h(B) \sum_{A\le x/B} g(A) + \sum_{B>\log^\ell(x)} h(B) \sum_{A\le x/B} g(A)\\ &:= S_1 + S_2.\end{aligned}$$

In S_1, we have $B \le \log^\ell(x)$, which implies $\log(x/B) = \log x\,(1 + O(\log\log x/\log x))$ and $\log\log(x/B) = \log\log x\,(1 + O(1/\log x))$. Since $\log^v(x/B) = \log^v(x)\left(1 + O_{v,\ell}(\log\log x/\log x)\right)$ and $(\log\log(x/B))^u = (\log\log x)^u\left(1 + O_{u,\ell}(1/\log x)\right)$ for any $u, v \in \mathbb{R}$, it follows that

$$P(\log\log(x/B), \log(x/B)) = P(\log\log x, \log x)\left(1 + O\left(\frac{\log\log x}{\log x}\right)\right). \quad (18)$$

By (18) we obtain that

$$\begin{aligned}S_1 &= \sum_{B\le\log^\ell(x)} h(B)(x/B)^\gamma P(\log\log(x/B), \log(x/B))\left(1 + O\left(\frac{(\log\log(x/B))^\beta}{\log^\alpha(x/B)}\right)\right)\\ &= \left(\sum_{B\le\log^\ell(x)} \frac{h(B)}{B^\gamma}\right) x^\gamma P(\log\log x, \log x)\left(1 + O\left(\frac{\log\log x}{\log x}\right)\right)\left(1 + O\left(\frac{(\log\log x)^\beta}{\log^\alpha(x)}\right)\right)\\ &= \left(\sum_B \frac{h(B)}{B^\gamma} + O\left(\frac{1}{\log^{\delta\ell}(x)}\right)\right) x^\gamma P(\log\log x, \log x)\left(1 + O\left(\frac{\log\log x}{\log x}\right) + O\left(\frac{(\log\log x)^\beta}{\log^\alpha(x)}\right)\right)\\ &= \left(\sum_B \frac{h(B)}{B^\gamma}\right) x^\gamma P(\log\log x, \log x)\left(1 + O\left(\frac{\log\log x}{\log x}\right) + O\left(\frac{(\log\log x)^\beta}{\log^\alpha(x)}\right)\right).\end{aligned}$$

For S_2, we have

$$S_2 \le \sum_{B>\log^{\ell}(x)} h(B) \sum_{A\le x} g(A) \ll \frac{x^{\gamma} P(\log\log x, \log x)}{\log^{\delta\ell}(x)}.$$

Combining the estimates above for S_1 and S_2 completes the proof. □

Next we prove two lemmas on Mertens-type sums.

Lemma 2.3 *Let $r \ge 1$ be an integer. Let $f : \mathbb{N}^r \to \mathbb{R}_{>0}$ be a function satisfying $\#\{(n_1,\dots,n_r) \in \mathbb{N}^r : f(n_1,\dots,n_r) \le a\}$ is finite for any $a \in \mathbb{R}_{>0}$ and*

$$\sum_{f(n_1,\dots,n_r)\ge x} \frac{1}{f(n_1,\dots,n_r)} \ll x^{-\delta}$$

for some $\delta > 0$. Then we have

$$\sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \frac{\log^s(f(n_1,\dots,n_r))}{f(n_1,\dots,n_r)^{1-\eta}} < +\infty$$

for any $0 \le \eta < \delta$ and $s \ge 0$. Furthermore, for any $0 \le \eta < \delta$ and $s \ge 0$, we have

$$\sum_{f(n_1,\dots,n_r)\ge x} \frac{\log^s(f(n_1,\dots,n_r))}{f(n_1,\dots,n_r)^{1-\eta}} \ll x^{\eta-\delta}\log^s(x). \tag{19}$$

Proof By our assumptions on f, it suffices to prove (19). We write $\mathbf{n} = (n_1,\dots,n_r) \in \mathbb{N}^r$. Since $\sum_{f(\mathbf{n})\ge x} 1/f(\mathbf{n}) = O\left(x^{-\delta}\right)$, we have

$$\sum_{x\le f(\mathbf{n})<2x} \frac{1}{f(\mathbf{n})} = O\left(x^{-\delta}\right).$$

This implies that $\#\{\mathbf{n} : x \le f(\mathbf{n}) < 2x\} = O(x^{1-\delta})$ and $\delta \le 1$. Then

$$\begin{aligned}
\sum_{f(n_1,\dots,n_r)\ge x} \frac{\log^s(f(n_1,\dots,n_r))}{f(n_1,\dots,n_r)^{1-\eta}} &= \sum_{k=0}^{\infty} \sum_{2^k x\le f(\mathbf{n})<2^{k+1}x} \frac{\log^s(f(\mathbf{n}))}{f(\mathbf{n})^{1-\eta}} \\
&\le \sum_{k=0}^{\infty} \frac{\log^s(2^{k+1}x)}{(2^k x)^{1-\eta}} \sum_{2^k x\le f(\mathbf{n})<2^{k+1}x} 1 \\
&\ll \sum_{k=0}^{\infty} \frac{\log^s(2^{k+1}x)}{(2^k x)^{1-\eta}} \cdot (2^k x)^{1-\delta} \\
&= x^{\eta-\delta} \sum_{k=0}^{\infty} \frac{\log^s(2^{k+1}x)}{2^{k(\delta-\eta)}} \\
&\ll x^{\eta-\delta} \sum_{k=0}^{\infty} \frac{\left(\log(2^{k+1})\log x\right)^s}{2^{k(\delta-\eta)}} \\
&\ll x^{\eta-\delta}\log^s(x),
\end{aligned}$$

which completes the proof. □

Lemma 2.4 *Let $f_i : \mathbb{N} \to \mathbb{R}_{\ge 1}$ be an increasing positive sequence satisfying $\sum_{f_i(n)\ge x} 1/f_i(n) \ll x^{-\delta_i}$ for some $\delta_i > 0$, $1 \le i \le r$. Then we have*

$$\sum_{f_1(n_1)\cdots f_r(n_r)\ge x} \frac{1}{f_1(n_1)\cdots f_r(n_r)} \ll x^{-\delta} \tag{20}$$

for some $\delta > 0$.

Proof Indeed, notice that $\sum_{n=1}^{\infty} 1/f_i(n) < \infty$ for each $1 \le i \le r$. We have

$$\begin{aligned}
&\sum_{f_1(n_1)\cdots f_r(n_r)\ge x^r} \frac{1}{f_1(n_1)\cdots f_r(n_r)} \\
&= \sum_{\substack{f_1(n_1)\cdots f_r(n_r)\ge x^r \\ f_1(n_1)\cdots f_{r-1}(n_{r-1})\ge x^{r-1}}} \frac{1}{f_1(n_1)\cdots f_r(n_r)} + \sum_{\substack{f_1(n_1)\cdots f_r(n_r)\ge x^r \\ f_1(n_1)\cdots f_{r-1}(n_{r-1})< x^{r-1}}} \frac{1}{f_1(n_1)\cdots f_r(n_r)} \\
&\le \sum_{f_1(n_1)\cdots f_{r-1}(n_{r-1})\ge x^{r-1}} \frac{1}{f_1(n_1)\cdots f_r(n_r)} + \sum_{f_r(n_r)\ge x} \frac{1}{f_1(n_1)\cdots f_r(n_r)} \\
&= \left(\sum_{f_1(n_1)\cdots f_{r-1}(n_{r-1})\ge x^{r-1}} \frac{1}{f_1(n_1)\cdots f_{r-1}(n_{r-1})}\right)\left(\sum_{n_r} \frac{1}{f_r(n_r)}\right) \\
&\quad + \left(\sum_{n_1,\ldots,n_{r-1}} \frac{1}{f_1(n_1)\cdots f_{r-1}(n_{r-1})}\right)\left(\sum_{f_r(n_r)\ge x} \frac{1}{f_r(n_r)}\right) \\
&\ll \sum_{f_1(n_1)\cdots f_{r-1}(n_{r-1})\ge x^{r-1}} \frac{1}{f_1(n_1)\cdots f_{r-1}(n_{r-1})} + x^{-\delta_r} \\
&\ll x^{-\delta_1} + \cdots + x^{-\delta_r},
\end{aligned}$$

where the last line above holds by induction on r. Thus, we may take $\delta = r^{-1}\min\{\delta_1, \ldots, \delta_r\}$ for (20). This completes the proof. □

The following two lemmas will be applied in the study of the average and variance of $\omega_k(x)$.

Lemma 2.5 *Let $k \ge 2$ be an integer. Then we have*

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty} \frac{1}{[a^k, b^k]} = \frac{\zeta^3(k)}{\zeta(2k)}. \tag{21}$$

Proof Set $d = (a, b)$, $a = da'$, $b = db'$, then we get $[a^k, b^k] = d^k a'^k b'^k$. It follows that

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty} \frac{1}{[a^k, b^k]} = \sum_{d=1}^{\infty}\sum_{\substack{a'=1 \\ (a',b')=1}}^{\infty}\sum_{b'=1}^{\infty} \frac{1}{d^k a'^k b'^k} = \zeta(k)\sum_{\substack{a'=1 \\ (a',b')=1}}^{\infty}\sum_{b'=1}^{\infty} \frac{1}{a'^k b'^k}. \tag{22}$$

Now, we compute the square of $\zeta(k)$ as follows:

$$\zeta^2(k) = \sum_{a=1}^{\infty}\sum_{b=1}^{\infty} \frac{1}{a^k b^k} = \sum_{d=1}^{\infty}\sum_{\substack{a'=1 \\ (a',b')=1}}^{\infty}\sum_{b'=1}^{\infty} \frac{1}{d^{2k} a'^k b'^k} = \zeta(2k)\sum_{\substack{a'=1 \\ (a',b')=1}}^{\infty}\sum_{b'=1}^{\infty} \frac{1}{a'^k b'^k}.$$

This implies that

$$\sum_{\substack{a'=1 \\ (a',b')=1}}^{\infty}\sum_{b'=1}^{\infty} \frac{1}{a'^k b'^k} = \frac{\zeta^2(k)}{\zeta(2k)}. \tag{23}$$

Then (21) follows by plugging (23) into (22). □

For the tail of the double series in (21) we have the following bound.

Lemma 2.6 *For any $\varepsilon > 0$, we have*

$$\sum_{[a^k,b^k]\ge x} \frac{1}{[a^k, b^k]} = O\left(x^{-1+\frac{1}{k}+\varepsilon}\right).$$

Proof By [21, (1.81)], we have $\tau_3(n) \ll n^{\varepsilon}$. Then we have

$$\sum_{[a^k,b^k]\geq x} \frac{1}{[a^k,b^k]} = \sum_{l^k\geq x,[a^k,b^k]=l^k} \frac{1}{l^k} \ll \sum_{l^k\geq x} \frac{\tau_3(l)}{l^k} \ll \sum_{l\geq x^{1/k}} \frac{l^{\varepsilon}}{l^k} \ll x^{-1+\frac{1}{k}+\varepsilon}.$$

□

3 Proofs of Theorem 1.1 and Corollary 2

In this section we prove Theorem 1.1 and Corollary 2. Indeed, if we take $A = p$, $B = f(n)$, $g(A) = 1_S(p)$, and $h(B) = 1$ in Lemma 2.2, where 1_S is the indicator function on S, then Theorem 1.1 follows immediately. Therefore, it suffices to prove Corollary 2, in which we apply Theorem 1.1.

Proof of Corollary 2 Using the fact that primes are asymptotically equi-distributed in the reduced residues mod 10, we calculate the asymptotic estimates for CP Numbers ending in 1 as follows. We have the following four cases: when $p \equiv 1 \pmod{10}$, we require a^3 to end in 1, thus a is congruent to 1 modulo 10; when $p \equiv 3 \pmod{10}$, we require a^3 to end in 7, thus a is congruent to 3 modulo 10; when $p \equiv 7 \pmod{10}$, we require a^3 to end in 3, thus a is congruent to 7 modulo 10; and finally when $p \equiv 9 \pmod{10}$, we require a^3 to end in 9, thus a is congruent to 9 modulo 10. From the definition of CP Numbers, we know that $a > 1$. Therefore, the number of pairs (p, a) such that $pa^3 \equiv 1 \pmod{10}$ with $pa^3 \leq x$ is

$$\begin{aligned}
&\#\left\{(p,k) : p \equiv 1 \pmod{10},\ p(10k+1)^3 \leq x,\ k \geq 1\right\} \\
&+\#\left\{(p,k) : p \equiv 3 \pmod{10},\ p(10k+3)^3 \leq x,\ k \geq 0\right\} \\
&+\#\left\{(p,k) : p \equiv 7 \pmod{10},\ p(10k+7)^3 \leq x,\ k \geq 0\right\} \\
&+\#\left\{(p,k) : p \equiv 9 \pmod{10},\ p(10k+9)^3 \leq x,\ k \geq 0\right\}.
\end{aligned}$$

Then from Theorem 1.1, we have

$$\begin{aligned}
&\#\left\{(p,k) : p \equiv 1 \pmod{10},\ p(10k+1)^3 \leq x,\ k \geq 1\right\} \\
&\sim \frac{x}{4\log x}\sum_{k=1}^{\infty}\frac{1}{(10k+1)^3} = \frac{x}{4000\log x}\left(\zeta\left(3,\frac{1}{10}\right) - 1000\right).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\#\left\{(p,k) : p \equiv 3 \pmod{10},\ p(10k+3)^3 \leq x,\ k \geq 0\right\} &\sim \frac{x}{4000\log x}\zeta\left(3,\frac{3}{10}\right), \\
\#\left\{(p,k) : p \equiv 7 \pmod{10},\ p(10k+7)^3 \leq x,\ k \geq 0\right\} &\sim \frac{x}{4000\log x}\zeta\left(3,\frac{7}{10}\right),
\end{aligned}$$

and

$$\#\left\{(p,k) : p \equiv 9 \pmod{10},\ p(10k+9)^3 \leq x,\ k \geq 0\right\} \sim \frac{x}{4000\log x}\zeta\left(3,\frac{9}{10}\right).$$

Hence the asymptotic formula (2) holds by adding these four asymptotic estimates up. □

4 Proofs of Theorem 1.2 and Equation (14)

In this section we prove Theorem 1.2 and Equation (14), in which we frequently apply Lemma 2.2.

4.1 Proof of Theorem 1.2

First, we prove (8). We write the partial sum in (8) as two parts:

$$\begin{aligned}
&\sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x} \frac{\log(p_1\cdots p_k f_1(n_1)\cdots f_r(n_r))}{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)}\\
&= \sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x} \frac{\log(p_1\cdots p_k)}{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)}\\
&\quad + \sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x} \frac{\log(f_1(n_1)\cdots f_r(n_r))}{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)}\\
&:= S_3 + S_4.
\end{aligned}$$

For S_3, we take $A = p_1p_2\cdots p_k$, $B = f_1(n_1)f_2(n_2)\cdots f_r(n_r)$, $g(A) = (\log A)/A$, and $h(B) = 1/B$ in Lemma 2.2. Combining with (6) and Lemma 2.4, we have

$$S_3 = \left(\prod_{i=1}^{r}\sum_{n=1}^{\infty}\frac{1}{f_i(n)}\right)\cdot F(\log\log x)\cdot\log x + O\left((\log\log x)^k\right). \tag{24}$$

For S_4, we take $A = p_1p_2\cdots p_k$, $B = f_1(n_1)f_2(n_2)\cdots f_r(n_r)$, $g(A) = 1/A$, and $h(B) = (\log B)/B$ in Lemma 2.2. Combining with 7, Lemma 2.3, and Lemma 2.4, we get

$$\begin{aligned}
S_4 &= \sum_{i=1}^{r}\sum_{n_i=1}^{\infty}\frac{\log(f_1(n_1)\cdots f_r(n_r))}{f_1(n_1)f_2(n_2)\cdots f_r(n_r)}\cdot P_k(\log\log x)\left(1+O\left(\frac{1}{\log\log x\cdot\log x}\right)\right)\\
&= O\left((\log\log x)^k\right).
\end{aligned} \tag{25}$$

Combining (24) and (25) together gives the desired formula (8).

Similarly, if we take $A = p_1\cdots p_k$, $B = f_1(n_1)\cdots f_r(n_r)$, $g(A) = 1/A$, and $h(B) = 1/B$ in Lemma 2.2, then by Lemma 2.4 we immediately obtain (9).

Finally, we use (9) to prove (10). Let

$$V_{k,r}(x) := \prod_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x}\left(1-\frac{1}{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)}\right).$$

After taking logarithm, we have

$$\begin{aligned}
-\log V_{k,r}(x) &= \sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x} -\log\left(1-\frac{1}{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x}\right)\\
&= \sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x}\sum_{t\ge1}\frac{1}{t(p_1\cdots p_k f_1(n_1)\cdots f_r(n_r))^t}\\
&= \sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x}\frac{1}{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)}\\
&\quad + \sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)\le x}\sum_{t\ge2}\frac{1}{t(p_1\cdots p_k f_1(n_1)\cdots f_r(n_r))^t}.
\end{aligned}$$

By our assumptions on the functions f_i, $1 \le i \le r$, the tail in the second sum satisfies

$$\begin{aligned}
&\sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)>x} \sum_{t\ge 2} \frac{1}{t(p_1\cdots p_k f_1(n_1)\cdots f_r(n_r))^t} \\
&\ll \sum_{p_1\cdots p_k f_1(n_1)\cdots f_r(n_r)>x} \frac{1}{(p_1\cdots p_k f_1(n_1)\cdots f_r(n_r))^2} \\
&\ll \sum_{p_1\ge x^{\frac{1}{2}}} \frac{1}{p_1^2} + \sum_{p_2\cdots p_k f_1(n_1)\cdots f_r(n_r)^2\ge x^{\frac{1}{2}}} \frac{1}{(p_1\cdots p_k f_1(n)\cdots f_r(n_r))^2} \\
&\ll \cdots\cdots \\
&\ll \sum_{i=1}^{k} \sum_{p_i\ge x^{\frac{1}{2^i}}} \frac{1}{p_i^2} + \sum_{j=1}^{r} \sum_{f_j(n_j)\ge x^{\frac{1}{2^{k+j}}}} \frac{1}{(f_j(n_j))^2} \\
&\ll x^{-\varepsilon}
\end{aligned}$$

for some constant $\varepsilon > 0$. Therefore, by (9) we obtain

$$-\log V_{k,r}(x) = \left(\prod_{i=1}^{r}\sum_{n=1}^{\infty}\frac{1}{f_i(n)}\right) P_k(\log\log x) - c(k, f_1, \cdots, f_r) + O\left(\frac{(\log\log x)^k}{\log x}\right)$$

for some constant $c(k, f_1, \cdots, f_r)$ that depends on k and the functions f_i, $1 \le i \le r$. Therefore, we know

$$V_{k,r}(x) = e^{-\left(\prod_{i=1}^{r}\sum_{n=1}^{\infty}\frac{1}{f_i(n)}\right)P_k(\log\log x)+c(k,f_1,\cdots,f_r)}\left(1 + O\left(\frac{(\log\log x)^k}{\log x}\right)\right).$$

We have finished the proof of Theorem 1.2.

4.2 Proof of Equation (14)

The proof of Equation (14) is similar to that of (8). We write

$$\sum_{pn^s\le x}\frac{\log(pn^s)}{pn^s} = \sum_{pn^s\le x}\frac{\log p}{pn^s} + \sum_{pn^s\le x}\frac{\log(n^s)}{pn^s} := S_5 + S_6.$$

For S_5, we take $A = p$, $B = n^s$, $g(A) = (\log A)/A$, and $h(B) = 1/B$ in Lemma 2.2. Combining with Mertens' first theorem, we obtain

$$S_5 = \zeta(s)\log x + O(1). \tag{26}$$

For S_6, we take $A = p$, $B = n^s$, $g(A) = 1/A$, and $h(B) = (\log B)/B$. Notice that

$$\sum_{n=1}^{\infty}\frac{\log(n^s)}{n^s} = -s\zeta'(s).$$

From Mertens' second theorem, similar to the argument of Lemma 2.2, we obtain

$$S_6 = -s\zeta'(s)\log\log x + O(1). \tag{27}$$

Combining (26) and (27), we obtain

$$\sum_{pn^s\le x}\frac{\log(pn^s)}{pn^s} = \zeta(s)\log x - s\zeta'(s)\log\log x + O(1),$$

which completes the proof of Equation (14).

5 Proof of Theorem 1.3

In this section we prove Theorem 1.3 by applying (12) in Corollary 3.

5.1 Proof of Equation (15)

By definition, we have $\omega_k(n) = \sum_{pm^k|n} 1$. Then

$$\begin{aligned}\sum_{n\le x}\omega_k(n) &= \sum_{n\le x}\sum_{pm^k|n} 1\\ &= \sum_{pm^k\le x}\sum_{n\le x/pm^k} 1\\ &= \sum_{pm^k\le x}\left\lfloor\frac{x}{pm^k}\right\rfloor\\ &= x\sum_{pm^k\le x}\frac{1}{pm^k} + O\left(\sum_{pm^k\le x} 1\right)\end{aligned} \tag{28}$$

On the one hand, from (12) we know

$$\sum_{pm^k\le x}\frac{1}{pm^k} = \zeta(k)\,(\log\log x + M) + O\left(\frac{\log\log x}{\log x}\right). \tag{29}$$

On the other hand, by Theorem 1.1 we have

$$\sum_{pm^k\le x} 1 \sim \zeta(k)\frac{x}{\log x}. \tag{30}$$

Combining (28), (29), and (30), we get

$$\sum_{n\le x}\omega_k(n) = \zeta(k)\log\log x + \zeta(k)Mx + O\left(\frac{x\log\log x}{\log x}\right),$$

which completes the proof of Equation (15).

5.2 Proof of Equation (16)

We first compute the second moment of $\omega_k(n)$. It is easy to see

$$\begin{aligned}\sum_{n\le x}\omega_k^2(n) &= \sum_{n\le x}\sum_{pa^k|n}\sum_{qb^k|n} 1\\ &= \sum_{pa^k\le x}\sum_{qb^k\le x}\sum_{\substack{n\le x\\ pa^k|n,qb^k|n}} 1\\ &= \sum_{pa^k\le x}\sum_{qb^k\le x}\left\lfloor\frac{x}{[pa^k,qb^k]}\right\rfloor\\ &= x\sum_{[pa^k,qb^k]\le x}\frac{1}{[pa^k,qb^k]} + O\left(\sum_{[pa^k,qb^k]\le x} 1\right)\\ &:= x\cdot S_7 + S_8.\end{aligned}$$

For the sum S_7, we will prove

$$S_7 = \frac{\zeta^3(k)}{\zeta(2k)} (\log\log x)^2 + O(\log\log x). \tag{31}$$

Lemma 2.5 and Mertens' second theorem imply

$$\begin{aligned}\sum_{p=q,[pa^k,qb^k]\le x} \frac{1}{[pa^k, qb^k]} &= \sum_{p[a^k,b^k]\le x} \frac{1}{p[a^k,b^k]}\\ &\le \sum_{[a^k,b^k]\le x} \frac{1}{[a^k,b^k]} \sum_{p\le x} \frac{1}{p}\\ &\ll \sum_{p\le x} \frac{1}{p}\\ &\ll \log\log x.\end{aligned}$$

So we only need to consider the terms with $p \neq q$. In the sum

$$S_7' = \sum_{p\neq q,[pa^k,qb^k]\le x} \frac{1}{[pa^k, qb^k]},$$

we let $a = p^{e_1}q^{e_2}a_1$ and $b = p^{f_1}q^{f_2}b_1$, where $(a_1, pq) = (b_1, pq) = 1$. Then $[pa^k, qb^k] = p^g q^h [a_1^k, b_1^k]$, where $g = \max\{ke_1 + 1, kf_1\} \ge 1$ and $h = \max\{ke_2, kf_2 + 1\} \ge 1$. Next we discuss the contributions of the terms according to their e_1, e_2, f_1, and f_2 values. If $e_1 \ge 1$, then $[pa^k, qb^k] \ge p^3 q[a_1^k, b_1^k]$, and thus

$$\sum_{p\neq q,[pa^k,qb^k]\le x,e_1\ge 1} \frac{1}{[pa^k, qb^k]} \ll \sum_{q\le x} \frac{1}{q} \sum_{p=1}^{\infty} \frac{1}{p^3} \sum_{a_1=1}^{\infty} \sum_{b_1=1}^{\infty} \frac{1}{[a_1^k, b_1^k]} \ll \log\log x. \tag{32}$$

If $f_1 \ge 1$, then $[pa^k, qb^k] \ge p^2 q[a_1^k, b_1^k]$, which implies

$$\sum_{p\neq q,[pa^k,qb^k]\le x,f_1\ge 1} \frac{1}{[pa^k, qb^k]} \ll \sum_{q\le x} \frac{1}{q} \sum_{p=1}^{\infty} \frac{1}{p^2} \sum_{a_1=1}^{\infty} \sum_{b_1=1}^{\infty} \frac{1}{[a_1^k, b_1^k]} \ll \log\log x. \tag{33}$$

Similarly, for the terms with $e_2 \ge 1$ or $f_2 \ge 1$, we also have

$$\sum_{\substack{[pa^k,qb^k]\le x\\ p\neq q, e_2\ge 1 \text{ or } f_2\ge 1}} \frac{1}{[pa^k, qb^k]} = O(\log\log x) \tag{34}$$

Noting that $e_1 = e_2 = f_1 = f_2 = 0$ implies $[pa^k, qb^k] = pq[a_1^k, b_1^k]$, we can combine the estimates (32), (33), and (34) to obtain

$$S_7' = \sum_{\substack{pq[a^k,b^k]\le x\\ p\neq q,(a,pq)=(b,pq)=1}} \frac{1}{pq[a^k, b^k]} + O(\log\log x).$$

In fact, we can remove the constrains $p \neq q$ and $(a, pq) = (b, pq) = 1$ in S_7' since

$$\begin{aligned}\sum_{pq[a^k,b^k]\le x,p=q} \frac{1}{pq[a^k, b^k]} &= \sum_{p^2[a^k,b^k]\le x} \frac{1}{p^2[a^k,b^k]}\\ &\le \sum_{p\le x} \frac{1}{p^2} \sum_{[a^k,b^k]\le x} \frac{1}{[a^k,b^k]}\\ &< +\infty,\end{aligned}$$

$$\sum_{\substack{pq[a^k,b^k]\le x,\, p\ne q\\ p|a,\, p|b}} \frac{1}{pq[a^k,b^k]} = \sum_{p^{k+1}q[a^k/p^k,b^k/p^k]\le x} \frac{1}{p^{k+1}q[a^k/p^k,b^k/p^k]}$$
$$\le \sum_{p\le x}\frac{1}{p^{k+1}} \sum_{[a^k,b^k]\le x}\frac{1}{[a^k,b^k]}\sum_{q\le x}\frac{1}{q}$$
$$\ll \log\log x,$$

and

$$\sum_{\substack{pq[a^k,b^k]\le x,\, p\ne q\\ p|a,\, p\nmid b}} \frac{1}{pq[a^k,b^k]} = \sum_{\substack{p^{k+1}q[a^k/p^k,b^k]\le x,\, p\ne q\\ p|a,\, p\nmid b}} \frac{1}{p^{k+1}q[a^k/p^k,b^k]} \ll \log\log x.$$

Therefore, we have proved

$$S_7 = \sum_{pq[a^k,b^k]\le x} \frac{1}{pq[a^k,b^k]} + O\left(\log\log x\right).$$

Now Equation (31) holds when we apply Lemma 2.2 with $A = pq$, $B = [a^k, b^k]$, $g(A) = 1/A$, and $h(B) = 1/B$, Lemma 2.5, and Lemma 2.6.

By similar arguments we can prove

$$S_8 = O\left(x\log\log x\right). \tag{35}$$

From (31) and (35) we know

$$\sum_{n\le x}\omega_k^2(n) = \frac{\zeta^3(k)}{\zeta(2k)}x\left(\log\log x\right)^2 + O\left(x\log\log x\right). \tag{36}$$

Combining (36) and (15), we obtain

$$\begin{aligned}
&\sum_{n\le x}\left(\omega_k(n) - \zeta(k)\log\log x\right)^2\\
&= \sum_{n\le x}\omega_k^2(n) - 2\zeta(k)\log\log x\cdot\sum_{n\le x}\omega_k(n) + \zeta^2(k)x\left(\log\log x\right)^2\\
&= \frac{\zeta^3(k)}{\zeta(2k)}x\left(\log\log x\right)^2 - 2\zeta^2(k)x\left(\log\log x\right)^2 + \zeta^2(k)x\left(\log\log x\right)^2 + O\left(x\log\log x\right)\\
&= \frac{\zeta^3(k) - \zeta^2(k)\zeta(2k)}{\zeta(2k)}x\left(\log\log x\right)^2 + O\left(x\log\log x\right),
\end{aligned}$$

which completes the proof of Equation (16).

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