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ORIGINAL RESEARCH

An efficient algorithm for evaluation of derivatives of refinable scale functions

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Abstract We present here an efficient algorithm for computing derivatives of refinable functions of compact support appearing in the bases of multiscale approximation of functions in $L^2(\mathbb{R})$. The scheme presented here provides values of admissible derivatives of refinable functions at dyadic rationals within their support. Results are demonstrated for the admissible derivatives of a variety of refinable functions often used for obtaining highly accurate approximate solution of physical problems.

Keywords Derivatives of refinable functions · Daubechies wavelets · Coiflets · Interpolating wavelets

Mathematics Subject Classification 42A45 · 15A18

1 Introduction

In the field of numerical analysis, wavelets are getting increasing attention in recent years [1–5]. The underlying theory of wavelet-based numerical scheme is multiresolution analysis (MRA) of function space which comprises basis generated by refinable functions and wavelets of compact support [6]. It can be regarded as a useful and efficient basis to obtain highly accurate approximate solutions of various problems arising in many areas of applied mathematics [7], physics [8] and engineering [9]. This can be attributed due to its elegant role of mathematical microscopy in the analysis of smoothness or regularity of a function.

While many applications (e.g. Galerkin approximation) use only the filter coefficients of the refinable scale functions of MRA, some (e.g. collocation scheme) require the values of the basis functions and their derivatives explicitly at different points within their support. For example, the bound state solution of non-relativistic (Schrödinger equation)/relativistic (Dirac equation) quantum mechanical problems (involving non-algebraic interaction potential function, in particular) can be obtained efficiently with the aid of interpolating wavelet collocation scheme. So, a straightforward efficient scheme for getting values of refinable functions and its derivatives at collocation points (dyadic rationals preferably) is highly desirable.

Lin et al. [10] presented an algorithm for computation of values of refinable functions in Daubechies family only and their admissible derivatives by solving a system of linear equations for a particular derivative of the basis

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functions. Here, we present an efficient algorithm for computing values of various refinable functions (Daubechies family [11], Coiflet family [12] and interpolating scale functions [13,14]) of compact support and their admissible derivatives appearing in the basis of multiscale approximation of functions in $L^2(\mathbb{R})$. The scheme proposed here finds the values of the refinable functions and their admissible derivatives at dyadic rationals within their support in the form of eigenvectors through direct solution a matrix eigenvalue problem. The results presented here may be useful for obtaining approximate solution of several problems of physical interest.

The work presented here is organized as follows. The algorithm for the evaluation of the derivatives of refinable functions of compact support at dyadic rationals has been presented in Sect. 2. Results obtained by the proposed scheme have been tested through some examples in Sect. 3. Some concluding remarks have been made in Sect. 4.

2 Algorithm

We begin with the refinable function $\phi(x)$ of compact support $[ML, MU]$, which satisfies the refinement equation

$$\phi(x) = \Lambda \sum_{l=ML}^{MU} h_l \, \phi(2x - l) \tag{1}$$

where $\Lambda = \sqrt{2}$ for ϕ in Daubechies wavelets or Coiflets family and $\Lambda = 1$ for interpolating scale functions. Certainly, ϕ covers the generators of wavelet basis in Daubechies family (support $ML = 0,\ MU = 2K - 1$), nearly interpolating wavelet family (Coiflets) (support $ML = -2K,\ MU = 4K - 1$), interpolating wavelet family (autocorrelation function generated by scale functions and wavelets in Daubechies family) (support $ML = -2K + 1,\ MU = 2K - 1$) and so on.

One of the most important properties of these refinable functions is that the rule of correspondence $x \to \phi(x)$ is not known explicitly. Instead, their values at any dyadic rational within their support may be obtained by regarding the refinable function ϕ as a fixed point of the functional relation (1). Consequently, finding values of ϕ and its admissible derivatives may be treated as finding a fixed point of the equation

$$\phi^{(p)}(x) = \Lambda \, 2^p \sum_{l=ML}^{MU} h_l \, \phi^{(p)}(2x - l) \tag{2}$$

where $\phi^{(p)}(x)$ denotes the p^{th} order derivative of $\phi(x)$. Due to the finite support of ϕ, evaluation of this relation at integer nodes $x_k = ML + k,\ k = 0, 1, \cdots, MU - ML$ leads to an eigenvalue problem

$$\mathbf{C} \begin{pmatrix} \phi^{(p)}(ML) \\ \phi^{(p)}(ML+1) \\ \vdots \\ \phi^{(p)}(MU-1) \\ \phi^{(p)}(MU) \end{pmatrix} = \lambda \begin{pmatrix} \phi^{(p)}(ML) \\ \phi^{(p)}(ML+1) \\ \vdots \\ \phi^{(p)}(MU-1) \\ \phi^{(p)}(MU) \end{pmatrix}. \tag{3}$$

Here,

$$\mathbf{C} = \{h_{k_2 - 2k_1},\ k_1, k_2 \in \{ML, ML+1, \cdots, MU\}\}. \tag{4}$$

The admissible eigenvalues for the determination of $\phi^{(p)}(x_k),\ x_k \in \{ML, ML+1, \cdots MU\}$ are

$$\lambda = \begin{cases} \frac{1}{2^{p+\frac{1}{2}}}, & \text{in case of scale functions in Daubechies family and Coiflets} \\ \frac{1}{2^p}, & \text{in case of autocorrelation scale functions.} \end{cases} \tag{5}$$

The values of $\phi^{(p)}(x_k)$ at integers $x_k \in \{ML, ML+1, \cdots, MU\}$ within their support for different order of derivatives p are obtained by multiplying the corresponding eigenvector v with an appropriate normalization constant which can be determined with the aid of formula (reproduction property of polynomials in the basis generated by the refinable functions)

$$\frac{1}{p!}\sum_{k=ML}^{MU} k^p \phi^{(p)}(-k) = 1 \tag{6}$$

for refinable functions in interpolating wavelet family. In case of refinable functions of Daubechies wavelet family or Coiflets, the equation for normalization constant is given by

$$\frac{1}{p!}\sum_{k=ML}^{MU} < x^p >_{\phi_k} \phi^{(p)}(-k) = 1 \tag{7}$$

where $< x^p >_{\phi_k}$ denotes $\int_{-\infty}^{\infty} x^p \phi_k(x)dx$.

It is informative to mention here that the smoothness of refinable functions in all three classes mentioned above is not so high as the number of eigenvalues (order of the matrix **C**). Consequently, all the eigenvectors may not correspond to the values of derivatives $\phi^{(p)}(x)$ at integers within their support. So, a searching process is involved in finding the eigenvector that corresponds to the value of the required derivatives. This may be attributed by the fact that the eigenvectors corresponding to the eigenvalues mentioned in (5) will provide the desired values of ϕ and its derivatives at integers within its support.

It seems worthy to mention here that this scheme provides values of ϕ and its admissible derivatives in a single mathematical operation instead of solving equations (2.3a) and (2.4) of Lin et al. [10] separately for each order of derivative p.

For the straightforward use of the scheme for interested users the program for the algorithm mentioned above may be available on request to the corresponding author of the article.

3 Illustrative examples

To test the scheme and for the use of the values of derivatives of refinable functions in all three families mentioned earlier, we present $\phi^{(p)}(x_k),\ x_k \in \{ML, ML+1, \cdots, MU\}$ for $p = 1, 2$ in Tables 1, 2, 3. Data presented in these tables may be used in the formula (2) to obtain the values of $\phi^{(p)}(x)$ at any dyadic rational within its support and their natures have been depicted in Figs. 1, 2, 3.

Due to the finite support of the refinable function ϕ, the representation of derivative ($p = 1, 2$) in any finite basis $\mathfrak{B}_j = \{\phi_{jk},\ k \in \Lambda_j(\text{index set})\}$ appears to be band diagonal as is evident from Figs. 4, 5, 6.

We now use the data of Tables 1, 2, 3 to test the efficiency of the proposed scheme by the evaluation of $f^{(p)}(x),\ p = 1, 2$ for any differentiable function $f(x)$ of admissible order with the help of the formula

$$f_j^{\text{Approx}(p)} = 2^{j\left(p+\frac{1}{2}\right)} \sum_{k\in\mathbb{Z}} < f(x) >_{\phi_{jk}} \phi^{(p)}(2^j x - k). \tag{8}$$

Here,

$$< f(x) >_{\phi_{jk}} = \int_{-\infty}^{\infty} f(x)\phi_{jk}(x)dx = \sum_{i=1}^{N} f\left(\frac{k+x_i}{2^j}\right) w_i \tag{9}$$

in the above expression provides the average value of $f(x)$ in the domain $\left[\frac{k-K+1}{2^j}, \frac{k+K}{2^j}\right]$ around the point $\frac{k}{2^j}$. Notations $x_i,\ w_i$ appearing in the quadrature rule are nodes and weights respectively of Gauss-Daubechies quadrature rules for the integral containing refinable function ϕ. The determination of these nodes and weights has been discussed in detail in reference [15].

Values of $f^{(p)}(x)$ obtained by using the formula (8) for $f(x) \in P_{K-1}(x)$ (for $\phi \in$ Sym K family) and $f(x) \in P_{2K-1}(x)$ (for ϕ in Coiflets or autocorrelation family) are found to be exact as expected, where $P_n(x)$ denotes the set of polynomials of degree $\leq n$. The data presented in Tables 1, 2, 3 is then applied to evaluate the values of $f^{(p)}(x),\ p = 1, 2$ for $f(x) \in P_n(x)$ ($n \geq K$ for SymK family and $n \geq 2K$ for CoifK or IntK family) and variation of errors with resolution j has been presented in Fig. 7a. It is observed that the accuracy of values obtained by using the data presented in the tables improves rapidly with an increase in the resolution (j). The same feature has been observed in case of the non-algebraic function $f(x) = e^{-x^2} \in L^2(\mathbb{R})$ as is evident from the Fig. 7b.

Table 1 $\phi^{(p)}(x)$ for Sym $K4$

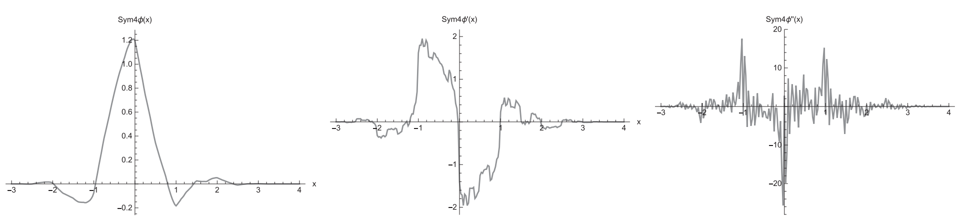
p	x			
	-3	-2	-1	0
1	0	$-\frac{1784132724639543}{5582138211102337}$	$\frac{24251818457948813}{15002979594784058}$	$-\frac{11984410407499897}{8516380964627954}$
p	x			
	1	2	3	4
1	$\frac{1596615487468447}{6836790259869942}$	$-\frac{860816142871031}{7604107339706451}$	$-\frac{214875402584725}{21568750058015097}$	0
p	x			
	-3	-2	-1	0
2	0	$\frac{15273960668436815}{17706598217739272}$	$-\frac{18807122134191627}{8002821377591870}$	$\frac{7381263635838980}{2646907163789751}$
p	x			
	1	2	3	4
2	$-\frac{5003649478674959}{2645077797311538}$	$\frac{8102831628317537}{16056953765294955}$	$\frac{1752413557236487}{20409447631794338}$	0

Table 2 $\phi^{(p)}(x)$ for Coif $K3$

p	x			
	-6	-5	-4	-3
1	0	$-\frac{237178935983}{15202904548398488}$	$\frac{18033051164891}{12681567013815645}$	$\frac{69196002004705}{12669455263648432}$
p	x			
	-2	-1	0	1
1	$-\frac{1619928600875452}{14429877767449201}$	$\frac{1988839493953891}{2920999407919733}$	$\frac{2101291624043483}{28197438277780851}$	$-\frac{1359712782525186}{1703584291307561}$
p	x			
	2	3	4	5
1	$\frac{366833306973491}{2171013565825122}$	$-\frac{331978820727228}{14881641606171193}$	$\frac{8022368033510}{4077171009666433}$	$-\frac{2708288075789}{5326501445849107}$
p	x			
	6	7	8	9
1	$\frac{145106628917}{6012737769630622}$	$\frac{15483699557}{13076759434911201}$	$\frac{309734144}{6643328096935315}$	$-\frac{1427341}{11382143985862942}$
p	x			
	10	11		
1	$\frac{145}{11817708573612093}$	0		
p	x			
	-6	-5	-4	-3
2	0	$-\frac{30163793175700}{11869649883528549}$	$\frac{1451374723354047}{12820398695731745}$	$\frac{1066581639280339}{3804126295482770}$
p	x			
	-2	-1	0	1
2	$-\frac{27243422389867633}{6886050978539917}$	$\frac{105983960130838991}{8366893229548656}$	$-\frac{78179965053189751}{4227291651705915}$	$\frac{216649608422337149}{16053949954154246}$
p	x			
	2	3	4	5
2	$-\frac{61932453349943718}{12864343097999509}$	$\frac{3823878710485623}{5643320493699589}$	$\frac{394251096409895}{11556064956540483}$	$-\frac{19031184964610}{14487653885056271}$
p	x			
	6	7	8	9
2	$\frac{18664162007545}{15077857845292086}$	$-\frac{1807942285581}{12924035715859411}$	$-\frac{41885295403}{68311031431030442}$	$\frac{223730045}{8077100498646822}$
p	x			
	10	11		
2	$-\frac{144371}{26640185997348554}$	0		

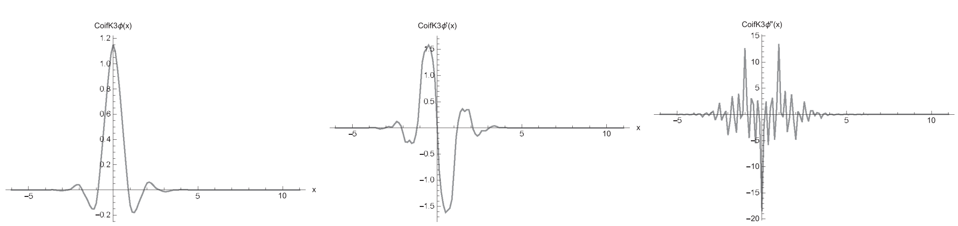
Table 3 $\phi^{(p)}(x)$ for interpolating wavelets for $K = 5$

p	x				
	-9	-8	-7	-6	-5
1	0	$\frac{5}{18545664272}$	$\frac{2048}{8113728119}$	$\frac{563818}{10431936153}$	$\frac{1386496}{5795520085}$
p	x				
	-4	-3	-2	-1	0
1	$-\frac{17297069}{2318208034}$	$\frac{735232}{13780629}$	$-\frac{265226398}{1159104017}$	$\frac{957310976}{1159104017}$	0
p	x				
	1	2	3	4	5
1	$-\frac{957310976}{1159104017}$	$\frac{265226398}{1159104017}$	$-\frac{735232}{13780629}$	$\frac{17297069}{2318208034}$	$-\frac{1386496}{5795520085}$
p	x				
	6	7	8	9	
1	$-\frac{563818}{10431936153}$	$-\frac{2048}{8113728119}$	$-\frac{5}{18545664272}$	0	
p	x				
	-9	-8	-7	-6	-5
2	0	$\frac{4375}{1236308743872}$	$\frac{32000}{19317324123}$	$\frac{148937594}{405663806583}$	$\frac{107449600}{135221268861}$
p	x				
	-4	-3	-2	-1	0
2	$-\frac{80883901277}{2704425377220}$	$\frac{367031529728}{2028319032915}$	$-\frac{439132551286}{676106344305}$	$\frac{1632655076608}{676106344305}$	$-\frac{2370618501415}{618154371936}$
p	x				
	1	2	3	4	5
2	$\frac{1632655076608}{676106344305}$	$-\frac{439132551286}{676106344305}$	$\frac{367031529728}{2028319032915}$	$-\frac{80883901277}{2704425377220}$	$\frac{107449600}{135221268861}$
p	x				
	6	7	8	9	
2	$\frac{148937594}{405663806583}$	$\frac{32000}{19317324123}$	$\frac{4375}{1236308743872}$	0	



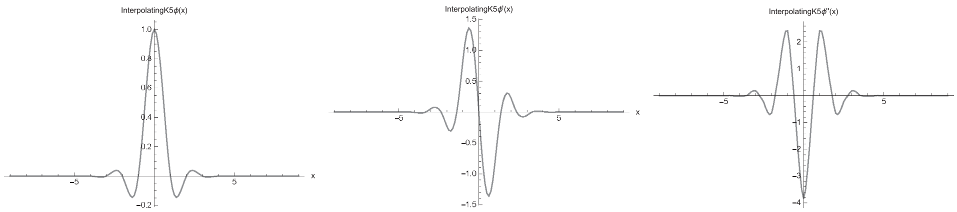
(a) Plot of $Sym4\phi(x)$ (b) Plot of $Sym4\phi'(x)$ (c) Plot of $Sym4\phi''(x)$

Fig. 1 Plots of refinable function in Daubechies family and its first and second order derivatives



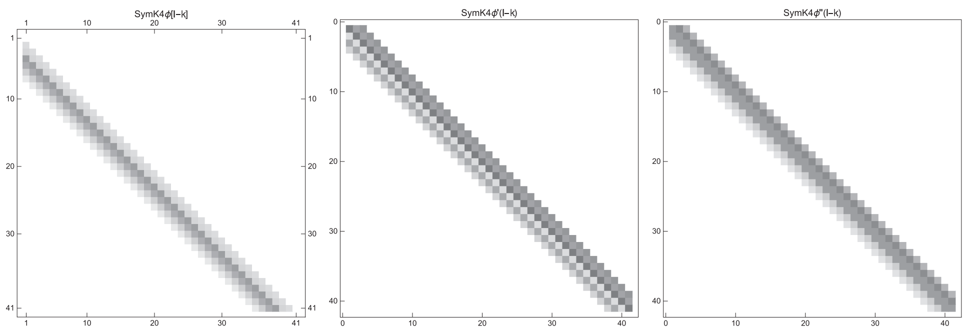
(a) Plot of $CoifK3\phi(x)$ (b) Plot of $CoifK3\phi'(x)$ (c) Plot of $CoifK3\phi''(x)$

Fig. 2 Plots of refinable function in Coiflet family and its first and second order derivatives



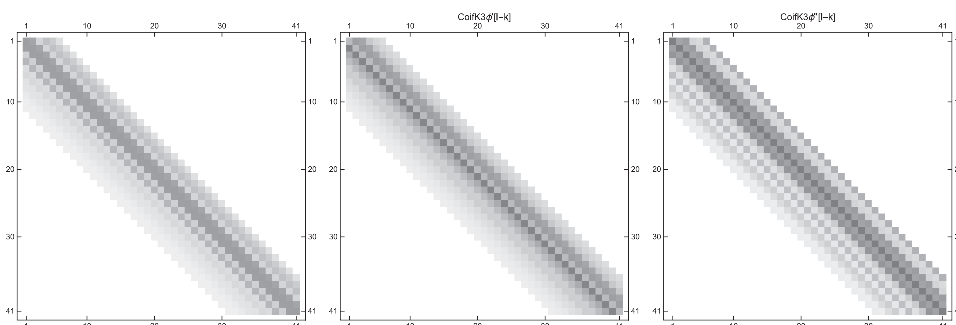
(a) Plot of $IntK5\phi(x)$ (b) Plot of $IntK5\phi'(x)$ (c) Plot of $IntK5\phi''(x)$

Fig. 3 Plots of refinable function in interpolating wavelet family and its first and second order derivatives



(a) Density plot of $Sym4\phi(x)$ (b) Density plot of $Sym4\phi'(x)$ (c) Density plot of $Sym4\phi''(x)$

Fig. 4 Density plots of $\phi_k(x), \phi'_k(x), \phi''_k(x)$ for $k = -30, \cdots, 30;\ x = -30, \cdots, 30$ in Daubechies family

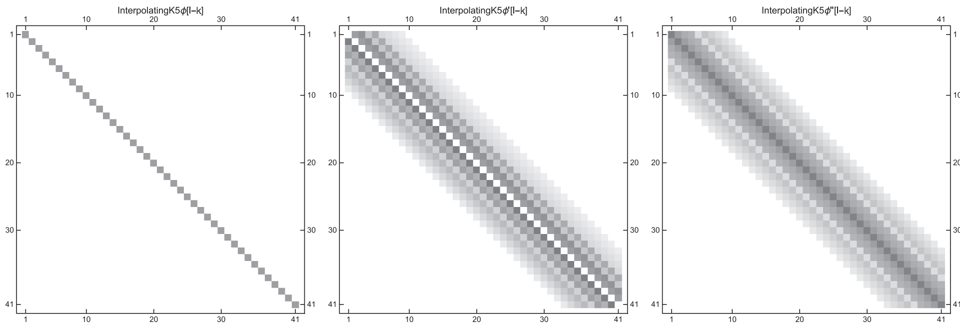


(a) Density plot of $CoifK3\phi(x)$ (b) Density plot of $CoifK3\phi'(x)$ (c) Density plot of $CoifK3\phi''(x)$

Fig. 5 Density plots of $\phi_k(x), \phi'_k(x), \phi''_k(x)$ for $k = -30, \cdots, 30;\ x = -30, \cdots, 30$ in Coiflet family

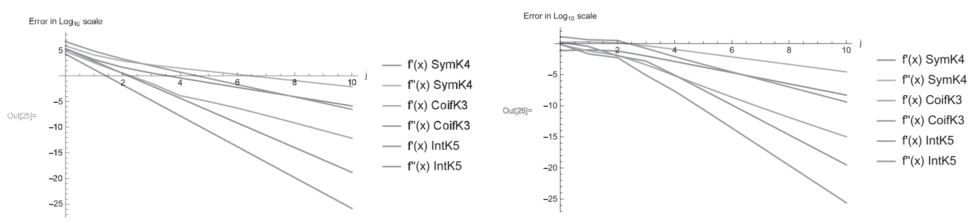
4 Conclusion

In the present work, the authors have presented an efficient scheme to obtain the values of refinable functions and their admissible derivatives at dyadic rationals within their support for three different wavelet bases viz. Daubechies family, Coiflet family and interpolating wavelet family. The scheme presented here finds the values of refinable functions and their admissible derivatives through a single mathematical operation by solving a matrix eigenvalue problem. The values of the first and second order derivatives of the refinable functions from the three wavelet families mentioned above are presented in Tables 1, 2, 3 and their natures are depicted in Figs. 1, 2, 3. To establish the efficiency of the proposed scheme, the obtained values of the derivatives of the refinable functions have been used to estimate $f^{(p)}(x)$ $(p = 1, 2)$ which are found to be exact for $f(x) \in P_{K-1}(x)$ for SymK family or $f(x) \in P_{2K-1}$ for Coiflet or autocorrelation family. We then compare the errors in approximating $f^{(p)}(x)$ $(p = 1, 2)$ for $f(x) = x^{11}$ and $x = e^{-5x^2}$ in bases comprising elements in Sym4, CoifK3 and IntK5 family at different resolution (j). It is found that the errors decrease rapidly as the resolution level (j) increases



(a) Density plot of $IntK5\phi(x)$ (b) Density plot of $IntK5\phi'(x)$ (c) Density plot of $IntK5\phi''(x)$

Fig. 6 Density plots of $\phi_k(x), \phi_k'(x), \phi_k''(x)$ for $k = -30, \cdots, 30;\ x = -30, \cdots, 30$ in interpolating wavelet family



(a) Errors for $f(x) = x^{11}$ (b) Errors for $f(x) = e^{-5x^2}$

Fig. 7 Absolute errors (in Log_{10} scale) in the approximation of $f^{(p)}(x)$ $[f(x) = x^{11}, e^{-5x^2};\ p = 1, 2]$ at $x = 1$ in three wavelet bases at different resolution (j)

which confirms the aptness of the values obtained with the aid of the scheme. From the comparison, one can see that the interpolating wavelet family can be treated to be more efficient than the Daubechies or Coiflet family to obtain approximate derivatives of various polynomials (in spite of not belonging to $L^2(\mathbb{R})$) and non-polynomial functions in $L^2(\mathbb{R})$. The results presented here have been employed directly to obtain highly accurate approximate solutions of several differential equations of physical importance, which will be reported shortly.

Declarations

Conflict of interest This work is supported by DST (Govt. of India) funded "INSPIRE" fellowship (Fellowship No: IF170200).

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