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ORIGINAL RESEARCH

A generalization of Piatetski–Shapiro sequences (II)

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Abstract Suppose that $\alpha, \beta \in \mathbb{R}$. Let $\alpha \geqslant 1$ and c be a real number in the range $1 < c < 12/11$. In this paper, it is proved that there exist infinitely many primes in the generalized Piatetski–Shapiro sequence, which is defined by $(\lfloor \alpha n^c + \beta \rfloor)_{n=1}^{\infty}$. Moreover, we also prove that there exist infinitely many Carmichael numbers composed entirely of primes from the generalized Piatetski–Shapiro sequences with $c \in (1, \frac{19137}{18746})$. The two theorems constitute improvements upon previous results by Guo and Qi [5].

Keywords Beatty sequence · Piatetski–Shapiro sequences · Arithmetic progression · Exponential sums

Mathematics Subject Classification 11N05 · 11L07 · 11N80 · 11B83

1 Introduction

For $1 < c \notin \mathbb{N}$, the Piatetski–Shapiro sequences are sequences of the form

$$\mathscr{N}^{(c)} := (\lfloor n^c \rfloor)_{n=1}^{\infty}.$$

For fixed real numbers α and β, the associated non–homogeneous Beatty sequence is the sequence of integers defined by

$$\mathcal{B}_{\alpha,\beta} := \big(\lfloor \alpha n + \beta \rfloor\big)_{n=1}^{\infty},$$

where $\lfloor t \rfloor$ denotes the integral part of any $t \in \mathbb{R}$. Such sequences are also called generalized arithmetic progressions. Let $\alpha \geqslant 1$ and β be real numbers. Denote $\mathscr{N}_{\alpha,\beta}^{(c)}$ by the generalized Piatetski–Shapiro sequences

$$\mathscr{N}_{\alpha,\beta}^{(c)} = \big(\lfloor \alpha n^c + \beta \rfloor\big)_{n=1}^{\infty}.$$

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Note that the special case $\mathscr{N}_{1,0}^{(c)}$ is the classical Piatetski–Shapiro sequences. Let

$$\pi(x;q,a) := \#\big\{p \leqslant x : p \equiv a \pmod q\big\}$$

and

$$\pi_{\alpha,\beta,c}(x;q,a) := \#\big\{p \leqslant x : p \in \mathscr{N}_{\alpha,\beta}^{(c)} \text{ and } p \equiv a \pmod q\big\}.$$

Recently, Guo and Qi [5] gave an asymptotic formula of $\pi_{\alpha,\beta,c}(x;q,a)$ for $1 < c < \frac{14}{13}$. Moreover, they also proved that there exist infinitely many Carmichael numbers composed entirely of primes from generalized Piatetski–Shapiro sequences with $1 < c < \frac{64}{63}$. In this paper, we shall continue to improve the range of c in these problems and establish the following two results by improving the estimate of the weighted exponential sum

$$\sum_{1\leqslant h\leqslant H}\Bigg|\sum_{x/2<n\leqslant x}\Lambda(n)\mathbf{e}(\theta h n^{\gamma} + \Xi n)\Bigg|,$$

where H, θ, γ, Ξ are positive numbers satisfying $H \geqslant 1$ and $0 < \theta, \gamma, \Xi < 1$.

Theorem 1.1 *Let a and q be coprime integers with $q \geqslant 1$. For fixed $1 < c < \frac{12}{11}$ and $\gamma = c^{-1}$, we have*

$$\begin{aligned}\pi_{\alpha,\beta,c}(x;q,a) &= \alpha^{-\gamma}\gamma x^{\gamma-1}\pi(x;q,a)\\ &\quad + \alpha^{-\gamma}\gamma(1-\gamma)\int_2^x u^{\gamma-2}\pi(u;q,a)\mathrm{d}u + O\big(x^{7\gamma/13+11/26+\varepsilon}\big).\end{aligned} \tag{1.1}$$

Moreover, define

$$\pi_{\alpha,\beta,c}(x) := \pi_{\alpha,\beta,c}(x;1,1) = \#\big\{p \leqslant x : p \in \mathscr{N}_{\alpha,\beta}^{(c)}\big\}.$$

Then we conclude that

Corollary 1.2 *Suppose that $\alpha \geq 1$ and β are real numbers. Then for $1 < c < \frac{12}{11}$, there holds*

$$\pi_{\alpha,\beta,c}(x) = \frac{x^{\gamma}}{\alpha^{\gamma}\log x} + O\bigg(\frac{x^{\gamma}}{\log^2 x}\bigg). \tag{1.2}$$

In the end, we improve the theorem in [5] related to Carmichael numbers, which are the composite natural numbers N with the property that $N|(a^N - a)$ for every integer a.

Theorem 1.3 *For every $c \in (1, \frac{19137}{18746})$, there are infinitely many Carmichael numbers composed entirely of the primes from the set $\mathscr{N}_{\alpha,\beta}^{(c)}$.*

2 Preliminaries

2.1 Notation

We denote by $\lfloor t\rfloor$ and $\{t\}$ the integral part and the fractional part of t, respectively. As usual, we put

$$\mathbf{e}(t) := e^{2\pi i t}.$$

Throughout this paper, we make considerable use of the sawtooth function, which is defined by

$$\psi(t) := t - \lfloor t\rfloor - \frac{1}{2} = \{t\} - \frac{1}{2}.$$

The letter p always denotes a prime. For the generalized Piatetski–Shapiro sequence $(\lfloor\alpha n^c + \beta\rfloor)_{n=1}^{\infty}$, we denote $\gamma := c^{-1}$ and $\theta := \alpha^{-\gamma}$. We use the notation of the form $m \sim M$ as an abbreviation for $M < m \leqslant 2M$.

Throughout the paper, implied constants in symbols O, $\ll$ and $\gg$ may depend (where obvious) on the parameters $\alpha, \beta, c, \varepsilon$ but are absolute otherwise. For given functions F and G, the notations $F \ll G$, $G \gg F$ and $F = O(G)$ are all equivalent to the statement that the inequality $|F| \leqslant \mathcal{C}|G|$ holds with some constant $\mathcal{C} > 0$.

2.2 Technical lemmas

We need the following well–known approximation of Vaaler [11].

Lemma 2.1 *For any $H \geqslant 1$, there exist numbers a_h, b_h such that*

$$\Big|\psi(t) - \sum_{0<|h|\leqslant H} a_h\,\mathbf{e}(th)\Big| \leqslant \sum_{|h|\leqslant H} b_h\,\mathbf{e}(th), \qquad a_h \ll \frac{1}{|h|}, \qquad b_h \ll \frac{1}{H}.$$

Lemma 2.2 *Let $z \geqslant 1$ and $k \geqslant 1$. Then, for any $n \leqslant 2z^k$, there holds*

$$\Lambda(n) = \sum_{j=1}^{k} (-1)^{j-1}\binom{k}{j} \sum_{\substack{n_1 n_2 \cdots n_{2j} = n \\ n_{j+1},\ldots,n_{2j} \leqslant z}} \cdots \sum (\log n_1)\mu(n_{j+1})\cdots\mu(n_{2j}).$$

Proof See the arguments on pp. 1366–1367 of Heath–Brown [6]. □

Lemma 2.3 *Suppose that*

$$L(H) = \sum_{i=1}^{m} A_i H^{a_i} + \sum_{j=1}^{n} B_j H^{-b_j},$$

where A_i, B_j, a_i and b_j are positive. Assume further that $H_1 \leqslant H_2$. Then there exists some $\mathscr{H}$ with $H_1 \leqslant \mathscr{H} \leqslant H_2$ and

$$L(\mathscr{H}) \ll \sum_{i=1}^{m} A_i H_1^{a_i} + \sum_{j=1}^{n} B_j H_2^{-b_j} + \sum_{i=1}^{m}\sum_{j=1}^{n} \big(A_i^{b_j} B_j^{a_i}\big)^{1/(a_i+b_j)}.$$

The implied constant depends only on m and n.

Proof See Lemma 3 of Srinivasan [10]. □

For real numbers θ, $\Xi \in [0, 1]$, the sum of the form

$$\sum_{0<|h|\leqslant H} \Bigg| \sum_{\substack{k\sim K \\ KL \asymp x}} \sum_{\ell\sim L} a_k b_\ell \mathbf{e}\big(\theta h (k\ell)^\gamma + \Xi k\ell\big)\Bigg|$$

with $|a_k| \ll x^\varepsilon$, $|b_\ell| \ll x^\varepsilon$ for every fixed $\varepsilon > 0$, it is usually called a "Type I" sum, denoted by $S_I(K, L)$, if $b_\ell = 1$ or $b_\ell = \log \ell$; otherwise it is called a "Type II" sum, denoted by $S_{II}(K, L)$.

Lemma 2.4 *Suppose that $f(x) : [a, b] \to \mathbb{R}$ has continuous derivatives of arbitrary order on $[a, b]$, where $1 \leqslant a < b \leqslant 2a$. Suppose further that*

$$\big|f^{(j)}(x)\big| \asymp \lambda_j, \qquad j \geqslant 1, \qquad x \in [a, b].$$

Then we have

$$\sum_{a<n\leqslant b} e\big(f(n)\big) \ll a\lambda_2^{1/2} + \lambda_2^{-1/2}, \tag{2.1}$$

and

$$\sum_{a<n\leqslant b} e\big(f(n)\big) \ll a\lambda_3^{1/6} + \lambda_3^{-1/3}. \tag{2.2}$$

Proof For (2.1), one can see Corollary 8.13 of Iwaniec and Kowalski [7], or Theorem 5 of Chapter 1 in Karatsuba [8]. For (2.2), one can see Corollary 4.2 of Sargos [9]. □

Lemma 2.5 *Suppose that* $|a_k| \ll 1$, $b_\ell = 1$ *or* $\log \ell$, $KL \asymp x$. *Then if* $K \ll x^{1/2}$, *there holds*

$$S_I(K, L) \ll H^{7/6} x^{\gamma/6+3/4} + H^{2/3} x^{1-\gamma/3}.$$

Proof Set $f(\ell) = \theta h(k\ell)^\gamma + \Xi k\ell$. It is easy to see that

$$f'''(\ell) = \gamma(\gamma-1)(\gamma-2)\theta h k^\gamma \ell^{\gamma-3} \asymp |h| K^\gamma L^{\gamma-3}.$$

If $K \ll x^{1/2}$, then by (2.2) of Lemma 2.4, we deduce that

$$\begin{aligned} x^{-\varepsilon} \cdot S_I(K, L) &\ll \sum_{0<|h|\leqslant H} \sum_{k\sim K} \Big| \sum_{\ell\sim L} \mathbf{e}\big(f(\ell)\big) \Big| \\ &\ll \sum_{0<|h|\leqslant H} \sum_{k\sim K} \Big(L\big(|h|K^\gamma L^{\gamma-3}\big)^{1/6} + \big(|h|K^\gamma L^{\gamma-3}\big)^{-1/3} \Big) \\ &\ll \sum_{0<|h|\leqslant H} \Big(|h|^{1/6} x^{\gamma/6+1/2} K^{1/2} + |h|^{-1/3} x^{1-\gamma/3} \Big) \\ &\ll H^{7/6} x^{\gamma/6+3/4} + H^{2/3} x^{1-\gamma/3}, \end{aligned}$$

which completes the proof of Lemma 2.5. □

Lemma 2.6 *Suppose that* $|a_k| \ll 1$, $|b_\ell| \ll 1$ *with* $k \sim K$, $\ell \sim L$ *and* $KL \asymp x$. *Then if* $x^{1/2} \ll K \ll x^{19/25}$, *there holds*

$$S_{II}(K, L) \ll H^{5/4} x^{\gamma/4+5/8} + H^{3/4} x^{1-\gamma/4} + H x^{22/25} + H^{7/6} x^{\gamma/6+3/4}.$$

Proof Let Q, which satisfies $1 < Q < L$, be a parameter which will be chosen later. By the Weyl–van der Corput inequality (see Lemma 2.5 of Graham and Kolesnik [4]), we have

$$\Big| \sum_{\substack{k\sim K \\ KL\asymp x}} \sum_{\ell\sim L} a_k b_\ell \mathbf{e}\big(\theta h(k\ell)^\gamma + \Xi k\ell\big) \Big|^2 \ll K^2 L^2 Q^{-1} + KLQ^{-1} \sum_{\ell\sim L} \sum_{0<|q|\leqslant Q} \big|\mathfrak{S}(q;\ell)\big|, \tag{2.3}$$

where

$$\mathfrak{S}(q;\ell) = \sum_{k\in\mathcal{I}(q;\ell)} \mathbf{e}\big(g(k)\big)$$

with

$$g(k) = \theta h k^\gamma \big(\ell^\gamma - (\ell+q)^\gamma\big) - \Xi kq.$$

It is easy to see that

$$g''(k) = \gamma(\gamma-1)\theta h k^{\gamma-2} \big(\ell^\gamma - (\ell+q)^\gamma\big) \asymp |h| K^{\gamma-2} L^{\gamma-1} |q|.$$

By (2.1) of Lemma 2.4, we have

$$\mathfrak{S}(q;\ell) \ll K\big(|h|K^{\gamma-2}L^{\gamma-1}|q|\big)^{1/2} + \big(|h|K^{\gamma-2}L^{\gamma-1}|q|\big)^{-1/2}. \tag{2.4}$$

Putting (2.4) into (2.3), we derive that

$$\begin{aligned} &\Big| \sum_{\substack{k\sim K \\ KL\asymp x}} \sum_{\ell\sim L} a_k b_\ell \mathbf{e}\big(\theta h(k\ell)^\gamma + \Xi k\ell\big) \Big|^2 \\ &\ll K^2 L^2 Q^{-1} + KLQ^{-1} \\ &\quad \times \sum_{\ell\sim L} \sum_{0<|q|\leqslant Q} \big(|h|^{1/2} K^{\gamma/2} L^{\gamma/2-1/2} |q|^{1/2} + |h|^{-1/2} K^{1-\gamma/2} L^{1/2-\gamma/2} |q|^{-1/2}\big) \\ &\ll K^2 L^2 Q^{-1} + KLQ^{-1}\big(|h|^{1/2} K^{\gamma/2} L^{\gamma/2+1/2} Q^{3/2} + |h|^{-1/2} K^{1-\gamma/2} L^{3/2-\gamma/2} Q^{1/2}\big) \\ &\ll K^2 L^2 Q^{-1} + |h|^{1/2} K^{1+\gamma/2} L^{\gamma/2+3/2} Q^{1/2} + |h|^{-1/2} K^{2-\gamma/2} L^{5/2-\gamma/2} Q^{-1/2}. \end{aligned}$$

By noting that $1 \leqslant Q \leqslant L$, it follows from Lemma 2.3 that there exists an optimal Q such that

$$\bigg|\sum_{\substack{k\sim K \\ KL\asymp x}}\sum_{\ell\sim L} a_k b_\ell \mathbf{e}\big(\theta h(k\ell)^\gamma + \Xi k\ell\big)\bigg|^2 \ll |h|^{1/2}x^{\gamma/2+3/2}K^{-1/2} + Kx + |h|^{-1/2}x^{2-\gamma/2} + |h|^{1/3}x^{\gamma/3+5/3}K^{-1/3} + K^{-1/2}x^2,$$

which implies

$$\bigg|\sum_{\substack{k\sim K \\ KL\asymp x}}\sum_{\ell\sim L} a_k b_\ell \mathbf{e}\big(\theta h(k\ell)^\gamma + \Xi k\ell\big)\bigg| \ll |h|^{1/4}x^{\gamma/4+3/4}K^{-1/4} + |h|^{-1/4}x^{1-\gamma/4} + K^{1/2}x^{1/2} + |h|^{1/6}x^{\gamma/6+5/6}K^{-1/6} + K^{-1/4}x.$$

Therefore, from the above estimate and the condition $x^{1/2} \ll K \ll x^{19/25}$, we obtain

$$\begin{aligned}\big|S_{II}(K,L)\big| &\ll \sum_{0<|h|\leqslant H}\Big(|h|^{1/4}x^{\gamma/4+3/4}K^{-1/4} + |h|^{-1/4}x^{1-\gamma/4} \\ &\qquad + K^{1/2}x^{1/2} + |h|^{1/6}x^{\gamma/6+5/6}K^{-1/6} + K^{-1/4}x\Big) \\ &\ll H^{5/4}x^{\gamma/4+3/4}K^{-1/4} + H^{3/4}x^{1-\gamma/4} + HK^{1/2}x^{1/2} \\ &\qquad + H^{7/6}x^{\gamma/6+5/6}K^{-1/6} + HK^{-1/4}x \\ &\ll H^{5/4}x^{\gamma/4+5/8} + H^{3/4}x^{1-\gamma/4} + Hx^{22/25} + H^{7/6}x^{\gamma/6+3/4},\end{aligned}$$

which completes the proof of Lemma 2.6. □

3 Proof of Theorem 1.1

By the same argument of [5, Section 3], we have

$$\pi_{\alpha,\beta,c}(x;q,a) = \Sigma_1(x) + \Sigma_2(x) + O(x^{\gamma-1}), \tag{3.1}$$

where

$$\Sigma_1(x) = \theta\gamma \sum_{\substack{p\leqslant x \\ p\equiv a \pmod q}} p^{\gamma-1},$$

and

$$\Sigma_2(x) = \sum_{\substack{p\leqslant x \\ p\equiv a \pmod q}} \Big(\psi\big(-\theta(p+1-\beta)^\gamma\big) - \psi\big(-\theta(p-\beta)^\gamma\big)\Big).$$

For $\Sigma_1(x)$, by partial summation, we get

$$\begin{aligned}\Sigma_1(x) &= \theta\gamma\int_2^x u^{\gamma-1}\mathrm{d}\Big(\sum_{\substack{p\leqslant u \\ p\equiv a \pmod q}} 1\Big) = \theta\gamma\int_2^x u^{\gamma-1}\mathrm{d}\pi(u;q,a) \\ &= \theta\gamma x^{\gamma-1}\pi(x;q,a) - \theta\gamma(\gamma-1)\int_2^x u^{\gamma-2}\pi(u;q,a)\mathrm{d}u.\end{aligned} \tag{3.2}$$

Next, we turn our attention to $\Sigma_2(x)$. Define

$$\mathcal{H}(x)=\sum_{\substack{n\leqslant x\\ n\equiv a\ (\mathrm{mod}\ q)}}\Lambda(n)\big(\psi(-\theta(n+1-\beta)^{\gamma})-\psi(-\theta(n-\beta)^{\gamma})\big),$$

$$\mathcal{J}(x)=\sum_{\substack{p\leqslant x\\ p\equiv a\ (\mathrm{mod}\ q)}}(\log p)\big(\psi(-\theta(p+1-\beta)^{\gamma})-\psi(-\theta(p-\beta)^{\gamma})\big).$$

Trivially, we have

$$\mathcal{H}(x)=\mathcal{J}(x)+O(x^{1/2}), \tag{3.3}$$

Moreover, it follows from partial summation that

$$\Sigma_2(x)=\int_2^x\frac{1}{\log u}\mathrm{d}\mathcal{J}(u)=\frac{\mathcal{J}(x)}{\log x}+\int_2^x\frac{\mathcal{J}(u)}{u\log^2 u}\mathrm{d}u. \tag{3.4}$$

In order to obtain the upper bound estimate of $\Sigma_2(x)$, it follows from (3.3) and (3.4) that we only need to derive the upper bound estimate of $\mathcal{H}(x)$. By a splitting argument, it is sufficient to give the upper bound estimate of the following sum

$$\mathcal{S}:=\sum_{\substack{x/2<n\leqslant x\\ n\equiv a\quad(\mathrm{mod}\ q)}}\Lambda(n)\Big(\psi\big(-\theta(n+1-\beta)^{\gamma}\big)-\psi\big(-\theta(n-\beta)^{\gamma}\big)\Big).$$

According to Vaaler's approximation, i.e. Lemma 2.1, we can write

$$\mathcal{S}=\mathcal{S}_1+O(|\mathcal{S}_2|), \tag{3.5}$$

where

$$\mathcal{S}_1=\sum_{\substack{x/2<n\leqslant x\\ n\equiv a\quad(\mathrm{mod}\ q)}}\Lambda(n)\sum_{0<|h|\leqslant H}a_h\big(\mathbf{e}(\theta h(n+1-\beta)^{\gamma})-\mathbf{e}(\theta h(n-\beta)^{\gamma})\big),$$

$$\mathcal{S}_2=\sum_{\substack{x/2<n\leqslant x\\ n\equiv a\quad(\mathrm{mod}\ q)}}\Lambda(n)\sum_{|h|\leqslant H}b_h\big(\mathbf{e}(\theta h(n+1-\beta)^{\gamma})+\mathbf{e}(\theta h(n-\beta)^{\gamma})\big).$$

Moreover, we split $\mathcal{S}_1$ into two parts

$$\mathcal{S}_1=\mathcal{S}_1^{(1)}+\mathcal{S}_1^{(2)}, \tag{3.6}$$

where

$$\mathcal{S}_1^{(1)}=\sum_{\substack{x/2<n\leqslant x\\ n\equiv a\quad(\mathrm{mod}\ q)}}\Lambda(n)\sum_{0<|h|\leqslant H}a_h\big(\mathbf{e}(\theta h(n+1-\beta)^{\gamma})-\mathbf{e}(\theta hn^{\gamma})\big),$$

$$\mathcal{S}_1^{(2)}=\sum_{\substack{x/2<n\leqslant x\\ n\equiv a\quad(\mathrm{mod}\ q)}}\Lambda(n)\sum_{0<|h|\leqslant H}a_h\big(\mathbf{e}(\theta hn^{\gamma})-\mathbf{e}(\theta h(n-\beta)^{\gamma})\big).$$

Firstly, we shall consider the upper bound of $\mathcal{S}_1^{(1)}$. Let

$$\phi_h(t):=\mathbf{e}\big(h((t+1-\beta)^{\gamma}-t^{\gamma})\big)-1.$$

Therefore, $\mathcal{S}_1^{(1)}$ is

$$=\sum_{\substack{x/2<n\leqslant x\\ n\equiv a\quad(\mathrm{mod}\ q)}}\Lambda(n)\sum_{0<|h|\leqslant H}a_h\phi_h(n)\mathbf{e}(\theta hn^{\gamma})=\sum_{0<|h|\leqslant H}a_h\sum_{\substack{x/2<n\leqslant x\\ n\equiv a\quad(\mathrm{mod}\ q)}}\Lambda(n)\phi_h(n)\mathbf{e}(\theta hn^{\gamma}),$$

which combined with the upper bound $a_h \ll |h|^{-1}$ yields

$$\mathcal{S}_1^{(1)} \ll \sum_{0<|h|\leqslant H} \frac{1}{|h|}\Bigg|\sum_{\substack{x/2<n\leqslant x\\ n\equiv a \pmod q}} \Lambda(n)\phi_h(n)\mathbf{e}(\theta hn^\gamma)\Bigg|.$$

It follows from partial summation and the bounds

$$\phi_h(t) \ll |h|t^{\gamma-1} \qquad \text{and} \qquad \frac{\partial\phi_h(t)}{\partial t} \ll |h|t^{\gamma-2}$$

that

$$\begin{aligned}
\mathcal{S}_1^{(1)} &\ll \sum_{0<|h|\leqslant H} \frac{1}{|h|}\Bigg|\int_{\frac{x}{2}}^{x} \phi_h(t)\mathrm{d}\Bigg(\sum_{\substack{x/2<n\leqslant t\\ n\equiv a \pmod q}} \Lambda(n)\mathbf{e}(\theta hn^\gamma)\Bigg)\Bigg| \\
&\ll \sum_{0<|h|\leqslant H} \frac{1}{|h|}\Big|\phi_h(x)\Big|\Bigg|\sum_{\substack{x/2<n\leqslant x\\ n\equiv a \pmod q}} \Lambda(n)\mathbf{e}(\theta hn^\gamma)\Bigg| \\
&\quad + \sum_{0<|h|\leqslant H} \frac{1}{|h|}\int_{\frac{x}{2}}^{x} \left|\frac{\partial\phi_h(t)}{\partial t}\right|\Bigg|\sum_{\substack{x/2<n\leqslant t\\ n\equiv a \pmod q}} \Lambda(n)\mathbf{e}(\theta hn^\gamma)\Bigg|\mathrm{d}t \\
&\ll x^{\gamma-1} \times \max_{x/2<t\leqslant x} \sum_{0<|h|\leqslant H}\Bigg|\sum_{\substack{x/2<n\leqslant t\\ n\equiv a \pmod q}} \Lambda(n)\mathbf{e}(\theta hn^\gamma)\Bigg|. \qquad (3.7)
\end{aligned}$$

For $\mathcal{S}_1^{(2)}$, by a similar argument with $\phi_h(t)$ replaced by $\Xi_h(t)$ defined by

$$\Xi_h(t) = 1 - \mathbf{e}\big(\theta h((t-\beta)^\gamma - t^\gamma)\big),$$

which satisfies

$$\Xi_h(t) \ll |h|t^{\gamma-1} \qquad \text{and} \qquad \frac{\partial\,\Xi_h(t)}{\partial t} \ll |h|t^{\gamma-2},$$

one can also derive that

$$\mathcal{S}_1^{(2)} \ll x^{\gamma-1} \times \max_{x/2<t\leqslant x} \sum_{0<|h|\leqslant H}\Bigg|\sum_{\substack{x/2<n\leqslant t\\ n\equiv a \pmod q}} \Lambda(n)\mathbf{e}(\theta hn^\gamma)\Bigg|. \qquad (3.8)$$

In order to prove Theorem 1.1, it is sufficient to give an upper bound estimate of the following sum

$$\max_{x/2<t\leqslant x} \sum_{0<|h|\leqslant H}\Bigg|\sum_{\substack{x/2<n\leqslant t\\ n\equiv a \pmod q}} \Lambda(n)\mathbf{e}(\theta hn^\gamma)\Bigg|. \qquad (3.9)$$

By using the well–known orthogonality

$$\frac{1}{q}\sum_{m=1}^{q}\mathbf{e}\bigg(\frac{(n-a)m}{q}\bigg) = \begin{cases} 1, & \text{if } q\,|\,n-a, \\ 0, & \text{if } q\nmid n-a, \end{cases}$$

we can represent the innermost sum in (3.9) as

$$\sum_{\substack{x/2<n\leqslant t\\ n\equiv a \pmod q}} \Lambda(n)\mathbf{e}(\theta hn^\gamma) = \frac{1}{q}\sum_{m=1}^{q}\sum_{x/2<n\leqslant t} \Lambda(n)\mathbf{e}\bigg(\theta hn^\gamma + \frac{(n-a)m}{q}\bigg). \qquad (3.10)$$

From (3.9) and (3.10), we know that it suffices to estimate

$$\max_{1\leqslant m\leqslant q}\max_{x/2<t\leqslant x}\sum_{0<|h|\leqslant H}\Bigg|\sum_{x/2<n\leqslant t}\Lambda(n)\mathbf{e}\big(\theta hn^{\gamma}+nmq^{-1}\big)\Bigg|.$$

By Heath–Brown's identity, i.e. Lemma 2.2, with $k=3$, one can see that the exponential sum

$$\max_{1\leqslant m\leqslant q}\max_{x/2<t\leqslant x}\sum_{0<|h|\leqslant H}\Bigg|\sum_{x/2<n\leqslant t}\Lambda(n)\mathbf{e}\big(\theta hn^{\gamma}+nmq^{-1}\big)\Bigg|$$

can be written as linear combination of $O(\log^6 x)$ sums, each of which is of the form

$$\mathcal{T}^*:=\sum_{0<|h|\leqslant H}\Bigg|\sum_{n_1\sim N_1}\cdots\sum_{n_6\sim N_6}(\log n_1)\mu(n_4)\mu(n_5)\mu(n_6)$$
$$\times\,\mathbf{e}\big(\theta h(n_1n_2\cdots n_6)^{\gamma}+(n_1n_2\cdots n_6)mq^{-1}\big)\Bigg|, \tag{3.11}$$

where $N_1N_2\cdots N_6\asymp x$; $2N_i\leqslant(2x)^{1/3}$, $i=4,5,6$ and some n_i may only take value 1. Therefore, it is sufficient for us to give upper bound estimate for each $\mathcal{T}^*$ defined as in (3.11). Next, we will consider three cases.

Case 1. If there exists an N_j such that $N_j\geqslant x^{1/2}$, then we must have $j\leqslant 3$ for the fact that $N_i\ll x^{1/3}$ with $i=4,5,6$. Let

$$k=\prod_{\substack{1\leqslant i\leqslant 6\\ i\neq j}}n_i,\qquad \ell=n_j,\qquad K=\prod_{\substack{1\leqslant i\leqslant 6\\ i\neq j}}N_i,\qquad L=N_j.$$

In this case, we can see that $\mathcal{T}^*$ is a sum of "Type I" satisfying $K\ll x^{1/2}$. By Lemma 2.5, we have

$$x^{-\varepsilon}\cdot\mathcal{T}^*\ll H^{7/6}x^{\gamma/6+3/4}+H^{2/3}x^{1-\gamma/3}.$$

Case 2. If there exists an N_j such that $x^{6/25}\leqslant N_j<x^{1/2}$, then we take

$$k=\prod_{\substack{1\leqslant i\leqslant 6\\ i\neq j}}n_i,\qquad \ell=n_j,\qquad K=\prod_{\substack{1\leqslant i\leqslant 6\\ i\neq j}}N_i,\qquad L=N_j.$$

Thus, $\mathcal{T}^*$ is a sum of "Type II" satisfying $x^{1/2}\ll K\ll x^{19/25}$. By Lemma 2.6, we have

$$x^{-\varepsilon}\cdot\mathcal{T}^*\ll H^{5/4}x^{\gamma/4+5/8}+H^{3/4}x^{1-\gamma/4}+Hx^{22/25}+H^{7/6}x^{\gamma/6+3/4}.$$

Case 3. If $N_j<x^{6/25}$ $(j=1,2,3,4,5,6)$, without loss of generality, we assume that $N_1\geqslant N_2\geqslant\cdots\geqslant N_6$. Let r denote the natural number j such that

$$N_1N_2\cdots N_{j-1}<x^{6/25},\qquad N_1N_2\cdots N_j\geqslant x^{6/25}.$$

Since $N_1<x^{6/25}$ and $N_6<x^{6/25}$, then $2\leqslant r\leqslant 5$. Thus, we have

$$x^{6/25}\leqslant N_1N_2\cdots N_r=(N_1\cdots N_{r-1})\cdot N_r<x^{6/25}\cdot x^{6/25}<x^{1/2}.$$

Let

$$k=\prod_{i=r+1}^{6}n_i,\qquad \ell=\prod_{i=1}^{r}n_i,\qquad K=\prod_{i=r+1}^{6}N_i,\qquad L=\prod_{i=1}^{r}N_i.$$

At this time, $\mathcal{T}^*$ is a sum of "Type II" satisfying $x^{1/2}\ll K\ll x^{19/25}$. By Lemma 2.6, we have

$$x^{-\varepsilon}\cdot\mathcal{T}^*\ll H^{5/4}x^{\gamma/4+5/8}+H^{3/4}x^{1-\gamma/4}+Hx^{22/25}+H^{7/6}x^{\gamma/6+3/4}.$$

Combining the above three cases, we derive that

$$x^{-\varepsilon}\cdot\mathcal{T}^*\ll H^{7/6}x^{\gamma/6+3/4}+H^{2/3}x^{1-\gamma/3}+H^{5/4}x^{\gamma/4+5/8}+H^{3/4}x^{1-\gamma/4}+Hx^{22/25},$$

which combined with (3.6), (3.7), and (3.8) yields

$$x^{-\varepsilon}\cdot\mathcal{S}_1\ll H^{5/4}x^{5\gamma/4-3/8}+H^{3/4}x^{3\gamma/4}+Hx^{\gamma-3/25}+H^{7/6}x^{7\gamma/6-1/4}. \tag{3.12}$$

Now, we focus on the upper bound of $\mathcal{S}_2$. The contribution from $h=0$ is

$$2b_0\sum_{\substack{x/2<n\leqslant x\\ n\equiv a \pmod q}}\Lambda(n)\ll\frac{b_0x}{\varphi(q)}\ll xH^{-1}. \tag{3.13}$$

On the other hand, by similar arguments of $\mathcal{S}_1$ with a shift of n, the contribution from $h\neq 0$ is

$$\begin{aligned}&\ll\sum_{\substack{x/2<n\leqslant x\\ n\equiv a \pmod q}}\Lambda(n)\sum_{0<|h|\leqslant H}b_h\mathbf{e}(\theta hn^{\gamma})=\sum_{0<|h|\leqslant H}b_h\sum_{\substack{x/2<n\leqslant x\\ n\equiv a \pmod q}}\Lambda(n)\mathbf{e}(\theta hn^{\gamma})\\ &\ll\frac{1}{H}\sum_{0<|h|\leqslant H}\left|\sum_{\substack{x/2<n\leqslant x\\ n\equiv a \pmod q}}\Lambda(n)\mathbf{e}(\theta hn^{\gamma})\right|,\end{aligned}$$

which can be treated as the process of (3.9) to give the upper bound

$$\ll H^{1/6}x^{\gamma/6+3/4}+H^{-1/3}x^{1-\gamma/3}+H^{1/4}x^{\gamma/4+5/8}+H^{-1/4}x^{1-\gamma/4}+x^{22/25}. \tag{3.14}$$

From (3.5), (3.12), (3.13) and (3.14), we obtain

$$\begin{aligned}x^{-\varepsilon}\cdot\mathcal{S}\ll\ &H^{5/4}x^{5\gamma/4-3/8}+H^{3/4}x^{3\gamma/4}+Hx^{\gamma-3/25}+H^{7/6}x^{7\gamma/6-1/4}+H^{1/6}x^{\gamma/6+3/4}\\ &+H^{1/4}x^{\gamma/4+5/8}+x^{22/25}+H^{-1/3}x^{1-\gamma/3}+H^{-1/4}x^{1-\gamma/4}+xH^{-1}.\end{aligned}$$

Since the above upper bound holds for any real $H\geqslant 1$, using Lemma 2.3 we deduce that

$$\begin{aligned}x^{-\varepsilon}\cdot\mathcal{S}\ll\ &x^{5\gamma/4-3/8}+x^{3\gamma/4}+x^{\gamma-3/25}+x^{7\gamma/6-1/4}+x^{\gamma/6+3/4}+x^{\gamma/4+5/8}\\ &+x^{22/25}+x^{5\gamma/9+7/18}+x^{3\gamma/7+3/7}+x^{\gamma/2+11/25}+x^{7\gamma/13+11/26}\\ &+x^{\gamma/7+11/14}+x^{\gamma/5+7/10}.\end{aligned} \tag{3.15}$$

By noting the fact that $\pi_{\alpha,\beta,c}(x;q,a)\ll x^{\gamma}$, the above bound is trivial unless the exponent of each term in the parentheses is strictly less than γ, which means that $\gamma>11/12$. Under this condition, after eliminating lower order terms, the previous bound of $\mathcal{S}$ in (3.15) simplifies to

$$\mathcal{S}\ll x^{7\gamma/13+11/26+\varepsilon}$$

for any $\varepsilon>0$. Therefore, we obtain $\mathcal{H}(x)\ll x^{7\gamma/13+11/26+\varepsilon}\ll x^{\gamma-\varepsilon}$ when $\gamma>11/12$, and thus $\mathcal{J}(x)\ll x^{7\gamma/13+11/26+\varepsilon}$. Moreover, from (3.4) we derive that

$$\Sigma_2(x)\ll x^{7\gamma/13+11/26+\varepsilon}\ll x^{\gamma-\varepsilon}. \tag{3.16}$$

Consequently, according to (3.1), (3.2) and (3.16), we derive the asymptotic formula (1.1). This completes the proof of Theorem 1.1.

4 Sketch of the proof of Theorem 1.3

By exactly the same argument as [5, Section 4], we conclude that:

Theorem 4.1 *Let $\alpha \geqslant 1$ and β be real numbers. Suppose that $c \in (1, \frac{12}{11})$. Then we have*

$$\vartheta_{\alpha,\beta,c}(x; q, a) = \alpha^{-\gamma}\gamma x^{\gamma-1}\vartheta(x; q, a) + \alpha^{-\gamma}\gamma(1-\gamma)\int_2^x u^{\gamma-2}\vartheta(u; q, a)\mathrm{d}u + O(x^{7\gamma/13+11/26+\varepsilon}),$$

where the implied constant depends only on α, β, c and ε.

The proof of our Theorem 1.3 is exactly the same as [5, Section 4] by switching the conditions

$$1 < c < \frac{14}{13} \qquad \text{and} \qquad -\frac{13}{35} + \frac{2\gamma}{5}$$

into

$$1 < c < \frac{12}{11} \qquad \text{and} \qquad -\frac{11}{26} + \frac{6\gamma}{13}.$$

Let $\pi(x, y)$ be the number of those for which $p - 1$ is free of prime factors exceeding y. Let $\mathcal{E}$ be the set of numbers E in the range $0 < E < 1$ for which

$$\pi(x, x^{1-E}) \geq x^{1+o(1)}$$

as $x \to \infty$, where the function implied by $o(1)$ depends on E. By the same argument in [5, Section 4], we conclude the following statement.

Lemma 4.2 *Let $\alpha \geq 1$ and β be real numbers. Suppose that $c \in \left(1, \frac{38}{37}\right)$. Let B, B_1 be positive real numbers such that $B_1 < B < -\frac{11}{26} + \frac{6\gamma}{13}$. For any $E \in \mathcal{E}$, there exists a number x_3 depending on c, B, B_1, E and ε, such that for any $x \geqslant x_1$ there exist at least $x^{EB+(1-B+B_1)(\gamma-1)-\varepsilon}$ Carmichael numbers up to x composed solely of primes from $\mathscr{N}_{\alpha,\beta}^{(c)}$.*

Taking B and B_1 arbitrarily close to $-\frac{11}{26} + \frac{6\gamma}{13}$, Lemma 4.2 implies that there are infinitely many Carmichael numbers composed entirely of the primes from $\mathscr{N}_{\alpha,\beta}^{(c)}$ with

$$\left(-\frac{11}{26} + \frac{6\gamma}{13}\right)E + \gamma - 1 > 0.$$

Taking $E = 0.7039$ from [3], we eventually have $\gamma > \frac{18746}{19137}$.

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References

1. W. R. Alford, A. Granville, C. Pomerance, There are infinitely many Carmichael numbers, Ann. of Math. (2), **139** (1994), no. 3, 703–722.
2. R. C. Baker, W. D. Banks, J. Brüdern, I. E. Shparlinski, A. J. Weingartner, Piatetski–Shapiro sequences, Acta Arith., **157** (2013), no. 1, 37–68.
3. R. C. Baker, G. Harman, Shifted primes without large prime factors, Acta Arith., **83** (1998), no. 4, 331–361.
4. S. W. Graham, G. Kolesnik, Van der Corput's method of exponential sums, Cambridge University Press, Cambridge, 1991.
5. V. Z. Guo, J. Qi, A Generalization of Piatetski–Shapiro Sequences, Taiwanese J. Math., **26** (2022), no. 1, 33–47.
6. D. R. Heath–Brown, Prime numbers in short intervals and a generalized Vaughan identity, Canadian J. Math., **34** (1982), no. 6, 1365–1377.
7. H. Iwaniec, E. Kowalski, Analytic number theory. American Mathematical Society, Providence, RI, 2004.
8. A. A. Karatsuba, Basic analytic number theory, Springer–Verlag, Berlin, 1993.

9. P. Sargos, Points entiers au voisinage d'une courbe, sommes trigonométriques courtes et paires d'exposants, Proc. London Math. Soc. (3), **70** (1995), no. 2, 285–312.
10. B. R. Srinivasan, The lattice point problem of many–dimensional hyperboloids II, Acta Arith., **8** (1963), 173–204.
11. J. D. Vaaler, Some extremal functions in Fourier analysis, Bull. Amer. Math. Soc. (N.S.), **12** (1985), no. 2, 183–216.

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