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ORIGINAL RESEARCH

Compressed Cayley graph of groups

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Abstract Let G be a group and let S be a subset of $G \setminus \{e\}$ with $S^{-1} \subseteq S$, where e is the identity element of G. The Cayley graph $\mathrm{Cay}(G, S)$ is a graph whose vertices are the elements of G and two distinct vertices $g, h \in G$ are adjacent if and only if $g^{-1}h \in S$. Let $S \subseteq Z(G)$. Then the relation $\sim$ on G, given by $a \sim b$ if and only if $Sa = Sb$, is an equivalence relation. Let G_E be the set of equivalence classes of $\sim$ on G and let $[a]$ be the equivalence class of the element a in G. Then G_E is a group with operation $[a].[b] = [ab]$. Also, let S_E be the set of equivalence classes of the elements of S. The compressed Cayley graph of G is introduced as the Cayley graph $\mathrm{Cay}(G_E, S_E)$, which is denoted by $\mathrm{Cay}_E(G, S)$. In this paper, we investigate some relations between $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}_E(G, S)$. Also, we prove that $\mathrm{Cay}(G, S)$ is a $\mathrm{Cay}_E(G, S)$-generalized join of certain empty graphs. Moreover, we describe the structure of the compressed Cayley graph of $\mathbb{Z}_n$ by introducing a subset S such that $\mathrm{Cay}_E(\mathbb{Z}_n, S)$ and $\mathrm{Cay}(\mathbb{Z}_n, S)$ are not isomorphic, and we describe the Laplacian spectrum of $\mathrm{Cay}(\mathbb{Z}_n, S)$.

Keywords Cayley graph · Compressed Cayley graph · Generalized join graph

Mathematics Subject Classification 05C25 · 05C50

1 Introduction

The investigation of graphs related to algebraic structures is a very large and growing area of research. One of the most important classes of graphs are Cayley graphs. These graphs have been considered, for example, in [1,2,13,16], and [17]. Let us refer the readers to the survey article [18] for extensive bibliography devoted to various applications of Cayley graphs. In particular, the Cayley graphs of semigroups are related to automata theory, as explained in [15] and the monograph [14]. Several other classes of graphs associated with algebraic structures have been also actively investigated. For example, zero-divisor graphs of rings have been investigated in [4–6,8], and [9]. Recently, Anderson and LaGrange [7] introduced and studied the concept of compressed zero-divisor graphs. (See also [3,20].)

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For a group G and a subset $S \subseteq G \setminus \{e\}$ with $S^{-1} \subseteq S$, where e is the identity element of G, the Cayley graph $\mathrm{Cay}(G, S)$ is a graph whose vertices are the elements of G and two distinct vertices $g, h \in G$ are adjacent if and only if $g^{-1}h \in S$. Now, let $S \subseteq Z(G)$. We define an equivalence relation on G. Given $a, b \in G$, we define $a \sim b$ if and only if $Sa = Sb$. It is easy to see that $\sim$ is an equivalence relation on G. Let G_E be the set of all equivalence classes of $\sim$ on G and let $[a]$ denote the equivalence class of the element a in G. For each two elements $[a], [b] \in G_E$, we define $[a] \cdot [b] = [ab]$. Then $(G_E, \cdot)$ forms a group. We set $S_E := \{[t] | \, t \in S\}$. Clearly $S_E^{-1} \subseteq S_E$. Now, if $[e] \in S_E$, then $[e] = [t]$ for some $t \in S$. Hence $Se = St$, and so there exists $s_1 \in S$ such that $te = s_1 t$, which implies that $s_1 = e$, and this is impossible. Hence $S_E \subseteq G_E \setminus \{[e]\}$. We introduce the compressed Cayley graph as follows. The compressed Cayley graph $\mathrm{Cay}_E(G, S)$ is a graph $\mathrm{Cay}(G_E, S_E)$, with vertex set G_E and two distinct vertices $[a]$ and $[b]$ are adjacent if and only if $[ba^{-1}] \in S_E$. The main purpose of this paper is to show that, in several cases, for studying $\mathrm{Cay}(G, S)$, it is sufficient to focus on study the compressed Cayley graph $\mathrm{Cay}_E(G, S)$. In Section 2, we study and investigate some relations between $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}_E(G, S)$. Also, We prove that $\mathrm{Cay}(G, S)$ is a $\mathrm{Cay}_E(G, S)$-generalized join of certain empty graphs. In Section 3, we describe the structure of the compressed Cayley graph of $\mathbb{Z}_n$ by introducing a subset S such that $\mathrm{Cay}_E(\mathbb{Z}_n, S)$ and $\mathrm{Cay}(\mathbb{Z}_n, S)$ are not isomorphic. In the last section, we describe the Laplacian spectrum of $\mathrm{Cay}(\mathbb{Z}_n, S)$.

We use the standard terminology of graphs following [10]. For a graph G by $E(G)$ and $V(G)$ we denote the set of all edges and vertices, respectively. For two distinct vertices a and b of G, we write $a - b$ if a and b are adjacent in G. We denote the set of all adjacent vertices to a vertex a in G, by $N(a)$, and the *degree* of a vertex a, which is denoted by $\deg(a)$, is defined to be $|N(a)|$. A graph is said to be r-*regular* whenever the degree of all vertices are equal to r. Recall that a graph is *connected* if there exists a path connecting any two distinct vertices. The *distance* between two distinct vertices a and b, denoted by $\mathrm{d}(a, b)$, is the length of the shortest path connecting them (if such a path does not exist, then we set $\mathrm{d}(a, b) = \infty$). The *diameter* of a graph G, denoted by $\mathrm{diam}(G)$, is equal to $\sup\{d(a, b) | a, b \in V(G)\}$. The graph G is called *complete* if $\mathrm{diam}(G) = 1$. We use K_n to denote the complete graph with n vertices. An r-partite graph is one whose vertex set can be partitioned into r subsets, so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is jointed to every vertex that is not in the same subset. The *complete bipartite* (i.e., 2-partite) graph with part sizes m and n is denoted by $K_{m,n}$. The *girth* of a graph G, denoted by $\mathrm{girth}(G)$, is the length of the shortest cycle in G. Also, we say that G is *empty* if no two vertices of G are adjacent. We denote an empty graph with n vertices by $\overline{K}_n$. Also, a cycle with n vertices is denoted by C_n. In a graph G, a vertex x is *isolated*, if no vertices of G is adjacent to x. For a graph G, let $\chi(G)$ denote the *chromatic number* of the graph G, that is, the minimal number of colors that can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. A *clique* of a graph is its complete subgraph and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the clique number of G. Obviously $\chi(G) \geq \omega(G)$. An *independent set* of G is a subset of the vertices of G such that no two vertices in the subset represent an edge of G. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of the largest independent set. Recall that for two graphs H_1 and H_2 with disjoint vertex sets, the *join* $H_1 \vee H_2$ of the graphs H_1 and H_2 is a graph obtained from the union of H_1 and H_2 by adding new edges from each vertex of H_1 to every vertex of H_2. The concept of join graph is generalized (in [21], it was called as a generalized composition graph). Assume that G is a graph on k vertices with $V(G) = \{v_1, v_2, \dots, v_k\}$, and let $H_1, H_2, \dots, H_k$ be k pairwise disjoint graphs. The G-*generalized join graph* $G[H_1, H_2, \dots, H_k]$ of $H_1, H_2, \dots, H_k$ is a graph formed by replacing each vertex v_i of G by the graph H_i and then joining each vertex of H_i to each vertex of H_j whenever $v_i - v_j$ in the graph G. Now, if the graph G consists of two adjacent vertices, then the G-generalized join graph $G[H_1, H_2]$ coincides with the join $H_1 \vee H_2$ of the graphs H_1 and H_2.

2 Basic properties

In the following lemmas, we investigate some relations between the adjacencies in $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}_E(G, S)$.

Lemma 2.1 *Let $[a]$ be an arbitrary vertex in $\mathrm{Cay}_E(G, S)$. Then for each $t, g \in [a]$, t and g are not adjacent.*

Proof Suppose that $t, g \in [a]$. Assume on the contrary that t is adjacent to g. Hence we have $gt^{-1} \in S$. Then there exists $s_1 \in S$ such that $gt^{-1} = s_1$. Since $t, g \in [a]$, we have $St = Sa = Sg$. Therefore there exists $s_2 \in S$ such that $s_1 t = s_2 g$. Now, since $s_1 = gt^{-1}$, we have $s_2 = e$, which is a contradiction. Thus the result holds. □

Lemma 2.2 *Let $[a]$ be an arbitrary vertex in $\mathrm{Cay}_E(G, S)$. Then for each $t, g \in [a]$, $N(t) = N(g)$.*

Proof Suppose that $b \in N(t)$. Then we have $bt^{-1} \in S$. Therefore, there exists $s_1 \in S$ such that $b = s_1 t$. Since $t, g \in [a]$ we have $St = Sa = Sg$, and so there exists $s_2 \in S$ such that $b = s_2 g$. It implies that $b \in N(g)$. Similarly $N(g) \subseteq N(t)$. □

Lemma 2.3 *Two vertices x and y in* $\mathrm{Cay}(G, S)$ *are adjacent if and only if* $[x]$ *is adjacent to* $[y]$ *in* $\mathrm{Cay}_E(G, S)$.

Proof First suppose that x is adjacent to y in $\mathrm{Cay}(G, S)$. Then $yx^{-1} \in S$. By Lemma 2.1, we have $[x] \neq [y]$. By the definition of S_E, we have $[yx^{-1}] \in S_E$. Then $[x]$ is adjacent to $[y]$ in $\mathrm{Cay}_E(G, S)$. Conversely, since $[x]$ is adjacent to $[y]$ in $\mathrm{Cay}_E(G, S)$, then $[yx^{-1}] \in S_E$ and so there exists $t \in S$ such that $[yx^{-1}] = [t]$. Then we have $Syx^{-1} = St$. There exists $s_2 \in S$ such that $tyx^{-1} = s_2 t$, and since $S \subseteq Z(G)$ we have $s_2 t = ts_2$. Therefore $yx^{-1} = s_2$ and so $yx^{-1} \in S$, which means that x is adjacent to y in $\mathrm{Cay}(G, S)$. □

In the following theorem, we state a necessary and sufficient condition under which the graphs $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}_E(G, S)$ are isomorphic.

Theorem 2.4 $\mathrm{Cay}_E(G, S) \cong \mathrm{Cay}(G, S)$ *if and only if* $|S| = |S_E|$.

Proof We know that Cayley graph $\mathrm{Cay}(G, S)$ is $|S|$-regular and $\mathrm{Cay}_E(G, S)$ is $|S_E|$-regular. If $\mathrm{Cay}_E(G, S) \cong \mathrm{Cay}(G, S)$, then it is clear that $|S| = |S_E|$.

Now, suppose that $|S| = |S_E|$. By Lemma 2.3, it is enough to show that $[g] = \{g\}$ for all $g \in G$. Assume on the contrary that $g' \neq g$ and that $g' \in [g]$. Hence $Sg' = Sg$. Then there exist distinct $s_1, s_2 \in S$ such that $s_1 g' = s_2 g$. Therefore, $s_1 g' g^{-1} = s_2$, and hence $Ss_2 = Ss_1 g' g^{-1}$. Since $S \subseteq Z(G)$, we have $Ss_1 g' g^{-1} = (Sg')s_1 g^{-1} = (Sg)s_1 g^{-1} = Ss_1 g g^{-1} = Ss_1$. It implies that $s_1 \in [s_2]$. Since $|S| = |S_E|$, it is a contradiction. Thus the result holds. □

Lemma 2.5 *If* $[x]$ *is adjacent to* $[y]$ *in* $\mathrm{Cay}_E(G, S)$*, then the induced subgraph on the set* $[x] \cup [y]$ *in* $\mathrm{Cay}(G, S)$ *is a complete bipartite graph.*

Proof By Lemma 2.1, each two vertices in $[x]$ or $[y]$ are not adjacent in $\mathrm{Cay}(G, S)$. Now, let $a \in [x]$ and let $b \in [y]$. Clearly $[a] = [x]$ and $[b] = [y]$. Since $[x]$ is adjacent to $[y]$, Lemma 2.3 implies that a and b are adjacent. □

The following theorem says that $\mathrm{Cay}(G, S)$ is a $\mathrm{Cay}_E(G, S)$-generalized join of some certain empty graphs.

Theorem 2.6 *Let* $G_t = \{[a_1], \ldots, [a_r]\}$. *Then*

$$\mathrm{Cay}(G, S) \cong \mathrm{Cay}_E(G, S)[\bar{k}_{|[a_1]|}, \bar{k}_{|[a_2]|}, \ldots, \bar{k}_{|[a_r]|}].$$

Proof Replace the vertex $[a_i]$ of $\mathrm{Cay}_E(G, S)$ by $\bar{k}_{|[a_i]|}$. Then the result is obtained by Lemmas 2.3 and 2.5. □

Theorem 2.7 *The Cayley graph* $\mathrm{Cay}(G, S)$ *is connected if and only if* $\mathrm{Cay}_E(G, S)$ *is connected.*

Proof First assume that $\mathrm{Cay}(G, S)$ is connected. Let $[a]$ and $[b]$ be two distinct arbitrary vertices in $\mathrm{Cay}_E(G, S)$. Consider two vertices a and b in $\mathrm{Cay}(G, S)$. Since $\mathrm{Cay}(G, S)$ is connected, there exists a path $a - t_1 - t_2 - \cdots - t_n - b$. By Lemmas 2.1 and 2.3, one can obtain a path from $[a]$ to $[b]$.

For the converse statement, assume that x and y are two distinct arbitrary vertices in $\mathrm{Cay}(G, S)$. We have the following cases

Case 1. $[x] = [y]$. Since $\mathrm{Cay}_E(G, S)$ is connected and $|S_E| \geq 1$, there exits a vertex $[z]$ such that $[x]$ is adjacent to $[z]$. Now, it is enough to consider the path $x - z - y$ from x to y in $\mathrm{Cay}(G, S)$.

Case 2. $[x] \neq [y]$. Since $\mathrm{Cay}_E(G, S)$ is connected, we have a path from $[x]$ to $[y]$. By Lemma 2.3, we obtain a path from x to y in $\mathrm{Cay}(G, S)$. Hence $\mathrm{Cay}(G, S)$ is connected.

□

In the following theorem, we investigate the relation between the clique number, chromatic number and independence number of the Cayley graphs $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}_E(G, S)$.

Theorem 2.8 *The following statements hold:*

1) $\omega\big(\mathrm{Cay}_E(G, S)\big) = \omega\big(\mathrm{Cay}(G, S)\big)$.
2) $\chi\big(\mathrm{Cay}(G, S)\big) = \chi\big(\mathrm{Cay}(G, S)\big)$.
3) $\alpha\big(\mathrm{Cay}_E(G, S)\big) \leq \alpha\big(\mathrm{Cay}(G, S)\big)$.

Proof 1) Clearly $\omega\big(\mathrm{Cay}_E(G,S)\big) \le \omega\big(\mathrm{Cay}(G,S)\big)$. Now suppose that α is a clique in $\mathrm{Cay}(G,S)$. Then Lemmas 2.1 and 2.3 imply that α is a clique in $\mathrm{Cay}_E(G,S)$, and so the result holds.

2) Let $\chi\big(\mathrm{Cay}_E(G,S)\big) = n$. Then we can color the vertices of $\mathrm{Cay}(G,S)$ with n colors such that each vertex has the same color as the representative of its class. Now assume that we can color $\mathrm{Cay}(G,S)$ with $n-1$ colors. Let $[x_1],\ldots,[x_k]$ be the vertices of $\mathrm{Cay}_E(G,S)$. We associate the color of the vertex x_i in $\mathrm{Cay}(G,S)$ to the vertex $[x_i]$ in $\mathrm{Cay}_E(G,S)$. Then we can color $\mathrm{Cay}_E(G,S)$ with at most $n-1$ colors, which is impossible.

3) Let $k = \{[k_1],[k_2],\ldots,[k_n]\}$ be the maximum independent set in $\mathrm{Cay}_E(G,S)$ such that $[k_i] \in G_E$ for $i = 1,2,\ldots,n$. We set $k' = \bigcup_{i=1}^{n}[k_i]$. By Lemma 2.1, k' is an independent set in $\mathrm{Cay}(G,S)$, and we have $|k'| \ge |k|$. Hence $\alpha\big(\mathrm{Cay}(G,S)\big) \ge \alpha(\mathrm{Cay}_E(G,S))$.

□

We end this section with the following theorem, which investigates the diameter and girth of $\mathrm{Cay}(G,S)$ and $\mathrm{Cay}_E(G,S)$.

Theorem 2.9 *The following statements hold.*

1) *If* $\mathrm{diam}\big(\mathrm{Cay}_E(G,S)\big) = 1$, *then* $\mathrm{diam}\big(\mathrm{Cay}(G,S)\big) \le 2$.
2) *If* $\mathrm{diam}\big(\mathrm{Cay}_E(G,S)\big) \ge 2$, *then* $\mathrm{diam}\big(\mathrm{Cay}(G,S)\big) = \mathrm{diam}\big(\mathrm{Cay}_E(G,S)\big)$.
3) *If* $\mathrm{gr}\big(\mathrm{Cay}_E(G,S)\big) \in \{3,4\}$, *then* $\mathrm{gr}\big(\mathrm{Cay}(G,S)\big) = \mathrm{gr}\big(\mathrm{Cay}_E(G,S)\big)$.

Proof 1) Let x and y be two distinct vertices in $\mathrm{Cay}(G,S)$. First assume that $[x] \ne [y]$. Since $d([x],[y]) = 1$, we have $d(x,y) = 1$. Now, let $[x] = [y]$. Consider a vertex $[z]$ in $\mathrm{Cay}_E(G,S)$ such that $[x] - [z]$. Then we have the path $x - z - y$ in $\mathrm{Cay}(G,S)$, which means that $d(x,y) = 2$.

2) Let $[x]$ and $[y]$ be two distinct vertices in $\mathrm{Cay}_E(G,S)$ such that $d([x],[y]) = \mathrm{diam}\big(\mathrm{Cay}_E(G,S)\big) = t \ge 2$. Then one can easily see that $d(x,y) \le t$ in $\mathrm{Cay}(G,S)$. Now, if $d(x,y) \le t-1$, then by Lemma 2.3, one can obtain a path of length less than t from $[x]$ to $[y]$ in $\mathrm{Cay}_E(G,S)$, which is impossible. Hence $d(x,y) = t$, and so the result holds.

3) If $\mathrm{gr}\big(\mathrm{Cay}_E(G,S)\big) = 3$, then clearly $\mathrm{gr}\big(\mathrm{Cay}(G,S)\big) = 3$. Now, let $\mathrm{gr}\big(\mathrm{Cay}_E(G,S)\big) = 4$. Then we have a cycle of length 4 in $\mathrm{Cay}(G,S)$. If there exists a triangle $x_1 - x_2 - x_3 - x_1$ in $\mathrm{Cay}(G,S)$, then by Lemma 2.3, we have the triangle $[x_1] - [x_2] - [x_3] - [x_1]$ in $\mathrm{Cay}_E(G,S)$, which is a contradiction.

□

3 On the structure of $\mathrm{Cay}_E(\mathbb{Z}_n, S)$

In this section, we investigate the structure of the compressed Cayley graph $\mathrm{Cay}_E(\mathbb{Z}_n,S)$. We denote the elements of $\mathbb{Z}_n$ by $0,1,2,\ldots,n-1$. For $i \in \mathbb{Z}_n$, $-i$ is the additive inverse of i. In the following, we assume that $n = pm$, where $m \notin \{1,p\}$ and p is the least prime integer that divides n. Also, we always assume that

$$S = \{p, -p, p+m, -(p+m), \ldots, p+(p-1)m, -(p+(p-1)m)\},$$

otherwise it is stated. In the next theorem, we study the structure of the compressed Cayley graph $\mathrm{Cay}_E(\mathbb{Z}_n,S)$, where $n \notin \{p^2, 2^3, 2^4\}$.

Theorem 3.1 *Let* $n = pm$, *where* $n \notin \{p, p^2, 2^3, 2^4\}$. *Then* $S_E = \{[p],[-p]\}$, $(\mathbb{Z}_n)_E = \{[0],[1],\ldots,[m-1]\}$, *and so* $\mathrm{Cay}_E(\mathbb{Z}_n,S)$ *is a 2-regular graph with m vertices.*

Proof It is clear that $S \subseteq \mathbb{Z}_n$ and that $S^{-1} \subseteq S$. Let $n = pm$, such that p is the least prim integer that divides n and let $n \notin \{p, p^2, 2^3, 2^4\}$. Let $S = A_1 \cup A_2$, where $A_1 = \{p+tm\}$ and $A_2 = \{-(p+tm)\}$, $0 \le t \le p-1$. First we show that $[p] = [p+tm]$. Suppose that $p+s_1$ and $p+s_2$ are two arbitrary elements in $p+S$, where $s_1 \in A_1$ and $s_2 \in A_2$. Since $s_1 \in A_1$ and $s_2 \in A_2$, there exist t_1 and t_2 in $\{0,1,\ldots,p-1\}$, such that $s_1 = p+t_1m$ and $s_2 = -(p+t_2m)$. Hence $p+s_1 = p+(p+t_1m) = (p+tm)+(p+(t_1-t)m)$. Since $p+(t_1-t)m \in S$, we have $p+s_1 \in (p+tm)+S$.

Also, $p+s_2 = p-(p+t_2m) = (p+tm)-(p+(t_2+t)m)$. Since $-(p+(t_2+t)m) \in S$, we have $p+s_2 \in (p+tm)+S$. Hence $p+S \subseteq (p+tm)+S$.

Now we show that $(p+tm)+S \subseteq p+S$. Suppose that $(p+tm)+s_1$ and $(p+tm)+s_2$ are two arbitrary elements in $(p+tm)+S$, where $s_3 \in A_1$ and $s_4 \in A_2$. Since $s_3 \in A_1$ and $s_4 \in A_2$, there exist $t_3, t_4 \in \{0,1,\ldots,p-1\}$ such that $s_3 = p+t_3m$ and $s_4 = -(p+t_4m)$. Then $(p+tm)+s_3 = (p+tm)+(p+t_3m) = p+(p+(t+t_3)m)$. Since

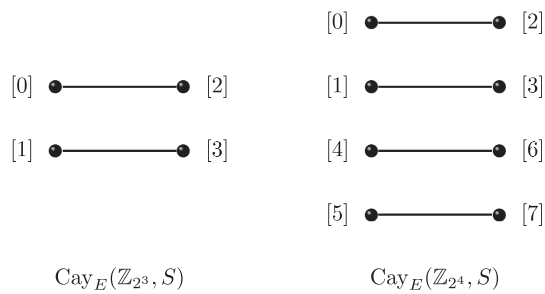

Fig. 1 Two compressed Cayley graphs

$p+(t+t_3)m \in S$, we have $(p+tm)+s_3 \in p+S$. Also $(p+tm)+S_4 = (p+tm)-(p+t_4m) = p-(p+(t_4-t)m)$. Since $-(p+(t_4-t)m) \in S$, we have $(p+tm)+s_4 \in p+S$. Hence $(p+tm)+S \subseteq p+S$.

Therefore $[p] = [p+tm]$.

Now, we show that $[-p] = [-(p+tm)],\ 0 \le t \le p-1$. Let $-p+s_5$ and $-p+s_6$ be two arbitrary elements in $-p+S$, where $s_5 \in A_1$ and $s_6 \in A_2$. Since $s_5 \in A_1$ and $s_6 \in A_2$, there exist $t_5, t_6 \in \{0, 1, \ldots, p-1\}$ such that $s_5 = p+t_5m$ and $s_6 = -(p+t_6m)$. We have $-p+s_5 = (-p)+(p+t_5m) = -(p+tm)+(p+(t_5+t)m)$, which implies that $-p+s_5 \in -(p+tm)+S$. Also $-p+s_6 = -p-(p+t_6m) = -(p+tm)-(p+(t_6-t)m)$, and so $-p+s_6 \in -(p+tm)+S$. Hence $-p+S \subseteq -(p+tm)+S$. Similarly, for two arbitrary elements $-(p+tm)+s_7$ and $-(p+tm)+s_8$ in $-(p+tm)+S$, where $s_7 \in A_1$ and $s_8 \in A_2$, we have that they belong to $-p+S$. Hence $S_E = \{[p], [-p]\}$ Also, one can see that the vertices of $\mathrm{Cay}_E(\mathbb{Z}_{pm}, S)$ read as follows:

$$
\begin{aligned}
&[0] = \{0, m, 2m, \ldots, (p-1)m\},\\
&[1] = \{0, m+1, 2m+1, \ldots, (p-1)m+1\},\\
&\vdots\\
&[m-1] = \{m-1, m+(m-1), \ldots, (p-1)m+(m-1)\}.
\end{aligned}
$$

Hence the compressed Cayley graph $\mathrm{Cay}_E(\mathbb{Z}_{pm}, S)$ is a 2-regular graph with m vertices. □

Proposition 3.2 *Let $n = pm$ and let $[a]$ be an arbitrary vertex in $\mathrm{Cay}_E(\mathbb{Z}_n, S)$. Then $[a]$ is adjacent to $[a-p]$ and $[a+p]$.*

Proof By Theorem 3.1, the compressed Cayley graph $\mathrm{Cay}_E(\mathbb{Z}_n, S)$ is a 2-regular graph. Hence, there exist two distinct vertices $[b], [c] \in (\mathbb{Z}_n)_E$, such that $[a]$ is adjacent to $[b]$ and $[c]$. Thus $[b]-[a] \in S_E$ and $[c]-[a] \in S_E$. Since $[b]-[a] \in S_E$, by Theorem 3.1, we have $[b]-[a] = [p]$ or $[b]-[a] = [-p]$. without loss of generality, we may assume that $[b]-[a] = [p]$. Therefore $[b] = [a]+[p] = [a+p]$. Now, since $[c]-[a] \in S_E$, we have $[c]-[a] = [p]$ or $[c]-[a] = [-p]$. If $[c]-[a] = [p]$, then $[c] = [p]+[a] = [b]$, which is impossible. Therefore $[c]-[a] = [-p]$, which implies that $[c] = [a-p]$. □

Remark 3.3 For each $a \in \mathbb{Z}_{pm}$, there exists an integer t with $0 \le t \le m-1$ such that $a \equiv t \bmod (m)$.

In the following example, we consider situations that $n \in \{2^3, 2^4\}$.

Example 3.4 (i) Let $n = 2^3$. Then $S = \{2, 6\}$ and $S_E = \{[2]\}$. The compressed Cayley graph $\mathrm{Cay}_E(\mathbb{Z}_{2^3}, S)$ is pictured in Figure 1.

(ii) Let $n = 2^4$. Then $S = \{2, 14, 10, 6\}$ and $S_E = \{[2]\}$. $\mathrm{Cay}_E(\mathbb{Z}_{2^4}, S)$ is pictured in Figure 1.
In the following example, we investigate the case that $n \in \{p, p^2\}$.

Example 3.5 (i) Let $n = p$ and $S = \{1, 2, \ldots, p-1\}$. Then $S_E = \{[1], \ldots, [p-1]\}$ and by Theorem 2.4, we have $\mathrm{Cay}(\mathbb{Z}_p, S) \cong k_p \cong \mathrm{Cay}_E(\mathbb{Z}_p, S)$.

(ii) Let $n = p^2$ and $S = \{p, -p, 2p, -2p, \ldots, \lceil \frac{p-1}{2} \rceil p, -\lceil \frac{p-1}{2} \rceil p\}$, where $\lceil x \rceil$ denotes the least integer that is greater than or equal to x for $x \in \mathbb{Z}$. Then $|S_E| = |S|$, and by Theorem 2.4, $\mathrm{Cay}(\mathbb{Z}_{p^2}, S) \cong \mathrm{Cay}_E(\mathbb{Z}_{p^2}, S)$. Moreover, $\mathrm{Cay}(\mathbb{Z}_{p^2}, S) = \cup_{i=1}^{p} k_p$.
The Cayley graph $\mathrm{Cay}(\mathbb{Z}_{3^2}, S)$ is pictured in Figure 2.

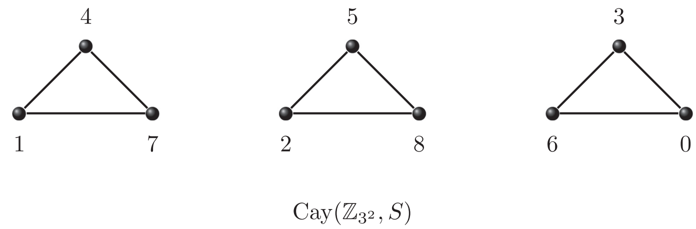

Fig. 2 A Cayley graph

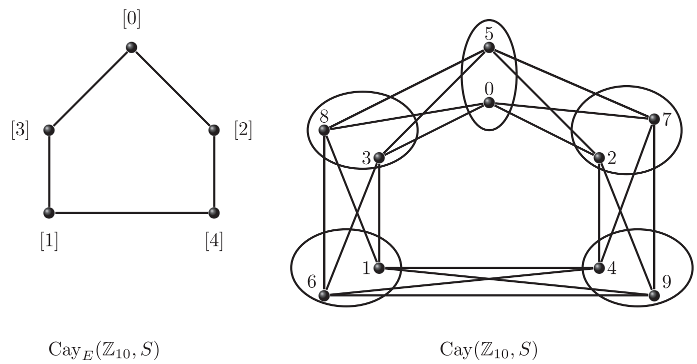

Fig. 3 A compressed Cayley graph and A Cayley graph

In the following proposition, we determine the structure of $\mathrm{Cay}_E(\mathbb{Z}_{pq}, S)$, where $n = pq$ and p, q are prime numbers with $p < q$.

Proposition 3.6 *Let $n = pq$, where p, q are prime numbers with $p < q$. Then the compressed Cayley graph $\mathrm{Cay}_E(\mathbb{Z}_{pq}, S)$ is isomorphic to the cycle of length q.*

Proof Note that $(\mathbb{Z}_{pq})_E$ is a cyclic group with $|(\mathbb{Z}_{pq})_E| = q$. Since q is a prime number, $[p]$ is a generator of $(\mathbb{Z}_{pq})_E$. By Theorem 3.1, $S_E = \{[p], [-p]\}$. Now, since $\mathrm{Cay}_E(\mathbb{Z}_{pq}, S)$ is a connected graph with q vertices and also it is 2-regular, we have $\mathrm{Cay}_E(\mathbb{Z}_{pq}, S)$ is isomorphic to the cycle C_q. □

Example 3.7 (i) Consider the group $\mathbb{Z}_{10}$. Then we have $p = 2$ and $q = 5$. By Theorem 3.1, $S = \{2, 8, 7, 3\}$. Hence $\mathrm{Cay}(\mathbb{Z}_{10}, S)$ is a 4-regular graph with 10 vertices. Now, consider $\mathrm{Cay}_E(\mathbb{Z}_{10}, S)$. By Theorem 3.1 and Remark 3.3, we have $S_E = \{[2], [3]\}$ and $(\mathbb{Z}_{10})_E = \{[0], [1], [2], [3], [4]\}$. The graphs $\mathrm{Cay}(\mathbb{Z}_{10}, S)$ and $\mathrm{Cay}_E(\mathbb{Z}_{10}, S)$ are pictured in Figure 3.

(ii) Consider the group $\mathbb{Z}_{21}$. we have $p = 3$ and $q = 7$. By Theorem 3.1, $S = \{3, 18, 10, 11, 17, 4\}$, $S_E = \{[3], [4]\}$ and $(\mathbb{Z}_{21})_E = \{[0], [1], \dots, [6]\}$. The $\mathrm{Cay}(\mathbb{Z}_{21}, S)$ is a 6-regular graph with 21 vertices. By Proposition 3.6, $\mathrm{Cay}_E(\mathbb{Z}_{21}, S)$ is a cycle of length 7.

Example 3.8 Assume that $n = p^2q$, where p, q are prime numbers with $p < q$. Then $\mathrm{Cay}_E(\mathbb{Z}_{p^2q}, S) \cong \cup_{i=1}^{p} C_q$. The compressed Cayley graph $\mathrm{Cay}_E(\mathbb{Z}_{20}, S)$, is pictured in Figure 4, where $S = \{2, 18, 12, 8\}$ and $S_E = \{[2], [8]\}$.

4 Laplacian spectrum

Assume that G is a simple graph with n vertices, whose vertex set is $V(G) = \{v_1, v_2, \dots, v_n\}$. For a vertex v of G, $\mathrm{N}_G(v)$ denotes the set of vertices of G that are adjacent to v in G, and we denote $|\mathrm{N}_G(v)|$ by $\deg(v)$. The *adjacency matrix* of G is an $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if $v_i - v_j$, and otherwise $a_{ij} = 0$. Also, $D(G)$ is the diagonal matrix with the (v, v)th entry having value $\deg v$. The *Laplacian matrix* $L(G)$ of G is defined by $L(G) = D(G) - A(G)$. The eigenvalues of $L(G)$ are called the *Laplacian eigenvalues* of G. Clearly, $L(G)$ is a real, symmetric, and positive semidefinite matrix, and so all its eigenvalues are real and nonnegative. Also, the sum of the entries in each row of $L(G)$ is zero, and hence the smallest eigenvalue of $L(G)$ is zero with corresponding eigenvector $[1, 1, \dots, 1]^T$.

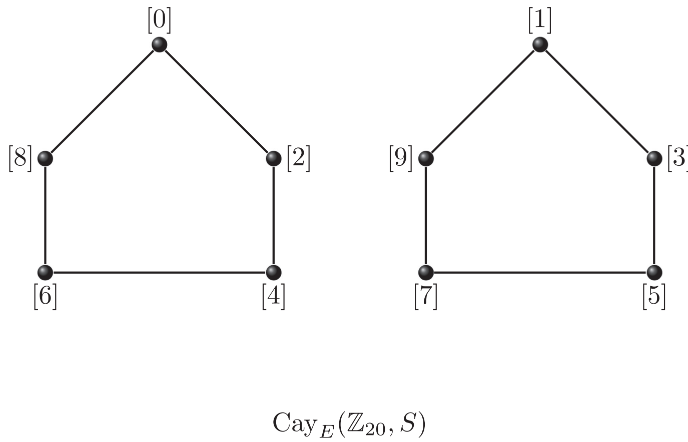

Fig. 4 A compressed Cayley graph

The *spectrum* of a square matrix B, denoted by $\sigma(B)$, is the multiset of all the eigenvalues of B. Let $\mu_1, \mu_2, \ldots, \mu_r$ be distinct eigenvalues of B with multiplicities $m_1, m_2, \ldots, m_r$, respectively. Then we denote the spectrum of B by

$$\sigma(B) = \begin{pmatrix} \mu_1 & \mu_2 & \ldots & \mu_r \\ m_1 & m_2 & \ldots & m_r \end{pmatrix}.$$

For a graph G, the spectrum of $L(G)$, is called the *Laplacian spectrum* and it is denoted by $\sigma_L(G)$. The Laplacian spectrum of graphs was widely studied in [19]. The Laplacian spectrum of the complete graph K_n and its complement $\overline{K_n}$ are known. Indeed,

$$\sigma_L(K_n) = \begin{pmatrix} 0 & n \\ 1 & n-1 \end{pmatrix} \text{ and } \sigma_L(\overline{K_n}) = \begin{pmatrix} 0 \\ n \end{pmatrix}.$$

Let $n = pm$, where p is the least prime integer that divides n, and let $m \notin \{1, p\}$. Note that, by Theorem 3.1, $|[t]| = p$ for each $[t] = (\mathbb{Z}_n)_E$ with $0 \leq t \leq m-1$. Therefore in view of Theorem 2.4, $\mathrm{Cay}(\mathbb{Z}_{pm}, S)$ is isomorphic to $\mathrm{Cay}_E(\mathbb{Z}_{pm}, S)$-generalized join graph

$$\mathrm{Cay}_E(\mathbb{Z}_{pm}, S)[\underbrace{\bar{k}_p, \bar{k}_p, \ldots, \bar{k}_p}_{m-times}].$$

The following theorem, which determines the Laplacian spectrum of the generalized join graphs, was proved in [11].

Theorem 4.1 *Let G be a graph on k vertices with $V(G) = \{v_1, v_2, \ldots, v_k\}$ and let $H_1, H_2, \ldots, H_k$ be k pairwise disjoint graphs on $m_1, m_2, \ldots, m_k$ vertices, respectively. Then the Laplacian spectrum of $G[H_1, H_2, \ldots, H_k]$ is given by*

$$\sigma_L(G[H_1, H_2, \ldots, H_k]) = (\bigcup_{j=1}^{k}(M_j + (\sigma_L(H_j) \setminus \{0\}))) \bigcup \sigma(\mathbb{L}(G)), \tag{1}$$

where

$$M_j = \begin{cases} \sum_{v_i - v_j} m_i & \text{if } N_G(v_j) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathbb{L}(G) = \begin{pmatrix} M_1 & -S_{1,2} & \ldots & -S_{1,k} \\ -S_{1,2} & M_2 & \ldots & -S_{2,k} \\ \vdots & & \ddots & \\ -S_{1,k} & -S_{2,k} & \ldots & M_k \end{pmatrix},$$

and

$$S_{i,j} = \begin{cases} \sqrt{n_i n_j} & \textit{if } v_i - v_j \text{ in } G, \\ 0 & \textit{otherwise.} \end{cases}$$

In (1), $\sigma_L(H_j)\backslash\{0\}$ means that one copy of the eigenvalue 0 is removed from the multiset $\sigma_L(H_j)$ and $M_j + (\sigma_L(H_j)\backslash\{0\})$ means that M_j is added to each element of $\sigma_L(H_j)\backslash\{0\}$.

Consider G as a vertex weighted graph by assigning the weight $m_i = |V(H_j)|$ to the vertex v_i of G for $1 \leqslant i \leqslant k$. Let $L(G) = (L_{i,j})$ be defined as follows:

$$L_{i,j} = \begin{cases} -m_j & \text{if } i \neq j \text{ and } v_i - v_j, \\ \sum_{v_i - v_r} m_r & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $L(G)$ is called the vertex weighted Laplacian matrix of G. In [12, p. 317], it is shown that $L(G)$ and $\mathbb{L}(G)$ are similar, and so $\sigma(\mathbb{L}(G)) = \sigma(L(G))$. Hence in Theorem 4.1, we can use $\sigma(L(G))$ instead of $\sigma(\mathbb{L}(G))$ for determining the Laplacian spectrum of $G[H_1, H_2, \ldots, H_k]$.

In the following Theorem, we describe the Laplacian spectrum of $cay(\mathbb{Z}_n, S)$, where $n = pm$ and $m \notin \{1, p\}$.

Theorem 4.2 *Let $n = pm$, where p is the least prime number that divides n and $m \notin \{1, p\}$. Then*

$$\sigma_L(\text{Cay}(\mathbb{Z}_{pm}, S)) = \begin{pmatrix} 2p \\ m(p-1) \end{pmatrix} \bigcup \sigma(L(\text{Cay}_E(\mathbb{Z}_{pm}, S))).$$

Proof Note that $\text{Cay}(\mathbb{Z}_{pm}, S) = \text{Cay}_E(\mathbb{Z}_{pm}, S)[\underbrace{\bar{k}_p, \bar{k}_p, \ldots, \bar{k}_p}_{m-times}]$.

Let $v_1, v_2, \ldots, v_m$ be the vertices of $G = \text{Cay}_E(\mathbb{Z}_{pm}, S)$.

By Theorem 3.1, G is a 2-regular graph, and so $M_j = 2p$ for each $1 \leq j \leq m$. Since the Laplacian spectrum of $\bar{k}_p$ is $\begin{pmatrix} 0 \\ p \end{pmatrix}$, we have $M_j + (\sigma_L(\bar{k}_p) \setminus \{0\}) = \begin{pmatrix} 2p \\ p-1 \end{pmatrix}$, and therefore

$$\bigcup_{j=1}^{m} M_j + (\sigma_L(\bar{k}_p) \setminus \{0\}) = \begin{pmatrix} 2p \\ m(p-1) \end{pmatrix}.$$

Now, in order to describe the Laplacian spectrum of $\text{Cay}(\mathbb{Z}_{pm}, S)$, it is enough to determine the spectrum of the matrix $L(G)$ with the following entries:.

$$L_{i,j} = \begin{cases} -p & \text{if } i \neq j \text{ and } v_i - v_j, \\ 2p & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\sigma_L(\text{Cay}(\mathbb{Z}_{pm}, S)) = \begin{pmatrix} 2p \\ m(p-1) \end{pmatrix} \bigcup \sigma(L(G)).$$

□

Example 4.3 Consider the Cayley graph $\text{Cay}(\mathbb{Z}_{10}, S)$, which is introduced in Example 3.7 (i). Theorem 2.6 implies that

$$\text{Cay}(\mathbb{Z}_{10}, S) = \text{Cay}_E(\mathbb{Z}_{10}, S)[\underbrace{\bar{k}_2, \ldots, \bar{k}_2}_{5-times}].$$

In view of Theorem 4.1, we have

$$L\big(\operatorname{Cay}_E(\mathbb{Z}_{10}, S)\big) = \begin{bmatrix} 4 & 0 & -2 & -2 & 0 \\ 0 & 4 & 0 & -2 & -2 \\ -2 & 0 & 4 & 0 & -2 \\ -2 & -2 & 0 & 4 & 0 \\ 0 & -2 & -2 & 0 & 4 \end{bmatrix}.$$

Therefore $\sigma\Big(L\big(\operatorname{Cay}_E(\mathbb{Z}_{10}, S)\big)\Big) = \begin{pmatrix} -2 & -4 \\ 4 & 1 \end{pmatrix}$. Now, Theorem 4.2 implies that

$$\sigma_L\big(\operatorname{Cay}(\mathbb{Z}_{10}, S)\big) = \begin{pmatrix} 4 & -2 & -4 \\ 5 & 4 & 1 \end{pmatrix}.$$

Recall that the largest eigenvalue of $L(G)$, denoted by $\lambda(G)$, is called the *Laplacian spectral radius* of G. The following theorem was proved in [22, Theorem 2.3].

Theorem 4.4 *[22] Let G be a connected graph on m vertices with maximal degree $\Delta(G)$. Then $\lambda(G) \geqslant \Delta(G)+1$, and the equality holds if and only if $\Delta(G) = m - 1$.*

Corollary 4.5 *Let $n = pq$, where p, q are prime numbers such that $p < q$. Then $\lambda(\operatorname{Cay}_E(\mathbb{Z}_{pq}, S)) \geq 3$ and the equality holds if and only if $n = 6$.*

Proof Let $G = \operatorname{Cay}_E(\mathbb{Z}_{pq}, S)$. By Proposition 3.6, G is isomorphic to the cycle of length q, and so $\Delta(G) = 2$. Hence $\lambda(G) \geq 3$. Now assume that $\lambda(G) = 3$. Thus, Theorem 4.4 implies that $\Delta(G) = 2 = q - 1$. Hence $q = 3$, and since $p < q$, we have $p = 2$. Therefore $n = 6$. Conversely, let $n = 6$. Since G is isomorphic to the cycle C_3, we have $m = 3$, and so $\Delta(G) = m - 1$, which implies that $\lambda(G) = \Delta(G) + 1 = 3$. □

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