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ORIGINAL RESEARCH

On mock theta functions of Gordon and McIntosh

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Abstract A new approach involving the concept of modular Ferrers diagram is employed to interpret the Gordon and McIntosh's mock theta functions in terms of $(n+t)$-color partitions. We then provide the generalized versions of the mock theta functions and their interpretations. We further found some arithmetic properties for two of the mock theta functions $\beta(q)$ and $\xi(q)$.

Keywords Mock theta functions · $(n+t)$-color partitions · Modular Ferrers diagram · Arithmetic properties

Mathematics Subject Classification 11P81 · 11P83 · 05A15 · 05A17

1 Introduction

In this paper, we study the combinatorics associated with Gordon and McIntosh's mock theta functions. The term 'mock theta functions' and their associated functions were introduced by Ramanujan. He introduced these without giving any formal definition and offered seventeen mock theta functions: four of order 3, ten of order 5 and three of order 7. Later, Watson found three more mock theta functions of order 3. He has shown that Ramanujan's mock theta functions together with defined by himself satisfy the strong approximation property according to which at each root of unity ζ, there is a θ-function $T_\zeta(q)$ such that $M(q) = T_\zeta(q) + O(1)$ as $q \mapsto \zeta$ radially, where $M(q)$ is mock theta function, for details see [13]. The strong approximation property was achieved by satisfying certain modular transformation laws. In a sequel, Gordon and McIntosh obtained modular transformation laws for two mock theta functions $\chi_3(q)$ and $\rho_3(q)$, given below, of order 3 due to Ramanujan and Watson, respectively. To do so, they found two more mock theta functions $\xi(q)$ and $\sigma(q)$ of the order 3 in [12], given below:

$$\chi_3(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q)_n}{(-q^3;q^3)_n}, \tag{1}$$

$$\rho_3(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(q;q^2)_{n+1}}{(q^3;q^6)_{n+1}}, \tag{2}$$

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$$\xi(q) = 1 + 2\sum_{n=1}^{\infty} \frac{q^{6n(n-1)+1}}{(q;q^6)_n(q^5;q^6)_n}, \tag{3}$$

$$\sigma(q) = \sum_{n=1}^{\infty} \frac{q^{3n(n-1)}}{(-q;q^3)_n(-q^2;q^3)_n}, \tag{4}$$

where

$$(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i) \qquad \text{and} \qquad (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i),$$

and a, q are complex numbers with $\mid q \mid < 1$.
Gordon and McIntosh further found identities relating third order mock theta functions that are considered as 'third order mock theta conjectures'. McIntosh [13] has introduced three additional mock theta functions of order 6, which are presented below and are appeared in modular transformation laws.

$$\beta(q) = \sum_{n=0}^{\infty} \frac{q^{3n^2+3n+1}}{(q;q^3)_{n+1}(q^2;q^3)_{n+1}}, \tag{5}$$

$$\upsilon_6(q) = \sum_{n=0}^{\infty} \frac{q^{n+1}(-q;q)_{2n+1}}{(q;q^2)_{n+1}}, \tag{6}$$

$$\xi_6(q) = \sum_{n=0}^{\infty} \frac{q^{n+1}(-q;q)_{2n}}{(q;q^2)_{n+1}}. \tag{7}$$

We turn finally to the tenth order mock theta functions from the Lost Notebook [3] as:

$$\phi_{10}(q) = \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(q;q^2)_{n+1}}, \tag{8}$$

$$\psi_{10}(q) = \sum_{n=1}^{\infty} \frac{q^{\binom{n+1}{2}}}{(q;q^2)_n}, \tag{9}$$

$$X_{10}(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}}, \tag{10}$$

$$\chi_{10}(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}. \tag{11}$$

Some relations connecting the above tenth order mock theta functions are given by Ramanujan that were proved by Choi [8]. For the interested reader concerning the study of mock theta functions with their classical and modern developments we refer [3]. Mock theta functions are closely connected to combinatorics, precisely partition theory.

In this paper, we study the combinatorial interpretations for coefficients of the mock theta functions given from (3)-(7) and (10)-(11) in terms of $(n+t)$-color partitions using modular Ferrers diagram (discussed in next section), these results are given in Section 2 and 3, respectively. We then provide the combinatorial interpretations of the generalized version of the mock theta functions in Section 4. Finally we explore some arithmetic properties for the two mock theta functions $\beta(q)$ and $\xi(q)$ in Section 5.

2 Interpretations of mock theta functions of Gordon and McIntosh

We start this section by recalling some basic definitions.

Definition 1 A Partition, $k = (k_1, k_2, \cdots, k_s)$ is a finite sequence of non-increasing positive integers. The elements k_i, $i \in \{1, 2, \cdots, s\}$ are called parts or summands. The sum of all parts, denoted by $\mid k \mid$, is called norm of partition k. The number of partitions of n is denoted by $p(n)$. By convention, $p(0) = 1$ and $p(n) = 0$, for n being a negative number.

Partitions have a graphical representation called a Ferrers diagram (or Young diagram), which is a left-justified array of cells where the i^{th} row has k_i cells as shown in Figure 1 that indicate the partition of 13 into parts (5, 4, 2, 2).

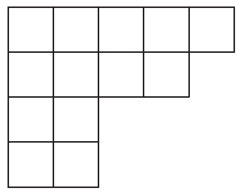

Fig. 1 Ferrers diagram for partition k=(5,4,2,2)

The notion of modular partitions first presented by MacMahon [15] and is also discussed by Berndt et al. [6] for the bijective proofs of Ramanujan's identities for a partial theta function.

Modular Ferrers diagram- These are also named as p-modular Ferrers diagram. For a partition k into parts k_i congruent to u modulo p where $0 < u \leq p$, its p-modular Ferrers diagram is the diagram in which the i^{th} row has $\lceil \frac{k_i}{p} \rceil$ cells, the cells in the last column have u and others have p. The sum of the numbers in all the cells equals $\mid k \mid$. For example, 2-modular Ferrers diagram for the partition (9, 7, 5, 5, 1) of 27 is shown in Figure 2. Here we also recall $(n+t)$-color partition which is another interesting tool to understand q-series combinatorially that was introduced by Agarwal and Andrews in [1].

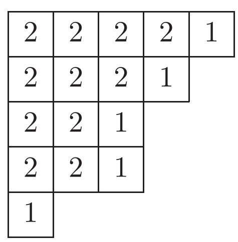

Fig. 2 2-modular Ferrers diagram for partition k=(9,7,5,5,1)

Definition 2 An $(n + t)-$color partition of a positive integer is a partition in which each part of size 'n', $n \geq 0$ can come with '$n + t$' different colors. Note that zeros are allowed if and only if $t > 0$ but zeros cannot repeat. For $t = 0$, $(n + t)$-color partitions are known as n-color partitions and the parts follow the order

$$1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < \cdots .$$

Example 1 Consider $(n + 1)$-color partitions of 2 as:

$$2_1, 2_2, 2_3, 2_1 0_1, 2_2 0_1, 2_3 0_1, 1_1 1_1, 1_2 1_1, 1_2 1_2, 1_1 1_1 0_1, 1_2 1_1 0_1, 1_2 1_2 0_1.$$

For simplicity we represent an $(n + t)$-color partition as

$$k = ((k_1)_{z_1}, (k_2)_{z_2}, \ldots, (k_s)_{z_s}),$$

a finite non-increasing sequence of colored positive integers into s parts, where k_i's are parts and z_i's are colors or subscripts. Here, $(k_s)_{z_s}$ will be considered as the least part or smallest part.
The weighted difference of two consecutive parts $(k_a)_{z_a}$ and $(k_b)_{z_b}$ (where $(k_a) \geq (k_b)$) in an $(n + t)$-color partition is $k_a - k_b - z_a - z_b$ and symbolize the weighted difference by W_a. For example, consider the partition

of 7 as $(4_3, 2_1, 1_1)$. Here $k_1 = 4, k_2 = 2, k_3 = 1$ and $z_1 = 3, z_2 = 1, z_3 = 1$, then the weighted difference for two consecutive parts is $W_1 = -2$ and $W_2 = -1$.

Fine [10] provided the combinatorial interpretation of third order mock theta function $\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}$, of Ramanujan as *'partition into odd parts without gaps'*. Then some more mock theta functions of different orders have been studied combinatorially in [2,7,17,18]. In 2012, Choi and Kim [9] have given the partition theoretic interpretations for some third and sixth order mock theta functions as generating functions of n-color overpartitions. Motivated from their work we provide the interpretations of mock theta functions given by Gordon and McIntosh of order 3 and 6.

To develop the proof in a systematic manner, we construct the modular Ferrers diagram with colors to represent an $(n+t)$-color partition, for instance shown in Figure 3. In Ferrers diagram of a partition k we label the exposed cell (the cell located at the end of the row) in each row k_i as done in [11].

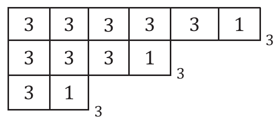

Fig. 3 Modular Ferrers diagram with colors for $k = (16_3, 10_3, 4_3)$

Now we consider

$$v_6(q) = \sum_{n=0}^{\infty} \frac{q^{n+1}(-q;q)_{2n+1}}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{n+1}(-q;q^2)_{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}},$$

and start with 2-modular Ferrers diagram as each term is of the form $(a;q^2)_n$. The term q^{n+1} produces the partition $k^{(1)}$ containing $(n+1)$ parts, each equal to 1. Allot the color 1 to each part so W_i between two consecutive parts is $-2 \ \forall \ 1 \le i < s$. The term $(-q;q^2)_{n+1}$ generates partition $k^{(2)}$ into distinct odd parts $\le 2n+1$. From the largest part of $k^{(2)}$, we append each part $k_j^{(2)}$ in the following manner: We append 2 starting from the first row to $((k_j^{(2)}-1)/2)^{th}$ row and attach 1 to $((k_j^{(2)}+1)/2)^{th}$ row. So W_j between the parts where different entries are attached increases by 1 and for rest of the cases it remains same. The factor $(-q^2;q^2)_n$ produces partition $k^{(3)}$ with distinct even parts $\le 2n$. Now we fix these parts to the earlier partition and attach 2 starting from the first row up to $(k_j^{(3)}/2)^{th}$ row. This step does not make any changes in W_i. At the end, $(q;q^2)_{n+1}^{-1}$ generates partition $k^{(4)}$ into odd parts $\le 2n+1$. We fix $k_j^{(4)}$ in a similar way as done for $k_j^{(2)}$. But now, we increase the color of $((k_j^{(4)}+1)/2)^{th}$ part by 1. So value of W_j remains unaltered.

Thus we see that $v_6(q)$ is the generating function for restricted n-color partition stated in the theorem given below:

Theorem 1 *$v_6(q)$ is the generating function for n-color partitions satisfying:*

(i) *smallest part is of the form $(k_s)_{k_s}$ or $(k_s+1)_{k_s}$,*

(ii) $W_i \ge \begin{cases} -2 \text{ if } k_i \equiv z_i \pmod 2, k_{i+1} \equiv z_{i+1} \pmod 2, \\ \ \ 0 \text{ if } k_i \not\equiv z_i \pmod 2, k_{i+1} \not\equiv z_{i+1} \pmod 2, \\ -1 \quad \text{otherwise}, \end{cases}$ *and $W_i \le 2 \ \forall \ 1 \le i < s$.*

For clarity of Theorem 1, we show the insertion steps in Figure 4, starting with partition $k^{(1)} = (1,1,1,1,1)$, followed by insertion of $k^{(2)} = (5,3)$, $k^{(3)} = (8,2)$ and $k^{(4)} = (9,5,5)$, respectively.
Another mock theta function $\xi_6(q)$ is interpreted as:

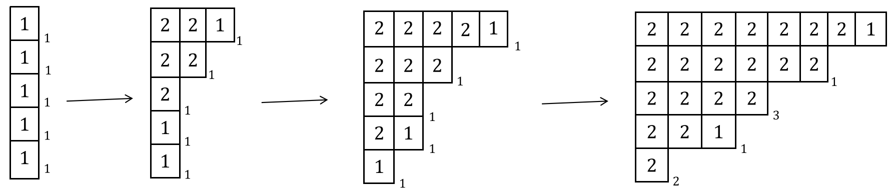

Fig. 4 Insertion of $k^{(2)} = (5, 3)$ into $k^{(1)} = (1, 1, 1, 1, 1)$ followed by the insertion of $k^{(3)} = (8, 2)$ and $k^{(4)} = (9, 5, 5)$

Theorem 2 *$\xi_6(q)$ is the generating function for n-color partitions where smallest part is of the form $(k_s)_{k_s}$ and second condition is same as in Theorem* 1.

Now we state the interpretation for $\beta(q)$ given below:

$$\beta(q) = \sum_{n=0}^{\infty} \frac{q^{3n^2+3n+1}}{(q; q^3)_{n+1}(q^2; q^3)_{n+1}}.$$

Theorem 3 *$\beta(q)$ is the generating function for $(n+1)$-color partitions satisfying:*

(i) $z_s \geq 2$ *and* $z_i \geq 3\ \forall\ i \neq s$,
(ii) $k_s \geq 1$ *and* $k_s - z_s \equiv 1 \pmod 2$,
(iii) $k_i \geq \begin{cases} 3\sum_{j=i+1}^{s} \frac{(-1)^{j-1}}{2^{j-i}} P_j & \text{if } i \text{ even}, \\ 3\sum_{j=i+1}^{s} \frac{(-1)^{j}}{2^{j-i}} P_j & \text{if } i \text{ odd}, \end{cases}$
where,

$$P_j = k_j + z_j + 1.$$

If equality holds then $z_i = 3$, if k_i greater than right hand side by A odd (even) units then $4 \leq z_i \leq 3 + A$ $(3 \leq z_i \leq 3 + A)$ and is always even (odd).

To further explain the conditions of the aforementioned theorem, consider the subsequent example.

Example 2 Note that $\beta(q) = q^1 + q^2 + 2q^3 + 2q^4 + \cdots + 46q^{21} + 50q^{22} + 60q^{23} + \cdots$ which implies that there are 50 partitions of the number 22 satisfying the conditions of Theorem 3 that are listed in the Table 1. To explain the relevant partitions and the conditions of Theorem 3 we consider all the relevant colored partitions for $|\ k\ | = 22$ into three parts $((k_1)_{z_1}, (k_2)_{z_2}, (k_3)_{z_3})$. From conditions (i) and (ii), let $(k_s)_{z_s} = 1_2$. From condition (iii),

$$k_2 \geq 3\sum_{j=3}^{s} \frac{(-1)^{j-1}}{2^{j-2}}(k_j + z_j + 1),$$
$$k_2 \geq 3\frac{(-1)^2}{2}(k_3 + z_3 + 1),$$
$$k_2 \geq 6.$$

If we take k_2 as 6 thus equality holds so $z_2 = 3$. Else if we take k_2 as 7 so k_2 greater than right hand side by 1 unit (odd), then $4 \leq z_2 \leq 3 + 1$ or $z_2 = 4$ and even. We can take $(k_2)_{z_2} = 7_4$. To find out $(k_1)_{z_1}$, consider two cases:

Case I When $(k_2)_{z_2} = 6_3$

$$k_1 \geq 3\sum_{j=2}^{s} \frac{(-1)^{j}}{2^{j-1}}(k_j + z_j + 1),$$
$$k_1 \geq 3\frac{(-1)^2}{2}(k_2 + z_2 + 1) + 3\frac{(-1)}{2^2}(k_3 + z_3 + 1),$$
$$k_1 \geq 12.$$

Now to reach at the partition of 22, k_1 should be 15, so $A = 3$ units (odd) greater than the right hand side. Also $4 \le z_1 \le 3 + 3$ and is always even. So relevant partitions are $(15_4 6_3 1_2)$, $(15_6 6_3 1_2)$.

Case II When $(k_2)_{z_2} = 7_4$

$$k_1 \ge 3 \sum_{j=2}^{s} \frac{(-1)^j}{2^{j-1}} (k_j + z_j + 1),$$
$$k_1 \ge 3 \frac{(-1)^2}{2} (k_2 + z_2 + 1) + 3 \frac{(-1)}{2^2} (k_3 + z_3 + 1),$$
$$k_1 \ge 15.$$

If we choose $k_1 = 15$, we get a partition of 23, so we discard 7_4 as $(k_2)_{z_2}$. Rest of the partitions of 22 into two parts can also be calculated in similar manner.

Table 1 50 partitions of $| k |= 22$ relevant to $\beta(q)$

s	Relevant colored partitions for $\| k \|= 22$
1	$22_3, 22_5, 22_7, 22_9, 22_{11}, 22_{13}, 22_{15}, 22_{17}, 22_{19}, 22_{21}, 22_{23}$.
2	$21_4 1_2, 21_6 1_2, 21_8 1_2, 21_{10} 1_2, 21_{12} 1_2, 21_{14} 1_2, 21_{16} 1_2, 21_{18} 1_2, 21_{18} 1_2, 20_4 2_3, 20_6 2_3, 20_8 2_3, 20_{10} 2_3, 20_{12} 2_3, 20_{14} 2_3, 19_3 3_2, 19_5 3_2, 19_7 3_2, 19_9 3_2, 19_{11} 3_2, 19_{13} 3_2, 19_4 3_4, 19_6 3_4, 19_8 3_4, 19_{10} 3_4, 18_3 4_3, 18_5 4_3, 18_7 4_3, 18_9 4_3, 18_4 4_5, 18_6 4_5, 17_4 5_2, 17_6 5_2, 17_8 5_2, 17_3 5_4, 17_5 5_4, 16_4 6_3, 15_3 7_2$.
3	$15_4 6_3 1_2, 15_6 6_3 1_2$.

Now we present the proof of Theorem 3.

Proof The proof follows from 3-modular Ferrers diagram as each term is of the form $(a; q^3)_n$. The factor q^{3n^2+3n+1} generates partition $k^{(1)}$ with $(n + 1)$ parts $(1, 6, 12, \cdots 6n)$. Allot the color 3 to each part and the smallest part is assigned with color 2 (color of $k_i^{(1)}$ is calculated using $[\frac{k_{i+1}^{(1)} - k_i^{(1)}}{2}]$) so $W_i = 0 \; \forall \; 1 \le i < s$.

Now we attach $k_i^{(2)}$ part from partition $k^{(2)}$ generated by $(q^2; q^3)_{n+1}^{-1}$ containing parts of the form $(3n + 2)$ and each part $\le (3n + 2)$. Append 3 from the largest part of the partition $k^{(2)}$ to the original partition up to $((k_i^{(2)} - 2)/3)^{th}$ row and 2 to $((k_i^{(2)} + 1)/3)^{th}$ row. In the upcoming step, we attach $k_i^{(3)}$ part from partition $k^{(3)}$ generated from $(q; q^3)_{n+1}^{-1}$ contains parts of the form $3n + 1$, where each part $\le (3n + 1)$. Fix 3 starting from the largest part of partition to $((k_i^{(3)} - 1)/3)^{th}$ row. Attach 1 to $((k_i^{(3)} + 2)/3)^{th}$ row and increase the color of the corresponding part by 1. By tracing the parts we get condition *(iii)* of the theorem. □

The next mock theta function under consideration is $\sigma(q)$, which can be rewritten as:

$$\sigma(q) = \sum_{n=1}^{\infty} \frac{q^{3n(n+1)-6n}}{(-q; q^3)_n (-q^2; q^3)_n}.$$

To obtain the combinatorial interpretation of the mock theta function $\sigma(q)$, we first consider the following function:

$$\sigma_1(q) = \sum_{n=1}^{\infty} \frac{q^{3n(n+1)}}{(-q; q^3)_n (-q^2; q^3)_n}, \tag{12}$$

which has combinatorial interpretation given in the following theorem.

Theorem 4 *$\sigma_1(q)$ is the generating function for n-color partitions satisfying:*

(i) $z_i \ge 3 \; \forall \; i$,
(ii) $k_s - z_s \equiv 1 \pmod 2$ *and* $k_s - z_s \ge 3$,

(iii) $k_i \geq \begin{cases} 3\sum_{j=i+1}^{s} \frac{(-1)^{j-1}}{2^{j-i}} P_j + 3\frac{(-1)^s}{2^{s-i-1}} & \text{if } i \text{ } even, \\ 3\sum_{j=i+1}^{s} \frac{(-1)^j}{2^{j-i}} P_j + 3\frac{(-1)^{s+1}}{2^{s-i-1}} & \text{if } i \text{ } odd, \end{cases}$

where,

$$P_j = k_j + z_j + 1,$$

If equality holds then $z_i = 3$, *if* k_i *greater than right hand side by A odd (even) units then* $4 \leq z_i \leq 3+A$ $(3 \leq z_i \leq 3+A)$ *and is always even (odd),*

(iv) *each partition is counted with* $(-1)^h$ *where* $h = \sum_{j=1}^{s} \frac{(-1)^{j+1}}{2^j}(P_j - 6) - 2s.$

Remark 1 The first three conditions in Theorem 4 contributes towards the count of the partitions enumerated by the unsigned version of $\sigma_1(q)$, that is, $\sigma_1(q) = \sum_{n=1}^{\infty} \frac{q^{3n(n+1)}}{(q;q^3)_n(q^2;q^3)_n}$. The condition (iv) including others, count the number of partitions generated by signed version, that is, $\sigma_1(q)$, here we employ the techniques used in [18].

Theorem 5 *$\sigma(q)$ is the generating function for $(n+3)$-color partitions satisfying:*

(i) $z_i \geq 3\ \forall\ i$,

(ii) $k_s - z_s \equiv 1 \pmod 2$ *and* $k_s - z_s \geq -3$,

(iii) $k_i \geq \begin{cases} 3\sum_{j=i+1}^{s} \frac{(-1)^{j-1}}{2^{j-i}} P_j & \text{if } i \text{ } even, \\ 3\sum_{j=i+1}^{s} \frac{(-1)^j}{2^{j-i}} P_j & \text{if } i \text{ } odd, \end{cases}$

where,

$$P_j = k_j + z_j + 1.$$

If equality holds then $z_i = 3$, *if* k_i *greater than right hand side by A odd (even) units then* $4 \leq z_i \leq 3+A$ $(3 \leq z_i \leq 3+A)$, *and is always even (odd),*

(iv) *each partition is counted with* $(-1)^h$ *where* $h = \sum_{j=1}^{s} \frac{(-1)^{j+1}}{2^j} P_j - 2s.$

Example 3 Consider the expansion of unsigned version of $\sigma(q)$ as:

$$\sum_{n=1}^{\infty} \frac{q^{3n(n-1)}}{(q;q^3)_n(q^2;q^3)_n} = 1 + q + 2q^2 + 2q^3 + \ldots + 7q^9 + 10q^{10} + 11q^{11} + \ldots.$$

As the coefficient of q^9 is 7 that shows that the relevant partitions which satisfies first three conditions given in Theorem 5 are 7 and in the expansion of signed version

$$\sum_{n=1}^{\infty} \frac{q^{3n(n-1)}}{(-q;q^3)_n(-q^2;q^3)_n} = 1 - q + q^4 - q^5 + q^6 - q^7 + q^8 - q^9 - q^{11} + \ldots$$

coefficient of q^9 is -1 and to arrive at the desired result we'll attach certain weight as defined in condition *(iv)*. And the seven relevant partitions and corresponding weight is shown in the Table 2.

Table 2 Partitions of $|\ k\ | = 9$ relevant to $\sigma(q)$

Partitions	h	Weight $= (-1)^h$
$9_4 0_3$	$\frac{1}{2}(9+4+1) - \frac{1}{2^2}(0+3+1) - 4 = 2$	1
$9_6 0_3$	$\frac{1}{2}(9+6+1) - \frac{1}{2^2}(0+3+1) - 4 = 3$	-1
9_4	$\frac{1}{2}(9+4+1) - 2 = 5$	-1
9_6	$\frac{1}{2}(9+6+1) - 2 = 6$	1
9_8	$\frac{1}{2}(9+8+1) - 2 = 7$	-1
9_{10}	$\frac{1}{2}(9+10+1) - 2 = 8$	1
9_{12}	$\frac{1}{2}(9+12+1) - 2 = 9$	-1
Total weight		-1

Now, we present the interpretation for $\xi(q)$.

Theorem 6 *$\xi(q)$ is the generating function for $(n+4)$-color partitions counted twice satisfying:*

(i) $z_i \geq 6 \;\forall\; i \neq s$,
(ii) $k_s - z_s \equiv 1 \pmod 5$, *and* $k_s - z_s \geq -4$,
(iii) $k_i \geq \begin{cases} 6\sum_{j=i+1}^{s} \frac{(-1)^{j-1}}{5^{j-i}} P_j - 18 & \text{if } i \text{ even}, \\ 6\sum_{j=i+1}^{s} \frac{(-1)^{j}}{5^{j-i}} P_j - 18 & \text{if } i \text{ odd}, \end{cases}$
where,

$$P_j = k_j + 4z_j + 4.$$

If equality holds then $z_i = 6$, if k_i greater than right hand side by '$5B + A$' units where $A, B \geq 0$ then $z_i = A + 6$.

Example 4 The relevant partitions of $\xi(q)$ counting twice satisfying Theorem 6 for $n = 21$ are $20_{14}1_5$, $20_9 1_5$, $19_7 2_6$, 21_{25}, 21_{20}, 21_{15}, 21_{10}, 21_5.

Sketch proof of Theorem 4-6 In the proof of $\sigma_1(q)$, we make use of 3-modular Ferrers diagram and rest of the proof is same as done for $\beta(q)$. From Theorem 4, we conclude Theorem 5 with the transition of n-color partition $\mapsto$ $(n+3)$-color partition. For $\xi(q)$, we consider 6-modular Ferrers diagram and remaining steps are similar as done for $\beta(q)$. Hence the details are omitted. □

3 Interpretations of tenth order mock theta functions $X_{10}(q)$, $\chi_{10}(q)$ of Ramanujan

This section of the paper includes the combinatorial interpretation of tenth order mock theta functions given in (10)–(11). We start with tenth order mock theta function:

$$X_{10}(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q^2;q^2)_n(-q;q^2)_{n+1}}.$$

Theorem 7 *$X_{10}(q)$ is the generating function for n-color partitions satisfying:*

(i) *parts and their subscripts have same parity,*
(ii) $W_i \geq 0$ *and even* $\forall\; 1 \leq i < s$,
(iii) *each partition is counted with* $(-1)^h$ *where* $h = \frac{k_1+z_1}{2}$.

Next tenth order mock theta function is given as:

$$\chi_{10}(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q^2;q^2)_n(-q;q^2)_{n+1}}.$$

Theorem 8 *$\chi_{10}(q)$ is the generating function for n-color partitions satisfying:*

(i) *smallest part is of the form* $(k_s)_{k_s}$,
(ii) *parts and their subscripts have same parity,*
(iii) $W_i \geq 0$ *and even* $\forall\; 1 \leq i < s$,
(iv) *each partition is counted with* $(-1)^h$ *where* $h = \frac{k_1+z_1}{2} - 1$.

The proof of above theorems follow from the previous section so we omit the details. In the next section, we give generalized version of all the mock theta functions discussed earlier.

4 Generalized versions

In the coming text, the variables l, m, r are positive integers.

4.1 Generalization of $\upsilon_6(q)$ and $\xi_6(q)$

Mock theta functions $\upsilon_6(q)$ and $\xi_6(q)$ are the members of following 3-parameter family of generating functions:

$$\mathcal{F}_1^{(l,m,r)}(q) = \sum_{n=0}^{\infty} \frac{q^{l(n+1)}(-q^r;q^r)_{2n+1}}{(q^m;q^{2m})_{n+1}}.$$

Theorem 9 *$\mathcal{F}_1^{(l,m,r)}(q)$ generates the n-color partition as:*

(i) *each part $\geq l$,*
(ii) *$k_s - z_s = l-1$ or $l+r-1$,*
(iii) $W_i \geq \begin{cases} -2 & \text{if } k_i - z_i \equiv l-1 \pmod 2, k_{i+1} - z_{i+1} \equiv l-1 \pmod 2, \\ 2r-2 & \text{if } k_i - z_i \not\equiv l-1 \pmod 2, k_{i+1} - z_{i+1} \not\equiv l-1 \pmod 2, \\ r-2 & \text{otherwise,} \end{cases}$
and $W_i \leq 4r-2 \ \forall\ 1 \leq i < s$.

Interpretations of $\upsilon_6(q)$ and $\xi_6(q)$ in terms of n-color partitions follows as $\mathcal{F}_1^{(1,1,1)}(q)$. A displacement of $2n \mapsto 2n+1$ in the factor $(q^r;q^r)_{2n+1}$ for $\upsilon_6(q)$ affects the smallest part condition.

4.2 Generalization of $\beta(q)$, $\sigma(q)$ and $\xi(q)$

We define a 2-parameter family of generating functions as follows:

$$\mathcal{F}_2^{(l,m)}(q) = \sum_{n=1}^{\infty} \frac{q^{2l\binom{n}{2}}}{(q^m;q^l)_n (q^{l-m};q^l)_n},$$

where $l > m$.

Theorem 10 *$\mathcal{F}_2^{(l,m)}(q)$ generates the $(n+l)$-color partitions as:*

(i) *$z_i \geq l \ \forall\ i$,*
(ii) *$k_s - z_s = (l-m)a - l$ where $a \geq 0$,*
(iii) $k_i \geq \begin{cases} l \sum_{j=i+1}^{s} \frac{m^{j-i-1}(-1)^{j-1}}{(l-m)^{j-i}} P_j & \text{if } i \text{ even}, \\ l \sum_{j=i+1}^{s} \frac{m^{j-i-1}(-1)^{j}}{(l-m)^{j-i}} P_j & \text{if } i \text{ odd}, \end{cases}$
where,

$$P_j = k_j + \frac{(l-2m)}{m} z_j + 2(l-m) - \frac{l(l-2m)}{m}.$$

If equality holds then $z_i = l$, if k_i greater than right hand side by '$(l-m)B + mA$' units where $A, B \geq 0$ then $z_i = mA + l$.

$\beta(q)$ is a member of the family $q\mathcal{F}_2^{(3,1)}(q)$ with displacement of $n \mapsto n-1$ and $\sigma(q)$ is a member of the family $\mathcal{F}_2^{(3,1)}(q)$ with one more condition for its signed version that is each partition is counted with $(-1)^h$, where

$$h = \sum_{j=1}^{s} \frac{(-1)^{j+1} m^{j-1}}{(l-m)^j} P_j - 2s,$$

and $\xi(q)$ is a member of the family $1 + 2q\mathcal{F}_2^{(6,1)}(q)$.

4.3 Generalization of $X_{10}(q)$ and $\chi_{10}(q)$

Mock theta functions $X_{10}(q)$ and $\chi_{10}(q)$ are the members of following 2-parameter family of generating functions:

$$\mathcal{F}_3^{(l,r)}(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{ln^2}}{(-q^r;q^r)_{2n}}.$$

Theorem 11 *$\mathcal{F}_3^{(l,r)}(q)$ generates the n-color partitions as:*

(i) *each part $\geq l$,*
(ii) $z_i \geq l \; \forall \, i$,
(iii) $k_i - z_i \equiv 2l(s-i) \pmod{2r} \; \forall \, i$,
(iv) $W_i \equiv 0 \pmod{2r}$ *and W_i is non-negative* $\forall \; 1 \leq i < s$,
(v) *each part is counted with $(-1)^h$ where* $h = \frac{k_1+z_1}{2r} + s\left(1 - \frac{l}{r}\right)$.

Interpretations of $X_{10}(q)$ and $\chi_{10}(q)$ in terms of n-color partitions follows as $\mathcal{F}_3^{(1,1)}(q)$ and a displacement of $2n \mapsto 2n+1$ in the factor $(q^r;q^r)_{2n+1}$ and $n \mapsto n+1$ in the numerator q^{ln^2} for $\chi_{10}(q)$ affects the smallest part condition.

5 Arithmetic Properties for $\beta(q)$ and $\xi(q)$

The exploration of arithmetic properties connected to different mock theta functions is also fascinating concept. In 2018, Zhang and Shi [20] gave some interesting arithmetic properties of the sixth order mock theta function $\beta(q)$ using the relation:

$$\Phi(q^3) + 2q^{-1}\Psi(q^3) + 2\beta(q) = \frac{f_2 f_3^5}{f_1^2 f_6^3},$$

where

$$\Phi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(-q;q)_{2n}}, \qquad \Psi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{(n+1)^2}}{(-q;q)_{2n+1}},$$

and

$$f_k = (q^k;q^k)_\infty,$$

for some positive integer k. The main theorem given in [20] is as follows:

Theorem 12 *We have*

$$\sum_{n=0}^{\infty} p_\beta(3n+1)q^n = \frac{f_3^3}{f_1^2}, \tag{13}$$

$$\sum_{n=0}^{\infty} p_\beta(9n+5)q^n = \frac{3f_3^6}{f_1^5}. \tag{14}$$

In continuation, Silva and Sellers have studied the arithmetic properties for the third order mock theta function $\xi(q)$ in [19], presented several infinite families of Ramanujan-like congruences. The proof of these congruences are based on the mock theta conjecture of $\xi(q)$ given by McIntosh [16] as:

$$\xi(q) = q^2 g_3(q^3,q^6) + \frac{f_2^4}{f_1^2 f_6},$$

where

$$g_3(x,q)=\sum_{n=0}^{\infty}\frac{(-q;q)_n q^{n(n+1)/2}}{(x;q)_{n+1}(x^{-1}q;q)_{n+1}}.$$

Some of the results from [19] are shown below:

$$\sum_{n=0}^{\infty}p_{\xi}(3n)q^n=\frac{f_2f_3^4}{f_1^2f_6^2}, \tag{15}$$

$$\sum_{n=0}^{\infty}p_{\xi}(9n)q^n=\frac{f_2^2f_3^6}{f_1^4f_6^3}, \tag{16}$$

$$\sum_{n=0}^{\infty}p_{\xi}(9n+6)q^n=4\frac{f_6^3}{f_1^2}. \tag{17}$$

Using the above relations, we proposed some recurrence relations connecting the coefficients of mock theta functions in arithmetic progressions and the restricted partition functions. Before proceeding forward, we need to define some basic definitions and notations. We recall Ramanujan's theta function [5,14]:

$$f(a,b)=\sum_{n=-\infty}^{\infty}a^{\frac{n(n+1)}{2}}b^{\frac{n(n-1)}{2}},\ \text{for } |ab|<1.$$

Some special cases of $f(a,b)$ are:

$$\phi(q):=f(q,q)=\sum_{n=-\infty}^{\infty}q^{n^2}=\frac{f_2^5}{f_1^2f_4^2},$$

$$\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty}q^{n(n+1)/2}=\frac{f_2^2}{f_1}.$$

The above function satisfy the following properties (see Entries 19, 20 in [4]).

$$f(a,b)=(-a,-b,ab;ab)_{\infty},\quad \text{(Jacobi's triple product identity)}.$$

The following infinite product is a useful consequence of Jacobi's triple product identity.

$$f_1^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}.$$

Now we present some restricted partitions and their corresponding generating functions. Let $\overline{p}_l(n)$ denote the number of overpartitions of n into parts with l copies. Then

$$\sum_{n=0}^{\infty}\overline{p}_l(n)q^n=\left(\frac{f_2}{f_1^2}\right)^l.$$

Let $p_{ld}(n)$ denote the number of partitions of n into distinct parts with l copies. Then

$$\sum_{n=0}^{\infty}p_{ld}(n)q^n=\left(\frac{f_2}{f_1}\right)^l.$$

Let $p_l(n)$ denote the number of partitions of n into parts with l copies. Then

$$\sum_{n=0}^{\infty}p_l(n)q^n=\frac{1}{f_1^l}.$$

Now we prove congruences modulo 15 for $p_{\beta}(n)$ where

$$\beta(q)=\sum_{n=0}^{\infty}p_{\beta}(n)q^n.$$

Theorem 13 *We have*

$$\sum_{n=0}^{\infty} p_\beta(45n+32)q^n \equiv 12\frac{f_3^6}{f_1} \pmod{15}. \tag{18}$$

Proof From (14), we have

$$\sum_{n=0}^{\infty} p_\beta(9n+5)q^n = \frac{3f_3^6}{f_1^5}$$

or

$$\sum_{n=0}^{\infty} p_\beta(9n+5)q^n = \frac{3f_3^5}{f_1^5} f_3.$$

From binomial theorem, for prime p, we have

$$f_1^p \equiv f_p \pmod{p}.$$

$$\sum_{n=0}^{\infty} p_\beta(9n+5)q^n \equiv \frac{3f_{15}}{f_5} f_3 \pmod{15}. \tag{19}$$

Recall Ramanujan's beautiful identity [4] (Eq.(7.4.1), p. 161)

$$f_1 = f_{25}(R^{-1}(q^5) - q - q^2R(q^5)), \tag{20}$$

where

$$R(q) = \frac{f(-q,q^4)}{f(-q^2,-q^3)}.$$

Substituting (20) in (19), we obtain

$$\sum_{n=0}^{\infty} p_\beta(9n+5)q^n \equiv \frac{3f_{15}f_{75}}{f_5}(R^{-1}(q^{15}) - q^3 - q^6R(q^{15})) \pmod{15}. \tag{21}$$

Equating the coefficients of q^{5n+3} from both sides, dividing both sides by q^3 and replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} p_\beta(45n+32)q^n \equiv \frac{-3f_3f_{15}}{f_1} \equiv \frac{-3f_3^6}{f_1} \pmod{15}. \tag{22}$$

□

Corollary 14 *For $n \geq 0$,*

$$p_\beta(45n+23) \equiv 0 \pmod{15}, \tag{23}$$

$$p_\beta(45n+41) \equiv 0 \pmod{15}. \tag{24}$$

Proof Comparing the coefficients of q^{5n+2} and q^{5n+4} from (21), dividing both sides by q^2 and q^4, respectively. Then replace q^5 by q to get (23) and (24). □

Theorem 15 *We have*

$$\begin{aligned} p_\beta(3n+1) = {} & p_2(n) - 3p_2(n-3) + 5p_2(n-9) - 7p_2(n-18) + 9p_2(n-30) + \cdots \\ & +(-1)^j(2j+1)p_2\left(n - \frac{3j(j+1)}{2}\right) + \cdots . \end{aligned} \tag{25}$$

Proof Consider (14) as:

$$
\begin{aligned}
\sum_{n=0}^{\infty} p_{\beta}(3n+1)q^n &= \frac{f_3^3}{f_1^2} = \frac{1}{f_1^2}\cdot f_3^3,\\
&= \left(\sum_{n=0}^{\infty} p_2(n)q^n\right)\left(\sum_{j=0}^{\infty}(-1)^j(2j+1)q^{3j(j+1)/2}\right),\\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j(2j+1)p_2(n)q^{n+3j(j+1)/2},\\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j(2j+1)p_2\left(n-\frac{3j(j+1)}{2}\right)q^n.
\end{aligned}
$$

Comparing the coefficients of q^n, we arrive at (25). □

Theorem 16 *We have*

$$
\sum_{j=0}^{\infty} p_{\xi}\left(3n-\frac{9j(j+1)}{2}\right) = \sum_{j=0}^{\infty}(-1)^j(2j+1)\overline{p}\left(n-\frac{j^2+j}{2}\right), \tag{26}
$$

$$
\sum_{j=0}^{\lfloor \frac{n}{3}\rfloor} p_{\xi}(9n-27j)\overline{p}_3(j) = \overline{p}_2(n), \tag{27}
$$

$$
\begin{aligned}
p_{\xi}(9n+6) = 4\overline{p}_2(n) - 12\overline{p}_2(n-6) + 20\overline{p}_2(n-18) - \cdots + 4(-1)^j(2j+1)\\
\overline{p}_2(n-3j(j+1)) + \cdots .
\end{aligned} \tag{28}
$$

Proof Consider (15) as:

$$
\begin{aligned}
\sum_{n=0}^{\infty} p_{\xi}(3n)q^n &= \frac{f_2}{f_1^2}\cdot\frac{f_3}{f_6^2}\cdot f_3^3,\\
\left(\sum_{n=0}^{\infty} p_{\xi}(3n)q^n\right)\frac{f_6^2}{f_3} &= \frac{f_2}{f_1^2}\cdot f_3^3,\\
\left(\sum_{n=0}^{\infty} p_{\xi}(3n)q^n\right)\psi(q^3) &= \left(\sum_{n=0}^{\infty}\overline{p}(n)q^n\right)f_3^3,\\
\left(\sum_{n=0}^{\infty} p_{\xi}(3n)q^n\right)\left(\sum_{j=0}^{\infty} q^{3j(j+1)/2}\right) &= \left(\sum_{n=0}^{\infty}\overline{p}(n)q^n\right)\left(\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{(k^2+k)/2}\right),\\
\sum_{n=0}^{\infty}\left(\sum_{j=0}^{\infty} p_{\xi}\left(3n-\frac{9j(j+1)}{2}\right)\right)q^n &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}(-1)^k(2k+1)\overline{p}\left(n-\frac{k^2+k}{2}\right)\right)q^n.
\end{aligned}
$$

Comparing the coefficients of q^n, we get (26). Consider (16),

$$
\begin{aligned}
\sum_{n=0}^{\infty} p_{\xi}(9n)q^n &= \frac{f_2^2 f_3^6}{f_1^4 f_6^3},\\
\left(\sum_{n=0}^{\infty} p_{\xi}(9n)q^n\right)\left(\frac{f_6}{f_3^2}\right)^3 &= \left(\frac{f_2}{f_1^2}\right)^2,\\
\left(\sum_{n=0}^{\infty} p_{\xi}(9n)q^n\right)\left(\sum_{k=0}^{\infty}\overline{p}_3(k)q^{3k}\right) &= \sum_{n=0}^{\infty}\overline{p}_2(n)q^n,
\end{aligned}
$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\lfloor \frac{n}{3}\rfloor} p_{\xi}(9(n-3k))\overline{p}_3(k)\right)q^n=\sum_{n=0}^{\infty}\overline{p}_2(n)q^n.$$

From the last equality, we arrive at (27). Consider (17),

$$\begin{aligned}\sum_{n=0}^{\infty}p_{\xi}(9n+6)q^n&=4\frac{f_6^3}{f_1^2},\\&=4\frac{1}{f_1^2}\left(\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{3k(k+1)}\right),\\ \sum_{n=0}^{\infty}p_{\xi}(9n+6)q^n&=4\left(\sum_{n=0}^{\infty}p_2(n)q^n\right)\left(\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{3k(k+1)}\right),\\&=4\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}(-1)^k(2k+1)p_2(n-3k(k+1))\right)q^n,\end{aligned}$$

which leads to (28). □

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