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ORIGINAL RESEARCH

Growth rate bounds and instability regions for azimuthal disturbances of inviscid swirling flows

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Abstract General analytical results on the azimuthal unstable modes of variable density swirling flows between coaxial cylinders are obtained in the present paper. In particular estimates for the growth rate of unstable modes and instability regions within which the complex phase velocities should lie are obtained. All the results obtained indicate an important role for the basic flow vorticity and its variation on the instability of the swirling flows. The general analytical results obtained are illustrated in figures for various basic angular velocity profiles and a family of density profiles.

Keywords Azimuthal instability · Swirling flows · Variable density · Growth rate bounds · Instability regions

1 Introduction

The linear stability analysis of rotating flows of inviscid incompressible homogeneous fluids to normal mode disturbances was first studied by Rayleigh [1]. In particular, the following problem was studied in Rayleigh [1] (cf. Chandrasekhar [2]). Consider the cylindrical polar coordinates system (r, θ, z). The flow domain is the annular region between two infinite rigid coaxial cylinders placed at $r = R_1, R_2$, where $0 < R_1 < R_2 < \infty$. The basic flow has velocity $(0, V(r), 0)$, constant density $\rho_\circ$, and a pressure $p_\circ(r)$ calculated from the Euler equations. A disturbance of the form $f(r)e^{im(\theta-ct)}$, is superposed on this flow. As is well known a disturbance of this form is an azimuthal (or two - dimensional) disturbance where the azimuthal wave number m is an integer, the (complex) phase velocity $c = c_r + ic_i$, and instability corresponds to the existence of an unstable mode with positive growth rate $\omega_i = mc_i$. It was shown in Rayleigh [1] (see also Drazin and Reid [3]) that a necessary condition for instability of the swirling flow is that DZ changes sign in (R_1, R_2) where $D = \frac{d}{dr}$ is the differential operator; $Z = rD\Omega + 2\Omega$ is the basic flow vorticity, and $\Omega = \frac{V}{r}$ is the angular velocity of the flow. As m appears in the stability equation as m^2, one can take m to be a positive integer without loss of generality. Consequently, instability of the swirling flow corresponds to the existence of a complex eigenvalue c with $c_i > 0$. Let $a = \Omega_{min}$ and $b = \Omega_{max}$. It was shown in Lalas [4] that the complex eigenvalues c corresponding to unstable disturbances lie inside a semicircle in the upper half of the c - plane with center $(\frac{a+b}{2}, 0)$ and radius $(\frac{b-a}{2})$.

The azimuthal instability of variable density swirling flows of inviscid fluids was considered first in Sakurai [5]. Motivated by a desire to understand instabilities in gas centrifuges a three-layer model of the basic flow was

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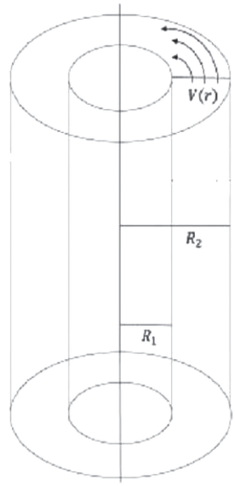

Fig. 1 Schematics of the Basic Flow

considered for detailed instability analysis. A systematic study of the azimuthal instability of variable density inviscid swirling flows with smooth angular and density profiles was initiated in Fung [6]. The basic azimuthal velocity $V(r)$ and density $\rho_\circ(r)$ are arbitrary functions of r while the basic pressure density $p_\circ(r)$ is related to these variables through the relation

$$Dp_\circ = \frac{\rho_\circ(r)V^2}{r}, \tag{1}$$

which ensures that the basic flows considered here are solutions to the Euler equations of fluid mechanics. A schematic of the basic flow velocity is given in Figure 1.

A local Richardson number $J(r)$ was defined and it was found that instability is possible only if $J_{min} \leq \frac{1}{4}$ where J_{min} is the minimum of the Richardson number $J(r) = \frac{\Omega^2(D\rho_\circ)}{r\rho_\circ(D\Omega)^2}$. Furthermore for basic flows satisfying the condition $ab(D\rho_\circ) \geq 0$ a semicircular instability region was obtained, which for the constant density case is the same as Lala's semicircle. A semielliptical instability region in which the major axis is the range $(b - a)$ of the angular velocity and the semi minor axis depends on J_{min} has also been obtained by Fung [6]. The three-layer model considered in Sakurai [5] was also reinvestigated and it was concluded that the stability boundary corresponds to $c_i = 0$ only. Azimuthal instability of inviscid incompressible flows with smoothly varying azimuthal velocity and density profile has been studied by showing the existence of normal modes with positive growth rates in many works like Sipp et al [7], and Dixit and Govindarajan [8]. A brief review of the analytical results known on this problem is also given by Sipp et al [7]. However, it must be noted here that the stability problem studied in these works is for flows without the rigid walls. It may be mentioned here that for the velocity profile considered in these works (DZ) is of definite sign throughout the flow domain and as such their instability to azimuthal disturbances is only due to density stratification. Further improvement of the instability regions have been obtained in Dattu and Subbiah [9] and Prakash and Subbiah [10]. A discussion of the Reynolds stress and neutral modes have been given in Dattu and Subbiah [11]. However, a missing point in all these studies is the role of the vorticity Z or its variation (DZ) on the instability of variable density swirling flows. Motivated by a desire to understand the role of (DZ) in the instability of variable density swirling flows we obtain analytical results on this problem that involve (DZ) explicitly. In particular, we obtain an estimate for the growth rate of arbitrary unstable modes. We have illustrated the growth rate estimate for basic flows with three different angular velocity profiles but with a family of density profiles. As our interest is in bringing out the role of (DZ) in the azimuthal instability of swirling flows we have chosen profiles with angular velocity such that (i) $(DZ) \equiv 0$; (ii) $(DZ) < 0$; and (iii) (DZ) changes sign in the annular region. The stability equation studied in Dixit and Govindarajan [8] is the one due to Fung [6] and which is also studied in the present paper. However, the flow domain in their study has no rigid cylinders as boundaries and so their flow domain is $0 \leq r < \infty$. However, the boundary conditions considered in Dixit and Govindarajan [8] are $u = 0$ at $r = 0, \infty$ and the wave numbers considered satisfy $m \geq 2$. So it may be remarked here that the analytical results obtained in the present paper are applicable to the unbounded case studied in that work. Moreover, it is seen that it gives a new estimate for the homogeneous case also. A semielliptical instability region that is different from the one given in Fung [6] is derived for a class of basic flows. The major axis of the semiellipse is $(b-a)$ as before but the semi-minor axis depends on $D(\rho_\circ Z)$. Further improvements of this instability region are obtained by the inclusion of the wave number m and the width $(R_2 - R_1)$ in the instability region. The instability regions obtained in the present paper are improvements over the already known instability regions as illustrated in Figures.

2 The eigenvalue problem

The Eigenvalue problem under consideration consists of the second order differential equation

$$D\left(\rho_\circ r^2 D_* u\right) + \left\{-\rho_\circ m^2 - \frac{rD\left[\rho_\circ D_*(r\Omega)\right]}{\Omega - c} + \frac{r\Omega^2 (D\rho_\circ)}{(\Omega - c)^2}\right\} u = 0, \tag{2}$$

and the boundary conditions are

$$u(R_1) = 0 = u(R_2), \tag{3}$$

where R_1 and R_2 are the radial positions of the cylindrical walls. Here it may be noted that the eigenfunction $u(r)$ comes from the radial component of the perturbation velocity which is given by $u(r)e^{im(\theta - ct)}$ and the operator D_* is given by $D_* = D + \frac{1}{r}$. The equation (2) along with the boundary condition (3) has been derived by Fung and Kurzweg [12]. A brief derivation of this problem has also been given in Prakash and Subbiah [10]. The growth rate bounds of unstable azimuthal modes are obtained by the use of the method of integral relations on the eigenvalue problem (2) - (3). For obtaining instability regions one needs the following equation and boundary conditions.

3 Estimates for the growth rate

Now we shall obtain some upper bounds for growth rate of an arbitrary unstable disturbance. For this purpose we adapt the parallel flow analysis of Subbiah and Jain [13] to the present context of cylindrical geometry.

Multiplying (2) by ru^* (where u^* is the complex conjugate of u) and integrating over (R_1, R_2) we get

$$\int \rho_\circ \left[r^2 |D_* u|^2 + m^2 |u|^2\right] r dr + \int \frac{r^2 D(\rho_\circ D_*(r\Omega)) |u|^2}{(\Omega - c)} dr - \int \frac{r^2 \Omega^2 (D\rho_\circ) |u|^2}{(\Omega - c)^2} dr = 0. \tag{4}$$

The real part of (4) gives the equation

$$\begin{aligned}\int \rho_\circ \left[r^2 |D_* u|^2 + m^2 |u|^2\right] r dr = &-\int \frac{r^2 D(\rho_\circ D_*(r\Omega)) |u|^2 (\Omega - c_r)}{|\Omega - c|^2} dr \\ &+ \int \frac{r^2 \Omega^2 (D\rho_\circ) |u|^2 \left[(\Omega - c_r)^2 - c_i^2\right] dr}{|\Omega - c|^4} dr.\end{aligned} \tag{5}$$

Taking into account that

$-D(\rho_\circ D_*(r\Omega))(\Omega - c_r) \le |D(\rho_\circ D_*(r\Omega))| \, |\Omega - c_r| \le |D(\rho_\circ D_*(r\Omega))|_{max} \, (\Omega_{max} - \Omega_{min})$ and $(\Omega - c_r)^2 - c_i^2 \le |\Omega - c|^2$ we have

$$\begin{aligned}\int \rho_\circ \left[r^2 |D_* u|^2 + m^2 |u|^2\right] r dr \le |D(\rho_\circ D_*(r\Omega))|_{max} \, (\Omega_{max} - \Omega_{min}) \int \frac{r^2 |u|^2}{|\Omega - c|^2} dr \\ + \int \frac{r^2 \Omega^2 (D\rho_\circ) |u|^2}{|\Omega - c|^2} dr;\end{aligned}$$

$$\begin{aligned}i.e., \int \rho_\circ \left[r^2 |D_* u|^2 + m^2 |u|^2\right] r dr \le |D(\rho_\circ D_*(r\Omega))|_{max} \, (b-a) \, R_2 \int \frac{r |u|^2}{|\Omega - c|^2} \\ + \left| r\Omega^2 (D\rho_\circ)\right|_{max} \int \frac{r |u|^2}{|\Omega - c|^2},\end{aligned} \tag{6}$$

where we have used the fact that $a \leq \Omega(r) \leq b$ and $|\Omega - c_r| \leq |\Omega_{max} - \Omega_{min}| = b - a$.
Using $\frac{1}{|\Omega-c|^2} \leq \frac{1}{c_i^2}$ in (6), we get the following estimate:

$$\int \rho_\circ r^3 |D_* u|^2 dr + m^2 \int \rho_\circ |u|^2 r dr \leq \frac{\left[|D(\rho_\circ D_*(r\Omega))|_{max}(b-a) R_2 + \left|r\Omega^2 (D\rho_\circ)\right|_{max}\right]}{c_i^2} \int r|u|^2 dr. \quad (7)$$

Dropping the first non negative term we get

$$\rho_{\circ_{min}} m^2 \int r|u|^2 dr \leq \frac{\left[|D(\rho_\circ D_*(r\Omega))|_{max}(b-a) R_2 + \left|r\Omega^2 (D\rho_\circ)\right|_{max}\right]}{c_i^2} \int r|u|^2 dr.$$

Dropping the integral we get the following result.

Theorem 1 *An upper bound for growth rate is given by*

$$\omega_i^2 \leq \frac{\left[|D(\rho_\circ D_*(r\Omega))|_{max}(b-a) R_2 + \left|r\Omega^2 (D\rho_\circ)\right|_{max}\right]}{\rho_{\circ_{min}}}. \quad (8)$$

If $\rho_\circ$ is constant then this reduces to

$$\omega_i^2 \leq |D(D_*(r\Omega))|_{max} R_2 (b-a). \quad (9)$$

We can improve the growth rate (8) by the use of well known Rayleigh-Ritz inequality (see, equation (22.9) of Drazin and Reid [3] page(134), and Pavithra and Subbiah [14]). Since $D_*(ru) = \frac{D(ru)}{r}$,

$$\begin{aligned}\int \rho_\circ r^3 |D_* u|^2 dr &= \int \rho_\circ r |D(ru)|^2 dr \\ &\geq \frac{\rho_{\circ_{min}} R_1 \pi^2}{(R_2-R_1)^2} \int r^2 |u|^2 dr \\ &\geq \frac{\rho_{\circ_{min}} R_1^2 \pi^2}{(R_2-R_1)^2} \int r|u|^2 dr. \end{aligned} \quad (10)$$

Using (10) to estimate the first term in (7) we get

$$\rho_{\circ_{min}} \left[\frac{R_1^2\pi^2}{(R_2-R_1)^2} + m^2\right] \int r|u|^2 dr \leq \frac{\left[|D(\rho_\circ D_*(r\Omega))|_{max}(b-a) R_2 + \left|r\Omega^2 (D\rho_\circ)\right|_{max}\right]}{c_i^2} \int r|u|^2 dr.$$

Since the integral is positive we can divide it out to get.

$$\rho_{\circ_{min}} \left[\frac{R_1^2\pi^2}{(R_2-R_1)^2} + m^2\right] \leq \frac{\left[|D(\rho_\circ D_*(r\Omega))|_{max}(b-a) R_2 + \left|r\Omega^2 (D\rho_\circ)\right|_{max}\right]}{c_i^2} \quad (11)$$

From (11) we get the following result.

Theorem 2 *An estimate for growth rate that depends on the width of the channel is given by*

$$\omega_i^2 \leq \frac{|D(\rho_\circ D_*(r\Omega))|_{max}(b-a) R_2 + \left|r\Omega^2 (D\rho_\circ)\right|_{max}}{\rho_{\circ_{min}} \left[\frac{R_1^2\pi^2}{m^2(R_2-R_1)^2} + 1\right]}. \quad (12)$$

If $\rho_\circ$ is constant then this reduces to

$$\omega_i^2 \leq \frac{|D(D_*(r\Omega))|_{max}(b-a) R_2}{\left[\frac{R_1^2\pi^2}{m^2(R_2-R_1)^2} + 1\right]}. \quad (13)$$

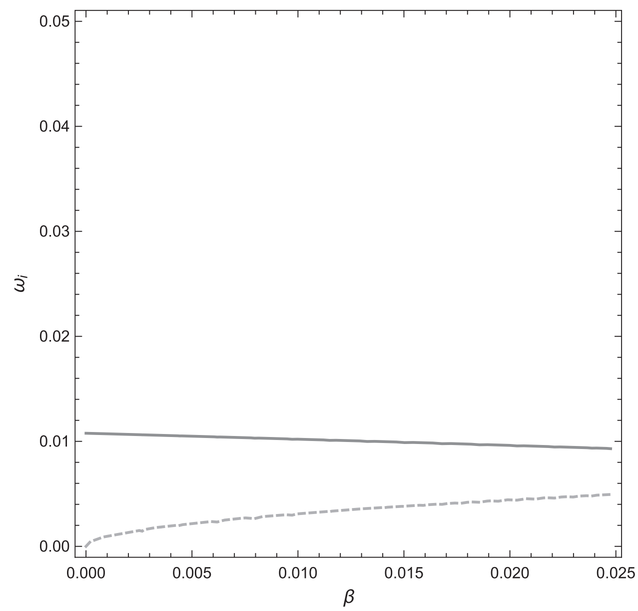

Fig. 2 ω_i verses β. Dashed line represent the new estimate given by (15), while the heavy line represents the estimate given by (14)

In the estimate (8) it is seen that density variation interacts with the vorticity variation through the first term and with the angular velocity through the second term. As remarked in Dixit and Govindarajan [8] density inhomogenity becomes important only in the presence of a non-zero angular velocity but once it is present it interacts with shear in the angular velocity also.

Remark 1 From equation (9) it is seen that if $(DZ) \equiv 0$ then $\omega_i^2 \leq 0$ which implies the stability of the flow. It may be noted that such a conclusion of stability of constant vorticity flows of homogeneous fluid cannot be made from the estimate for growth rate given in Dattu and Subbiah [9]. Consequently if a flow with $(DZ) = 0$ is unstable then its instability is only due to density stratification.

Remark 2 From (12) it is seen that $(R_2 - R_1) \to 0$ implies that the growth rate $\omega_i \to 0$. Moreover it is seen from a comparison of the growth rates given by (8) and (12) that the presence of the cylinders at $r = R_1, R_2$ have a stabilizing effect and that a small value of the width $(R_2 - R_1)$ of the annular region reduces the estimate for growth rate further.

Remark 3 It may be noted here that the stability of the Couette flow of a viscous incompressible homogeneous fluid to infinitesimal azimuthal disturbances has been demonstrated in Ali and Herron [15]. It means that a flow with $(DZ) \equiv 0$ is azimuthally stable in the viscous case also. Thus the only cause of azimuthal instability for a flow with constant basic vorticity is density stratification.

Now we shall compare the growth rate bound obtained here with those obtained in Dattu and Subbiah [9], for a particular basic flow and we demonstrate that our bounds are better ones .

Example 1 Consider the basic flow with velocity $\Omega(r) = \frac{1}{r^2}$ and density $\rho_\circ = e^{\beta r}$ where β is the positive constant in the interval $[R_1, R_2]$. The growth rate bound (5.1) of Dattu and Subbiah [9] yields the estimate

$$\omega_i^2 \leq \left(\frac{1}{4} - \frac{\beta R_1}{4}\right) R_2^2 \frac{4}{R_1^6}. \tag{14}$$

For the same flow the growth rate bound given by (8) of the present paper gives

$$\omega_i^2 \leq \frac{\beta e^{\beta R_2}}{R_1^3 e^{\beta R_1}}. \tag{15}$$

It is to be noted that we are considering a family of basic flows here since β is an arbitrary positive constant. So we can plot the growth rate bounds for different values of β. This is done in Figure 2 and it is seen that

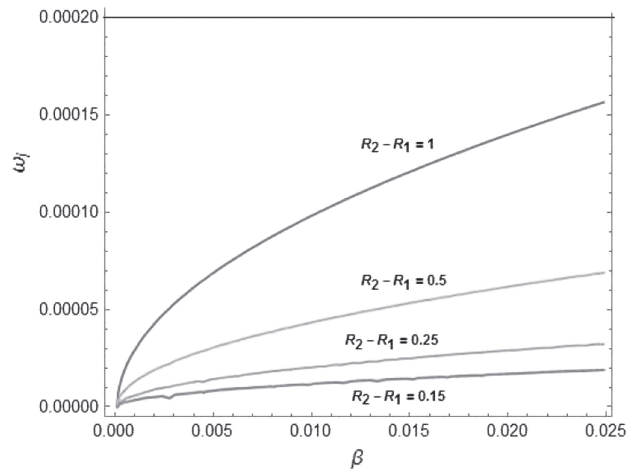

Fig. 3 ω_i verses β. Growth Rate bounds (17) with $R_2 = 11.1$, $m = 1$ and different values of width of the annulus

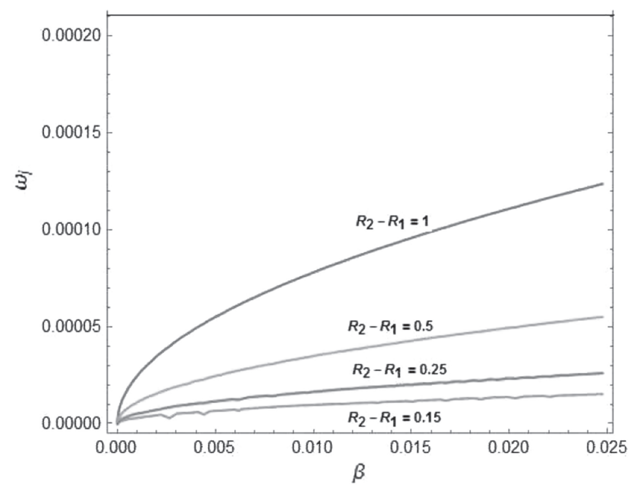

Fig. 4 ω_i verses β. Growth Rate bounds (17) with $R_2 = 12.1$, $m = 1$ and different values of width of the annulus

our bounds are the better ones. For the same basic flow the growth rate bound (5.2) of Dattu and Subbiah [9] yields the estimate. For a given basic velocity profile $\Omega(r)$ and density profile $\rho_\circ(r) = e^{\beta r}$, $\frac{D\rho_\circ}{\rho_\circ} = \beta$ and the Richardson number $J(r)$ increases with increase in β. So the large values of β will make the flow stable.

$$\omega_i^2 \leq \frac{e^{\beta(R_2-R_1)}\left(\frac{1}{4} - \frac{\beta R_1}{4}\right) R_2^3 \, (D\Omega)^2_{max}}{\left(\frac{R_1^3\pi^2}{m^2(R_2-R_1)^2} + R_1\right)}, \tag{16}$$

while the bound given in (12) of the present paper gives the estimate

$$\omega_i^2 \leq \frac{\frac{\beta e^{\beta R_2}}{R_1^3}}{e^{\beta R_1}\left[\frac{R_1^2\pi^2}{m^2(R_2-R_1)^2} + 1\right]}. \tag{17}$$

It is seen from (17) that the upper bounds of the growth rate depend on the radii R_1 and R_2 of the cylinders individually and also on the width of the annulus $(R_2 - R_1)$. Now we plot the estimates by varying $R_2 - R_1$ and azimuthal wave number m in Figure 3, Figure 4 and Figure 5. It is seen from Figure 3, Figure 4 and Figure 5 that the growth rate bounds for different valued of R_1 and R_2 are different even when the width $(R_2 - R_1)$ is same.

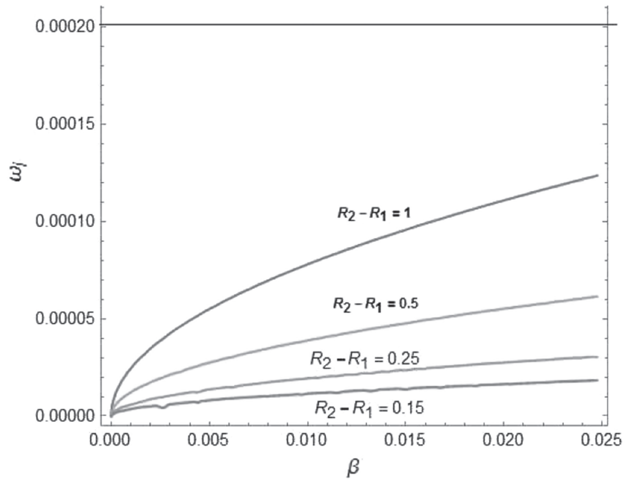

Fig. 5 ω_i verses β. Growth Rate bounds (17) with $R_1 = 11.1$, $m = 1$ and different values of width of the annulus

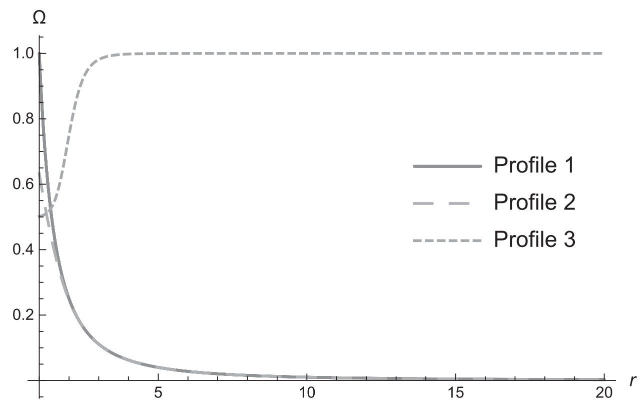

Fig. 6 Plot of angular velocity profile $\Omega(r)$

Table 1 Basic flow velocity profiles

Profile	Angular Velocity $\Omega(r)$	R_1	R_2	Density
1	$\Omega(r) = \frac{1}{r^2}$	1	3	$e^{\beta r}$, $\beta > 0$
2	$\Omega(r) = \frac{1}{r^2}\left[1 - e^{-r^2}\right]$	1	3	$e^{\beta r}$, $\beta > 0$
3	$\Omega(r) = (\frac{1.5}{2}) + (\frac{0.5}{2}) tanh(4 log(\frac{r}{2}))$	1	3	$e^{\beta r}$, $\beta > 0$

As mentioned in the introduction we shall illustrate the growth rate estimate (8) for basic flows with three different angular velocity profiles and with a particular family of density profiles.

Since we are interested in understanding the role of vorticity variation on the instability of swirling flow we consider angular velocity profiles with differing characteristics of Z.

These angular velocity profiles are plotted in Figure 6 and the corresponding vorticity variations are plotted in Figure 7. It is seen from Figure 7 that (DZ) changes sign in the third example as is necessary for instability in the homogeneous case.

For the basic flows considered for illustration of our analytical results one can see the Table 1.

Example 2 We consider here a flow with constant vorticity Z. It is well known that a flow satisfying $DZ = 0$ has angular velocity $\Omega(r) = \frac{C_1}{r^2} + C_2$ (cf. Chandrasekhar [2]) where C_1 and C_2 are constants.

A special case of this Couette flow is the one with $\Omega(r) = \frac{1}{r^2}$ which is considered for illustration of our growth rate bound.

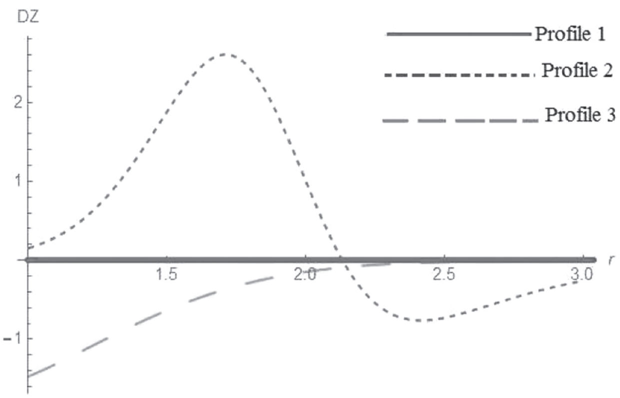

Fig. 7 Plot of vorticity variation (DZ)

Example 3 In this example we consider the flow with angular velocity $\Omega(r) = \frac{1}{r^2}\left[1 - e^{-r^2}\right]$ ($cf.$ DiPierro and Abid [16]). In this case the vorticity is not constant, but, it is easily seen that $(DZ) = -4re^{-r^2} \leq 0$ and so this flow is also stable in the absence of density stratification.

Example 4 Now we shall consider a swirl flow which is known to be unstable in the homogeneous case. Consider the flow with angular velocity $\Omega = (\frac{1.5}{2}) + (\frac{0.5}{2})tanh(4log(\frac{r}{2}))$ ($cf.$ Shukhman [17]). It is seen that vorticity $Z = sech^2(4log(\frac{r}{2})) + (1.5) + 0.5tanh(4log(\frac{r}{2}))$, and $DZ = \frac{2}{r}sech^2(4log(\frac{r}{2}))\left[1 - 4tanh(4log(\frac{r}{2}))\right]$. It has been shown in Shukhman [17] that this flow is unstable in the linear homogeneous case and the nonlinear instability has also been studied by Shukhman [17].

We choose the density profile $\rho = e^{\beta r}$, where β is a positive constant and we have plotted in Figure 8 the growth rate bound given by (8) for the three angular velocity profiles discussed above. It is seen from Figure 8 that the growth rate bounds indicate the destabilizing effect of the vorticity variation.

Example 5 In this example we consider the angular velocity profile $\Omega(r) = \frac{q}{r^2}\left[1 - e^{-r^2}\right]$, where the constant q is the swirl parameter and we choose the density profile given by $\rho_\circ = \frac{1}{2}\left[(\rho_1 + \rho_2) - (\rho_1 - \rho_2)tanh(r - \frac{R_1+R_2}{2})\right]$. It is to be noted here that angular velocity profile considered here is the one in Batchelor q-vortex system. As we are considering only azimuthal instability of flows the axial velocity does not play any role and as such we have not specified the axial velocity profile here. The instability of the Batchelor q-vortex flow is important in aeronautics and this has been studied in many works like Dixit and Govindarajan [8] and DiPierro and Abid [16]. We fix the value of ρ_2 at $\rho_2 = 1$ and plot in Figure 9 the growth rate bound versus the swirl parameter curves for different values of ρ_1. It is seen that the growth rate bound increases with increasing values of q for all values of ρ_1 considered.

The growth rates of azimuthal unstable modes of the Batchelor has been computed and plotted in Figure 1 of Dipierro and Abid [16]; the density profile chosen is $\rho_\circ = \rho_\infty\left[1 + (s-1)exp(-r^2)\right]$ where ρ_∞ is a constant and s is the Atwood number. The growth rates of azimuthal $\Omega = \frac{q}{r^2}[1 - e^{-r^2}]$, $Z = 2qe^{-r^2}$ unstable modes have also been computed in Dixit and Govindarajan [8]. However in these studies the growth rates are computed for different values of the Atwood number rather than the Richardson number. The Atwood number s is defined by $s = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}$, where ρ_1, and ρ_2 are values of density at two different radial positions, say, $\rho_1 = \rho(R_1)$ and $\rho_2 = \rho(R_2)$.

4 Semielliptical instability region

Now we shall derive a new instability region. Transforming the stability equation and boundary condition by substituting $u = (\Omega - c)F$ in (2) and (3) we get

$$D\left(\rho_\circ r^2 D_*\left[(\Omega - c)F\right]\right) + \left\{-m^2\rho_\circ(\Omega - c) - rD\left(\rho_\circ D_*(r\Omega)\right) + \frac{r\Omega^2(D\rho_\circ)}{(\Omega - c)}\right\}F = 0, \tag{18}$$

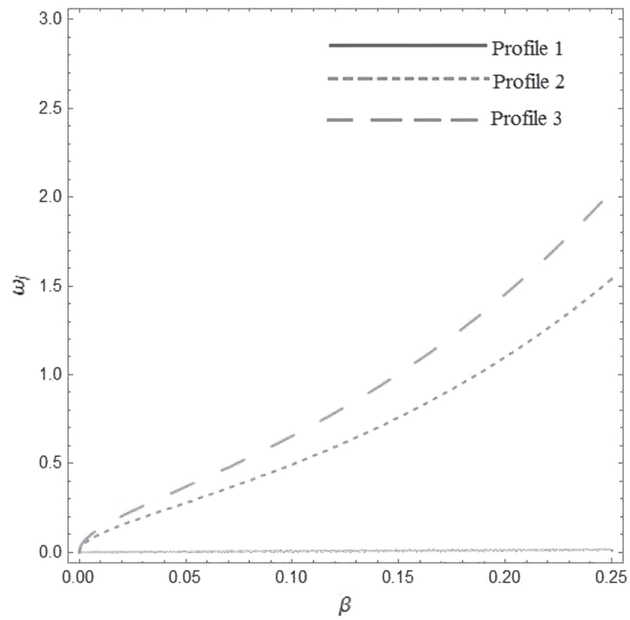

Fig. 8 Growth rate bounds obtained by varying β for the profiles

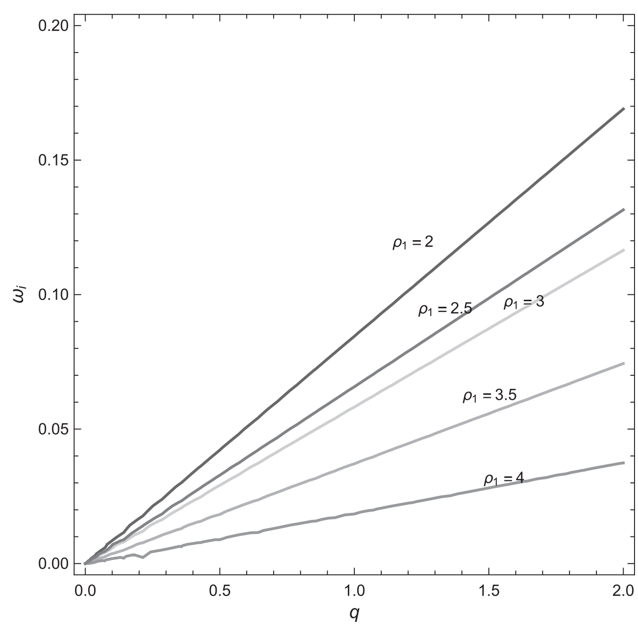

Fig. 9 Growth rate bounds obtained by varying the values of q and density profiles for different values of ρ_1

with boundary conditions

$$F(R_1) = 0 = F(R_2). \tag{19}$$

If the equation (18) is multiplied by rF^* (F^* the complex conjugate of F) and integrated over (R_1, R_2) then its real and imaginary part together with certain manipulations give

$$\left[c_r^2 + c_i^2 - (a+b)c_r + ab\right] \int r\rho_\circ Q dr + \int ab\,(D\rho_\circ)\,r^2\,|F|^2\,dr \leq 0, \tag{20}$$

where

$$Q = \left[r^2\,|D_* F|^2 + m^2\,|F|^2\right]. \tag{21}$$

First, Fung [6] considered flows satisfying the condition $ab(D\rho_\circ) \geq 0$ for which the last term in (20) can be dropped to get the semicircle theorem. Subsequently he used the approach of Kochar and Jain [18] from the parallel flow theory to obtain an estimate for the second term in (20) to obtain the semiellipse theorem for the same class of flows. In this paper we adapt the method of Subbiah and Jain [13] to find a new estimate for the last term in (20) to get a new semiellipse theorem depending on $D\left(\rho_\circ D_*\left(r\Omega\right)\right)$ and the minimum Richarson number J_{min}.

From the relation $u = (\Omega - c)\, F$ we have

$D_* u = (\Omega - c)\,(D_* F) + (D\Omega)\, F$.

From this we have the inequality

$|D_* u| \geq |\Omega - c|\, |D_* F| - |D\Omega|\, |F|$.

From this we have

$|D_* u|^2 \geq |\Omega - c|^2\, |D_* F|^2 + |D\Omega|^2\, |F|^2 - 2\, |\Omega - c|\, |D\Omega|\, |D_* F|\, |F|$.

A consequence of the above inequality is that

$$\begin{aligned}\int r\rho_\circ \left[r^2\, |D_* u|^2 + m^2\, |u|^2\right] dr &\geq \int r\rho_\circ\, |\Omega - c|^2 \left[r^2\, |D_* F|^2 + m^2\, |F|^2\right] dr \\ &+ \int r\rho_\circ r^2\, |D\Omega|^2\, |F|^2\, dr - 2\int r^3 \rho_\circ\, |\Omega - c|\, |D_* F|\, |D\Omega|\, |F|\, dr. \end{aligned} \tag{22}$$

In the following analysis we need the quantities stated below:

$A^2 = \int \rho_\circ\, |\Omega - c|^2\, r^3\, |D_* F|^2\, dr$,

$B^2 = \int r^3 \rho_\circ\, (D\Omega)^2\, |F|^2\, dr$,

$C^2 = \int r\rho_\circ\, |\Omega - c|^2\, |F|^2\, dr$,

$E^2 = \int r\rho_\circ\, |\Omega - c|^2 \left[r^2\, |D_* F|^2 + m^2\, |F|^2\right] dr = \int r\rho_\circ\, |\Omega - c|^2\, Q dr$.

It may be noted that $E^2 = A^2 + m^2 C^2$.

The inequality (22) can be written as

$$\int r\rho_\circ \left[r^2\, |D_* u|^2 + m^2\, |u|^2\right] dr \geq E^2 + B^2 - 2\int r^2 \rho_\circ\, |\Omega - c|\, |D_* F|\, |D\Omega|\, r\, |F|\, dr. \tag{23}$$

The last term in (23) can be estimated by Cauchy-Schwarz inequality to get

$$2\int r^2 \rho_\circ\, |\Omega - c|\, |D_* F|\, |D\Omega|\, r\, |F|\, dr \leq 2\left(\int r\rho_\circ r^2\, |\Omega - c|^2\, |D_* F|^2\right)^{\frac{1}{2}} \left(\int r\rho_\circ r^2\, |D\Omega|^2\, |F|^2\right)^{\frac{1}{2}}.$$

This gives

$$2\int r^2 \rho_\circ\, |\Omega - c|\, |D_* F|\, |D\Omega|\, r\, |F|\, dr \leq 2AB. \tag{24}$$

Since $A \leq E$ we also have

$$2\int r^2 \rho_\circ\, |\Omega - c|\, |D_* F|\, |D\Omega|\, r\, |F|\, dr \leq 2EB. \tag{25}$$

Using the estimate (25) in (23) we get

$$\int r\rho_\circ \left[r^2\, |D_* u|^2 + m^2\, |u|^2\right] dr \geq (E - B)^2. \tag{26}$$

Now using (6) we get

$$(E - B)^2 \leq R_2 \left[\left|D(\rho_\circ D_*\left(r\Omega\right))\right|_{max}\, (b - a)\, R_2 + \left|r\Omega^2\, (D\rho_\circ)\right|_{max}\right] \int |F|^2\, dr;$$

i.e.,

$$(E - B)^2 \leq A_1^2 \int |F|^2\, dr; \tag{27}$$

where $A_1^2 = R_2 \left[|D(\rho_\circ D_* (r\Omega))|_{max} (b-a) R_2 + \left| r\Omega^2 (D\rho_\circ) \right|_{max} \right]$.
Consider the basic flows with $(D\Omega)^2_{min} \neq 0$. For such flows we have

$$\int |F|^2 \, dr \leq \int \frac{r^3 \rho_\circ |D\Omega|^2 |F|^2}{R_1^3 \rho_{\circ_{min}} |D\Omega|^2_{min}} dr. \tag{28}$$

Using this estimate in the inequality (27) we get

$$(E-B)^2 \leq A_2^2 \int r^3 \rho_\circ |D\Omega|^2 |F|^2 \, dr, \tag{29}$$

where $A_2^2 = \frac{A_1^2}{R_1^3 \rho_{\circ_{min}} |D\Omega|^2_{min}}$.

The above inequality (29) can be rewritten as $(E-B)^2 \leq A_2^2 B^2$,
which implies the inequality

$$\left(\frac{E}{B} - 1 \right)^2 \leq A_2^2.$$

This implies the inequalities

$$1 - A_2 \leq \frac{E}{B} \leq 1 + A_2.$$

Taking the right hand inequality and squaring one gets the bound

$$E^2 \leq (1 + A_2)^2 B^2. \tag{30}$$

From the definition of the Richardson number we have

$$J_{min} \leq \frac{\Omega^2 (D\rho_\circ)}{r\rho_\circ (D\Omega)^2},$$

where the J_{min} is the minimum of $J(r)$ taken over $[R_1, R_2]$. From this we have

$$J_{min} \leq \frac{b^2 (D\rho_\circ)}{r\rho_\circ (D\Omega)^2},$$

As a consequence we have

$$(D\rho_\circ) \geq \frac{r\rho_\circ (D\Omega)^2 J_{min}}{b^2}.$$

Considering the last term of the inequality (20) we have

$$\begin{aligned} \int ab\, (D\rho_\circ)\, r^2 |F|^2 \, dr &\geq ab \frac{\int r^3 \rho_\circ (D\Omega)^2 J_{min} |F|^2 \, dr}{b^2} \\ &\geq \frac{a J_{min}}{b} B^2. \end{aligned}$$

On using (30) we get

$$\int ab\, (D\rho_\circ)\, r^2 |F|^2 \, dr \geq \frac{a J_{min} E^2}{b\, (1 + A_2)^2}$$

On using the relation between E^2 and Q (i.e., $E^2 = \int r\rho_\circ |\Omega - c|^2 \, Q dr \geq c_i^2 \int r\rho_\circ Q dr$) we get

$$\int ab\, (D\rho_\circ)\, r^2 |F|^2 \, dr \geq \frac{a J_{min} c_i^2}{b\, (1 + A_2)^2} \int r\rho_\circ Q dr.$$

Thus we have the following result

Lemma 1 *For an arbitrary unstable mode,we have*

$$\int ab\,(D\rho_\circ)\,r^2\,|F|^2\,dr \geq \frac{aJ_{min}c_i^2}{b\,(1+A_2)^2}\int r\rho_\circ Q dr, \tag{31}$$

provided $(D\Omega)^2_{min} \neq 0$.

By making use of estimate given by above lemma in the inequality (20) we get

$$\left(c_r^2 + c_i^2 - (a+b)c_r + ab + \frac{aJ_{min}c_i^2}{b\,(1+A_2)^2}\right)\int r\rho_\circ Q dr \leq 0.$$

Since the integral term is positive for all unstable modes it follows that

$$\left(c_r^2 + c_i^2 - (a+b)c_r + ab + \frac{aJ_{min}c_i^2}{b\,(1+A_2)^2}\right) \leq 0.$$

This can be put in the following way

Theorem 3 *For basic flows with* $(D\Omega)^2_{min} \neq 0$, *the complex wave velocity* $c = c_r + ic_i$ *for any unstable mode must lie inside the semielliptical region*

$$\left(c_r - \frac{a+b}{2}\right)^2 + \left(1 + \frac{aJ_{min}}{b\,(1+A_2)^2}\right)c_i^2 \leq \left(\frac{b-a}{2}\right)^2. \tag{32}$$

The condition $(D\Omega)^2_{min} \neq 0$ means we are considering flows with monotonic angular velocity. For a homogenous flow for which $J_{min} = 0$, this semiellipse theorem reduces to Fung's semicircle theorem. It is seen that the above semielliptical region is different from the one given in Fung [6]. However Fung's semielliptical region has been further improved in Dattu and Subbiah [9] and Prakash and Subbiah [10]. We shall show that similar improvements can be obtained for our instability region also.

Note: In the limit $R_2 \to \infty$ we have $A_1, A_2 \to \infty$. In this limit the instability region given by (32) reduces to the semicircular instability region of Fung [6].

5 Semiellipse-type instability regions

Now we shall improve the instability region given by Theorem 3 by obtaining an instability region depending on the growth rate mc_i. For this purpose we follow the parallel flow analysis of Makov and Stepanyants [19] to the problem under consideration. Using (24) in (23) we get

$$\int r\rho_\circ\left[r^2\,|D_*u|^2 + m^2\,|u|^2\right]dr \geq E^2 + B^2 - 2AB.$$

i.e.,

$$\int r\rho_\circ\left[r^2\,|D_*u|^2 + m^2\,|u|^2\right]dr \geq A^2 + m^2C^2 + B^2 - 2AB. \tag{33}$$

By using the inequality (6) in (33) we get

$$A^2 + m^2C^2 + B^2 - 2AB \leq A_2^2\int r^3\rho_\circ\,|D\Omega|^2\,|F|^2\,dr \leq A_2^2B^2. \tag{34}$$

Now (34) can be written as

$$A^2 - 2AB + m^2C^2 + B^2(1 - A_2^2) \leq 0. \tag{35}$$

Solving the above inequality with respect to A we get

$$B - \sqrt{B^2A_2^2 - m^2C^2} \leq A \leq B + \sqrt{B^2A_2^2 - m^2C^2}.$$

Our aim is to obtain a lower bound for the last term of (20) to improve the semicircular instability region. This can be done by taking the upper bound of A. If we take the lower bound of A no improvement of the instability region can be obtained.

The upper bound of A gives

$$A^2 + m^2C^2 \le 2B^2\left[1+\sqrt{A_2^2-\frac{m^2C^2}{B^2}}-\frac{1-A_2^2}{2}\right]. \tag{36}$$

Now we see that

$$\begin{aligned}\frac{C^2}{B^2} &\ge \frac{c_i^2\int r\rho_\circ\,|F|^2}{R_2^2\,|D\Omega|_{max}^2\int r\rho_\circ\,|F|^2}\\ &= \frac{c_i^2}{R_2^2\,|D\Omega|_{max}^2}.\end{aligned} \tag{37}$$

So

$$\begin{aligned}A^2+m^2C^2 &= \int r^3\rho_\circ\,|\Omega-c|^2\,|D_*F|^2\,dr + m^2\int r\rho_\circ\,|\Omega-c|^2\,|F|^2\,dr\\ &\ge c_i^2\left[\int r\rho_\circ\,|D_*F|^2\,r^2dr + m^2\int r\rho_\circ\,|F|^2\,dr\right]\\ &= c_i^2\int r\rho_\circ Q dr.\end{aligned} \tag{38}$$

Combining inequalities (38) and (36) we have

$$c_i^2\int r\rho_\circ Q dr \le 2\left[1+\sqrt{A_2^2-\frac{m^2C^2}{B^2}}-\frac{1-A_2^2}{2}\right]\int r^3\rho_\circ\,|D\Omega|^2\,|F|^2\,dr.$$

This gives the estimate

$$\int r^3\rho_\circ\,|D\Omega|^2\,|F|^2\,dr \ge \frac{c_i^2\int r\rho_\circ Q dr}{2\left[1+\sqrt{A_2^2-\frac{m^2c_i^2}{R_2^2|D\Omega|_{max}^2}}-\frac{1-A_2^2}{2}\right]}. \tag{39}$$

The estimate for the last term on the left hand side of (20) yields

$$\begin{aligned}ab\int (D\rho_\circ)r^2\,|F|^2\,dr &\ge \frac{aJ_{min}}{b}\int r^3\rho_\circ\,|D\Omega|^2\,|F|^2\,dr\\ &\ge \frac{aJ_{min}}{b}\frac{c_i^2\int r\rho_\circ Q dr}{\left[1+2\sqrt{A_2^2-\frac{m^2c_i^2}{R_2^2|D\Omega|_{max}^2}}+A_2^2\right]},\end{aligned} \tag{40}$$

on using (39). Using the estimate (40) in (20) we get

$$\left[\left(c_r-\frac{a+b}{2}\right)^2+c_i^2-\left(\frac{b-a}{2}\right)^2+\frac{a}{b}\frac{J_{min}c_i^2}{\left[1+2\sqrt{A_2^2-\frac{m^2c_i^2}{R_2^2|D\Omega|_{max}^2}}+A_2^2\right]}\right]\int r\rho_\circ Q dr \le 0.$$

Since the integral is positive, we have the following generalized semi ellipse theorem.

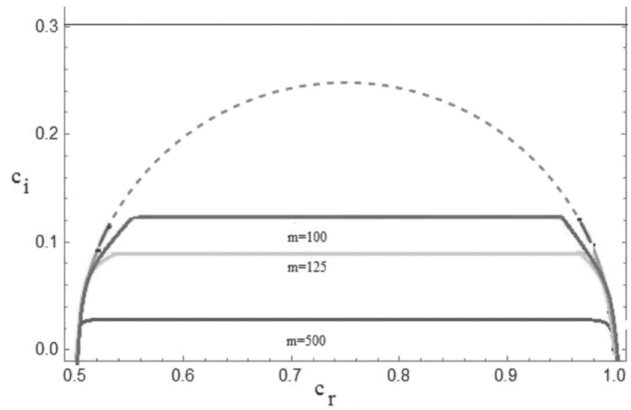

Fig. 10 Instability region given by (41) for different values of m. Dashed line represents the semi ellipse obtained from (32)

Theorem 4 *For basic flows with* $(D\Omega)^2_{min} \neq 0$ *the complex phase velocity c of any unstable azimuthal mode must lie inside the semi-ellipse type region given by*

$$\left(c_r - \frac{a+b}{2}\right)^2 + \left(1 + \frac{a}{b}\frac{J_{min}}{\left[1 + 2\sqrt{A_2^2 - \frac{m^2 c_i^2}{R_2^2 |D\Omega|^2_{max}}} + A_2^2\right]}\right) c_i^2 \leq \left(\frac{b-a}{2}\right)^2. \tag{41}$$

It is to be noted that $A_2^2 - \frac{m^2 c_i^2}{R_2^2 |D\Omega|^2_{max}} \geq 0$ is necessary for the above result to be valid and that this is true follows from the estimate for growth rate given in Theorem 1. In fact the result of Theorem 1 can be stated as $m^2 c_i^2 \leq \frac{A_1^2}{R_2 \rho_{\circ_{min}}}$. As a consequence we have $A_2^2 = \frac{A_1^2}{R_1^3 \rho_{\circ_{min}} |D\Omega|^2_{min}} \geq \frac{\rho_{\circ_{min}} R_2 (m^2 c_i^2)}{R_1^3 \rho_{\circ_{min}} |D\Omega|^2_{max}} \geq \frac{m^2 c_i^2}{R_2^2 |D\Omega|^2_{max}}$. Furthermore, note that the instability region explicitly depends on minimum local Richardson number J_{min}, radius of the outer cylinder R_2 and on growth rate mc_i. In the absence of outer cylinder that is when R_2 is infinite, the generalized semi ellipse theorem (41) reduces to semi ellipse theorem (32). When outer cylinder is present at $r = R_2 < \infty$, the generalized semi-elliptical region (41) lies inside the semi-ellipse region of Fung [6] and also inside the semi-ellipse region given (32). Since, $m \neq 0$ for azimuthal modes the semi-ellipse type instability region given by (41) is a definite improvement over the instability region given by (32). When $J_{min} \to 0$, (41) reduces to Fung [6] semicircle theorem .

For better understanding of the result we plot the curves by taking c_r along $x-axis$ and c_i along $y-axis$ for the angular velocity profile $\Omega(r) = \left(\frac{1.5}{2}\right) + \left(\frac{0.5}{2}\right) tanh\left(4log\left(\frac{r}{2}\right)\right)$ and density $\rho_\circ(r) = \frac{1}{2}\left[13 + tanh\left(\frac{r-2.5}{0.1}\right)\right]$. The angular velocity profile chosen is a special case of the Shukhman [17]and the density profile is chosen from Dixit and Govindarajan [8]. By choosing $R_1 = 1$ and $R_2 = 4$ we get $a = 0.5041946$ and $b = 0.998054$. The dashed line in Figure 10 shows the semi ellipse obtained by the estimate (32) and the heavy line in Figure 10 is given for generalized semi ellipse given by Theorem 4 for different values of m. The instability region given by (41) is a semiellipse - type regions rather than semielliptical regions. This is because of the presence of c_i in the denominator of the second term. Moreover it is necessary that the expression inside the square root is non-negative and this is the reason for part of the curves being horizontal lines. It can be seen from Figure 10 that the instability regions given by (41) shrink in size when the values of the azimuthal wave m is increased. This is an indication of azimuthal modes with larger m being less unstable than those with smaller values of m.

It is to be noted that the instability region obtained in Theorem 4 depends on the growth rate mc_i rather than on the wave number m. Now we shall obtain an instability region depending on m independently. For this purpose we adapt the parallel flow work of Jain and Kochar [20] to the swirling flow stability problem.

Use of the relation $u = (\Omega - c) F$ in (6) gives

$$\int \rho_\circ \left[r^2 |D_* u|^2 + m^2 |u|^2\right] r dr \leq \left[|D(\rho_\circ D_*(r\Omega))|_{max} (b-a) R_2 + \left|r\Omega^2 (D\rho_\circ)\right|_{max}\right] R_2 \int |F|^2 dr.$$

Use of (28) gives the relation

$$\int \rho_\circ \left[r^2 \left| D_* u \right|^2 + m^2 \left| u \right|^2 \right] r dr \leq \frac{A_1^2}{R_1^3 \rho_{\circ_{min}} \left| D\Omega \right|_{min}^2} \int r^3 \rho_\circ \left| D\Omega \right|^2 \left| F \right|^2 dr = A_2^2 B^2. \tag{42}$$

By use of well known Rayleigh-Ritz inequality we have

$$\frac{\int \rho_\circ r^3 \left| D_* u \right|^2 dr}{\int r \rho_\circ \left| u \right|^2 dr} \geq \frac{\rho_{\circ_{min}} \int r^3 \left| D_* u \right|^2 dr}{\rho_{\circ_{max}} \int r \left| u \right|^2 dr} \geq \frac{\rho_{\circ_{min}}}{\rho_{\circ max}} \left[\frac{R_1^2 \pi^2}{(R_2 - R_1)^2} \right] = \lambda_1^2. \tag{43}$$

By the use of the estimate of λ_1^2 in (42),we get

$$\begin{aligned} \lambda_1^2 + m^2 &\leq A_2^2 \frac{\int r^3 \rho_\circ \left| D\Omega \right|^2 \frac{|u|^2}{|\Omega - c|^2} dr}{\int r \rho_\circ \left| u \right|^2 dr} \\ &\leq A_2^2 \frac{\left| D\Omega \right|_{max}^2 R_2^2}{c_i^2}, \end{aligned} \tag{44}$$

This gives

$$c_i^2 \leq \frac{A_2^2 \left| D\Omega \right|_{max}^2 R_2^2}{\lambda_1^2 + m^2}. \tag{45}$$

If

$$\nu_\circ^2 = \frac{\int r^3 \rho_\circ \left| D_* F \right|^2 dr}{\int r \rho_\circ \left| F \right|^2 dr}, \tag{46}$$

then

$$\frac{\int r \rho_\circ Q dr}{\int r \rho_\circ \left| F \right|^2 dr} = \nu_\circ^2 + m^2 \leq \left(\frac{\nu_\circ^2 + m^2}{\lambda_1^2 + m^2} \right) \frac{\left| D\Omega \right|_{max}^2 A_2^2 R_2^2}{c_i^2}. \tag{47}$$

Now we have

$$\begin{aligned} \frac{E^2}{B^2} &= \frac{\int r \rho_\circ \left| \Omega - c \right|^2 Q dr}{\int r^2 \rho_\circ \left(D\Omega \right)^2 r \left| F \right|^2 dr}, \\ &\geq \frac{c_i^2 \int r \rho_\circ Q dr}{R_2^2 \left| D\Omega \right|_{max}^2 \int r \rho_\circ \left| F \right|^2 dr}, \\ &\geq \frac{c_i^2}{R_2^2 \left| D\Omega \right|_{max}^2} \left(\nu_\circ^2 + m^2 \right). \end{aligned} \tag{48}$$

which can be written as

$$\nu_\circ^2 + m^2 \leq \frac{R_2^2 \left| D\Omega \right|_{max}^2}{c_i^2} \left(\frac{E^2}{B^2} \right).$$

By making use of (30) we get

$$\nu_\circ^2 + m^2 \leq \frac{R_2^2 \left| D\Omega \right|_{max}^2}{c_i^2} \left(1 + A_2 \right)^2. \tag{49}$$

Inequalities (47) and (49) give

$$\frac{\int r \rho_\circ Q dr}{\int r \rho_\circ \left| F \right|^2 dr} \leq \frac{A_2^2}{\left(\lambda_1^2 + m^2 \right)} \left(1 + A_2 \right)^2 \frac{R_2^4 \left| D\Omega \right|_{max}^4}{c_i^4},$$

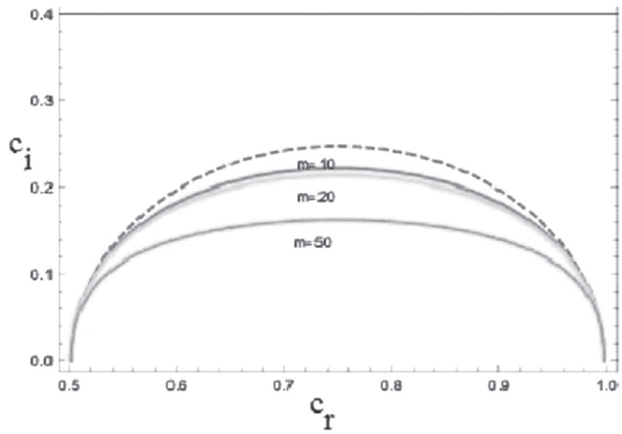

Fig. 11 Instability region given by (52) for different values of m. Dashed line represents the semi ellipse obtained from (32)

which gives

$$\int r\rho_\circ |F|^2\, dr \geq \frac{\left(\lambda_1^2+m^2\right) c_i^4 \int r\rho_\circ Q dr}{A_2^2 \,|D\Omega|_{max}^4\, R_2^4\,(1+A_2)^2}. \tag{50}$$

Using this we have

$$\int ab\,(D\rho_\circ)\, r^2\, |F|^2 \geq \frac{a}{b} J_{min} R_1^2 \left[\frac{\left(\lambda_1^2+m^2\right) c_i^4 \int r\rho_\circ Q dr}{A_2^2 \,|D\Omega|_{max}^2\, R_2^4\,(1+A_2)^2}\right]. \tag{51}$$

Using this estimate in (20) we have

$$\left[\left(c_r-\frac{a+b}{2}\right)^2 + c_i^2 - \left(\frac{b-a}{2}\right)^2 + \frac{a}{b} J_{min} R_1^2 \left[\frac{\left(\lambda_1^2+m^2\right) c_i^4}{A_2^2 \,|D\Omega|_{max}^2\, R_2^4\,(1+A_2)^2}\right]\right] \int r\rho_\circ Q dr \leq 0,$$

Since the integral $\int r\rho_\circ Q dr$ is positive we have,

$$\left(c_r-\frac{a+b}{2}\right)^2 + c_i^2 + \frac{a J_{min} R_1^2}{b}\left(\frac{\left(\lambda_1^2+m^2\right) c_i^4}{A_2^2 \,|D\Omega|_{max}^2\, R_2^4\,(1+A_2)^2}\right) \leq \left(\frac{b-a}{2}\right)^2.$$

Thus we have arrived at the following instability region.

Theorem 5 *For basic flows with $(D\Omega)_{min}^2 \neq 0$, the complex phase velocity c of any unstable azimuthal mode must lie inside the generalized semi-ellipse type region given by*

$$\left(c_r-\frac{a+b}{2}\right)^2 + c_i^2 + \frac{a J_{min} R_1^2}{b}\left(\frac{\left(\lambda_1^2+m^2\right) c_i^4}{A_2^2 \,|D\Omega|_{max}^2\, R_2^4\,(1+A_2)^2}\right) \leq \left(\frac{b-a}{2}\right)^2. \tag{52}$$

.

It is seen that the present instability region depends on the width of the annular region $d = R_2 - R_1$ through the term λ_1^2. When $d \to 0$, $\lambda_1^2 \to \infty$ and it follows from (52) that $c_i^2 \to 0$ or $c_i \to 0$ in this limit. This is illustrated in Figure 11 by considering the angular velocity profile $\Omega(r) = \left(\frac{1.5}{2}\right) + \left(\frac{0.5}{2}\right) tanh\left(4log\left(\frac{r}{2}\right)\right)$ and density $\rho_\circ(r) = \frac{1}{2}\left[13 + tanh\left(\frac{r-2.5}{0.1}\right)\right]$. We choose $R_1 = 1$ and $R_2 = 4$. The dashed line in Figure 11 shows the semi ellipse obtained by the estimate (32) and the heavy line in Figure 11 is given for generalized semi ellipse given by Theorem 5 for different values of m.

It is interesting to note that the instability region given in Theorem 5 depends also on the width of the annulus also. This is illustrated in Figure 12.

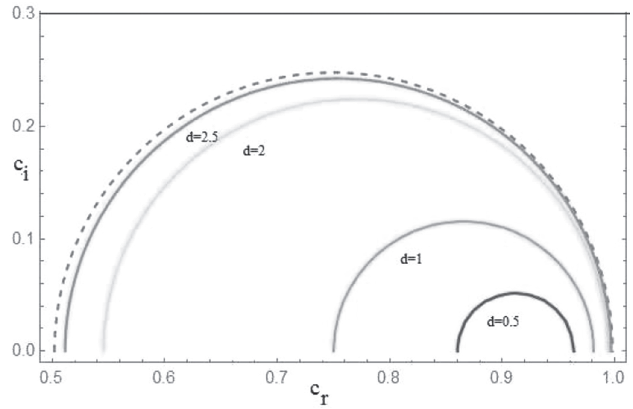

Fig. 12 Instability region of Theorem 5 for different values of width d. Dashed line represents the semi ellipse obtained from (32)

6 Concluding remarks

For a swirling flow of inviscid incompressible homogeneous fluid between coaxial cylinders with angular velocity $\Omega(r)$ a necessary condition for instability to infinitesimal azimuthal disturbances is that (DZ) should change sign atleast once in the flow domain where $Z = r(D\Omega) + 2\Omega$ is the basic flow vorticity. If we consider the same flow with radially varying density $\rho_\circ(r)$ then whether the vorticity and its variation would play such an important role in the azimuthal instability is unknown. With a desire to understand the role of (DZ) in the instability of variable density swirling flows to azimuthal disturbances we have tried to obtain general analytical results depending on (DZ). We consider flows satisfying the condition $ab(D\rho_\circ) \geq 0$ for which a semicircle theorem and a semiellipse theorem have been proved in Fung [6]. We have obtained an estimate for growth rate that depends on $D(\rho_\circ Z)$ and $(D\rho_\circ)$. This estimate is further improved to get a growth rate bound that depends on the width $(R_2 - R_1)$ of the annular flow region. If $(R_2 - R_1) \to 0$ then it follows that the growth rate $\omega_i \to 0$. We have compared this growth rate with an estimate found in Dattu and Subbiah [9] and it is seen that our estimate is a better one for a particular flow. We have plotted the growth rate bound curves for flows with different angular velocity profiles and a particular family of density profiles and it is seen from the figures that change in sign of (DZ) corresponds to a higher growth rate bound. Moreover, we have plotted the growth rate bound curve for the q-vortex flow with a particular family of hyperbolic density profiles and it is seen that the growth rate bound increases with the value of the swirl parameter q. In addition to the growth rate bound for this problem, we have obtained various instability regions. The main point to be noted here is that in our semielliptical instability region given in our Theorem 3 the minor axis of the semiellipse depends on $D(\rho_\circ Z)$ in addition to depending on the stratification parameter J_{min}. This semielliptical instability region has been further improved in two theorems giving semiellpise-type regions that depend on the vorticity variation through $D(\rho_\circ Z)$ and on the wave number m and width of the channel. The instability region in Theorem 4 depends on the minimum Richardson number J_{min}, growth rate mc_i, and in addition, it depends on the minimum and maximum of the angular velocity. The general instability region obtained in Theorem 5 depends on the width of the annulus region through λ_1^2 and on the wave number. From this instability region, we can conclude that $m^2 \to \infty$ implies $c_i \to 0_+$. Similarly $d \to 0$ implies $\lambda_1 \to \infty$ and so $c_i \to 0_+$.

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Declarations

Conflict of interest The authors declares that they have no conflict of interest in the submitted work.

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