Indian J Pure Appl Math (2025) 56:1361–1366
https://doi.org/10.1007/s13226-024-00611-6

ORIGINAL RESEARCH

Nonhomogeneous Wavelet Bi-frames for Reducing Subspaces of $H^s(K)$ and their Characterization

M. Younus Bhat

Received: 8 February 2022 / Accepted: 24 May 2024 / Published online: 18 June 2024
© The Indian National Science Academy 2024

Abstract Bhat in *Annal. Univ. Craiova, Math. Comp. Scien. Series* 49: 2(2022), 401-410 has studied nonhomogeneous wavelet bi-frames in Sobolev spaces on local fields of positive characteristic. In this paper, we used the same platform to characterize nonhomogeneous wavelet bi-frames for the reducing subspaces of Sobolev spaces over local fields of positive characteristics.

Keywords Local field · Nonhomogeneous · Wavelet Bi-frames · Sobolev Spaces

Mathematics Subject Classification 42C40 · 42C15 · 43A70 · 11S85 · 47A25

1 Introduction

To begin with it is pertain to mention that a refinable structure generates the classical nonhomogeneous systems with some restrictions on them [16,22–24] which are technical in nature. The generated wavelet systems will have too fast wavelet transform. But the correspondence between these systems is not exact. Thanks to Han [22,23] who showed that the nonstationary wavelets and nonhomogeneous wavelet systems are related so closely. Later on tremendous work has been done nonhomogeneous wavelet systems. we refer to [29–34] and the references therein. Keeping in view these constructions, our aim in this paper is to characterize nonhomogeneous wavelet bi-frames (NWBFs) on Sobolev spaces over local fields of positive characteristic.
Let us now move to the frame theory. It were Duffin and Schaeffer [21] who introduced frames in non-harmonic Fourier series in 1952. The frame theory did not end up here. They were again visualized in 1986 by Daubechies and this process continues. Frames and the dual frames have an important role to play in the characterization of signal, image and video processing, sampling theory, function spaces and many other fields. From the mathematical point of view, a frame is defined as a sequence of functions $\{f_k\}_{k=1}^{\infty}$ of Hilbert space $\mathbb{H}$ if there exist constants $\mathcal{A}, \mathcal{B} > 0$ such that for all $f \in \mathbb{H}$,

$$\mathcal{A}\|f\|_2^2 \leq \sum_{k=1}^{\infty} \left|\langle f, f_k\rangle\right|^2 \leq \mathcal{B}\|f\|_2^2,$$

where $\mathcal{A}$ is lower bound and $\mathcal{B}$ is the upper one. If $\mathcal{A} = \mathcal{B}$, then we have a *tight frame*. If $\mathcal{A} = \mathcal{B} = 1$, then we end up with a *normalized tight frame*. For more details about frames, we refer to [18–20].

Communicated by N. M. Bujurke.

M. Younus Bhat (✉)
Department of Mathematical Sciences, Islamic University of Science and Technology, Kashmir 192122, India
E-mail: gyounusg@gmail.com

From 1980 onwards a progressive growth in the construction wavelets and its associates on local fields of positive characteristic has occured. Local fields can be broadly divided into two fields; fields of zero characteristics and fields of positive characteristics. Although they have similar topology, their MRA (multiresolution analysis) and wavelet theories are quite different. The construction of wavelets, wavelet frames, MRA and other related works on local fields of positive characteristics (LFPC)have been studied by various well known mathematicians in the series of works like [2–15, 17, 25–27]. However the research on these fields is still in its infancy. Recently Ahamd and Shiekh [1] have studied dual wavelet frames in Sobolev spaces on local fields of positive characteristic, where as Bhat [4] introduced the reducing subspaces of Sobolev spaces over local fields of prime characteristics. Continuing our study of frames on local fields, we introduced a comprehensive characterization of nonhomogeneous wavelet bi-frames in Sobolev spaces our local fields of positive characteristic. For the safeguard of the results, we had put this whole work on arXiv [3].
The rest of the paper is structured as follows. In Section 2, we recall some basic Fourier analysis on local fields and also some results which are required in the subsequent sections. In section 3, Characterization of NWBFs over Local Fields is provided.

2 Preliminaries on Local Fields

It is well known that a field K is local if it comprises of the properties of locally compactness, non-discreteness and totally disconnectedness. Local fields have been widely divided into the fields of zero and positive characteristics. When characteristic is zero, it becomes a field of p-adic numbers $\mathbb{Q}_p$ or its finite extension. For K having positive characteristic, it becomes a formal Laurent series over a finite field $GF(p^c)$. The ring of integers over local fields can be defined as $\mathfrak{D} = \{x \in K : |x| \leq 1\}$. Due to the locally compactness, it has Haar measure dx for K^+. For the ring defined above, the prime ideal is like $\mathfrak{B} = \{x \in K : |x| < 1\}$. The residue space thus formed $\mathfrak{D}/\mathfrak{B}$ will be isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$. Due to the disconnectedness of local fields we have a prime element $\mathfrak{p}$ of K such that $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. We closely follow the results and notations of the Taibleson's book [28]. In the rest of this paper, we use the symbols $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.
For any signal or function $f \in L^1(K)$, the novel Fourier transform denoted by $\widehat{f}(\xi)$ can be defined as

$$\mathcal{F}\big\{f(x)\big\} = \widehat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}\,dx.$$

It is pertain to note that

$$\widehat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)}dx = \int_K f(x)\chi(-\xi x)\,dx.$$

Definition 2.1 For $k \in \mathbb{N}_0$, the *translation operator* $T_{u(k)} : L^2(K) \to L^2(K)$ as usual is defined as

$$T_{u(k)}\psi(\cdot) = \psi\,(\cdot - u(k))$$

and the corresponding *dilation operator* $D\psi(\cdot) : L^2(K) \to L^2(K)$ as

$$D\psi\,(\cdot) = \psi\left(\mathfrak{p}^{-1}\cdot\right).$$

For $s \in K$, we define the Sobolev space $H^s(k)$ as the space of all tempered distributions f such that

$$\|f\|^2_{H^s(k)} = \int_K |\hat{f}(\xi)|^2\left(1 + \|\xi\|^2\right)^2 d\xi < \infty,$$

where $\|\cdot\|$ denotes the usual Euclidean norm on K. The inner product for $f, g \in H^s(k)$ is given by

$$\langle f, g\rangle_{H^s(k)} = \int_K \hat{f}(\xi)\overline{\hat{g}(\xi)}\left(1 + \|\xi\|^2\right)^2 d\xi,$$

It is to be noted that for each $f \in H^s(k)$ and $g \in H^{-s}(k)$, we have

$$\langle f, g\rangle_{H^s(k)} = \int_K \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi.$$

The spaces $H^s(k)$ and $H^{-s}(k)$ form a pair of dual spaces over local fields of positive characteristic. For a distribution f, $j \in \mathbb{N}_0, s, k \in K$, one can write

$$f_{j,k} = q^{j/2} f\left(\mathfrak{p}^{-j}\xi - u(k)\right) \quad and \quad f_{j,k} = q^{-js/2} f\left(\mathfrak{p}^{-j}\xi - u(k)\right)$$

Given $L \in \mathbb{N}$, let us suppose that $\psi_0 \in H^s(K)$ be a tempered distribution and $\Psi = \{\psi_1, \cdots, \psi_L\} \subset H^s(K)$ be a finite set of tempered distributions. We denote the homogeneous wavelet system $X^s(\Psi)$ and the nonhomogeneous wavelet system $X^s(\psi_0, \Psi)$ in $H^s(K)$ respectively by

$$X^s(\Psi) = \left\{\psi^s_{\ell,j,k} : j \in \mathbb{N}_0, k \in K, 1 \le \ell \le L\right\}$$

and

$$X^s(\psi_0, \Psi) = \left\{\psi_{0,0,k} : k \in K\right\}\left\{\psi^s_{\ell,j,k} : j \in \mathbb{N}_0, k \in K, 1 \le \ell \le L\right\}$$

3 Characterization of NWBFs over Local Field

In this section, we provide the characterization of NWBFs in $(FH^s(\xi), FH^{-s}(\xi))$. Firstly we need following two propositions whose proofs are given in [4].

Proposition 3.1 *Given $s \in K$, let $\{T_{u(k)}\psi_0 : k \in \mathbb{N}_0\} \cup \{T_{u(k)}\psi_\ell : k \in \mathbb{N}_0,\ 1 \le \ell \le L\}$ be a Bessel sequence in $H^s(K)$. Then*

$$\begin{aligned}
&\sum_{k\in\mathbb{N}_0} \left|\langle g, \psi_{0,0,k}\rangle\right|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0} \left|\langle g, \psi^s_{\ell,j,k}\rangle\right|^2 \\
&= \int_K \left|\hat{g}(\xi)\right|^2 \left(\left|\hat{\psi}_0(\xi)\right|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} m^{-2js}\left|\hat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\right|^2\right) d\xi \\
&+ \int_K \overline{\hat{g}(\xi)} \sum_{k\in\mathbb{N}_0} \hat{g}\left(\xi + u(k)\right) \\
&\times \left(\hat{\psi}_0(\xi)\overline{\hat{\psi}_0\left(\xi + u(k)\right)} + \sum_{\ell=1}^{L}\sum_{j=0}^{k} m^{-2js}\hat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\hat{\psi}_\ell\left(\xi + u(k)\right)}\right)
\end{aligned} \tag{3.1}$$

for $g \in \mathcal{D}$

Proposition 3.2 *Given $s \in K$, let $X^s(\psi_0; \Psi)$ and $X^{-s}(\tilde{\psi}_0; \tilde{\Psi})$ be a Bessel sequences in $H^s(K)$ and $H^{-s}(K)$, respectively. Then for all $f, g \in \mathcal{D}$, we have*

$$\begin{aligned}
&\sum_{k\in\mathbb{N}_0} \langle f, \tilde{\psi}_{0,0,k}\rangle\langle\psi_{0,0,k}, g\rangle + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0} \langle f, \tilde{\psi}^{-s}_{\ell,j,k}\rangle\langle\psi^s_{\ell,j,k}, g\rangle \\
&= \int_K \hat{f}(\xi)\overline{\hat{g}(\xi)}\left(\hat{\psi}_0(\xi)\overline{\hat{\tilde{\psi}}_0(\xi)} + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\hat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}\xi)}\right) d\xi \\
&+ \int_K \overline{\hat{g}(\xi)} \sum_{k\in\mathbb{N}_0} \hat{f}(\xi + u(k)) \\
&\times \left(\hat{\psi}_0(\xi)\overline{\hat{\tilde{\psi}}_0(\xi + u(k))} + \sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(k)}\hat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}(\xi + u(k))}\right) d\xi.
\end{aligned} \tag{3.2}$$

We are now in a position to present a characterization of NWBFs in $(FH^s(\xi), FH^{-s}(\xi))$ in the form of the following theorem.

Theorem 3.1 *Given $s \in K$, let $FH^s(\xi)$ and $FH^{-s}(\xi)$ be reducing subspaces of $H^s(K)$ and $H^{-s}(K)$, respectively, $\psi_0 \in H^s(K)$, $\tilde{\psi}_0 \in H^{-s}(K)$ and $\Psi \in H^s(K)$, $\tilde{\Psi} \in H^{-s}(K)$. Suppose that $X^s(\psi_0, \Psi)$ and $X^{-s}(\tilde{\psi}_0, \tilde{\Psi})$ are Bessel sequences in $FH^s(\xi)$ and $FH^{-s}(\xi)$, respectively. Then $X^s(\psi_0, \Psi); X^{-s}(\tilde{\psi}_0, \tilde{\Psi})$ is an NWBFs in $(FH^s(\xi), FH^{-s}(\xi))$ if and only if*

$$\hat{\psi}_0(\cdot)\overline{\hat{\tilde{\psi}}_0(\cdot + u(k))} \sum_{\ell=1}^{L} \sum_{j=0}^{\kappa(k)} \hat{\psi}_\ell(\mathfrak{p}^{-j}\cdot)\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}(\cdot + u(k)))} = \delta_{0,k} \quad a.e. \text{ on } \xi. \tag{3.3}$$

Proof As $\mathcal{D} \cap FH^s(\xi)$ is dense in $FH^s(\xi)$, then

$$X^s(\psi_0, \Psi); X^{-s}(\tilde{\psi}_0, \tilde{\Psi})$$

ia an NWBF's in $(FH^s(\xi), FH^{-s}(\xi))$ iff for $f \in \mathcal{D} \cap FH^s(\xi)$ and $g \in \mathcal{D} \cap FH^{-s}(\xi)$

$$\sum_{k\in\mathbb{N}_0} \langle f, \tilde{\psi}_{0,0,k}\rangle\langle\psi_{0,0,k}, g\rangle + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0} \langle f, \tilde{\psi}^{-s}_{\ell,j,k}\rangle\langle\psi^s_{\ell,j,k}, g\rangle = \langle f, g\rangle$$

This is equivalent to

$$\begin{aligned}&\sum_{k\in\mathbb{N}_0} \langle(\hat{f}\chi_\xi), \tilde{\psi}_{0,0,k}\rangle\langle\psi_{0,0,k}, (\hat{g}\chi_\xi)\rangle + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0} \langle(\hat{f}\chi_\xi), \tilde{\psi}^{-s}_{\ell,j,k}\rangle\langle\psi^s_{\ell,j,k}, (\hat{g}\chi_\xi)\rangle \\ &= \langle(\hat{f}\chi_\xi), (\hat{g}\chi_\xi)\rangle\end{aligned} \tag{3.4}$$

with $f, g \in \mathcal{D}$ as $\mathcal{D} \cap FH^s(\xi) = \{(\hat{h}\chi_\xi) : h \in \mathcal{D}\}$. The expression of (3.4) is well defined as $X^s(\psi_0, \Psi)$ and $X^{-s}(\tilde{\psi}_0, \tilde{\Psi})$ are Bessel sequences in $H^s(K)$ and $H^{-s}(K)$. Hence using Lemma 3.2, we can write expression (3.4) as

$$\begin{aligned}&\int_K \hat{f}(\xi)\overline{\hat{g}(\xi)}\chi_\xi(\xi)\left(\hat{\psi}_0(\xi)\overline{\hat{\tilde{\psi}}_0(\xi)} + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\hat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}\xi)}\right)d\xi \\ &+ \int_K \overline{\hat{g}(\xi)}\chi_\xi(\xi)\sum_{k\in\mathbb{N}_0}(\hat{f}\chi_\xi)(\xi + u(k)) \\ &\times\left(\hat{\psi}_0(\xi)\overline{\hat{\tilde{\psi}}_0(\xi + u(k))} + \sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(k)}\hat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}\xi + u(k))}\right)d\xi \\ &= \int_K \hat{f}(\xi)\overline{\hat{g}(\xi)}\chi_\xi(\xi)\end{aligned} \tag{3.5}$$

with $f, g \in \mathcal{D}$. Hence the expression (3.3) leads (3.5). It remains only to prove the converse statement. Suppose (3.5) hold. Using Cauchy-Schwartz inequality, we have

$$\begin{aligned}&|\hat{\psi}_0(\cdot)\overline{\hat{\tilde{\psi}}_0(\cdot + u(k))}| + \sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(k)}|\hat{\psi}_\ell(\mathfrak{p}^{-j}\cdot)\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}\cdot + u(k))}| \\ &\le \left(|\hat{\psi}_0(\cdot)|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}m^{-2js}|\hat{\psi}_\ell(\mathfrak{p}^{-j}\cdot)|^2\right)^{\frac{1}{2}} \\ &+ \left(|\hat{\psi}_0(\cdot + u(k))|^2 + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty}m^{-2js}|\hat{\psi}_\ell(\mathfrak{p}^{-j}(\cdot + u(k)))|^2\right)^{\frac{1}{2}} \\ &\le B_1B_2(1 + |\cdot|^2)^{-s}(1 + |\cdot + u(k)|^2)^s\end{aligned}$$

$$= C_k < \infty$$

for each $k \in \mathbb{N}_0$ using Lemma 3.2. Hence the series $\hat{\psi}_0(\cdot)\overline{\hat{\tilde{\psi}}_0(\cdot + u(k))} + \sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(k)} \hat{\psi}_\ell(\mathfrak{p}^{-j}\cdot)\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}\cdot + u(k))}$ converges absolutely a. e. on K and is contained in $L^\infty(K)$. Therefore almost all points in K are its Lebesgue points. Next we consider two cases. When $k = 0$. Let $\xi_0 \neq 0$ be a Lebesgue point of $\hat{\psi}_0(\cdot)\overline{\hat{\tilde{\psi}}_0(\cdot)} + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} \hat{\psi}_\ell(\mathfrak{p}^{-j}\cdot)\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}\cdot)}$ and $\chi_\xi(\cdot)$. Fix f and g for $0 < \epsilon < u(1)$, we have

$$\hat{f}(\cdot) = \hat{g}(\cdot) = \frac{\chi_{B(\xi_0,\epsilon)}}{\sqrt{|B(\xi_0,\epsilon)|}}$$

in (3.5), where $B(\xi_0, \epsilon)$ is an open ball centred at ξ_0 and radius ϵ. Therefore

$$\frac{1}{|B(\xi_0,\epsilon)|}\int_{B(\xi_0,\epsilon)} \chi_\xi(\xi)\left(\hat{\psi}_0(\xi)\overline{\hat{\tilde{\psi}}_0(\xi)} + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} \hat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}\xi)}\right) d\xi$$
$$= \frac{1}{|B(\xi_0,\epsilon)|}\int_{B(\xi_0,\epsilon)} \chi_\xi(\xi) d\xi$$

Letting $\epsilon \to 0$, we have

$$\hat{\psi}_0(\xi_0)\overline{\hat{\tilde{\psi}}_0(\xi_0)} + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} \hat{\psi}_\ell(\mathfrak{p}^{-j}\xi_0)\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}\xi_0)}$$

For $k \neq 0$, we fix $k_0 \in \mathbb{N}_0$ and take f and g

$$\hat{f}(\cdot + u(k)) = \hat{g}(\cdot) = \frac{\chi_{B(\xi_0,\epsilon)}}{\sqrt{|B(\xi_0,\epsilon)|}}$$

in (3.5), with $0 < \epsilon < \frac{1}{2}$. Therefore

$$\frac{1}{|B(\xi_0,\epsilon)|}\int_{B(\xi_0,\epsilon)} \chi_\xi(\xi)$$
$$\times\left(\hat{\psi}_0(\xi)\overline{\hat{\tilde{\psi}}_0(\xi + u(k))} + \sum_{\ell=1}^{L}\sum_{j=0}^{\kappa(k)} \hat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}\xi + u(k))}\right) d\xi = 0$$

Letting $\epsilon \to 0$ and using Lebesgue Differentiation theorem, we get

$$\hat{\psi}_0(\xi_0 + u(k_0))\overline{\hat{\tilde{\psi}}_0(\xi_0)} + \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} \hat{\psi}_\ell(\mathfrak{p}^{-j}\xi_0)\overline{\hat{\tilde{\psi}}_\ell(\mathfrak{p}^{-j}\xi_0 + u(k_0))}$$

Hence we obtain (3.3) by using the arbitrariness of ξ_0 and k_0, which completes the proof of the theorem. □

Acknowledgements We are deeply indebted to the referee for his/her valuable suggestions which greatly improved the presentation of this paper.

Funding Not applicable

Availability of data and materials The data is provided on the request to the authors.

Author contributions

All the authors equally contributed towards this work.

Declarations

Competing interests The authors have no competing interests.

Ethical Approval Not applicable.

References

1. Ahmad I. and Sheikh N.A, Dual wavelet frames in Sobolev spaces on local fields of positive characteristic, *Filomat* **34** (2020) 2091-2099
2. Ahmad O, Bhat M. Y, and Sheikh N. A, Construction of Parseval framelets associated with GMRA on local fields of positive characteristic, *Num. Funct. Anal Optimiz.* **42**(3)(2021) 344-370.
3. Bhat M. Y, Characterization of nonhomogeneous wavelet bi-frames for reducing subspaces of Sobolev spaces over on local fields of prime characteristic, axXiv:2110.11243v1 [math.FA] 15 Oct 2021.
4. Bhat M. Y, On the nonhomogeneous wavelet bi-frames for reducing subspaces of $H^s(K)$, *Annal. Univ. Craiova, Math. Comp. Scien. Series*, **49**(2)(2022), 401-410
5. Bhat M. Y, Characterization and Wavelet Packets Associated with VN-MRA on $L^2(K, C^N)$, *Azer. J. Math.* **11**(2)(2021), 3-24.
6. Bhat M. Y, Multiwavelets on Local Fields of Positive Characteristic, *Annal. Univ. Craiova, Math. Comp. Scien. Series*, **47**(2)(2020), 276-284
7. Bhat M. Y, A Short Note on Wavelet Frames Based on FMRA on Local Fields, *J. Math.*, (2020), Article ID 3957064, 5 pages.
8. Bhat M. Y, Nonstationary Multiresolution Analysis on Local Fields of Prime Characteristic, *Acta Scien. Math.*, **86**(2020), 303-320.
9. Bhat M. Y, Nonuniform Discrete Wavelets on Local Fields of Positive Characteristic, *Comp. Anal. Oper. Theory*, **13**(2019), 2203-2208.
10. Bhat M. Y, Dual Wavelets Associated with Nonuniform MRA, *Tamk. J. Math.* **50**(2) (2019), 119-132.
11. Bhat M. Y, Pair of Dual Wavelet Frames on Local Fields, *Acta Scien. Math.* **85** (2019), 271-289.
12. Bhat M. Y, Tight Affine, Quasi-Affine Wavelet Frames on Local Fields of Positive Characteristic, *Inter. J. Func. Anal. Oper. Theory Appl.* **11**(1) (2019), 13-31.
13. Bhat M. Y, Necessary Condition and Sufficient Conditions for Nonuniform Wavelet Frames on $L^2(K)$, *Inter. J. Wavelets, Multires. Info. Process*, **16**(1) (2018).
14. Behera B and Jahan Q, Multiresolution analysis on local fields and characterization of scaling functions, *Adv. Pure Appl. Math.* **3** (2) (2012) 181-202.
15. Behera B and Jahan Q, Characterization of wavelets and MRA wavelets on local fields of positive characteristic, *Collect. Math.* **66** (1) (2015) 33-53.
16. Benedetto J.J and Li s, The theory of multiresolution analysis frames and applications to filter banks *App. Comput. Harmon. Anal.* **5** (1998) 389-427.
17. Benedetto J.J and . Benedetto R. L, A wavelet theory for local fields and related groups. *J. Geom. Anal.* **14** (2004) 423-456.
18. Christensen O. *An introduction to frames and Riesz bases*. New York: Springer; 2003.
19. Chui C. K, Shi X. *Orthonormal wavelets and tight frames with arbitrary real dilations*. Appl. Comput. Harmon. Anal. **9**(2000) 243-264.
20. Daubechies I. *Ten lecture on wavelets*. Philadelphia(PA):SIAM; (1992).
21. Duffin R. J. and Shaeffer A. C., A class of nonharmonic Fourier series. *Trans. A. M. Soc.* **72** (1952)341-366.
22. Han B, Pairs of frequency based nonhomogeneous dual wavelet frames in the distribution space, *Appl. Comput. Harmon. Anal.* **29** (2010) 330-353
23. Han B. and Shen Z., Nonhomogeneous wavelet systems in high dimensions, *Appl. Comput. Harmon. Anal.* **32** (2012) 169-196.
24. Jia H. F. and Zhang J., A characterization of nonhomogeneous wavelet bi-frames for reducing subspaces of Sobolev spaces, *J. Ineq. Appl* (**55**2021)
25. Li. D and Jiang H. K, The necessary condition and sufficient conditions for wavelet frame on local fields, *J. Math. Anal. Appl.* **345** (2008) 500-510.
26. Shukla N. K, Maury S. C, Mittal S, Semi-orthogonal Parseval Wavelets Associated with GMRAs on Local Fields of Positive Characteristic, *Mediterr. J. Math.* (2019) 16: 120 https://doi.org/10.1007/s00009-019-1383-1.
27. Shukla N. K, Maury S. C, Super-wavelets on local fields of positive characteristic. *Math. Nachr.* **291** (4) (2018) 704-719.
28. Taibleson M. H, . *Fourier Analysis on Local Fields*, Princeton University Press, Princeton, NJ 1975.
29. Zhang Y, Walsh shift-invariant sequences and p-adic nonhomogeneous dual wavelet frames in $L^2(\mathbb{R}^+)$, *Results Math.* **74** (2019), Paper No. 111, 26 pp.
30. Zhang Y, Li Y, Weak nonhomogeneous wavelet dual frames for Walsh reducing subspace of $L^2(\mathbb{R}^+)$, *Int. J. Wavelets Multiresolut. Inf. Process.* **20** (2022), no. 1, Paper No. 2150040, 19 pp.
31. Li Y, Zhang J, Nonhomogeneous dual wavelet frames and mixed oblique extension principles in Sobolev spaces, *Appl. Anal.* **97**(2018), 1049-1073.
32. Zhang J, Li Y, On a class of weak nonhomogeneous affine bi-frames for reducing subspaces of $L^2(\mathbb{R}^d)$ *Acta Math. Sin. (Engl. Ser.)* **33** (2017), 1339-1351.
33. Li Y, Jia H, Weak nonhomogeneous wavelet bi-frames for reducing subspaces of Sobolev spaces, *Numer. Funct. Anal. Optim.* **38** (2017), 181-204.
34. Zhang J, Li Y, A characterization of nonhomogeneous wavelet dual frames in Sobolev spaces, *J. Inequal. Appl.* 2016, Paper No. 288.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.