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ORIGINAL RESEARCH

Pointwise and weighted estimates for Bernstein-Kantorovich type operators including beta function

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Abstract We establish a Kantorovich variant of Bernstein operators including the beta function inspired by the King-type operators which preserve certain functions. We provide some pointwise, weighted, and direct approximation theorems to examine the approximation properties of the introduced sequence of operators. Furthermore, by graphical and numerical examples, we demonstrate the convergence of defined operators with the help of the MATLAB program.

Keywords Pointwise convergence · King type operators · Beta function · Weighted approximation · Error estimation

Mathematics Subject Classification 41A10 · 41A25 · 41A36

1 Introduction

Many approximating operators have been introduced under certain conditions in order to approximate continuous functions [1–4,24–31]. It is known that most of the approximating operators say, $\mathfrak{L}_n$, preserve $e_m(v) = v^m$ $(m = 0, 1)$, i.e., $\mathfrak{L}_n(e_0; v) = e_0(v)$ and $\mathfrak{L}_n(e_1; v) = e_1(v)$, $n \in \mathbb{N}$. For example, this preservation holds for the Bernstein, λ-Bernstein (see, [6,8,12,23]) and $\alpha-$Bernstein polynomials (see [7,9–11]) as well as Baskakov and Szász-Mirakjan operators (see [13,14]). For each of such operators $\mathfrak{L}_n(e_2; v) \neq e_2(v) = v^2$. First time any non-trivial sequence $\{\mathfrak{S}_m\}$ of linear positive operators was presented by King [15] to approximate continuous functions on [0, 1] although they preserve e_0 and e_2 functions, i.e., $\mathfrak{S}_m : C[0, 1] \to C[0, 1]$, if there is any $m \in \mathbb{N}$ and regardless of the function $f \in C[0, 1]$, is given as below:

$$\mathfrak{S}_m(f; y) = \sum_{k=0}^{m} f\left(\frac{k}{m}\right)\binom{m}{k}\left(r_m^*(y)\right)^k\left(1 - r_m^*(y)\right)^{m-k}, \quad 0 \leqslant y \leqslant 1, \tag{1.1}$$

where $r_m^*(y) : [0, 1] \to [0, 1]$ is defined by

$$r_m^*(y) = \begin{cases} y^2 & \text{if } m = 1 \\ -\frac{1}{2(m-1)} + \sqrt{\frac{m}{m-1}y^2 + \frac{1}{4(m-1)^2}} & \text{if } m = 2, 3, \ldots. \end{cases} \tag{1.2}$$

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When we replace $r_m^*(y)$ by e_1, the operators given by (1.1) are reduced to the Bernstein polynomials. This sequence satisfies $\mathfrak{S}_m(e_1; y) = r_m^*(y)$ and preserves the test functions e_0 and e_2. One can observe that the operators $\mathfrak{S}_m$ provide a better rate of convergence than the classical Bernstein polynomials so long as $0 \leqslant y \leqslant 1/3$. Operators preserving different test functions have been investigated by various researchers, e.g. [16], [18], [19].

Recently, Bhatt et al. [17] established a new sequence of Bernstein type operators including beta function as follows:

$$\mathfrak{C}_m(f; y) = \sum_{j=0}^{m} \varrho_{m,j}(y) f\left(\frac{j}{m}\right) \tag{1.3}$$

for $\sigma \in C[0, 1]$ and $y \in [0, 1]$, where

$$\varrho_{m,j}(y) = \binom{m}{j} \frac{\beta(my + j + 1,\ 2m - j - my + 1)}{\beta(my + 1,\ m - my + 1)}.$$

Here $\beta(u, v)$ is the Beta function defined as

$$\beta(u, v) = \int_0^1 t^{u-1}(1-t)^{v-1} \mathrm{d}t \quad (u, v > 0).$$

The present study deals with certain estimates of the Bernstein-Kantorovich type operators based on the beta function. In section 2, we introduce our sequence of operators. Section 3 is devoted to proving some pointwise and weighted approximation theorems. In section 4, we give a Lipschitz theorem, estimate the rate of convergence of defined operators, and obtain a direct approximation theorem. In section 5, we establish an asymptotic and a quantitative formula. In the last section, we demonstrate certain computer graphics to show the convergence of defined operators.

2 Preliminary results

Our purpose here is to introduce Kantorovich-type Bernstein operators based on beta functions preserving linear mapping. Let the sequence of real-valued continuous function $\theta_n(y)$ be defined as

$$\theta_n(y) = \left(1 + \frac{2}{n}\right) y - \frac{3}{2n} - \frac{1}{n^2},$$

with $0 \leqslant \theta_n(y) < \infty$, then we define our operators as follows:

$$\zeta_n(f; y) = n \sum_{j=0}^{n} Q_{n,j}\left(\theta_n(y)\right) \int_{\frac{j}{n}}^{\frac{j+1}{n}} f(t)\mathrm{d}t, \tag{2.1}$$

where

$$Q_{n,j}(y) = \binom{n}{j} \frac{\beta(ny + j + 1, 2n - j - ny + 1)}{\beta(ny + 1, n - ny + 1)}.$$

It can be observed that $\lim_{n\to\infty} \theta_n(y) = y$.

Moments of the proposed operators, which are needed throughout the study, are given in the next lemma.

Lemma 2.1 *We have the moments for the defined operators* (2.1) *as follows:*

(i) $\zeta_n(1; y) = 1$;

(ii) $\zeta_n(t; y) = y$;

(iii) $\zeta_n(t^2; y) = \frac{(n-1)}{n} \frac{(n\theta_n(y)+1)}{n+2} \frac{(n\theta_n(y)+2)}{n+3} + \frac{2}{n} \frac{(n\theta_n(y)+1)}{n+2} + \frac{1}{3n^2}$;

(iv)

$$\zeta_n(t^3; y) = \frac{(n-1)(n-2)}{n^2} \frac{(n\theta_n(y)+3)}{n+4} \frac{(n\theta_n(y)+2)}{n+3} \frac{(n\theta_n(y)+1)}{n+2} + \frac{9}{2} \frac{(n-1)}{n^2} \frac{(n\theta_n(y)+2)}{n+3} \frac{(n\theta_n(y)+1)}{n+2} + \frac{1}{4n^3};$$

(v)

$$\zeta_n(t^4;y) = \frac{(n-1)(n-2)(n-3)}{n^3}\frac{(n\theta_n(y)+4)}{n+5}\frac{(n\theta_n(y)+3)}{n+4}\frac{(n\theta_n(y)+2)}{n+3}\frac{(n\theta_n(y)+1)}{n+2}$$
$$+\frac{8(n-1)(n-2)}{n^3}\frac{(n\theta_n(y)+3)}{n+4}\frac{(n\theta_n(y)+2)}{n+3}\frac{(n\theta_n(y)+1)}{n+2}$$
$$+\frac{15(n-1)}{n^3}\frac{(n\theta_n(y)+2)}{n+3}\frac{(n\theta_n(y)+1)}{n+2}+\frac{6}{n^3}\frac{(n\theta_n(y)+1)}{n+2}+\frac{1}{5n^4}.$$

Corollary 2.2 *The central moments for* (2.1) *are given below:*

(i) $\zeta_n(t-y;y)=0$*;*

(ii) $\zeta_n((t-y)^2;y) = \frac{2n(n+1)(\theta_n(y)-\theta_n^2(y))}{(n+2)^2(n+3)} + \frac{5n^2(5n+11)+40n+12}{12n^2(n+2)^2(n+3)}.$

3 Pointwise and Weighted Approximation Results

In this part, we focus on pointwise and weighted approximation behaviors of operators $\zeta_n(f;y)$.

We start by recalling the Lipschitz condition to obtain our first result related to pointwise convergence.

Assume that $0<\kappa\le 1$, $T\subset\mathbb{R}_+=[0,\infty)$ and $C(\mathbb{R}_+)$ denotes the space of all continuous functions f on $\mathbb{R}_+$ then, a function f in $C_B(\mathbb{R}_+)$ belongs to $Lip_N(\kappa)$ if

$$|f(t)-f(y)| \le K_{f,\kappa}|t-y|^\kappa \qquad (t\in T,\ y\in\mathbb{R}_+)$$

holds, where the constant $K_{f,\kappa}$ depends on both κ and f.

Theorem 3.1 *Let* $0<\kappa\le 1$, $f\in C_B(\mathbb{R}_+)$ *and* $T\subset\mathbb{R}_+$ *then, for each* $y\in\mathbb{R}_+$,

$$|\zeta_n(f;y)-f(y)| \le K_{f,\kappa}\left\{2d^\kappa(y,T) + \frac{2n(n+1)(\theta_n(y)-\theta_n^2(y))}{(n+2)^2(n+3)} + \frac{5n^2(5n+11)+40n+12}{12n^2(n+2)^2(n+3)}\right\},$$

where $d(y,T)$ *is the distance between* y *and* T *given as*

$$d(y,T)=\inf\{|t-y|:t\in T\}.$$

Proof Let $y\in\bar{T}$ such that $|y-z|=d(y,T)$, where $\bar{T}$ is a closure of T, then the relation

$$|f(t)-f(y)| \le |f(y)-f(y)| + |f(t)-f(y)| \quad (y\in\mathbb{R}_+)$$

is satisfied. With the help of relation

$$|\zeta_n(f;y)-f(y)| \le \zeta_n(|f(y)-f(y)|;y) + \zeta_n(|f(t)-f(y)|;y)$$

we have

$$\begin{aligned}|\zeta_n(f;y)-f(y)| &\le K_{f,\kappa}\left\{|y-z|^\kappa + \zeta_n(|t-y|^\kappa;y)\right\}\\ &\le K_{f,\kappa}\left\{|y-z|^\kappa + \zeta_n(|t-y|^\kappa + |y-z|^\kappa;y)\right\}\\ &= K_{f,\kappa}\left\{2|y-z|^\kappa + \zeta_n(|t-y|^\kappa;y)\right\}.\end{aligned}$$

The following relations are satisfied by applying Hölder's inequality to the above inequality for $\alpha=2/\kappa$ and $\theta=2/(2-\kappa)$:

$$\begin{aligned}|\zeta_n(f;y)-f(y)| &\le K_{f,\kappa}\left\{2d^\kappa(y,T) + \zeta_n^{\frac{1}{\alpha}}(|t-y|^{\alpha\kappa};y)\zeta_n^{\frac{1}{\theta}}(1^\theta;y)\right\}\\ &= K_{f,\kappa}\left\{2d^\kappa(y,T) + \zeta_n^{\frac{\kappa}{2}}(|t-y|^2;y)\right\},\end{aligned}$$

taking into Lemma 2.1 account we complete the proof. □

Let $y \in \mathbb{R}_+$ and $0 < \kappa \leq 1$, then Lipschitz-type maximal function of order κ is

$$\omega_\kappa(f; y) = \sup_{y \neq x, y \in \mathbb{R}_+} \frac{|f(y) - f(x)|}{|y - x|^\kappa}, \tag{3.1}$$

[22]. We provide a local direct estimate of $\zeta_n(f; y)$ by the next theorem.

Theorem 3.2 *Let $f \in C_B(\mathbb{R}_+)$ and $0 < \kappa \leq 1$, then, we have*

$$|\zeta_n(f; y) - f(y)| \leq \omega_\kappa(f; y) \left\{ \frac{2n(n+1)(\theta_n(y) - \theta_n^2(y))}{(n+2)^2(n+3)} + \frac{5n^2(5n+11) + 40n + 12}{12n^2(n+2)^2(n+3)} \right\}^{\frac{\kappa}{2}}$$

for all $y \in \mathbb{R}_+$.

Proof We have the following relations

$$\begin{aligned} |\zeta_n(f; y) - f(y)| &= |\zeta_n(f; y) - f(y)\zeta_n(1; y)| \\ &\leq \zeta_n(|f(t) - f(y)|; y) \\ &\leq \omega_\kappa(f; y)\zeta_n(|t - y|^\kappa; y) \end{aligned}$$

by the help of (3.1). Also applying Hölder inequality to the last inequality for

$$\alpha = 2/\kappa \quad \text{and} \quad \theta = 2/(2 - \kappa)$$

we observe that

$$|\zeta_n(f; y) - f(y)| \leq \omega_\kappa(f; y)\zeta_n^{\frac{\kappa}{2}}(|t - y|^2; y). \tag{3.2}$$

We obtain the desired result from the last inequality together with Lemma 2.1 and the relation in (3.1). □

The set of all functions f on $\mathbb{R}_+$ having the property

$$|f(y)| \leq \psi(y) K_f$$

is called the weighted space $B_\psi(\mathbb{R}_+)$, where a constant $K_f > 0$ depending on f, and $\psi(y) = 1 + y^2$ is a weight function. Recall that the weighted space $B_\psi(\mathbb{R}_+)$ is a Banach space equipped with the norm

$$\|f\|_\psi = \sup_{y \in \mathbb{R}_+} \frac{|f(y)|}{\psi(y)}.$$

The subspace of all continuous functions in $B_\psi(\mathbb{R}_+)$ is denoted by $C_\psi(\mathbb{R}_+)$ and defined as

$$C_\psi^*(\mathbb{R}_+) = \left\{ f \in C_\psi(\mathbb{R}_+) : \lim_{y \to \infty} \frac{|f(y)|}{\psi(y)} < \infty \right\}.$$

Theorem 3.3 *Assume that $f \in C_\psi^*(\mathbb{R}_+)$, then the following relation is satisfied for $\psi(y) = 1 + y^2$:*

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}_+} \frac{|\zeta_n(f; y) - f(y)|}{\psi^{1+\theta}(y)} = 0. \tag{3.3}$$

Proof We have the following relations for any fixed $\gamma > 0$:

$$\begin{aligned} \sup_{y \in \mathbb{R}_+} \frac{|\zeta_n(f; y) - f(y)|}{\psi^{1+\theta}(y)} &\leq \sup_{x \leq \gamma} \frac{|\zeta_n(f; y) - f(y)|}{\psi^{1+\theta}(y)} + \sup_{x \geq \gamma} \frac{|\zeta_n(f; y) - f(y)|}{\psi^{1+\theta}(y)} \\ &\leq \|\zeta_n(f; y) - f\|_{C[0,\gamma]} + \|f\|_\psi \sup_{x \geq \gamma} \frac{|\zeta_n(1 + t^2; y)|}{\psi^{1+\theta}(y)} \\ &\quad + \sup_{x \geq \gamma} \frac{|f(y)|}{\psi^{1+\theta}(y)}. \end{aligned} \tag{3.4}$$

Using the fact $|f(y)| \leq \psi(y)N$ we have

$$\sup_{x\geq\gamma} \frac{|f(y)|}{\psi^{1+\theta}(y)} \leq \frac{\|f(y)\|_{\psi}}{(1+\gamma^2)^{1+\theta}}.$$

The parameter γ can be chosen to be so large that the following inequality is satisfied:

$$\frac{\|f(y)\|_{\psi}}{(1+\gamma^2)^{1+\theta}} < \epsilon/3, \tag{3.5}$$

where $\epsilon > 0$ is given. By the help of Corollary 2.2, we obtain

$$\|f\|_{\psi} \frac{|\zeta_n(1+t^2; y)|}{\psi^{1+\theta}(y)} \to 0 \ (n \to \infty).$$

Also, if we choose γ as large as enough, we have

$$\|f\|_{\psi} \sup_{x\geq\gamma} \frac{|\zeta_n(1+t^2; y)|}{\psi^{1+\theta}(y)} < \epsilon/3. \tag{3.6}$$

Moreover, bearing in mind Korovkin's theorem, the expression on the right-hand side of inequality (3.4) becomes

$$\|\zeta_n(f; y) - f\|_{C[0,\gamma]} < \epsilon/3. \tag{3.7}$$

Combining the results in (3.5)-(3.7), the proof is completed. □

Theorem 3.4 *Assume that $f \in C^*_{\psi}(\mathbb{R}_+)$, then the following relation is satisfied for $\psi(y) = 1 + y^2$:*

$$\lim_{n\to\infty} \|\zeta_n(f; y) - f(y)\|_{\psi} = 0.$$

Proof We must prove that the following equality for $r = 0, 1, 2$:

$$\lim_{n\to\infty} \left\|\zeta_n\left(t^r; y\right) - y^r\right\|_{\psi} = 0. \tag{3.8}$$

The Equality (3.8) is satisfied when $r = 0$ since $\zeta_n(1; y) = 1$. Since

$$\|\zeta_n(t; y) - y\|_{\psi} = \sup_{y\in\mathbb{R}_+} \frac{|\zeta_n(t; y) - y|}{\psi(y)} = 0$$

$\|\zeta_n(t; y) - y\|_{\psi} \to 0$ as $n \to \infty$. Hence, the Equality (3.8) is satisfied for $r = 1$.
Using the same idea, we obtain

$$\begin{aligned}
&\left\|\zeta_n(t^2; y) - y^2\right\|_{\psi} \\
&= \sup_{y\in\mathbb{R}_+} \left|\frac{(n-1)}{n}\frac{(n\theta_n(y)+1)}{n+2}\frac{(n\theta_n(y)+2)}{n+3} + \frac{2}{n}\frac{(n\theta_n(y)+1)}{n+2} + \frac{1}{3n^2} - y^2\right| \times \frac{1}{\psi(y)} \\
&= \sup_{y\in\mathbb{R}_+} \left|\frac{2(n^2+n)(y-y^2)}{(n+2)^2(n+3)} - \frac{(n+1)(y+4y^2)}{(n+2)^2(n+3)} - \frac{8(n+1)(2y+y^2)}{n(n+2)^2(n+3)}\right. \\
&\quad \left. - \frac{(n+1)(6+8y)}{n^2(n+2)^2(n+3)} - \frac{(n+1)(6n^3+13n^2+4)}{2n(n+2)^2(n+3)}\right| \times \frac{1}{\psi(y)} \\
&\leqslant \frac{2(n^2+n)}{(n+2)^2(n+3)} \sup_{y\in\mathbb{R}_+}\left|\frac{y-y^2}{\psi(y)}\right| + \frac{n+1}{(n+2)^2(n+3)} \sup_{y\in\mathbb{R}_+}\left|\frac{y+4y^2}{\psi(y)}\right| \\
&\quad + \frac{8(n+1)}{n(n+2)^2(n+3)} \sup_{y\in\mathbb{R}_+}\left|\frac{2y+y^2}{\psi(y)}\right| + \frac{n+1}{n^2(n+2)^2(n+3)} \sup_{y\in\mathbb{R}_+}\left|\frac{6+8y}{\psi(y)}\right| \\
&\quad + \frac{(n+1)(6n^3+13n^2+4)}{2n(n+2)^2(n+3)} \sup_{y\in\mathbb{R}_+}\left|\frac{1}{\psi(y)}\right|.
\end{aligned}$$

Hence, we have the desired result with the help of [5, Theorem 1]. □

4 Certain Approximation Properties

Here, we give a Lipschitz theorem, estimate the rate of convergence, and obtain a direct approximation theorem for operators (2.1).

Theorem 4.1 *The following equality is satisfied*

$$\lim_{n\to\infty} \|\zeta_n(f) - f\| = 0$$

for $f \in C[0, 1]$.

Proof The following equalities are satisfied

$$\lim_{n\to\infty} \zeta_n(e_0; y) = e_0(y),$$

$$\lim_{n\to\infty} \zeta_n(e_1; y) = e_1(y)$$

via Lemma 2.1, and similarly

$$\lim_{n\to\infty} \|\zeta_n(e_2) - e_2\| = 0.$$

Hence, the Korovkin's theorem provides the desired result. □

Assume that $t, y \in [0, 1]$, $f \in C[0, 1]$, N is a positive number and $r \in (0, 1]$. When the following inequality is satisfied:

$$|f(t) - f(y)| \leqslant N|t - y|^r,$$

then a function f is said to be in the Lipschitz class $\mathrm{Lip}_N(r)$.

Theorem 4.2 *Assume that* $f \in \mathrm{Lip}_N(r)$, *where* $r \in (0, 1]$, *then*

$$|\zeta_n(f; y) - f(y)| \leqslant N\nu_n^r(y)$$

is satisfied, where $\nu_n(y) = \left(\zeta_n\left((t - y)^2; y\right)\right)^{\frac{1}{2}}$.

Proof Since operators (2.1) are monotonic, the following relations are satisfied:

$$\begin{aligned}
|\zeta_n(f; y) - f(y)| &\leqslant \sum_{j=0}^{n} Q_{n,j}(y) \left| f\left(\frac{j}{n}\right) - f(y) \right| \\
&\leqslant N \sum_{j=0}^{n} Q_{n,j}(y) \left| \frac{j}{n} - y \right|^r \\
&= N \sum_{j=0}^{n} Q_{n,j}^{\frac{2-r}{2}}(y) \left(Q_{n,j}(y) \left(\frac{j}{n} - y \right)^2 \right)^{\frac{r}{2}}.
\end{aligned}$$

We obtain the following relations if we apply Hölder's inequality for the above series

$$\begin{aligned}
|\zeta_n(f; y) - f(y)| &\leqslant N \left(\sum_{j=0}^{n} Q_{n,j}(y) \right)^{\frac{2-r}{2}} \left(\sum_{j=0}^{n} Q_{n,j}(y) \left(\frac{j}{n} - y \right)^2 \right)^{\frac{r}{2}} \\
&= N \left(\sum_{j=0}^{n} Q_{n,j}(y) \left(\frac{j}{n} - y \right)^2 \right)^{\frac{r}{2}} \\
&= N \left(\zeta_n \left((t - y)^2; y \right) \right)^{\frac{r}{2}}.
\end{aligned}$$

□

The usual modulus of continuity (MC) for a function f is given as

$$\omega\left(f;\nu^{1/2}\right)=\sup_{0<h\leq\nu}\sup_{y,y+h\in[0,1]}|f(y+h)-f(y)|.$$

The rate of convergence of operators (2.1) is given via MC in the next step.

Theorem 4.3 *Let $f\in C[0,1]$, then we have*

$$|\zeta_n(f;y)-f(y)|\leqslant 2\omega(f,\nu_n^{1/2}),$$

where $\nu_n=\zeta_n\left((t-y)^2;y\right)$.

Proof With the help of the monotonicity of operators (2.1), we have

$$\begin{aligned}|\zeta_n(f;y)-f(y)| &\leqslant \sum_{j=0}^{n}Q_{n,j}(y)\left|f\left(\frac{j}{n}\right)-f(y)\right|\\ &\leqslant \sum_{j=0}^{n}Q_{n,j}(y)\omega_f\left(\frac{j}{n}-y\right)\\ &\leqslant \sum_{j=0}^{n}Q_{n,j}(y)\left(1+\nu^{-2}\left(\frac{j}{n}-y\right)^2\right)\omega_f(\nu)\\ &=\left[1+\nu^{-2}\sum_{j=0}^{n}Q_{n,j}(y)\left(\frac{j}{n}-y\right)^2\right]\omega_f(\nu)\\ &=\left[1+\nu^{-2}\zeta_n\left((t-y)^2;y\right)\right]\omega_f(\nu).\end{aligned}$$

Choosing $\nu=\zeta_n\left((t-y)^2;y\right)$, we obtain the desired result. □

Since we want to obtain a direct approximation result for operators (2.1) we use Peetre's K functional and the second-order MC. Assuming $f\in C[a,b]$, $\nu>0$ and $\mathcal{W}^2=\left\{h\in C[a,b]:h',h''\in C[a,b]\right\}$ the Peetre's K -functional is defined as

$$K_2(f,\nu)=\inf_{h\in W^2}\left\{\|f-h\|+\nu\left\|h''\right\|\right\}.$$

On the other hand, the second-order MC is defined as

$$\omega_2(f,\nu)=\sup_{0<h<\nu}\sup_{a\leqslant y<b}\{|f(y+2h)-2f(y+h)+f(y)|\}.$$

The following inequality constructs a link between Peetre's K-functional and second order MC (see [20]):

$$K_2(f,\nu)\leqslant N\cdot\omega_2(f,\nu).$$

Theorem 4.4 *Assuming $f\in C[0,1]$ the following inequality is satisfied for a constant $N>0$ and for each $n\in\mathbb{N}$:*

$$|\zeta_n(f;y)-f(y)|\leqslant N\cdot\omega_2\left(f,\nu_n(y)\right),$$

where $\nu_n(y)=\left(\zeta_n\left((t-y)^2;y\right)\right)^{\frac{1}{2}}$.

Proof Set $\mathcal{W}^2$ for [0, 1]. For $h \in \mathcal{W}^2$, employ the Taylor's expansion to obtain

$$g(t) = g(y) + h'(y)(t-y) + \int_y^t (t-u)h''(u)\mathrm{d}u.$$

The following relations are satisfied if we apply Lemma 2.1:

$$\zeta_n(h;y) = g(y) + h'(y)\zeta_n(t-y) + \zeta_n\left(\int_y^t (t-u)h''(u)\mathrm{d}u; y\right)$$

$$\left|\zeta_n(h;y) - h(y) - h'(y)\zeta_n(t-y)\right| \leqslant \zeta_n\left(\int_y^t |t-u|\left|h''(u)\right|\mathrm{d}u; y\right)$$

$$\left|\zeta_n(h;y) - h(y) - h'(y)\zeta_n(t-y)\right| \leqslant \zeta_n\left((t-y)^2; y\right)\left\|h''\right\|.$$

Hence, we get

$$\begin{aligned}&\left|\zeta_n(h;y) - h(y) - h'(y)\zeta_n(t-y)\right| \\ &\quad\leqslant \left(\frac{2n(n+1)(\theta_n(y) - \theta_n^2(y))}{(n+2)^2(n+3)} + \frac{5n^2(5n+11)+40n+12}{12n^2(n+2)^2(n+3)}\right)\left\|h''\right\|.\end{aligned}$$

Now,

$$\begin{aligned}&\left|\zeta_n(f;y) - f(y) - h'(y)\zeta_n(t-y)\right| \\ &\quad\leqslant |\zeta_n(f-h;y) - (f-h)(y)| + \left|\zeta_n(h;y) - h(y) - h'(y)\zeta_n(t-y)\right|.\end{aligned}$$

Using the relation $|\zeta_n(f;y)| \leqslant \|f\|$, we obtain

$$\begin{aligned}&\left|\zeta_n(f;y) - f(y) - h'(y)\zeta_n(t-y)\right| \\ &\quad\leqslant \|f-h\| + \left(\frac{2n(n+1)(\theta_n(y) - \theta_n^2(y))}{(n+2)^2(n+3)} + \frac{5n^2(5n+11)+40n+12}{12n^2(n+2)^2(n+3)}\right)\left\|h''\right\|.\end{aligned}$$

We obtain the following inequality when we take the infimum of the right side of the above inequality over all $h \in \mathcal{W}^2$:

$$\left|\zeta_n(f;y) - f(y) - h'(y)\zeta_n(t-y)\right| \leqslant K_2\left(f, \nu_n^2(y)\right),$$

where $\nu_n^2(y) = \zeta_n\left((t-y)^2; y\right)$. We obtain the desired result via Peetre's K-functional property. □

5 An Asymptotic and a Quantitative Formula

Here, we establish a Voronovskaja-type asymptotic formula and obtain a quantitative Voronovskaja-type theorem for (2.1). In order to achieve this aim, the following notions are needed.

Assuming $f \in C[0,1]$ and $\phi(y) = \sqrt{y(1-y)}$ Ditzian-Totik modulus of smoothness [21] is defined as follows:

$$\omega_\phi(f,\nu) := \sup_{0<|\lambda|\leqslant\nu}\left\{\left|f\left(y + \frac{\lambda\phi(y)}{2}\right) - f\left(y - \frac{\lambda\phi(y)}{2}\right)\right|, y \pm \frac{\lambda\phi(y)}{2} \in [0,1]\right\}.$$

Let us denote the class of all absolutely continuous functions defined on the closed interval [0, 1] by $AC_{loc}[0,1]$, then the corresponding Peetre's K-functional is defined as

$$K_\phi(f,\nu) = \inf_{h\in W_\phi[0,1]}\left\{||f-h|| + \nu||\phi h'|| : h \in C^1[0,1], \nu > 0\right\},$$

where

$$W_\phi[0,1] = \{h : \ h \in AC_{loc}[0,1], \ \|\phi h'\| < \infty\}.$$

Also, there exists a constant $N > 0$ so that the following inequality is satisfied:

$$K_\phi(f,\nu) \leqslant N\,\omega_\phi(f,\nu).$$

Theorem 5.1 *Let $y \in [0,1]$ and $f, f', f'' \in C[0,1]$. Then, the following inequality is satisfied for sufficiently large n:*

$$\left|\zeta_n(f;y) - f''(y)\mu_n(y) - f(y)\right| \leqslant \frac{N}{n}\phi^2(y)\omega_\phi\left(f'', \sqrt{\frac{1}{n}}\right).$$

Here

$$\mu_n(y) = \frac{2n(n+1)(\theta_n(y) - \theta_n^2(y))}{2(n+2)^2(n+3)} + \frac{5n^2(5n+11)+40n+12}{24n^2(n+2)^2(n+3)}.$$

Proof Assuming $f \in C[0,1]$, the following relation is obtained:

$$f(t) - f(y) - (t-y)f'(y) = \int_y^t (t-u)f''(u)\mathrm{d}u.$$

This relation implies the following inequality

$$f(t) - f(y) - (t-y)f'(y) - \frac{f''(y)}{2}(t-y)^2 \leqslant \int_y^t (t-u)[f''(u) - f''(y)]\mathrm{d}u. \tag{5.1}$$

If we apply the operators ζ_n to the each side of (5.1), we have

$$\begin{aligned}&\left|\zeta_n(f;y) - g(y) - \zeta_n((t-y);y)f'(y) - \frac{f''(y)}{2}\zeta_n((t-y)^2;y)\right|\\ &\leqslant \zeta_n\left(\left|\int_y^t |t-u|\,|f''(u) - f''(y)|\,\mathrm{d}u\right|;y\right).\end{aligned} \tag{5.2}$$

The following quantity which is given in (5.2) is estimated as

$$\left|\int_y^t |t-u|\,|f''(u) - f''(y)|\,\mathrm{d}u\right| \leqslant 2\|f'' - f\|(t-y)^2 + 2\|\phi f'\|\phi^{-1}(y)|t-y|^3,$$

where $f \in W_\phi[0,1]$. For sufficiently large n, there exists $N > 0$ so that the following two inequalities are satisfied:

$$\begin{aligned}\zeta_n((t-y)^2;y) &\leqslant \frac{(\theta_n(y) - \theta_n^2(y))^2 N}{n} \quad \text{and}\\ \zeta_n((t-y)^4;y) &\leqslant \frac{(\theta_n(y) - \theta_n^2(y))^4 N}{n^2}.\end{aligned} \tag{5.3}$$

Taking into relations (5.2)-(5.3) account and Cauchy-Schwartz inequality, the following relations are satisfied:

$$\begin{aligned}&\left|\zeta_n(f;y) - f(y) - \zeta_n((t-y);y)f'(y) - \frac{f''(y)}{2}\big(\zeta_n((t-y)^2;y) + \zeta_n(1;y)\big)\right|\\ &\leqslant 2\|f'' - f\|\zeta_n((t-y)^2;y) + 2\|\phi f'\|\phi^{-1}(y)\zeta_n(|t-y|^3;y)\\ &\leqslant \frac{N}{n}y(1-y)\|f'' - f\| + 2\|\phi f'\|\phi^{-1}(y)\{\zeta_n((t-y)^2;y)\}^{1/2}\{\zeta_n((t-y)^4;y)\}^{1/2}\\ &\leqslant \frac{N}{n}\phi^2(y)\left\{\|f'' - f\| + n^{-1/2}\|\phi f'\|\right\}.\end{aligned}$$

We deduce the desired result if we take the infimum of the right-hand side over all $f \in W_\phi[0,1]$. □

By Theorem 5.1, following result is satisfied:

Corollary 5.2 *Assuming $f, f', f'' \in C[0,1]$, we have the following result:*

$$\lim_{n\to\infty} n\left\{\zeta_n(f;y) - f(y) - f''(y)\mu_n(y)\right\} = 0.$$

Theorem 5.3 *Assume that $f, f', f'' \in C[0, 1]$ and $y \in [0, 1]$. Then the following relation is satisfied:*

$$\lim_{n\to\infty} n\left(\zeta_n(f; y) - f(y)\right) = (y - y^2) f''(y).$$

Proof Assume that $y \in [0, 1]$ and $f, f', f'' \in C[0, 1]$. We have the following relation by Taylor's expansion:

$$f(t) = f(y) + (t - y) f'(y) + \frac{(t - y)^2}{2} f''(y) + f(t)(t - y)^2, \tag{5.4}$$

where $f(t) \in C[0, 1]$ is the Peano's type remainder such that $\lim_{t\to x} f(t) = 0$. We have the following relations for every $y \in [0, 1]$ by Corollary 2.2:

$$\lim_{n\to\infty} n \cdot \zeta_n((t - y); y) = 0$$

and

$$\lim_{n\to\infty} n \cdot \zeta_n\left((t - y)^2; y\right) = 2(y - y^2).$$

The following equality is satisfied:

$$\begin{aligned} &n\left(\zeta_n(f; y) - f(y)\right) \\ &= n \cdot f'(y)\zeta_n((t - y); y) + \frac{n}{2} \cdot f''(y)\zeta_n\left((t - y)^2; y\right) + n \cdot \zeta_n\left(f(t)(t - y)^2; y\right) \end{aligned} \tag{5.5}$$

using the linearity of operators ζ_n in Eq. (5.4).

We have the following relations by Cauchy-Schwartz inequality:

$$\begin{aligned} \zeta_n\left(f(t)(t - y)^2; y\right) &\leqslant \sqrt{\zeta_n\left(h^2(t); y\right)} \cdot \sqrt{\zeta_n\left((t - y)^4; y\right)}; \\ n \cdot \zeta_n\left(f(t)(t - y)^2; y\right) &\leqslant \sqrt{\zeta_n\left(h^2(t); y\right)} \cdot \sqrt{n^2 \cdot \zeta_n\left((t - y)^4; y\right)}. \end{aligned} \tag{5.6}$$

The limit $\lim_{n\to\infty} n^2 \cdot \zeta_n\left((t - y)^4; y\right)$ is non-negative and finite for $y \in [0, 1]$ by Corollary 2.2. Using uniform convergence of the operators ζ_n for $f(t) \in C[0, 1]$, we have

$$\lim_{n\to\infty} \zeta_n\left(h^2(t); y\right) = h^2(y) = 0$$

uniformly for $y \in [0, 1]$ since $f(t) \in C[0, 1]$ and $\lim f(t) = 0$. So, we have

$$\lim_{n\to\infty} n \cdot \zeta_n\left(f(t)(t - y)^2; y\right) = 0$$

by Eq. (5.6). The following equality, which completes the proof, is satisfied by Eq. (5.5):

$$\lim_{n\to\infty} n\left(\zeta_n(f; y) - f(y)\right) = (y - y^2) f''(y).$$

$\square$

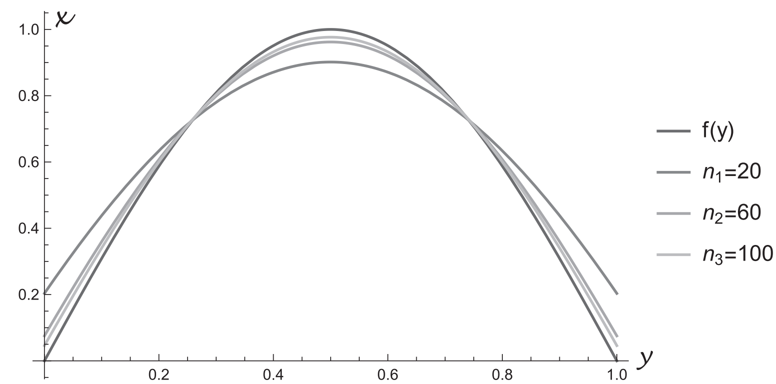

Fig. 1 Approximation of operators $\zeta_n(f; y)$ with different n values

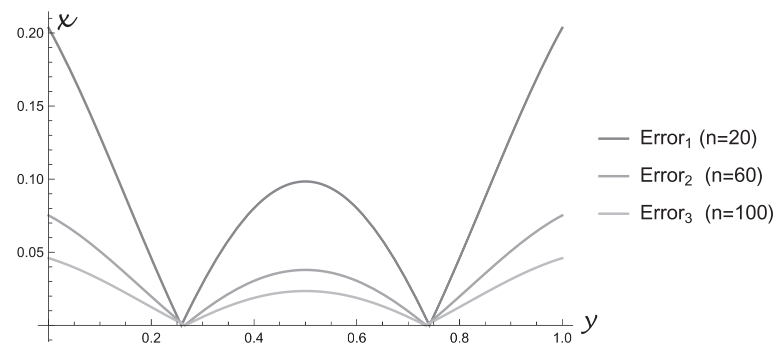

Fig. 2 Error of approximation with different n values

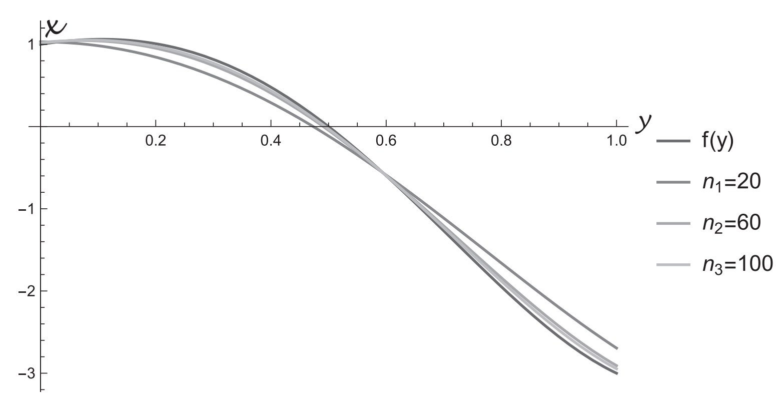

Fig. 3 Approximation of operators $\zeta_n(f; y)$ with different n values

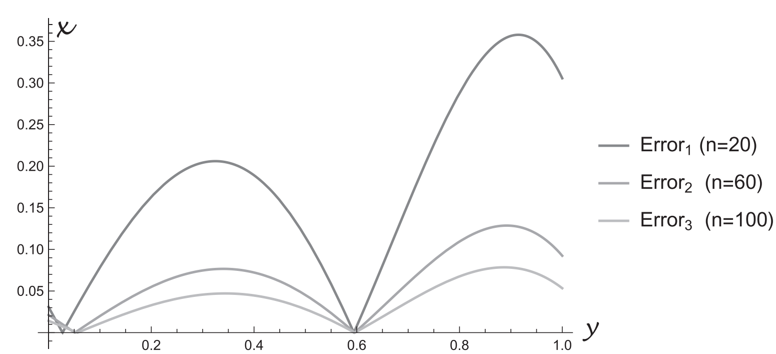

Fig. 4 Error of approximation with different n values

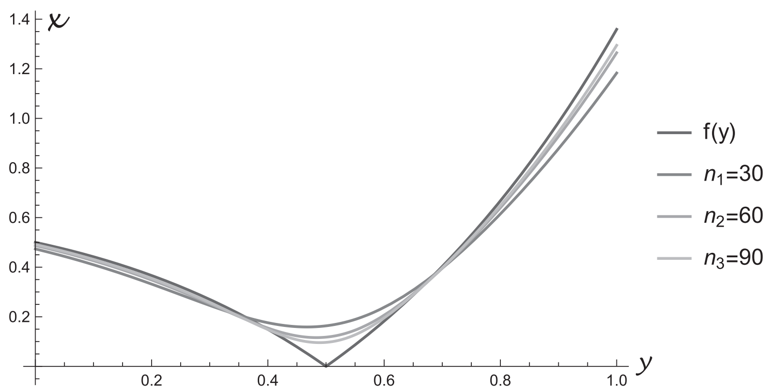

Fig. 5 Approximation of operators $\zeta_n(f; y)$ with different n values

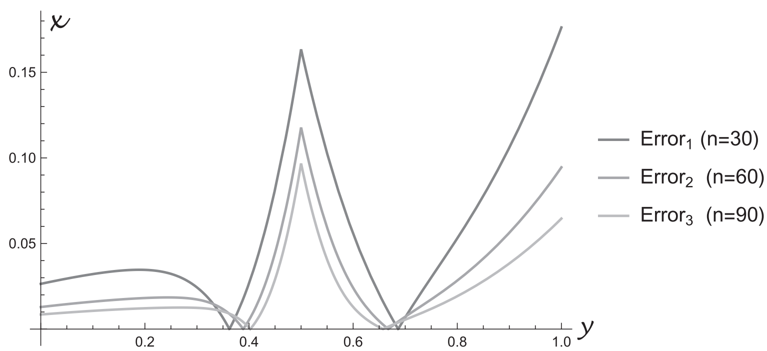

Fig. 6 Error of approximation with different n values

6 Computer Graphics

Here, we demonstrate certain graphics to show the proposed operators $\zeta_n(f; y)$ approximate functions with different parameters. We present the convergence and error of convergence of operators $\zeta_n(f; y)$ to some functions with three examples.

Example 6.1 We examine convergence of approximating operators $\zeta_n(f; y)$ towards the $f(y) = \sin(2\pi y)$ for $n = 20, 60, 100$. In Figure 1, we demonstrate the convergence of our operators to the function given in this example for some n values. Figure 2 provides an error of approximation for the information given in Figure 1. It is clear from Figure 1-2 that operators $\zeta_n(f; y)$ provide a better rate of convergence when n increases.

Example 6.2 We consider the function $f(y) = 3^y \cos(\pi y)$ to see the accuracy of approximation of our operators $\zeta_n(f; y)$ for $n = 20, 60, 100$. Figure 3 and Figure 4 provide convergence and error of convergence of our operators, respectively. It can be seen from Figure 3-4 that operators $\zeta_n(f; y)$ provide a better rate of convergence when n increases.

Example 6.3 We examine approximation of operators $\zeta_n(f; y)$ to $f(y) = |y - 0.5|\, e^y$ for $n = 30, 60, 90$. We demonstrate convergence and error of convergence of our operators in Figure 5 and Figure 6, respectively. We see from Figure 5-6 that operators $\zeta_n(f; y)$ provide a better rate of convergence when n increases.

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References

1. S. Rahman, M. Mursaleen, A. Khan, A Kantorovich variant of Lupaş-Stancu operators based on Pólya distribution with error estimation, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, (2020), 114:75.

2. A. Kajla, S.A. Mohiuddine, A. Alotaibi, M. Goyal, K.K. Singh, Approximation by υ-Baskakov-Durrmeyer-Type Hybrid Operators. Iran J Sci Technol Trans Sci, 44, (2020), 1111-1118. https://doi.org/10.1007/s40995-020-00914-3
3. A. Naaz, M. Mursaleen, Some Approximation Results on Compact Sets by (p, q)-Bernstein-Faber Polynomials, $q>p>1$. Iran J Sci Technol Trans Sci 43, (2019), 2585-2593.
4. F. Usta, On New Modification of Bernstein Operators: Theory and Applications. Iran J Sci Technol Trans Sci 44, (2020), 1119-1124. https://doi.org/10.1007/s40995-020-00919-y
5. A.D. Gadjiev, Theorems of Korovkin type, Mat. Zamekti, 20(5), (1976), 781-786.
6. F. Özger, On new Bézier bases with Schurer polynomials and corresponding results in approximation theory, Commun Fac Sci Univ Ank Ser Math Stat. 69(1), , 376-393. (2020)
7. S.A. Mohiuddine, F. Özger, Approximation of functions by Stancu variant of Bernstein-Kantorovich operators based on shape parameter α, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas RACSAM, 114(70), (2020).
8. F. Özger, Weighted statistical approximation properties of univariate and bivariate λ-Kantorovich operators, Filomat, 2019; 33(11):3473-3486.
9. F. Özger, H.M. Srivastava, S.A. Mohiuddine, Approximation of functions by a new class of generalized Bernstein-Schurer operators, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM (2020) 114:173.
10. S.A. Mohiuddine, N. Ahmad, F. Özger, et al. Approximation by the parametric generalization of Baskakov-Kantorovich operators linking with Stancu operators, Iran J. Sci. Technol. Trans. Sci. 45, (2021), 593-605. https://doi.org/10.1007/s40995-020-01024-w
11. A. Alotaibi, F. Özger, S.A. Mohiuddine et al. Approximation of functions by a class of Durrmeyer-Stancu type operators which includes Euler's beta function, Adv. Differ. Equ. 13, (2021), 2021. https://doi.org/10.1186/s13662-020-03164-0
12. F. Özger, Applications of generalized weighted statistical convergence to approximation theorems for functions of one and two variables, Numerical Functional Analysis and Optimization, 41(16), (2020), 1990-2006. https://doi.org/10.1080/01630563.2020.1868503.
13. F. Altomare, M. Campiti, Korovkin-type approximation theory and its applications, de Gruyter Stud. Math., vol. 17, de Gruyter & Co., Berlin, 1994.
14. P.P. Korovkin, Linear Operators and Approximation Theory, Hindustan Publishing Corporation, Delhi, 1960.
15. J.P. King, Positive linear operators which preserve y^2, Acta Math. Hungar. 99(3) , 203-208.(2003)
16. M. Mursaleen, K.J. Ansari, Approximation of q-Stancu-Beta operators which preserve x^2, Bull. Malays. Math. Sci. Soc. 40, 1479-1491 (2017). https://doi.org/10.1007/s40840-015-0146-9.
17. D.J. Bhatt, V.N. Mishra, R.K. Jana, On a new class of Bernstein type operators based on beta function, Khayyam J. Math. 6(1) (2020), 1-15.
18. M. Mursaleen, S. Rahman, Dunkl generalization of q-Szász-Mirakjan operators which preserve y^2, Filomat, 32(3), (2018), 733-747.
19. M. Mursaleen, A. Naaz, A. Khan, Improved approximation and error estimations by King type (p, q)-Szász-Mirakjan Kantorovich operators, Appl. Math. Comp. 348, (2019) 175-185.
20. R.A. DeVore, G.G. Lorentz, Constructive Approximation, Springer, Berlin, 1993.
21. Z. Ditzian, V. Totik, Moduli of Smoothness, Springer Series in Computational Mathematics, 8. Springer-Verlag, New York, 1987.
22. B. Lenze, Bernstein-Baskakov-Kantorovic operators and Lipschitz-type maximal functions, in Approximation Theory (Kecskemet, 1990) 469-496, Colloq. Math. Soc. Janos Bolyai, 58 North-Holland, Amsterdam.
23. H.M. Srivastava, K.J. Ansari, F. Özger, Z.Ö. Özger, A link between approximation theory and summability methods via four-dimensional infinite matrices, Mathematics, 9, (2021), 1895. https://doi.org/10.3390/math9161895
24. Md. Nasiruzzaman, Adem Kilicman and Mohammad Ayman Mursaleen, Construction of q-Baskakov operators by wavelets and approximation properties, Iranian Journal of Science and Technology, Transactions A: Science, 46(5) (2022) 1495-1503.
25. Ming-Yu Chen, Md. Nasiruzzaman, M.A. Mursaleen, N. Rao, A. Kiliçman, On shape parameter α based approximation properties and q-statistical convergence of Baskakov-Gamma operators, Journal of Mathematics, 2022 (2022) 4190732.
26. K.J. Ansari, F. Özger, Z. Ödemiş Özger, Numerical and theoretical approximation results for Schurer-Stancu operators with shape parameter λ. Comp. Appl. Math. 41, 181 (2022).
27. Ansari, K. J., Karakılıç, S., and Özger, F. Bivariate Bernstein-Kantorovich operators with a summability method and related GBS operators. Filomat, 36(19),(2022) 6751-6765.
28. K.J. Ansari, M. Civelek, F. Usta, Jain's Operator: A New Construction and Applications in Approximation Theory, Math. Meth. Appl. Sci., (2023), https://doi.org/10.1002/mma.9311.
29. N.I. Mahmudov, P. Sabancıgil, Approximation Theorems for q-Bernstein Kantorovich Operators, Filomat, vol.27:4, pp. 721-730, 2013.
30. H. Hamal, P. Sabancıgil, Some Approximation Properties of new Kantorovich type q-analogue of Balazs Szabados Operators, Journal of Inequalities and Applications, 159, 2020.
31. H. Hamal, P. Sabancıgil, Kantorovich Type Generalization of Berstein Type Rational Functions Based on (p, q)-integers, Symmetry, 14(5), 1054, 2022.