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ORIGINAL RESEARCH

Treatment of fractional multi-order/multi-term differential equations: utilizing fractional shifted Lucas polynomials

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Abstract In this work, novel type of fractional polynomials are proposed as the generalization to shifted Lucas polynomials, which are called as fractional shifted Lucas polynomials. Useful operational matrices are developed here utilizing the newly established analytical formula for the construction of the numerical scheme. Further, a theorem for the calculation of Caputo fractional derivative is proved and an useful remark is provided for integer order derivative. At the core of the work, the task is to use collocation points for tackling the multi-term differential equations/multi-order differential equations (MODEs/MTDEs). To develop the strong background for the proposed scheme, error analysis is performed. The algorithm of the scheme is examined through some test examples of MODEs/MTDEs, and comparisons are made with other existing methods.

Keywords Fractional shifted Lucas polynomials · Analytical formula · Fractional weight function · Operational matrices · Multi-order differential equations · Multi-term differential equations

1 Introduction and basic preliminaries

Three decades ago, fractional calculus was not much popular among scientists and applied mathematicians due to the non-equivalence of various definitions, reliable applications. But the importance of fractional calculus was realized in the works of Podlubny [1] and other important concepts in [2,3]. Though most of the phenomena are well explained by employing fractional calculus, yet there are some phenomena such as complex multi rate physical process [4], viscoelastic models [5] etc. which can not be accurately provided by single order differential equations. These require the modeling through multi-order and multi-terms differential equations [6].

The generic form of MODE/MTDE is given below [7]:

$$D^{\lambda}\varpi(\xi) = G\left(\xi, \varpi(\xi), D^{\sigma_1}\varpi(\xi), ..., D^{\sigma_n}\varpi(\xi)\right), n = 1, 2, \cdots, 0 < \xi < 1, n-1 < \lambda \leq n, \tag{1}$$

with

$$\varpi^{(j)}(0) = 0, j = 0, 1, ..., n-1. \tag{2}$$

In equations (1), $\lambda > 0$, and assume that $0 < \sigma_1 < ... < \sigma_n < \lambda$. $D^{\sigma_1}, ..., D^{\sigma_n}$ are the Caputo's fractional derivatives, $\varpi(\xi)$ is unknown function. Generally, equation (1) appears in the vibration equation, fractional Riccati equation, fractional Van Der Pol equation [7] etc.

In the context of the solutions for MODEs/MTDEs, orthogonal polynomials serve as an extensively used popular tool. For instance, Chebyshev polynomials [8], Boubaker polynomials [9], Legendre polynomials [10],

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Jacobi polynomials [11], fractional Chebyshev polynomials [12], fractional Legendre polynomials [13], fractional Jacobi polynomials [14] etc.

B-spline functions were used for the numerical solution of MTDEs by Lakestani et al. [15], whereas Tau approach based scheme was constructed using Legendre polynomials for the treatment of MODEs by Saadatmandi and Dehghan [16]. Spectral aspects of Chebyshev polynomials were manipulated in an attempt to achieve the solution to MODEs by Doha et al. [17]. Bhrawy et al. [18] constructed an entirely novel expression for fractional integrals of shifted Chebyshev polynomials to solve MTDEs. The uniqueness and existence for MTDEs are explored in [19]. In [20], a numerical strategy is presented using spectral collocation algorithm for the solutions of MODEs with multiple delay. Recently, the applications of operational matrices of Chelyshkov polynomials have been seen in [21] to solve MODEs. Others are given in [22–24].

A general way to solve these kind of equations is to present the solution as a linear combination of scalars and basis functions. The types of functions may or may not be orthogonal. Utilization of orthogonal functions in approximation significantly influence the solution in various aspects of engineering and science [25]. This objective can be fulfilled only by utilizing operational matrices and truncated series of orthogonal basis function. Main purpose of adopting an orthogonal basis is that it can reduce the considered problem into a set-up of linear/nonlinear algebraic equations. The orthogonal function may be classified into four category. The first category consists of sets of piecewise constant orthogonal functions like walsh function, block-pulse function etc. Walsh function somewhat behaves like usual sine, cosine orthogonal functions and their application can be seen in coding, number theory, statistical analysis, and medical systems [26] whereas block- pulse functions (BPF) are known to control system of electrical engineers for the analysis [27]. Disjointness, completeness, and orthogonality properties in block-pulse functions made them more popular in computational methods. Second includes orthogonal polynomials like Chebyshev, Legendre, Lucas etc. The applications of orthogonal polynomials have been already discussed in earlier chapters. In the third category, sine-cosine added which are usually involved in the expansion of Fourier series and orthogonal wavelets comes under fourth category.

In this paper, fractional orthogonal polynomials are developed for the construction of numerical scheme. The residual function appearing in numerical scheme is calculated by adopting collocation points. Residual function can also be computed by choosing Galerkin, tau, least square method etc. The structure of Galerkin approach depends on the certain combinations of orthogonal polynomials that satisfy the given conditions, then enforcing the residual to be orthogonal with the chosen basis functions. This approach is also suitable for dealing with linear problems [28]. Next is tau approach. It is a specific class of Petrov-Galerkin method. As compared to Galerkin approach, tau approach has freedom in choosing the basis functions. Least square method is uses to compute the residual function by minimizing the functional. Collocation method has power to vanish the residual at the assigned nodes or points. It is more effective technique to solve nonlinear problems.

Our prime focus is to present a numerical scheme for the fractional solution of MTDEs. To the best of our knowledge no one has shown fractional solutions through the three dimensional plots for MTDEs. For getting the fractional solution, FSLPs are used as basis functions. The operational matrices for FSLPs are developed by introducing the novel analytical expression of FSLPs. Moreover, a theorem is proved to compute the fractional derivative and a beneficial remark is presented for integer order derivative of FSLPs. The proposed structure of work is made interesting by choosing the collocation points which converts the considered problem into set of algebraic equations. Consequently, error analysis is also performed.

The rest content of this chapter is arranged as: Definition of fractional derivative is given in Sect. 2. Some informational facts about fractional shifted Lucas polynomials are presented in Sect. 3. Definition of function approximation and operational matrices are mentioned in Sect. 4. In next Section, numerical scheme is described. Section 6 deals with error analysis. In Sect. 7, superiority of the present algorithm is shown by taking some examples. Then conclusion is given.

2 Fractional derivative

If $D^{\sigma_n} > 0$, then [1]:

$$D^{\sigma_n}\varpi(\xi) = \begin{cases} \dfrac{1}{\Gamma(n-\sigma_n)}\displaystyle\int_0^{\xi}(\xi - T)^{n-\sigma_n-1}\varpi^{(n)}(T)dT, & n-1 < \sigma_n < n, n \in \mathbb{N}, \\ \dfrac{d^{(n)}}{d\xi^{(n)}}\varpi(\xi), & \sigma_n = n, \end{cases} \tag{3}$$

is recognized as Caputo's fractional derivative.

3 Fractional shifted Lucas polynomials (FSLPs)

Using transformation $T = \xi^{\nu}$, fractional recurrence formula for FSLPs is given by:

$$Q_{\gamma}^{\nu}(\xi) = (2\xi^{\nu} - 2\iota)Q_{\gamma-1}^{\nu}(\xi) + Q_{\gamma-2}^{\nu}(\xi), \gamma \in 2, 3, ..., \nu \in (0, 1], \tag{4}$$

with $Q_0^{\nu}(\xi) = 2$, $Q_1^{\nu}(\xi) = (2\xi^{\nu} - 2\iota)$.

The transformed analytic form of $Q_{\gamma}^{\nu}(\xi)$ is given by:

$$Q_{\gamma}^{\nu}(\xi) = \sum_{k=0}^{[\frac{\gamma}{2}]} \frac{\gamma}{\gamma - k}\binom{\gamma - k}{k}(2\xi^{\nu} - 2\iota)^{\gamma-2k}, \gamma \geqslant 1, Q_0^{\nu}(\xi) = 2, \tag{5}$$

$$\left[\frac{\gamma}{2}\right] = \begin{cases} \frac{\gamma}{2}, \gamma \text{ is even,} \\ \frac{(\gamma - 1)}{2}, \gamma \text{is odd.} \end{cases} \tag{6}$$

Remark 3.1 If $\nu = 1$ in equations (4) and (5), then we obtain ordinary SLPs which have following analytic formula:

$$Q_{\gamma}(\xi) = \sum_{k=0}^{[\frac{\gamma}{2}]} \frac{\gamma}{\gamma - k}\binom{\gamma - k}{k}(2\xi - 2\iota)^{\gamma-2k}, \gamma \geqslant 1, \tag{7}$$

with $Q_0(\xi) = 2$ and $\left[\frac{\gamma}{2}\right]$ is given in equation (6).

Lemma 3.1 *If $Q_{\gamma}^{\nu}(\xi)$, $Q_m^{\nu}(\xi)$ are orthogonal with respect to fractional weight function* $W^{\nu}(\xi) = \frac{1}{\iota\xi^{1-\nu}\sqrt{(2\xi^{\nu} - 2\iota)^2 + 4}}$ *defined on $\hat{R}_1 = [0, 2\iota]$, then*

$$\frac{1}{2}\int_{\hat{R}_1} \frac{Q_{\gamma}(\xi^{\nu})\overline{Q_m(\xi^{\nu})}}{\iota\xi^{1-\nu}\sqrt{(2\xi^{\nu} - 2\iota)^2 + 4}}d\xi = \begin{cases} 0, \textit{ if } \gamma \neq m, \\ \pi/\nu, \textit{ if } \gamma = m = 0, \\ \pi/2\nu, \textit{ if } \gamma = m \geqslant 1. \end{cases} \tag{8}$$

Proof Since,

$$\frac{1}{2}\int_{\hat{R}_1} \frac{Q_{\gamma}(T)\overline{Q_m(T)}}{\iota\sqrt{(2T - 2\iota)^2 + 4}}dT = \begin{cases} 0, \text{ if } \gamma \neq m, \\ \pi, \text{ if } \gamma = m = 0, \\ \pi/2, \text{ if } \gamma = m \geqslant 1. \end{cases}$$

Let $T = \xi^{\nu}$ and $dT = \nu\xi^{\nu-1}d\xi$, then

$$\frac{1}{2}\int_{\hat{R}_1} \frac{Q_{\gamma}(\xi^{\nu})\overline{Q_m(\xi^{\nu})}}{\iota\xi^{1-\nu}\sqrt{(2\xi^{\nu} - 2\iota)^2 + 4}}d\xi = \begin{cases} 0, \text{ if } \gamma \neq m, \\ \pi/\nu, \text{ if } \gamma = m = 0, \\ \pi/2\nu, \text{ if } \gamma = m \geqslant 1. \end{cases}$$

Hence desired result is proved. □

4 Function approximation and operational matrices

Here the function $\varpi(\xi) \in L^2$ is approximated as linear combination of FSLPs as:

$$\varpi(\xi) = \sum_{\gamma=0}^{\infty} e_\gamma Q_\gamma^\nu(\xi), \tag{9}$$

Suppose that $\left\{Q_0^\nu(\xi), Q_1^\nu(\xi), ..., Q_N^\nu(\xi)\right\} \subseteq L^2$ is the set of FSLPs and Ω=span$\left\{Q_0^\nu(\xi), Q_1^\nu(\xi), ..., Q_N^\nu(\xi)\right\}$. Let $\varpi(\xi)$ be an arbitrary element of L^2 and since Ω is finite dimensional subspace of L^2, then $\varpi_N(\xi)$ is said to be the best approximation to $\varpi(\xi)$ if

$$||\varpi(\xi) - \varpi_N(\xi)|| \leq ||\varpi(\xi) - y(\xi)||, \forall y \in \Omega.$$

Definition 4.1 If $\varpi_N(\xi) \in L^2$, then truncated form of $\varpi(\xi)$ is:

$$\varpi_N(\xi) = \sum_{\gamma=0}^{N} e_\gamma Q_\gamma^\nu(\xi) = E^T \Psi(\xi), \tag{10}$$

where

$$\Psi(\xi) = \left(Q_0^\nu(\xi), Q_1^\nu(\xi), ..., Q_N^\nu(\xi)\right)^T, \text{and} E^T = (e_0, e_1, ..., e_N). \tag{11}$$

Let

$$\Psi(\xi) = \mathcal{G}\underline{Z}^\nu(\xi), \tag{12}$$

where,

$$\underline{Z}^\nu(\xi) = \begin{pmatrix} 1 \\ \xi^\nu \\ \xi^{2\nu} \\ \vdots \\ \xi^{N\nu} \end{pmatrix}_{(N+1)\times 1} \quad \text{and } \mathcal{G} = \mathcal{B}C. \tag{13}$$

If N is odd

$$\mathcal{B} = \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0}2^{1-0} & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1}2^{2-2} & 0 & \frac{2}{2}\binom{2}{0}2^{2-0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \binom{N}{\frac{N+1}{2}}\binom{\frac{N+1}{2}}{\frac{N-1}{2}}2^{N-2(N-1)/2} & 0 & \cdots & \binom{N}{N}\binom{N}{0}2^{N-0} \end{pmatrix}_{(N+1)\times(N+1)}, \tag{14}$$

if N is even

$$\mathcal{B} = \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0}2^{1-0} & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1}2^{2-2} & 0 & \frac{2}{2}\binom{2}{0}2^{2-0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{N}{\frac{N}{2}}\binom{\frac{N}{2}}{\frac{N}{2}}2^{N-2N/2} & 0 & \binom{N}{\frac{N+2}{2}}\binom{\frac{N+2}{2}}{\frac{N-2}{2}}2^{N-2(N/2-1)} & \cdots & \binom{N}{N}\binom{N}{0}2^{N-0} \end{pmatrix}_{(N+1)\times(N+1)}, \tag{15}$$

and

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \binom{1}{0}(-\iota)^{1-0} & \binom{1}{1}(-\iota)^{1-1} & 0 & \cdots & 0 \\ \binom{2}{0}(-\iota)^{2-0} & \binom{2}{1}(-\iota)^{2-1} & \binom{2}{2}(-\iota)^{2-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{N}{0}(-\iota)^{N-0} & \binom{N}{1}(-\iota)^{N-1} & \binom{N}{2}(-\iota)^{N-2} & \cdots & \binom{N}{N}(-\iota)^{N-N} \end{pmatrix}_{(N+1)\times(N+1)}. \tag{16}$$

From equation (12), we have

$$\underline{Z}^{\nu}(\xi) = \mathcal{G}^{-1}\Psi(\xi). \tag{17}$$

Theorem 4.1 *Prove that the fractional derivative of FSLPs $\Psi(\xi)$ in Caputo sense is*

$$D^{\sigma_n}\Psi(\xi) = H\Psi(\xi), n-1 < \sigma_n \leq n, \quad n \in \mathbb{N}, \tag{18}$$

where, $H = \mathcal{G}M_n\mathcal{G}^{-1}$.

Proof Since,

$$\Psi(\xi) = \mathcal{G}\underline{Z}^{\nu}(\xi).$$

Applying Caputo's fractional derivative D^{σ_n} (as defined in equation (3)) on both sides,

$$D^{\sigma_n}\Psi(\xi) = D^{\sigma_n}(\mathcal{G}\underline{Z}^{\nu}(\xi)) = \mathcal{G}D^{\sigma_n}\underline{Z}^{\nu}(\xi), (\mathcal{G}\text{is given in equation(13)}).$$

$$D^{\sigma_n}\Psi(\xi) = \mathcal{G}D^{\sigma_n}\begin{pmatrix} 1 \\ \xi^{\nu} \\ \xi^{2\nu} \\ \vdots \\ \xi^{N\nu} \end{pmatrix} \text{by means of the equation(13), } weobtain \tag{19}$$

$$= \mathcal{G}\begin{pmatrix} 0 \\ \dfrac{\Gamma(\nu+1)}{\Gamma(\nu+1-\sigma_n)}\xi^{\nu-\sigma_n} \\ \dfrac{\Gamma(2\nu+1)}{\Gamma(2\nu+1-\sigma_n)}\xi^{2\nu-\sigma_n} \\ \vdots \\ \dfrac{\Gamma(N\nu+1)}{\Gamma(N\nu+1-\sigma_n)}\xi^{N\nu-\sigma_n} \end{pmatrix} \tag{20}$$

$$= \mathcal{G}\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \dfrac{\Gamma(\nu+1)}{\Gamma(\nu+1-\sigma_n)}\xi^{-\sigma_n} & 0 & \cdots & 0 \\ 0 & 0 & \dfrac{\Gamma(2\nu+1)}{\Gamma(2\nu+1-\sigma_n)}\xi^{-\sigma_n} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dfrac{\Gamma(N\nu+1)}{\Gamma(N\nu+1-\sigma_n)}\xi^{-\sigma_n} \end{pmatrix}\begin{pmatrix} 1 \\ \xi^{\nu} \\ \xi^{2\nu} \\ \vdots \\ \xi^{N\nu} \end{pmatrix}$$

Hence

$$D^{\sigma_n}\Psi(\xi) = \mathcal{G}M_n\underline{Z}^{\nu}(\xi),$$

where

$$M_n = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\sigma_n)}\xi^{-\sigma_n} & 0 & \dots & 0 \\ 0 & 0 & \frac{\Gamma(2\nu+1)}{\Gamma(2\nu+1-\sigma_n)}\xi^{-\sigma_n} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \frac{\Gamma(N\nu+1)}{\Gamma(N\nu+1-\sigma_n)}\xi^{-\sigma_n} \end{pmatrix}.$$

From equation (17), we obtained

$$D^{\sigma_n}\Psi(\xi) = \mathcal{G}M_n\mathcal{G}^{-1}\Psi(\xi) = H\Psi(\xi), \tag{21}$$

and $H = \mathcal{G}M_n\mathcal{G}^{-1}$.

Hence complete the proof. □

Remark 4.1 If $\sigma_n = \lambda = n$, $n \in \mathbb{N}$ then Theorem (4.1) can be used as integer order derivative of FSLPs.

5 Numerical scheme

The equation (1) can be approximated by using equation (10) in the following form:

$$\sum_{\gamma=0}^{N} e_\gamma D^\lambda Q_\gamma^\nu(\xi) = G\left(\xi, \sum_{\gamma=0}^{N} e_\gamma Q_\gamma^\nu(\xi), \sum_{\gamma=0}^{N} e_\gamma D^{\sigma_1} Q_\gamma^\nu(\xi), ..., \sum_{\gamma=0}^{N} e_\gamma D^{\sigma_n} Q_\gamma^\nu(\xi)\right).$$

Further, utilization of equation (21) in above equation will give us

$$\sum_{\gamma=0}^{N} e_\gamma \mathcal{G}M_\lambda\mathcal{G}^{-1} Q_\gamma^\nu(\xi) = G\left(\xi, \sum_{\gamma=0}^{N} e_\gamma \mathcal{G}\underline{Z}^\nu(\xi), \sum_{\gamma=0}^{N} e_\gamma \mathcal{G}M_1\mathcal{G}^{-1} Q_\gamma^\nu(\xi), ..., \sum_{\gamma=0}^{N} e_\gamma \mathcal{G}M_n\mathcal{G}^{-1} Q_\gamma^\nu(\xi)\right). \tag{22}$$

Similarly, equation (2) is:

$$\sum_{\gamma=0}^{N} e_\gamma \left(\mathcal{G}M_j\mathcal{G}^{-1}\right) Q_\gamma^\nu(0) = 0,\ j = 0, 1, ..., n-1. \tag{23}$$

Let the residual function for equation (1) be define as:

$$\Re(\xi) = \sum_{\gamma=0}^{N} e_\gamma \mathcal{G}M_\lambda\mathcal{G}^{-1} Q_\gamma^\nu(\xi) - G\left(\xi, \sum_{\gamma=0}^{N} e_\gamma \mathcal{G}\underline{Z}^\nu(\xi), \sum_{\gamma=0}^{N} e_\gamma \mathcal{G}M_1\mathcal{G}^{-1} Q_\gamma^\nu(\xi), ..., \sum_{\gamma=0}^{N} e_\gamma \mathcal{G}M_n\mathcal{G}^{-1} Q_\gamma^\nu(\xi)\right),$$

which can be determine by employing the collocation points. To collocate the above equation, we choose the collocation points as:

$$\xi_i = \frac{2i+1}{2(N+1)},\ i = 0, 1, 2, ..., N-n. \tag{24}$$

So, the collocated residual function is

$$\begin{aligned} R(\xi_i) = \sum_{\gamma=0}^{N} e_\gamma \mathcal{G}M_\lambda\mathcal{G}^{-1} Q_\gamma^\nu(\xi_i) - G\Bigg(&\xi_i, \sum_{\gamma=0}^{N} e_\gamma \mathcal{G}\underline{Z}^\nu(\xi_i), \sum_{\gamma=0}^{N} e_\gamma \mathcal{G}M_1\mathcal{G}^{-1} Q_\gamma^\nu(\xi_i), \\ &\sum_{\gamma=0}^{N} e_\gamma \mathcal{G}M_2\mathcal{G}^{-1} Q_\gamma^\nu(\xi_i), ..., \sum_{\gamma=0}^{N} e_\gamma \mathcal{G}M_n\mathcal{G}^{-1} Q_\gamma^\nu(\xi_i)\Bigg). \end{aligned} \tag{25}$$

Utilizing the collocation point theory, the algebraic equations are obtained from equations (23) and (25). Further, computing these equations simultaneously with Mathematica software, we can receive the values of the unknown scalars e_γ. Accordingly, we received the approximate solution for equation (1).

6 Error analysis

Theorem 6.1 *If $D^{\gamma\sigma_n}\varpi(\xi)$ for $\gamma = 0, 1, 2..., N+1$, is continuously differentiable function and $\varpi_N(\xi)$ be the best approximation to $\varpi(\xi)$, then*

$$\delta_N(\xi) = ||\varpi(\xi) - \varpi_N(\xi)|| \leq \frac{M^2 S^{2(N+1)\sigma_n}}{[\Gamma((N+1)\sigma_n + 1)]^2}\left(\frac{|\arccos(1+\iota(2\iota)^{\nu})|}{2\nu}\right), \nu > 0,$$

where $M = \max(D^{(N+1)\sigma_n}\varpi(\xi))$ on $\hat{R}_1$ and $S = \max[\xi_0 - 2\iota - \xi_0]$.

Proof Assume that $P(\xi)$ be the fractional Taylor's formula of $\varpi(\xi)$, then [1]:

$$P_N(\xi) = \sum_{\gamma=0}^{N} \frac{(\xi - \xi_0)^{\gamma\sigma_n}}{\Gamma(\gamma\sigma_n + 1)} D^{\gamma\sigma_n}\varpi(0^+), \xi \in \hat{R}_1,$$

and error bound is

$$|\varpi(\xi) - P_N(\xi)| = \left|\frac{(\xi - \xi_0)^{(N+1)\sigma_n}}{\Gamma((N+1)\sigma_n + 1)} D^{(N+1)\sigma_n}\varpi(z)\right|, z \in (0, \xi].$$

Since $\varpi_N(\xi) = \sum_{\gamma=0}^{N} e_\gamma Q_\gamma^{\nu}(\xi)$ is the best approximation to $\varpi(\xi)$. So, we attain

$$\begin{aligned} ||\varpi(\xi) - \varpi_N(\xi)||^2 &\leq ||\varpi(\xi) - P_N(\xi)||^2 \\ &= \int_{\hat{R}_1} [\varpi(\xi) - P_N(\xi)]^2 W^{\nu} d\xi \\ &= \int_{\hat{R}_1} \left[\frac{(\xi - \xi_0)^{(N+1)\sigma_n}}{\Gamma((N+1)\sigma_n + 1)} D^{(N+1)\sigma_n}\varpi(z)\right]^2 W^{\nu} d\xi \\ &\leq \frac{M^2}{[\Gamma((N+1)\sigma_n + 1)]^2} \int_{\hat{R}_1} (\xi - \xi_0)^{2(N+1)\sigma_n} W^{\nu} d\xi \end{aligned}$$

Let $S = \max[\xi_0, 2\iota - \xi_0]$, then we can write

$$||\varpi(\xi) - \varpi_N(\xi)||^2 \leq \frac{M^2 S^{2(N+1)\sigma_n}}{[\Gamma((N+1)\sigma_n + 1)]^2}\left|\int_{\hat{R}_1}\left(\frac{1}{\iota\xi^{1-\nu}\sqrt{(2\xi^{\nu} - 2\iota)^2 + 4}}\right) d\xi\right|.$$

Hence

$$||\varpi(\xi) - \varpi_N(\xi)|| \leq \frac{M^2 S^{2(N+1)\sigma_n}}{[\Gamma((N+1)\sigma_n + 1)]^2}\left(\frac{|\arccos(1+\iota(2\iota)^{\nu})|}{2\nu}\right).$$

Since, $\nu > 0$. So, $|\nu| = \nu$. Hence we proved the desired result. □

7 Test examples

In this section, we consider some MODEs/MTDEs and solve them using the introduced technique. All the calculations of numerical examples have done with Mathematica software.

Problem 1 Consider the initial condition associated problem as [29]:

$$\Lambda_1 D^{\sigma_1}\varpi(\xi) + \Lambda_2(\xi) D^{\sigma_2}\varpi(\xi) + \Lambda_3(\xi) D^{\sigma_3}\varpi(\xi) + \Lambda_4(\xi) D^{\sigma_4}\varpi(\xi) + \Lambda_5(\xi)\varpi(\xi) = \Upsilon(\xi), \quad (26)$$

$$\varpi(\xi)|_{\xi=0} = 2, D\varpi(\xi)|_{\xi=0} = 0, \quad (27)$$

where

$$\Upsilon(\xi)=-1-\frac{\Lambda_2(\xi)}{\Gamma(3-\sigma_2)}\xi^{2-\sigma_2}-\frac{\Lambda_3(\xi)}{\Gamma(3-\sigma_3)}\xi^{2-\sigma_3}-\Lambda_4(\xi)\frac{\xi^{2-\sigma_4}}{\Gamma(3-\sigma_4)}+\Lambda_5(\xi)\left(2-\frac{\xi^2}{2}\right),$$
$$\Lambda_1=1,\ \Lambda_2(\xi)=\sqrt{\xi},\ \Lambda_3(\xi)=\xi^{1/3},\ \Lambda_4(\xi)=\xi^{1/4},\ \Lambda_5(\xi)=\xi^{1/5},$$
$$\sigma_1=2,\ \sigma_2=\frac{1}{3},\ \sigma_3=\frac{1}{4},\ \sigma_4=\frac{1}{5},\ 0<\sigma_2,\sigma_3,\sigma_4\leq 1.$$

The closed form solution at $\nu=1$ is $\varpi(\xi)=2-\dfrac{\xi^2}{2}$ [29] and for other values of ν, closed form solution is not known. In order to compute the solution of considered problem, we choose $N=2$ in equation (10). Further, utilizing O.M. and theorem (4.1) we get:

$$\frac{\xi^{11/5}}{2}+\frac{\xi^{13/6}}{\Gamma\left(\frac{8}{3}\right)}+\frac{\xi^{25/12}}{\Gamma\left(\frac{11}{4}\right)}+\frac{\xi^{41/20}}{\Gamma\left(\frac{14}{5}\right)}+\left[8+4\xi^{(1/5+2\nu)}\right]e_2+4$$
$$\left[\frac{\xi^{(1/6+2\nu)}}{\Gamma\left(\frac{2}{3}+2\nu\right)}+\frac{\xi^{(1/12+2\nu)}}{\Gamma\left(\frac{3}{4}+2\nu\right)}+\frac{\xi^{(1/20+2\nu)}}{\Gamma\left(\frac{4}{5}+2\nu\right)}\right]e_2\Gamma(1+2\nu)=-1. \tag{28}$$

Now, we are defining collocated residual using collocation points (defined in equation (24)) as:

$$R(\xi_i)=\frac{\xi_i^{11/5}}{2}+\frac{\xi_i^{13/6}}{\Gamma\left(\frac{8}{3}\right)}+\frac{\xi_i^{25/12}}{\Gamma\left(\frac{11}{4}\right)}+\frac{\xi_i^{41/20}}{\Gamma\left(\frac{14}{5}\right)}+\left[8+4\xi_i^{(1/5+2\nu)}\right]e_2+4$$
$$\left[\frac{\xi_i^{(1/6+2\nu)}}{\Gamma\left(\frac{2}{3}+2\nu\right)}+\frac{\xi_i^{(1/12+2\nu)}}{\Gamma\left(\frac{3}{4}+2\nu\right)}+\frac{\xi_i^{(1/20+2\nu)}}{\Gamma\left(\frac{4}{5}+2\nu\right)}\right]e_2\Gamma(1+2\nu)+1. \tag{29}$$

The initial conditions gives following values of scalars

$$e_0=1-3e_2,\ e_1=4\iota e_2. \tag{30}$$

Utilizing above equation, equation (29) is solved at collocation point $\xi_0=\dfrac{1}{6}$ and we get

$$e_2=\frac{1.05343}{(2^{1.8-2\nu})(3^{-0.2-2\nu})+A\Gamma(1+2\nu)} \tag{31}$$

where

$$A=\left[\frac{(2^{4-2\nu})(3^{2-2\nu})}{\Gamma(-1+2\nu)}+\frac{(2^{1.83333-2\nu})(3^{-0.166667-2\nu})}{\Gamma(0.666667+2\nu)}+\frac{(2^{1.91667-2\nu})(3^{-0.0833333-2\nu})}{\Gamma(0.75+2\nu)}+\frac{(2^{1.95-2\nu})(3^{-0.05-2\nu})}{\Gamma(0.8+2\nu)}\right].$$

Thus,

$$\varpi_2(\xi)=\begin{pmatrix}e_0 & e_1 & e_2\end{pmatrix}\begin{pmatrix}2 & 0 & 0\\ -2\iota & 2 & 0\\ -2 & -8\iota & 16\end{pmatrix}\begin{pmatrix}1\\ \xi^{\nu}\\ \xi^{2\nu}\end{pmatrix} \tag{32}$$
$$=2-\frac{4.21371\xi^{2\nu}}{(2^{1.8-2\nu})(3^{-0.2-2\nu})+A\Gamma(1+2\nu)}. \tag{33}$$

We can easily see that when $\nu=1$, we attained exact solutions. The approximate solutions at $\nu=0.7, 0.8, 0.9$ and $\nu=1$ are plotted in Fig. (1) and three dimensional picture of approximate solution is graphically shown in Fig. (2). Also, the normed error at $\nu=1, 0.5$ are performed in Table (4).

Problem 2 Suppose the nonlinear MTDE as [29]:

$$\Lambda_1 D^{\sigma_1}\varpi(\xi) + \Lambda_2 D^{\sigma_2}\varpi(\xi) D^{\sigma_3}\varpi(\xi) + \Lambda_3\varpi^2(\xi) = \Upsilon(\xi) \tag{34}$$

with

$$\begin{aligned}&\Lambda_1 = \Lambda_2 = \Lambda_3 = 1,\ \Upsilon(\xi) = \xi^6 + \frac{6}{\Gamma(4-\sigma_1)}\xi^{3-\sigma_1} + \frac{36}{\Gamma(4-\sigma_2)\Gamma(4-\sigma_3)}\xi^{6-\sigma_2-\sigma_3},\\ &\sigma_1 = 2.5, \sigma_2 = 1.5, \sigma_3 = 0.9, 2 < \sigma_1 \le 3, 1 < \sigma_2 \le 2, 0 < \sigma_1 \le 1,\end{aligned}$$

and

$$\varpi^{(\gamma)}(\xi)|_{\xi=0} = 0, \gamma = 0, 1, 2. \tag{35}$$

Which has closed form solution $\varpi(\xi) = \xi^3$ at only $\nu = 1$. For $N = 3$, the collocated residual equation is

$$\begin{aligned}R(\xi_i) =& -6.77028\xi_i^{0.5} - 12.3229\xi_i^{3.6} - \xi_i^6 + 64e_3^2\xi_i^{6\nu} + \frac{8e_3\xi_i^{-2.5+3\nu}\Gamma(1+3\nu)}{\Gamma(-1.5+3\nu)}\\ &+ \frac{64e_3^2\xi_i^{-2.4+6\nu}\Gamma(1+3\nu)^2}{\Gamma(-0.5+3\nu)\Gamma(0.1+3\nu)}.\end{aligned} \tag{36}$$

The reduced initial conditions are

$$e_0 = e_2 + \iota(e_1 - e_3), e_1 = 4\iota e_2 + 9e_3, e_2 = 6\iota e_3.$$

Employing these values of scalars into equation (36) at collocation point $\xi_0 = \frac{1}{8}$, we get

$$e_3 = \frac{0.5 \times 8^{6\nu}\left(-\frac{1448.15 \times 8^{-3\nu}\Gamma(1+3\nu)}{\Gamma(-1.5+3\nu)}\right) + \sqrt{B_0 + (9.60228 \times 8^{-6\nu})B_1}}{B_1},$$

where

$$B_0 = \frac{2.09715 \times 10^6 \times 8^{-6\nu}\Gamma(1+3\nu)^2}{\Gamma(-1.5+3\nu)^2},\ B_1 = 64 + \frac{9410.14\Gamma(1+3\nu)^2}{\Gamma(-0.5+3\nu)\Gamma(0.1+3\nu)}.$$

Hence

$$\varpi_3(\xi) = \begin{pmatrix} e_0 & e_1 & e_2\end{pmatrix}\begin{pmatrix} 2 & 0 & 0\\ -2\iota & 2 & 0\\ -2 & -8\iota & 16\end{pmatrix}\begin{pmatrix}1\\ \xi^\nu\\ \xi^{2\nu}\end{pmatrix}$$

$$= \frac{4 \times 8^{6\nu}\xi^{3\nu}(B_2 + \sqrt{B_0 + (9.60228 \times 8^{-6\nu})B_1})}{B_1},$$

$$B_2 = -\frac{1448.15 \times 8^{-3\nu}\Gamma(1+3\nu)}{\Gamma(-1.5+3\nu)}.$$

Which is approximate solution. If we substitute $\nu = 1$ in above equation, then we get closed form solution. For other values of ν, the approximate solutions are plotted in Fig. (3) at $N = 3$ and normed error at $\nu = 1, 0.5$ are performed in Table (4).

Problem 3 Consider the following equation as [30]:

$$D^2\varpi(\xi) + \Lambda_1(\xi)D^{\sigma_1}\varpi(\xi) + \varpi^2(\xi) = \Upsilon(\xi), \tag{37}$$

where

$$\Upsilon(\xi) = \left(1 + \frac{4}{\sqrt{\pi}}\right)\xi^4 + 2, \sigma_1 = \frac{3}{2}, 1 < \sigma_1 \leq 2 \text{and} \Lambda_1(\xi) = \xi^{\frac{7}{2}},$$

with $\varpi^{(\gamma)}(\xi)|_{\xi=0} = 0, \gamma = 0, 1$ and has closed form solution $\varpi(\xi) = \xi^2$ [30]. The approximate solution can be computed by taking $N = 2$ in proposed method. The other computational techniques are similar as previous example. In this case, we received following approximate solution:

$$\varpi_2(\xi) = \left[-\frac{18}{\Gamma(-1+2\nu)} - \frac{0.0138889}{\Gamma(-0.5+2\nu)}\right] \times 6^{2\nu}\xi^{2\nu}\Gamma(1+2\nu) + 0.125 \times 6^{4\nu}\xi^{2\nu}$$
$$\sqrt{128.161 \times 6^{-4\nu} + 6^{-4\nu}\left(\frac{144}{\Gamma(-1+2\nu)} + \frac{0.111111}{\Gamma(-0.5+2\nu)}\right)^2 \Gamma(1+2\nu)^2}.$$

Fig. (4) portraits the clear picture of approximate solutions at $\nu = 0.75, 0.85, 0.95, 1$ and at $\nu = 1$ the approximate solution converges to the exact solution. In fig. (5), we have plotted the approximate solution in three dimensional form. Moreover, the comparison of absolute errors are listed in Table (1) and normed error at $\nu = 1, 0.5$ are performed in Table (4).

Problem 4 Consider the MTDE as [31]:

$$D^{\sigma_1}\varpi(\xi) + 1.3D^{\sigma_2}\varpi(\xi) + 2.6\varpi(\xi) = \Upsilon(\xi), \tag{38}$$

where

$$\sigma_1 = 2.2, \sigma_2 = 1.5, 2 < \sigma_1 \leq 3, 1 < \sigma_2 \leq 2, \Upsilon(\xi) = \sin(2\xi),$$

with initial conditions

$$\varpi(\xi)|_{\xi=0} = D\varpi(\xi)|_{\xi=0} = D^2\varpi(\xi)|_{\xi=0} = 0. \tag{39}$$

Employing the scheme as we described in Section (5) at $N = 8$, we obtained the approximate solutions at $\nu = 0.7, 0.8, 0.9, 1$ which have presented in Table (2).

Problem 5 Suppose the following MODEs [32,33] as :

$$D^{\sigma_1}\varpi(\xi) + \varpi^2(\xi) = 1, \xi \in (0, 1), \sigma_1 \in (0, 1], \tag{40}$$

with

$$\varpi(0) = 0. \tag{41}$$

The exact solution for equation (40) is $\varpi(\xi) = \dfrac{e^{2\xi} - 1}{e^{2\xi} + 1}$. This closed form solution is valid only for $\sigma_1, \nu = 1$. For $(\sigma_1, \nu) \in (0, 1)$ the closed form solution is not known. We demonstrate the comparison of current approximate solution at $N = 8$ and UWCM [32], S5CMTM [32] solutions at distinct values of σ_1 and $\nu = 1$ in Table (3). The comparison shows that current method provides the good solution at $N = 8$. Further in Fig. (6), we have plotted approximate solutions for different and same values of σ, ν. Graph conclude that approximate solutions have very small variation (Table 4).

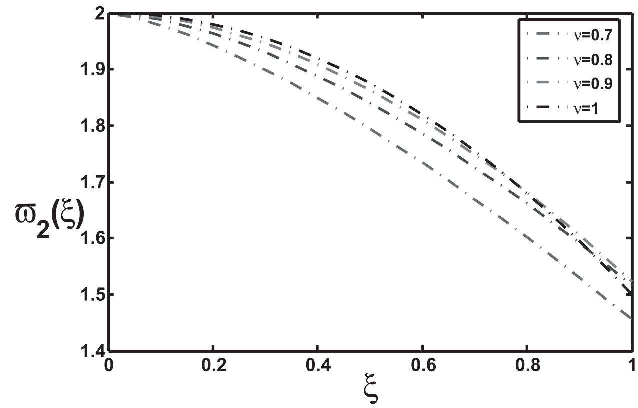

Fig. 1 Behavior of $\varpi(\xi)$ at $N = 2$ with respect to ξ at distinct values of ν for equation (26)

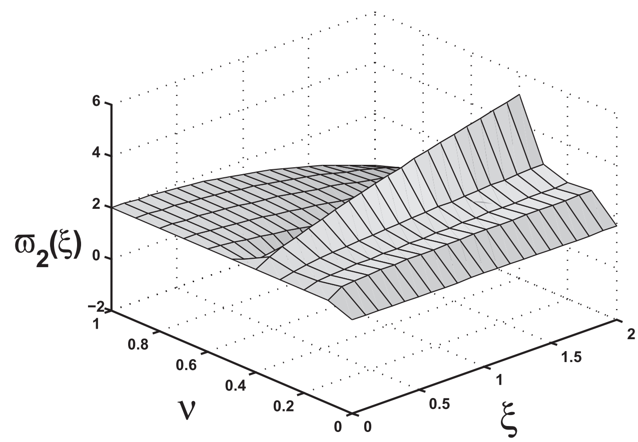

Fig. 2 Three dimensional graph of $\varpi(\xi)$ at $N = 2$ with respect to ν and ξ for equation (26)

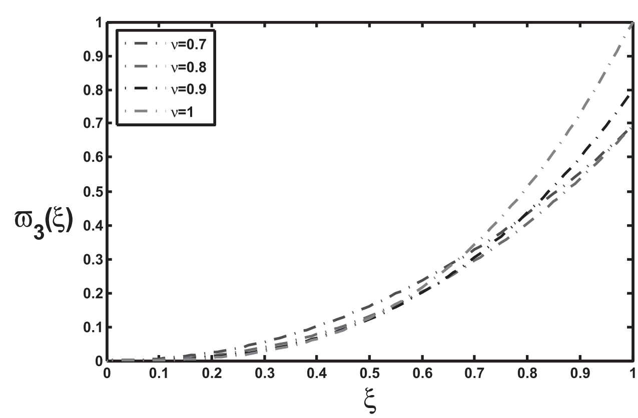

Fig. 3 Plot of approximate solutions $\varpi(\xi)$ at $N = 3$ with respect to ξ at various values of ν for equation (34)

8 Conclusion

This work is mainly related to the construction of a numerical approach based on FSLPs and operational matrices. We all know that in the literature, there exist Lucas polynomials and shifted Lucas polynomials but no one have mentioned about FSLPs anywhere in the literature. In this regard, shifted Lucas polynomials are proposed in previous paper and in continuation to that work, the FSLPs are proposed in this paper. The set up of algebraic equations in the scheme are obtained by utilizing collocation points. The obtained solutions are compared

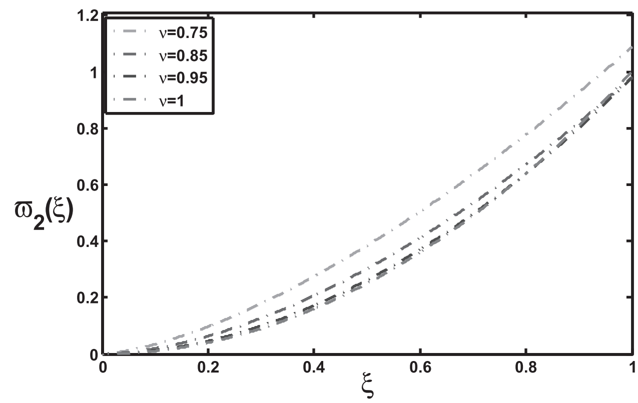

Fig. 4 Graph of $\varpi(\xi)$ for equation (37) at $N = 2$ with respect to $\nu = 0.75, 0.85, 0.95, 1$

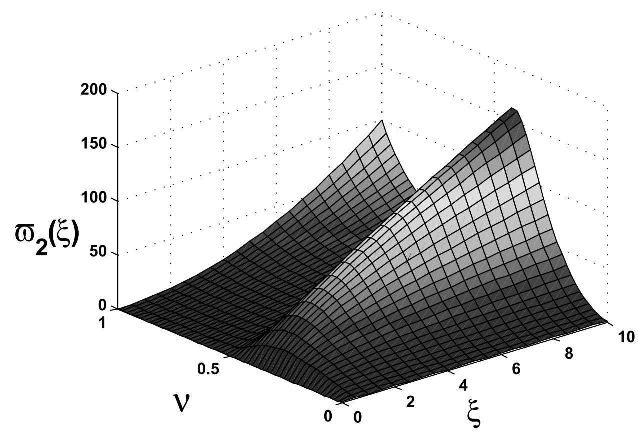

Fig. 5 Graph of $\varpi(\xi)$ for equation (37) with respect to ξ, ν at $N = 2$

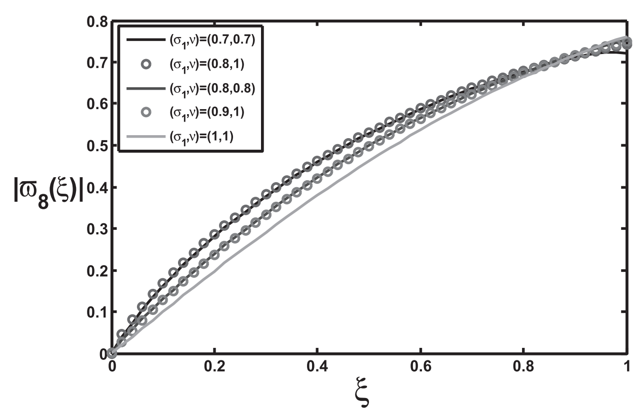

Fig. 6 Plots of approximate solutions $|\varpi_8(\xi)|$ with respect to ξ and σ_1, ν at $N = 8$ for equation (40)

Table 1 Comparison of absolute error by current method at $N = 2$, $\nu = 1$ and technique given in [30] at $N = 4$ for equation (37)

ξ	$\lvert\varpi(\xi) - \varpi_4(\xi)\rvert$ [30]	$\lvert\varpi(\xi) - \varpi_2(\xi)\rvert$ $\nu = 1$
0.2	8.8×10^{-4}	6.2×10^{-15}
0.4	7.1×10^{-3}	2.4×10^{-14}
0.6	3.5×10^{-3}	7.0×10^{-14}
0.8	2.8×10^{-2}	9.9×10^{-14}
1.0	5.4×10^{-2}	2.2×10^{-13}

Table 2 Approximate solutions with respect to ξ at $\nu = 0.7, 0.8, 0.9, 1$ and $N = 8$ for equation (38)

ξ	$\nu = 0.7$	$\nu = 0.8$	$\nu = 0.9$	$\nu = 1$
0.2	0.00014012	0.00036710	0.00074400	0.00128224
0.4	0.00317555	0.00526097	0.00773029	0.01045350
0.6	0.01802145	0.02352779	0.02884897	0.03385765
0.8	0.05744226	0.06399931	0.06955553	0.07435024
1.0	0.12953575	0.12977894	0.13162401	0.13575775

Table 3 Comparison of approximate solution between current method and methods in [32] at various values of σ_1 with respect to $\nu = 1$ and $N = 8$ for equation (40)

ξ		$\sigma_1 = 0.7$			$\sigma_1 = 0.8$			$\sigma_1 = 0.9$	
	Elhameed and Youssri [32]	S5CMTQM [32]	Current method	Elhameed and Youssri [32]	S5CMTQM [32]	Current method	Elhameed and Youssri [32]	S5CMTQM [32]	Current method
0.1	0.209216	0.171333	0.204679	0.165498	0.138091	0.161796	0.129138	0.101874	0.127189
0.3	0.429549	0.393889	0.421432	0.383197	0.351883	0.377776	0.336448	0.295195	0.333747
0.5	0.556331	0.537862	0.553819	0.530743	0.510705	0.528357	0.498915	0.465588	0.497407
0.7	0.643854	0.636781	0.642183	0.637215	0.628023	0.635499	0.624307	0.606359	0.623088
0.9	0.707567	0.706583	0.700279	0.714519	0.713562	0.711081	0.717972	0.716482	0.716514

Table 4 Error obtained through best approximation

Equation no.	Normed error at $\nu = 1$	CPU time (s)	Normed error at $\nu = 0.5$	CPU time (s)
(26) at $N = 2$	0	0.062500	9.39×10^{-3}	0.203125
(34) at $N = 3$	6.84×10^{-15}	0.234375	8.13×10^{-3}	0.218750
(37) at $N = 2$	9.85×10^{-14}	0.062500	1.94×10^{-2}	0.187500

with results exiting in literature. The validity of the proposed algorithm is checked through various types of MODEs/MTDEs. The obtained solutions are compared with the results existing in literature. It is noticed that, current algorithm offers a good approximation, and may be easily applied to any problems arising in applied science.

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