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ORIGINAL RESEARCH

Green–Lazarsfeld property N_p for Segre product of Hibi rings

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In memory of Prof. C. S. Seshadri

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Abstract In this article, we prove that if a Hibi ring satisfies property N_2, then its Segre product with a polynomial ring in finitely many variables also satisfies property N_2. When the polynomial ring is in two variables, we also prove the above statement for N_3. Moreover, we study the minimal Koszul relations of the second syzygy module of Hibi rings.

Keywords Distributive lattices · Hibi rings · Green–Lazarsfeld property N_p · Minimal resolution · Syzygies

Mathematics Subject Classification 05E40 · 13C05 · 13D02

1 Introduction

A classical problem in commutative algebra is to study the graded minimal free resolution of graded modules over polynomial rings. Let S be a standard graded polynomial ring in finitely many variables over a field K and I be a graded S-ideal. To study the graded minimal free resolution of S/I, Green and Lazarsfeld [9] defined property N_p for $p \in \mathbb{N}$. The ring S/I satisfies property N_p if S/I is normal and the graded minimal free resolution of S/I over S is linear upto pth position. In this article, we study the Green–Lazarsfeld property N_p for Segre product of Hibi rings and the minimal Koszul sygygies of the Hibi ideals and the initial Hibi ideals.

Let L be a finite distributive lattice and $P = \{p_1, \dots, p_n\}$ be the subposet of join-irreducible elements of L. Let K be a field and let $R = K[y_1, \dots, y_n, z_1, \dots, z_n]$ be a polynomial ring over K. The Hibi ring associated with L, denoted by $R[L]$, is the subring of R generated by the monomials $u_\alpha = (\prod_{p_i \in \alpha} y_i)(\prod_{p_i \notin \alpha} z_i)$ where $\alpha \in L$. Hibi [14] showed that $R[L]$ is a normal Cohen–Macaulay domain of dimension $\#P + 1$, where $\#P$ is the cardinality of P. Let $K[L] = K[\{x_\alpha : \alpha \in L\}]$ be the polynomial ring over K and $\pi : K[L] \to R[L]$ be the K-algebra homomorphism with $x_\alpha \mapsto u_\alpha$. The ideal $I_L = (x_\alpha x_\beta - x_{\alpha\wedge\beta} x_{\alpha\vee\beta} : \alpha, \beta \in L$ and α, β incomparable) is the kernel of the map π. It is called the Hibi ideal associated to L.

In past, various authors have studied minimal free resolution of Hibi rings. Ene et al. [6] provided a combinatorial formula for the regularity of Hibi rings. In [5,8], the authors have characterized all Hibi rings with linear resolution. Ene [7] characterizes all simple planar distributive lattices for which the associated Hibi ring satisfies property N_2. Das and Mukherjee [4] described the generators of second syzygy module of simple planar

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distributive lattices. Additionally, the author [25] has proved several results regarding property N_p of Hibi rings, including a characterization of those Hibi rings that satisfy property N_p for $p \geq 4$.

The Segre product of polynomial rings may be viewed as a Hibi ring. Let $A = K[x_{1,0}, \ldots, x_{1,n_1}] * \cdots * K[x_{r,0}, \ldots, x_{r,n_r}]$ be the Segre product of r polynomial rings, where $n_i \geq 1$ and $n_i \in \mathbb{N}$ for all i. Sharpe [21] proved in 1964 that if $r = 2$, then A satisfies property N_2. For $r = 2$, Lascoux [16] and Pragacz-Weyman [18] proved that A satisfies property N_3 if K contains the rational field $\mathbb{Q}$. Hashimoto [11] showed that if $r = 2$, $n_1, n_2 \geq 4$ and characteristic of the field K is 3, then A does not satisfy property N_3. Rubei [19,20] proved that if $r \geq 3$ and $\text{char}(K) = 0$, then A satisfies property N_3 but it does not satisfy property N_4.

It is known that if a poset in disconnected, then the associated Hibi ring is the Segre product of the Hibi rings associated to each individual connected piece of the poset. Let P be a poset that is disjoint union of a poset P_1 and an isolated point. In this article, we prove that if $R[\mathcal{I}(P_1)]$ satisfies property N_p for $p \leq 3$, then so does $R[\mathcal{I}(P)]$. Our proof follows the argument of Rubei [19] which is combinatorial in nature. The core idea of Rubei's approach is as follows: in [3,23], it was shown that multigraded Betti numbers of affine semigroup rings can be computed using homologies of squarefree divisor complexes. Utilizing the results of [3,23], it is suffices to prove that for $p = 2$, the first homology of certain squarefree divisor complexes vanishes, whereas for $p = 3$, the second homology of certain squarefree divisor complexes vanishes. For $p = 2$ (resp. $p = 3$), we begin by showing that every 1-cycle (resp. 2-cycle) γ of the simplicial complex Δ is homologous to an 1-cycle (resp. a 2-cycle) of a subcomplex of Δ. We then use this result to demonstrate that γ is, in fact, a boundary of Δ.

We generalize the above result for $p = 2$ to the case when P is the disjoint union of a poset P_1 and a chain. More precisely, we show that if $R[\mathcal{I}(P_1)]$ satisfies property N_2, then $R[\mathcal{I}(P)]$ also satisfies property N_2. To prove this, we use induction on the length of the chain. The above result serves as the base case for the induction. This proof is motivated from Rubei [20]. Following the combinatorial argument outlined in the previous result, we first show that every 1-cycle γ of the simplicial complex Δ is homologous to an 1-cycle γ' of a simplicial complex of X, which contains Δ as a subcomplex. We then note that γ' is homologous to 0 is X using the hypothesis $R[\mathcal{I}(P_1)]$ satisfies property N_2. Finally, we consider different cases to show that γ is homologous to 0 in Δ, which is equivalent to saying that γ is a boundary of Δ.

Next, we inquire the following: if a Hibi ring does not satisfy property N_2, then which Koszul relations will be in the minimal generating set of its second syzygy module. We provide a partial answer this question. In the process, we characterize the Koszul relations pairs of the initial Hibi ideals under a fixed monomial order.

The article is organized as follows. In Sect. 2, we recall some basic notions of algebra and combinatorics. Section 3 is about the results on property N_p of Segre products of Hibi rings. Finally, in Sect. 4, we study the Koszul relation pairs of the Hibi ideals and initial Hibi ideals.

2 Preliminaries

We start by defining some notions of posets and distributive lattices. For more details and examples, we refer the reader to [22, Chapter 3]. Throughout this article, all posets and distributive lattices will be finite.

Let P be a poset. We say that two elements x and y of P are *comparable* if $x \leq y$ or $y \leq x$; otherwise x and y are *incomparable*. For $x, y \in P$, we say that y *covers* x if $x < y$ and there is no $z \in P$ with $x < z < y$. We denote it by $x \lessdot y$. A poset is completely determined by its cover relations. A *chain* C of P is a totally ordered subset of P. The *length* of a chain C of P is $\#C - 1$. A subset α of P is called an *order ideal* of P if it satisfies the following condition: for any $x \in \alpha$ and $y \in P$, if $y \leq x$, then $y \in \alpha$. Define $\mathcal{I}(P) := \{\alpha \subseteq P : \alpha \text{ is an order ideal of } P\}$. It is easy to see that $\mathcal{I}(P)$, ordered by inclusion, is a distributive lattice under union and intersection. $\mathcal{I}(P)$ is called the *ideal lattice* of the poset P.

Let P and Q be two posets. The *ordinal sum* $P \oplus Q$ of the disjoint posets P and Q is the poset on the set $P \cup Q$ with the following order: if $x, y \in P \oplus Q$, then $x \leq y$ if either $x, y \in P$ and $x \leq y$ in P or $x, y \in Q$ and $x \leq y$ in Q or $x \in P$ and $y \in Q$. Let P, Q be posets on disjoint sets. The *disjoint union* of posets P and Q is the poset $P + Q$ on the set $P \cup Q$ with the following order: if $x, y \in P + Q$, then $x \leq y$ if either $x, y \in P$ and $x \leq y$ in P or $x, y \in Q$ and $x \leq y$ in Q. A poset which can be written as direct sum of two posets is called *disconnected*. Otherwise, P is *connected*.

Let L be a distributive lattice. An element $x \in L$ is called *join-irreducible* if x is not the minimal element of L and whenever $x = y \vee z$ for some $y, z \in L$, we have either $x = y$ or $x = z$. Let P be the subposet of join-irreducible elements of L. By Birkhoff's fundamental structure theorem [22, Theorem 3.4.1], L is isomorphic to the ideal lattice $\mathcal{I}(P)$. Write $P = \{p_1, \ldots, p_n\}$ and let $R = K[y_1, \ldots, y_n, z_1, \ldots, z_n]$ be a polynomial ring in $2n$ variables over a field K. The *Hibi ring* associated with L, denoted by $R[L]$, is the subring of R generated by the monomials $u_\alpha = (\prod_{p_i \in \alpha} y_i)(\prod_{p_i \notin \alpha} z_i)$ where $\alpha \in L$.

Let $K[L] = K[\{x_\alpha : \alpha \in L\}]$ be the polynomial ring over K and $\pi : K[L] \to R[L]$ be the K-algebra homomorphism with $x_\alpha \mapsto u_\alpha$. Let $I_L = (x_\alpha x_\beta - x_{\alpha\wedge\beta}x_{\alpha\vee\beta} : \alpha, \beta \in L$ and α, β incomparable) be an $K[L]$-ideal.

Let $<$ be a total order on the variables of $K[L]$ with the property that one has $x_\alpha < x_\beta$ if $\alpha < \beta$ in L. Consider the graded reverse lexicographic order $<$ on $K[L]$ induced by this order of the variables.

Theorem 2.1 [12, Theorem 6.19] *The generators of I_L described above forms a Gröbner basis of* $\ker(\pi)$ *with respect to $<$. In particular,* $\ker(\pi) = I_L$.

The ideal I_L is called the *Hibi ideal* of L. By Theorem 2.1, it follows that

$$\mathrm{in}_<(I_L) = (x_\alpha x_\beta : \alpha, \beta \in L \text{ and } \alpha, \beta \text{ incomparable}).$$

It is easy to see that if $\mathcal{P} = \{p_1, \ldots, p_n\}$ is a chain with $p_1 \lessdot \cdots \lessdot p_n$, then $\mathcal{I}(\mathcal{P}) = \{\emptyset, \{p_1\}, \{p_1, p_2\}, \ldots, \{p_1, \ldots, p_n\}\}$. Thus, the Hibi ring $R[\mathcal{I}(\mathcal{P})] = K[z_1 \cdots z_n, y_1 z_2 \cdots z_n, \ldots, y_1 \cdots y_n]$, which is a polynomial ring in $n + 1$ variables. Let P_1 and P_2 be two posets and P be their disjoint union. It was observed in [13] that $R[\mathcal{I}(P)] \cong R[\mathcal{I}(P_1)] * R[\mathcal{I}(P_2)]$, where $*$ denotes the Segre product.

2.1 Green–Lazarsfeld property

Let $S = K[x_1, \ldots, x_n]$ be a standard graded polynomial ring in n variables over a field K and M be a graded S-module. Let $\mathbb{F}$ be the graded minimal free resolution of M over S:

$$\mathbb{F} : 0 \to \bigoplus_j S(-j)^{\beta_{rj}} \to \cdots \to \bigoplus_j S(-j)^{\beta_{1j}} \to \bigoplus_j S(-j)^{\beta_{0j}}.$$

The numbers β_{ij} are called the minimal graded Betti numbers of the module M.

Let I be a graded S-ideal. We say that S/I satisfies *Green–Lazarsfeld property N_p* if S/I is normal and $\beta_{ij}(S/I) = 0$ for $i \neq j + 1$ and $1 \leq i \leq p$. Therefore, S/I satisfies property N_0 if and only if it is normal; it satisfies property N_1 if and only if it is normal and I is generated by quadratics; it satisfies property N_2 if and only if it satisfies property N_1 and I is linearly presented and so on. We know that the Hibi rings are normal and the Hibi ideals are generated by quadratics. Hence, the Hibi rings satisfy property N_1.

2.2 Squarefree divisor complexes

Let $H \subset \mathbb{N}^n$ be an affine semigroup and $K[H]$ be the semigroup ring attached to it. Suppose that $h_1, \ldots, h_m \in \mathbb{N}^n$ is the unique minimal set of generators of H. We consider the polynomial ring $T = K[t_1, \ldots, t_n]$ in n variables. Then $K[H]$ is the subring of T generated by the monomials $u_i = \prod_{j=1}^n t_j^{h_i(j)}$ for $1 \leq i \leq m$, where $h_i(j)$ denotes the jth component of the integer vector h_i. Consider a K-algebra map $S = K[x_1, \ldots, x_m] \to K[H]$ with $x_i \mapsto u_i$ for $i = 1, \ldots, m$. Let I_H be the kernel of this K-algebra map. Set $\deg x_i = h_i$ to assign a $\mathbb{Z}^n$-graded ring structure to S. Let $\mathfrak{m}$ be the graded maximal S-ideal. Then $K[H]$ as well as I_H become $\mathbb{Z}^n$-graded S-modules. Thus, $K[H]$ admits a minimal $\mathbb{Z}^n$-graded S-resolution $\mathbb{F}$.

Given $h \in H$, we define the *squarefree divisor complex* Δ_h as follows:

$$\Delta_h := \left\{ F \subseteq [m] : \prod_{i \in F} u_i \text{ divides } t_1^{h(1)} \cdots t_n^{h(n)} \text{ in } K[H] \right\}.$$

We denote the ith reduced simplicial homology of a simplicial complex Δ with coefficients in K by $\widetilde{H}_i(\Delta, K)$.

Proposition 2.2 [3, Proposition 1.1], [23, Theorem 12.12] *With the notation and assumptions introduced one has* $\mathrm{Tor}_i(K[H], K)_h \cong \widetilde{H}_{i-1}(\Delta_h, K)$. *In particular,*

$$\beta_{ih}(K[H]) = \dim_K \widetilde{H}_{i-1}(\Delta_h, K).$$

Definition 2.3 Let $H \subset \mathbb{N}^n$ be an affine semigroup generated by $h_1, \ldots, h_m$. An affine subsemigroup $H' \subset H$ generated by a subset of $\{h_1, \ldots, h_m\}$ will be called a *homologically pure* subsemigroup of H if for all $h \in H'$ and all h_i with $h - h_i \in H$, it follows that $h_i \in H'$.

Let H' be a subsemigroup of H generated by a subset $\mathcal{X}$ of $\{h_1, \ldots, h_m\}$, and let $S' = K[\{x_i : h_i \in \mathcal{X}\}] \subseteq S$. Furthermore, let $\mathbb{F}'$ be the $\mathbb{Z}^n$-graded free S'-resolution of $K[H']$. Then, since S is a flat S'-module, $\mathbb{F}' \otimes_{S'} S$ is a $\mathbb{Z}^n$-graded free S-resolution of $S/I_{H'}S$. The inclusion $S'/I_{H'}S \otimes_{S'} S \to S/I_H S$ induces a $\mathbb{Z}^n$-graded S-module complex homomorphism $\mathbb{F}' \otimes_{S'} S \to \mathbb{F}$. Applying $\_ \otimes_S K$ on this complex homomorphism with $K = S/\mathfrak{m}$, we obtain the following sequence of isomorphisms and natural maps of $\mathbb{Z}^n$-graded K-modules

$$\begin{aligned}\operatorname{Tor}_i^{S'}(K[H'], K) \cong H_i(\mathbb{F}' \otimes_{S'} K) \cong H_i(\mathbb{F}' \otimes_{S'} S) \otimes_S K) \to \\ H_i(\mathbb{F} \otimes_S K) \cong \operatorname{Tor}_i^S(K[H], K).\end{aligned}$$

We need the following proposition several times in this paper.

Proposition 2.4 [5, Corollary 2.4] *Let H' be a homologically pure subsemigroup of H. If $\mathbb{F}'$ is the minimal $\mathbb{Z}^n$-graded free S'-resolution of $K[H']$ and $\mathbb{F}$ is the minimal $\mathbb{Z}^n$-graded free S-resolution of $K[H]$, then the complex homomorphism $\mathbb{F}' \otimes S \to \mathbb{F}$ induces an injective map $\mathbb{F}' \otimes K \to \mathbb{F} \otimes K$. Hence,*

$$\operatorname{Tor}_i^{S'}(K[H'], K) \to \operatorname{Tor}_i^S(K[H], K)$$

is injective for all i. In particular, any minimal set of generators of $\operatorname{Syz}_i(K[H'])$ *is part of a minimal set of generators of* $\operatorname{Syz}_i(K[H])$. *Moreover, $\beta_{ij}(K[H']) \leq \beta_{ij}(K[H])$ for all i and j.*

Let us now define the semigroup ring structure on Hibi rings. Let $L = \mathcal{I}(P)$ be a distributive lattice with $P = \{p_1, \ldots, p_n\}$. For $\alpha \in L$, define a $2n$-tuple h_α such that for $1 \leq i \leq n$,

$$\begin{cases} 1 & \text{at } ith \text{ position if } \quad p_i \in \alpha, \\ 0 & \text{at } ith \text{ position if } \quad p_i \notin \alpha, \\ 0 & \text{at } (n+i)th \text{ position if } \quad p_i \in \alpha, \\ 1 & \text{at } (n+i)th \text{ position if } \quad p_i \notin \alpha. \end{cases}$$

Let H be the affine semigroup generated by $\{h_\alpha : \alpha \in L\}$. Then, we have $K[H] = R[L]$. Let $\beta, \gamma \in L$ such that $\beta \leq \gamma$. Define $L_1 = \{\alpha \in L : \beta \leq \alpha \leq \gamma\}$. Clearly, L_1 is a sublattice of L. Let H_1 be the affine subsemigroup of H generated by $\{h_\alpha : \alpha \in L_1\}$.

Proposition 2.5 *Let H and H_1 be as defined above. Then H_1 is a homologically pure subsemigroup of H.*

Proof We show that if $\alpha \notin L_1$ then $h - h_\alpha \notin H$ for all $h \in H_1$. Suppose $\alpha \notin L_1$ then either $\alpha \nleq \gamma$ or $\alpha \ngeq \beta$.

If $\alpha \nleq \gamma$, then there exists a $p_i \in \alpha$ such that $p_i \notin \gamma$. So i^{th} entry of h_α is 1 but for any $\alpha' \in L_1$, i^{th} entry of $h_{\alpha'}$ is 0. Hence, $h - h_\alpha \notin H$ for all $h \in H_1$.

If $\alpha \ngeq \beta$, then there exists a $p_j \in \beta$ such that $p_j \notin \alpha$. So $(n+j)^{th}$ entry of h_α is 1 but for any $\alpha' \in L_1$, $(n+j)^{th}$ entry of $h_{\alpha'}$ is 0. Hence, $h - h_\alpha \notin H$ for all $h \in H_1$. □

3 Property N_p for Segre products of Hibi rings

In this section, we discuss the property N_p of Segre products of Hibi rings for $p \in \{2, 3\}$. Let P_1 and P_2 be two posets and P be their disjoint union. From Sect. 2, we know that the Segre product of Hibi rings $R[\mathcal{I}(P_1)]$ and $R[\mathcal{I}(P_2)]$ is isomorphic to the Hibi ring $R[\mathcal{I}(P)]$. We also know that a poset $\mathcal{P}$ is a chain if and only if $R[\mathcal{I}(\mathcal{P})]$ is a polynomial ring. From Sect. 2.2, recall the definition of the semigroup associated to a Hibi ring. For $i \in \{1, 2\}$, let H_i be the affine semigroup generated by $\{h_\alpha : \alpha \in \mathcal{I}(P_i)\}$ and let H be the affine semigroup associated to the Hibi ring $R[\mathcal{I}(P)]$. Since $\mathcal{I}(P) = \{(\alpha, \beta) : \alpha \in \mathcal{I}(P_1) \text{ and } \beta \in \mathcal{I}(P_2)\}$, it is easy to see that H is generated by $\{(h_\alpha, h_\beta) : \alpha \in \mathcal{I}(P_1) \text{ and } \beta \in \mathcal{I}(P_2)\}$.

3.1 Segre product of a Hibi ring with a polynomail ring in two variables

Theorem 3.1 *Let P_1, P_2, P, H_1, H_2 and H be as above. Then, for each $l \in \{1, 2\}$, H_l is isomorphic to a homologically pure subsemigroup of H. In particular, if $\beta_{ij}(R[\mathcal{I}(P_l)]) \neq 0$ for some $l \in \{1, 2\}$, then $\beta_{ij}(R[\mathcal{I}(P)]) \neq 0$.*

Proof By symmetry, it suffices to prove the theorem for $l = 1$. Consider the subsemigroup G_1 of H generated by $\{(h_\alpha, h_\emptyset) : \alpha \in \mathcal{I}(P_1)\}$, where $\emptyset$ is the minimal element of $\mathcal{I}(P_2)$. It is easy to see that G_1 is isomorphic to the semigroup H_1. Also, observe that $\delta = (\emptyset, \emptyset)$ and $\gamma = (P_1, \emptyset)$ are the order ideals of H. The subsemigroup G_1 is generated by $\{h_\eta : \delta \leq \eta \leq \gamma\}$. So by Proposition 2.5, G_1 is a homologically pure subsemigroup of H. The second part of the theorem follows from Proposition 2.4. Hence the proof. □

3.2 Segre product of a Hibi ring with a polynomail ring

Corollary 3.2 *Let P be a poset such that it is a disjoint union of two posets P_1 and P_2. If $R[\mathcal{I}(P)]$ satisfies property N_p for some $p \geq 2$, then so does $R[\mathcal{I}(P_1)]$ and $R[\mathcal{I}(P_2)]$.*

Proof The proof follows from Theorem 3.1. □

Lemma 3.3 *Let $R[\mathcal{I}(P)]$ be a Hibi ring associated to a poset P. Then the following statements hold:*

(a) *If $\beta_{24}(R[\mathcal{I}(P)]) = 0$, then $R[\mathcal{I}(P)]$ satisfies property N_2.*
(b) *If $R[\mathcal{I}(P)]$ satisfies property N_2 and $\beta_{35}(R[\mathcal{I}(P)]) = 0$, then it satisfies property N_3.*

Proof (a) Hibi rings are algebra with straightening laws (ASL) and straightening relations are quadratic [14, Sect. 2]. ASL with quadratic straightening relations are Koszul [15]. So by [15, Lemma 4], $\beta_{2j}(R[\mathcal{I}(P)]) = 0$ for all $j \geq 5$. This concludes the proof.
(b) The proof follows from [1, Theorem 6.1].

□

In this subsection, we prove Theorem 3.4.

Theorem 3.4 *Let P_1 be a poset, $P_2 = \{b\}$ and $p \in \{2, 3\}$. Let P be the disjoint union of P_1 and P_2. If $R[\mathcal{I}(P_1)]$ satisfies property N_p , then so does $R[\mathcal{I}(P)]$.*

The proof of the above theorem follows the argument of Rubei [19]. Let P_1 and P_2 be as in theorem. So $\mathcal{I}(P) = \{(\alpha, \beta) : \alpha \in \mathcal{I}(P_1),\ \beta \in \mathcal{I}(P_2)\}$. Let H be the affine semigroup generated by $\{(h_\alpha, h_\beta) : \alpha \in \mathcal{I}(P_1),\ \beta \in \mathcal{I}(P_2)\}$. In order to prove the above theorem, by Proposition 2.2 and Lemma 3.3, it is enough to show that for $p \in \{2, 3\}$, if $h = (h_1, h_2) \in H$ with $deg(h) = p + 2$, then $\widetilde{H}_{p-1}(\Delta_h) = 0$. For $i = 1, 2$, let H_i be the affine semigroup generated by $\{h_\alpha : \alpha \in \mathcal{I}(P_i)\}$. Observe that H_2 is generated by two elements $h_\emptyset$ and $h_{\{b\}}$. For simplicity, we denote $h_{\{b\}}$ by h_b.

Let Δ be a simplicial complex on a vertex set V. The *support* of a simplex σ in Δ is the set of all vertices $v \in V$ such that $v \in \sigma$. Let $\alpha = \sum_i a_i \sigma_i$ where $c_i \in \mathbb{Z}$, be a chain in Δ. The *support* of α, denoted by $sp(\alpha)$, is the union of the support of the simplexes σ_i. We denote the *i-skeleton* of Δ by $sk^i(\Delta)$.

Notation 3.5 *Let $g \in H_1$ with $deg(g) = d$.*

(a) *Denote $g_\epsilon = (g, g')$, where $g' = (d - \epsilon)h_\emptyset + \epsilon h_b \in H_2$ and $\epsilon \in \{0, \dots, d\}$.*
(b) *For $0 \leq l \leq d - 1$, let*

$$F^l(\Delta_g) = \bigcup_{\substack{g_1,\dots,g_d \\ s.t.\ g_1+\cdots+g_d=g}} \ \bigcup_{i_0,\dots,i_l \in \{1,\dots,d\}} \langle (g_{i_0}, h_\emptyset), \dots, (g_{i_l}, h_\emptyset) \rangle.$$

Lemma 3.6 *Under the notations of Notation 3.5.*

(a) *For all $i \leq l - 1$, $\widetilde{H}_i(F^l(\Delta_g)) \cong \widetilde{H}_i(\Delta_g)$.*
(b) *For $\epsilon \in \{1, 2\}$, $F^l(\Delta_g) \subseteq \Delta_{g_\epsilon}$ if and only if $l \leq d - \epsilon - 1$.*

Proof (a) The proof follows from $F^l(\Delta_g) \cong sk^l(\Delta_g)$.
(b) Let $g_1, \dots, g_d \in H_1$ be such that $\sum_{i=1}^d g_i = g$. Observe that for any $\{i_0, ..., i_l\} \subseteq [d]$, $\{(g_{i_0}, h_\emptyset), \dots, (g_{i_l}, h_\emptyset)\}$ is a simplex in Δ_{g_ϵ} if and only if $l \leq d - \epsilon - 1$.

□

Let $g \in H_1$ with $deg(g) = d$ and let $\epsilon \in \{0, \ldots, d\}$. Note that $\Delta_{g_\epsilon} \cong \Delta_g$ for all $\epsilon \in \{0, d\}$. Also, we have $\Delta_{g_\epsilon} \cong \Delta_{g_{d-\epsilon}}$. Thus, to prove the theorem, it suffices to consider the cases $h_2 = (p + 2 - \epsilon)h_\emptyset + \epsilon h_b$, where $\epsilon \in \{1, 2\}$ and $p \in \{2, 3\}$.

Remark 3.7 Let $g \in H_1$ with $deg(g) = d$ and $\epsilon \in \{0, \ldots, d\}$. Let $g_1, ..., g_d \in H_1$ be such that $g = \sum_{i=1}^{d} g_i$. Observe that $\sigma = \{(g_{i_1}, h_\emptyset), \ldots, (g_{i_{d-\epsilon+1}}, h_\emptyset)\} \notin \Delta_{g_\epsilon}$ for any $i_1, \ldots, i_{d-\epsilon+1} \in \{1, \ldots, d\}$. For $l \in \{1, \ldots, d\}$ with $l \neq i_j$, $j \in \{1, \ldots, d - \epsilon + 1\}$, let

$$\sigma' = \sum_{j=1}^{d-\epsilon+1} (-1)^{j-1} \Big\{(g_{i_l}, h_b), (g_{i_1}, h_\emptyset), \ldots, \widehat{(g_{i_j}, h_\emptyset)}, \ldots, (g_{i_{d-\epsilon+1}}, h_\emptyset)\Big\}$$

be a $(d - \epsilon)$-chain in Δ_{g_ϵ}. Then $\partial\sigma = \partial\sigma'$.

Definition 3.8 For any $g \in H_1$ with $deg(g) = d$ and $\epsilon \in \{1, \ldots, d\}$, we define $R_{g,\epsilon}$ to be the following simplicial complex:

$$\bigcup_{\substack{g_1,\ldots,g_d \in H_1 \\ \text{s.t. } g_1+\ldots+g_d=g}} \; \bigcup_{\substack{i_1,\ldots,i_{d-1} \in \{1,\ldots,d\} \\ i_l \neq i_m}} \big\langle (g_{i_1}, h_b), \ldots, (g_{i_{\epsilon-1}}, h_b), (g_{i_\epsilon}, h_\emptyset), \ldots, (g_{i_{d-1}}, h_\emptyset)\big\rangle.$$

Lemma 3.9 *Let $g \in H_1$ with $deg(g) = d$ and $\epsilon \in \{1, 2\}$. Assume that*

(a) $(i, d) \in \{(0, 3), (1, 4)\}$ *and*
(b) $\widetilde{H}_i(\Delta_{g_{\epsilon-1}}) = 0$.

Then $\widetilde{H}_i(R_{g,\epsilon}) = 0$.

Proof Observe that $R_{g,\epsilon} \subseteq \Delta_{g_{\epsilon-1}}$. If $\epsilon = 1$, then $sk^2(\Delta_{g_{\epsilon-1}}) \subseteq sk^2(R_{g,\epsilon})$. Thus, $\widetilde{H}_i(\Delta_{g_{\epsilon-1}}) = \widetilde{H}_i(R_{g,\epsilon})$ for $i = 0, 1$. So we only have to consider the case $\epsilon = 2$. Let β be an i-cycle in $R_{g,\epsilon}$. Since $\widetilde{H}_i(\Delta_{g_{\epsilon-1}}) = 0$, there exists an $(i + 1)$-chain η in $\Delta_{g_{\epsilon-1}}$ such that $\partial\eta = \beta$. Suppose $\eta = \sum_j c_j\sigma_j$, where σ_j is an $(i + 1)$-simplex in $\Delta_{g_{\epsilon-1}}$. Now consider an $(i + 1)$-chain ψ in $R_{g,\epsilon}$ such that $\psi = \sum_j c_j\sigma'_j$, where $\sigma'_j = \sigma_j$ if $\sigma_j \in R_{g,\epsilon}$ else σ'_j is as defined in Remark 3.7 corresponding to σ_j. Then $\partial\psi = \beta$. □

Lemma 3.10 *Let $g \in H_1$ with $deg(g) = 4$ and $\epsilon \in \{1, 2\}$. Every 1-cycle γ in Δ_{g_ϵ} is homologous to an 1-cycle in $F^1(\Delta_g)$ $(\subseteq \Delta_{g_\epsilon})$.*

Proof We prove the lemma by induction on the cardinality of $(sp(\gamma) \cap sk^0(\Delta_{g_\epsilon})) \setminus F^1(\Delta_g)$.

Let $(f, h_b) \in sp(\gamma)$. Let $\mathcal{S}_{(f,h_b)}$ be the set of 1-simplexes of γ with vertex (f, h_b). For $\sigma = \{v, (f, h_b)\} \in \mathcal{S}_{(f,h_b)}$, let $\sigma' = \{v, (f, h_\emptyset)\}$. Clearly, σ' is an 1-simplex of Δ_{g_ϵ}. Let $\alpha = \sum_{\sigma \in \mathcal{S}_{(f,h_b)}} (-\sigma + \sigma')$ be the 1-cycle in Δ_{g_ϵ}.

Now, we show that for a vertex v in Δ_{g_ϵ}, if $\langle v, (f, h_b)\rangle \subseteq \Delta_{g_\epsilon}$, then $v \in R_{g-f,\epsilon}$. Observe that

$$R_{g-f,1} = \bigcup_{\substack{g_1,g_2,g_3 \in H_1 \\ \text{s.t. } g_1+g_2+g_3=g-f}} \; \bigcup_{\substack{i_1,i_2 \in \{1,2,3\} \\ i_1 \neq i_2}} \big\langle (g_{i_1}, h_\emptyset), (g_{i_2}, h_\emptyset)\big\rangle$$

and

$$R_{g-f,2} = \bigcup_{\substack{g_1,g_2,g_3 \in H_1 \\ \text{s.t. } g_1+g_2+g_3=g-f}} \; \bigcup_{\substack{i_1,i_2 \in \{1,2,3\} \\ i_1 \neq i_2}} \big\langle (g_{i_1}, h_b), (g_{i_2}, h_\emptyset)\big\rangle.$$

If $\epsilon = 1$, then $v = (f', h_\emptyset)$ for some $f' \in H_1$ such that $g - (f + f') \in H_1$. If $\epsilon = 2$, then either $v = (f', h_\emptyset)$ or $v = (f', h_b)$ for some $f' \in H_1$ such that $g - (f + f') \in H_1$. In both cases, $v \in R_{g-f,\epsilon}$.

So we obtain that $sp(\alpha) \subseteq C$, where C is the union of the cones $\langle (f, h_b), R_{g-f,\epsilon}\rangle$ and $\langle (f, h_\emptyset), R_{g-f,\epsilon}\rangle$. Notice that $C \subseteq \Delta_{g_\epsilon}$. Since Hibi rings satisfy property N_1, we have $\widetilde{H}_0(\Delta_{(g-f)_{\epsilon-1}}) = 0$. Thus, $\widetilde{H}_0(R_{g-f,\epsilon}) = 0$ by Lemma 3.9. Since $\widetilde{H}_i(C) = \widetilde{H}_{i-1}(R_{g-f,\epsilon})$, we have $\widetilde{H}_1(C) = 0$. Thus, α is homologous to 0 which implies that σ is homologous to $\sigma + \alpha$.

Observe that $\#((sp(\gamma + \alpha) \cap sk^0(\Delta_{g_\epsilon})) \setminus F^1(\Delta_g)) < \#((sp(\gamma) \cap sk^0(\Delta_{g_\epsilon})) \setminus F^1(\Delta_g))$. Hence, we conclude the proof by induction. □

Proof of Theorem 3.4 *for* N_2 We have to show that if $h = (h_1, h_2) \in H$ with $deg(h) = 4$, then $\widetilde{H}_1(\Delta_h) = 0$. We consider the following cases:

(1) Assume that $h_2 = 3h_\emptyset + h_b$. Let γ be an 1-cycle in Δ_h. By Lemma 3.10, γ is homologous to an 1-cycle γ' of $F^1(\Delta_{h_1}) \subset \Delta_h$. In other words, there exists a 2-chain μ in Δ_h such that $\partial\mu = \gamma - \gamma'$. Also, $\widetilde{H}_1(F^2(\Delta_{h_1})) \cong \widetilde{H}_1(\Delta_{h_1}) = 0$, where the isomorphism is due to Lemma 3.6(a) and the equality is by hypothesis. As $F^1(\Delta_{h_1}) \subset F^2(\Delta_{h_1})$, there exists a 2-chain μ' in $F^2(\Delta_{h_1})$ such that $\partial\mu' = \gamma'$. Since $F^2(\Delta_{h_1}) \subseteq \Delta_h$, μ' is a 2-chain in Δ_h. Therefore, $[\gamma'] = 0$ in $\widetilde{H}_1(\Delta_h)$. So $[\gamma] = 0$ in $\widetilde{H}_1(\Delta_h)$.

(2) Consider $h_2 = 2h_\emptyset + 2h_b$. Every 1-cycle γ in Δ_h is homologous to an 1-cycle γ' in $F^1(\Delta_{h_1})$ by Lemma 3.10. But in this case, $F^2(\Delta_{h_1})$ is not contained in Δ_h. Since $\widetilde{H}_1(F^2(\Delta_{h_1})) = 0$, there exists a 2-chain μ in $F^2(\Delta_{h_1})$ such that $\partial\mu = \gamma'$. Let $\mu = \sum_i c_i\sigma_i$, where σ_i is a 2-simplex in $F^2(\Delta_{h_1})$. Consider a 2-chain ψ in Δ_h such that $\psi = \sum_i c_i\sigma_i'$, where $\sigma_i' = \sigma_i$ if $\sigma_i \in \Delta_h$ else σ_i' is as defined in Remark 3.7 corresponding to σ_i. Then $\partial\psi = \gamma'$. Therefore, $[\gamma'] = 0$ in $\widetilde{H}_1(\Delta_h)$. So $[\gamma] = 0$ in $\widetilde{H}_1(\Delta_h)$. Hence the proof.

□

Lemma 3.11 *Let $g \in H_1$ with $deg(g) = 5$ and $\epsilon \in \{1, 2\}$. Every 2-cycle γ in Δ_{g_ϵ} is homologous to a 2-cycle in $F^2(\Delta_g)$ $(\subset \Delta_{g_\epsilon})$.*

Proof We prove the result by induction on the cardinality of $(sp(\gamma) \cap sk^0(\Delta_{g_\epsilon})) \setminus F^2(\Delta_g)$.

Let $(f, h_b) \in sp(\gamma)$. Let $\mathcal{S}_{(f,h_b)}$ be the set of 2-simplexes of γ with vertex (f, h_b). For $\sigma = \{v, u, (f, h_b)\} \in \mathcal{S}_{(f,h_b)}$, let $\sigma' = \{v, u, (f, h_\emptyset)\}$. Clearly, σ' is a 2-simplex of Δ_{g_ϵ}. Let $\alpha = \sum_{\sigma\in\mathcal{S}_{(f,h_b)}}(-\sigma + \sigma')$ be the 2-cycle in Δ_{g_ϵ}.

Now, we show that for vertexes v, u in Δ_{g_ϵ}, if $\langle v, u, (f, h_b)\rangle \subseteq \Delta_{g_\epsilon}$, then $\langle v, u\rangle \subseteq R_{g-f,\epsilon}$. Observe that

$$R_{g-f,1} = \bigcup_{\substack{g_1,\ldots,g_4\in H_1\\ \text{s.t. } g_1+\cdots+g_4=g-f}} \ \bigcup_{\substack{i_1,i_2,i_3\in\{1,\ldots,4\}\\ i_l\neq i_k}} \big\langle (g_{i_1}, h_\emptyset), (g_{i_2}, h_\emptyset)(g_{i_3}, h_\emptyset)\big\rangle$$

and

$$R_{g-f,2} = \bigcup_{\substack{g_1,\ldots,g_4\in H_1\\ \text{s.t. } g_1+\ldots+g_4=g-f}} \ \bigcup_{\substack{i_1,i_2,i_3\in\{1,\ldots,4\}\\ i_l\neq i_k}} \big\langle (g_{i_1}, h_b), (g_{i_2}, h_\emptyset)(g_{i_3}, h_\emptyset)\big\rangle.$$

If $\epsilon = 1$, then $\langle v, u\rangle = \langle (f_1, h_\emptyset), (f_2, h_\emptyset)\rangle$ for some $f_1, f_2 \in H_1$ such that $g - (f + f_1 + f_2) \in H_1$. If $\epsilon = 2$, then either $\langle v, u\rangle = \langle (f_1, h_\emptyset), (f_2, h_\emptyset)\rangle$ or $\langle v, u\rangle = \langle (f_1, h_b), (f_2, h_\emptyset)\rangle$ for some $f_1, f_2 \in H_1$ such that $g - (f + f_1 + f_2) \in H_1$. In both cases, $\langle v, u\rangle = \langle (f_1, h_\emptyset), (f_2, h_\emptyset)\rangle \subseteq R_{g-f,\epsilon}$.

So we obtain that $sp(\alpha) \subseteq C$, where C is the union of the cones $\langle (f, h_b), R_{g-f,\epsilon}\rangle$ and $\langle (f, h_\emptyset), R_{g-f,\epsilon}\rangle$. Notice that $C \subseteq \Delta_{g_\epsilon}$. Since $R[\mathcal{I}(P)]$ satisfies property N_2, we have $\widetilde{H}_1(\Delta_{(g-f)_{\epsilon-1}}) = 0$. Thus by Lemma 3.9, $\widetilde{H}_1(R_{g-f,\epsilon}) = 0$. Since $\widetilde{H}_i(C) = \widetilde{H}_{i-1}(R_{g-f,\epsilon})$, we have $\widetilde{H}_2(C) = 0$. Thus, α is homologous to 0. So σ is homologous to $\sigma + \alpha$.

Observe that $\#((sp(\gamma+\alpha)\cap sk^0(\Delta_{g_\epsilon}))\setminus F^2(\Delta_g)) < \#((sp(\gamma)\cap sk^0(\Delta_{g_\epsilon}))\setminus F^2(\Delta_g))$. Hence, we conclude the proof by induction. □

Proof of Theorem 3.4 *for* N_3 We have to show that if $h = (h_1, h_2) \in H$ with $deg(h) = 5$, then $\widetilde{H}_2(\Delta_h) = 0$. We consider the following cases:

(1) Consider $h_2 = 4h_\emptyset + h_b$. Let γ be a 2-cycle in Δ_h. By Lemma 3.11, γ is homologous to a 2-cycle γ' of $F^3(\Delta_{h_1}) \subset \Delta_h$. In other words, there exists a 3-chain μ in Δ_h such that $\partial\mu = \gamma - \gamma'$. Also, $\widetilde{H}_2(F^3(\Delta_{h_1})) \cong \widetilde{H}_2(\Delta_{h_1}) = 0$, where the isomorphism is due to Lemma 3.6 and the equality holds because $R[\mathcal{I}(P_1)]$ satisfies property N_3. As $F^2(\Delta_{h_1}) \subset F^3(\Delta_{h_1})$, there exists a 3-chain μ' in $F^3(\Delta_{h_1})$ such that $\partial\mu' = \gamma'$. Since $F^3(\Delta_{h_1}) \subseteq \Delta_h$, μ' is a 3-chain in Δ_h. Therefore, $[\gamma'] = 0$ in $\widetilde{H}_1(\Delta_h)$. So $[\gamma] = 0$ in $\widetilde{H}_1(\Delta_h)$.

(2) Assume $h_2 = 3h_\emptyset + 2h_b$. By Lemma 3.11, every 2-cycle γ in Δ_h is homologous to a 2-cycle γ' in $F^2(\Delta_{h_1})$. But in this case, $F^3(\Delta_{h_1}) \nsubseteq \Delta_h$. Since $\widetilde{H}_2(F^3(\Delta_{h_1})) = 0$, there exists a 3-chain μ in $F^3(\Delta_{h_1})$ such that $\partial\mu = \gamma'$. Let $\mu = \sum_i c_i\sigma_i$, where σ_i is a 3-simplex in $F^3(\Delta_{h_1})$. Consider a 3-chain ψ in Δ_h such that $\psi = \sum_i c_i\sigma_i'$, where $\sigma_i' = \sigma_i$ if $\sigma_i \in \Delta_h$ else σ_i' is as defined in Remark 3.7 corresponding to σ_i. Then $\partial\psi = \gamma'$. Therefore, $[\gamma'] = 0$ in $\widetilde{H}_2(\Delta_h)$. So $[\gamma] = 0$ in $\widetilde{H}_2(\Delta_h)$. Hence the proof.

□

In this subsection, we show that if a Hibi ring satisfies property N_2, then its Segre product with a polynomial ring in finitely many variables also satisfies property N_2.

Proposition 3.12 *Let P be a poset such that it is a disjoint union of a poset P_1 and a chain $P_2 = \{a_1, \dots, a_n\}$ with $a_1 \lessdot \dots \lessdot a_n$. Let $\{x\}$ be a poset and P_2' be the ordinal sum $P_2 \oplus \{x\}$. Let Q be the disjoint union of the posets P_1 and P_2'. If $R[\mathcal{I}(P)]$ satisfies property N_2, then so does $R[\mathcal{I}(Q)]$.*

Theorem 3.13 *Let P be a poset such that it is a disjoint union of a poset P_1 and a chain P_2. If $R[\mathcal{I}(P_1)]$ satisfies property N_2, then so does $R[\mathcal{I}(P)]$.*

Proof The proof follows from Theorem 3.4 and Proposition 3.12. □

Now, this subsection is dedicated to the proof of the Proposition 3.12. The proof of Proposition 3.12 is motivated from Rubei [20].

Let P be a poset such that it is a disjoint union of two posets P_1 and P_2. Let $\{x\}$ be a poset and P_2' be the ordinal sum $P_2 \oplus \{x\}$. Let Q be the disjoint union of posets P_1 and P_2'. Let H and H' be the affine semigroups generated by $\{h_\alpha : \alpha \in \mathcal{I}(Q)\}$ and $\{h_\beta : \beta \in \mathcal{I}(P)\}$ respectively. For $i \in \{1, 2\}$, let H_i be the affine semigroup generated by $\{h_\alpha : \alpha \in \mathcal{I}(P_i)\}$. For $\alpha \in Q$, the first entry of h_α is 1 if $x \in \alpha$ and the second entry of h_α is 1 if $x \notin \alpha$.

Note : Let $h \in H$ with $deg(h) = d$. In this subsection, we either denote h by $(\epsilon, d-\epsilon, h')$, where $h' \in H'$ or we denote it by $(\epsilon, d-\epsilon, h_2, h_1)$, where $h_i \in H_i$ for all $i = 1, 2$.

Let $h \in H$ with $deg(h) = d$. Then $h = (\epsilon, d-\epsilon, h')$, where $h' \in H'$, $\epsilon \le d$, $\epsilon \in \mathbb{N}$. Let X_h be the following simplicial complex:

$$X_h := \Delta_h \cup \Delta_{(\epsilon-1, d-\epsilon+1, h')} \cup \dots \cup \Delta_{(0,d,h')}.$$

Observe that $\Delta_{(0,d,h')} \cong \Delta_{h'} \cong \Delta_{(d,0,h')}$. Let γ be an 1-cycle in X_h. For every vertex $v \in \gamma$, let $\mathcal{S}_{v,\gamma}$ be the set of simplexes of γ with vertex v and $\mu_{v,\gamma}$ be the 0-cycle such that $v * \mu_{v,\gamma} = \sum_{\tau \in \mathcal{S}_{v,\gamma}} \tau$, where $*$ denotes the joining.

Proposition 3.14 *Let $h \in H$ with $deg(h) = 4$. Let γ be an 1-cycle in Δ_h. Then there exists an 1-cycle γ' in $\Delta_{(0,4,h')}$ such that γ is homologous to γ' in X_h.*

Proof Let $h_{\alpha_1}, \dots, h_{\alpha_m}$ be the vertices of γ with non-zero first entry. In other words, these are all the vertices h_α of γ such that $x \in \alpha$. For $1 \le i \le m$, let $\beta_i = \alpha_i \setminus \{x\}$. Observe that $\mu_{h_{\alpha_1},\gamma}$ is in $\Delta_{h-h_{\alpha_1}}$ and

$$\widetilde{H}_1\left(h_{\alpha_1} * \Delta_{h-h_{\alpha_1}} \cup h_{\beta_1} * \Delta_{h-h_{\alpha_1}}\right) = \widetilde{H}_0\left(\Delta_{h-h_{\alpha_1}}\right) = 0,$$

where the last equality holds because $R[\mathcal{I}(Q)]$ satisfies property N_1. So $h_{\alpha_1} * \mu_{h_{\alpha_1},\gamma} - h_{\beta_1} * \mu_{h_{\alpha_1},\gamma}$ is homologous to 0 in X_h. Hence, $\gamma_1 := \gamma - (h_{\alpha_1} * \mu_{h_{\alpha_1},\gamma} - h_{\beta_1} * \mu_{h_{\alpha_1},\gamma})$ is homologous to γ in X_h. Informally speaking, we have got γ_1 from γ by replacing the vertex h_{α_1} with h_{β_1}. Inductively, define

$$\gamma_i := \gamma_{i-1} - (h_{\alpha_i} * \mu_{h_{\alpha_i},\gamma_{i-1}} - h_{\beta_i} * \mu_{h_{\alpha_i},\gamma_{i-1}})$$

for $2 \le i \le m$. Since all the vertexes of γ_m have first entry zero, we have $\gamma_m \in \Delta_{(0,d,h')}$. We set $\gamma' = \gamma_m$ and prove that γ_m is homologous to γ in X_h. To prove this, it suffices to show that $h_{\alpha_i} * \mu_{h_{\alpha_i},\gamma_{i-1}} - h_{\beta_i} * \mu_{h_{\alpha_i},\gamma_{i-1}}$ is homologous to 0 for $2 \le i \le m$.

Let θ_0 be the sum of simplexes τ of $\mu_{h_{\alpha_i},\gamma_{i-1}}$ such that τ is a vertex of $\Delta_{h-h_{\alpha_i}}$ and let θ_1 be the sum of simplexes τ of $\mu_{h_{\alpha_i},\gamma_{i-1}}$ such that τ is not a vertex of $\Delta_{h-h_{\alpha_i}}$. Observe that $\mu_{h_{\alpha_i},\gamma_{i-1}} = \theta_0 + \theta_1$ and θ_1 is a 0-cell in $\Delta_{(\epsilon-1,5-\epsilon,h')-h_{\alpha_i}}$. Since $\mu_{h_{\alpha_i},\gamma}$ is a 0-cycle, θ_0 is a 0-cycle of $\Delta_{h-h_{\alpha_i}}$ and θ_1 is a 0-cycle of $\Delta_{(\epsilon-1,5-\epsilon,h')-h_{\alpha_i}}$. Furthermore, since $R[\mathcal{I}(Q)]$ satisfies property N_1, θ_0 is homologous to 0 in $\Delta_{h-h_{\alpha_i}}$ and θ_1 is homologous to 0 in $\Delta_{(\epsilon-1,5-\epsilon,h')-h_{\alpha_i}}$. Thus, they are homologous to 0 in X_h. Therefore, $h_{\alpha_i} * \mu_{h_{\alpha_i},\gamma_{i-1}} - h_{\beta_i} * \mu_{h_{\alpha_i},\gamma_{i-1}}$ is homologous to 0 in X_h. This concludes the proof. □

Lemma 3.15 *Let $\mathcal{P} = \{q_1, \dots, q_r\}$ be a chain such that $q_1 \lessdot \dots \lessdot q_r$ and let $\mathcal{H}$ be the semigroup corresponding to $R[\mathcal{I}(\mathcal{P})]$. Let $h = \sum_{i=1}^{d} h_{\alpha_i} \in \mathcal{H}$. Then*

$$\Delta_h = \langle h_{\alpha_1}, \dots, h_{\alpha_d} \rangle.$$

Proof It is enough to show that for some $\alpha \in \mathcal{I}(\mathcal{P})$, if $h_\alpha \notin \{h_{\alpha_1}, \ldots, h_{\alpha_d}\}$, then h_α is not a vertex of Δ_h. If $\alpha = \emptyset$, then the entry corresponding to "$q_1 \notin \alpha$" in $h - h_\alpha$ will be -1. So h_α is not a vertex of Δ_h. If $\alpha_i \leq \alpha$ for all $i \in \{1, \ldots, d\}$, then $h - h_\alpha \notin \mathcal{H}$. Hence, h_α is not a vertex of Δ_h. Now suppose for all $i \in \{1, \ldots, d\}$, $\alpha_i \nleq \alpha$. Let $\{h_{\alpha_{i_1}}, \ldots, h_{\alpha_{i_m}}\}$ be the subset of $\{h_{\alpha_1}, \ldots, h_{\alpha_d}\}$ such that $h_\alpha < h_{\alpha_{i_j}}$ for all $j = 1, \ldots, m$. Let $\alpha = \{q_1, \ldots, q_s\}$, where $1 \leq s \leq r-1$. Observe that the entries corresponding to q_s and q_{s+1} in h are m. But the entries corresponding q_s and q_{s+1} in $h - h_\alpha$ are $m-1$ and m respectively. Hence, $h - h_\alpha \notin \mathcal{H}$. This completes the proof. □

From now onwards, let P be a poset such that it is a disjoint union of a poset P_1 and a chain $P_2 = \{a_1, \ldots, a_n\}$ with $a_1 \lessdot \cdots \lessdot a_n$. Let P_2' be the ordinal sum $P_2 \oplus \{x\}$. Furthermore, let Q be the disjoint union of posets P_1 and P_2'. Let $H_{P_2'}$ be the semigroup associated to $R[\mathcal{I}(P_2')]$.

Lemma 3.16 *Let $h = \sum_{i=1}^{d} h_{\alpha_i} \in H$ with $h = (\epsilon, d-\epsilon, h')$, $\epsilon \geq 1$. For $r < d$, assume that there are exactly r number of i's with $\alpha_i = \alpha_i^1 \cup P_2$, where $\alpha_i^1 \in \mathcal{I}(P_1)$. Let $\tau = \{h_{\beta_1}, \ldots, h_{\beta_{m+1}}\}$ be an m-simplex of X_h. Then $\tau \in \Delta_h$ if and only if there are atmost r number of β_i's with $\beta_i = \beta_i^1 \cup P_2$, where $\beta_i^1 \in \mathcal{I}(P_1)$.*

Proof If we write $h = (\epsilon, d-\epsilon, h_2, h_1)$, where $h_i \in H_i$ for all $i = 1, 2$, then $(\epsilon - 1, d - \epsilon + 1, h') = (\epsilon - 1, d - \epsilon + 1, h_2, h_1)$. For $1 \leq i \leq d$, let $\alpha_i = (\alpha_i^2, \alpha_i^1)$, where $\alpha_i^2 \in \mathcal{I}(P_2')$ and $\alpha_i^1 \in \mathcal{I}(P_1)$, So we can write $h_{\alpha_i} = (1, 0, h_{P_2}, h_{\alpha_i^1})$ if $x \in \alpha_i$ and $h_{\alpha_i} = (0, 1, h_{\alpha_i^2}, h_{\alpha_i^1})$ if $x \notin \alpha_i^2$, where $h_{\alpha_i^2} \in H_2$ and $h_{\alpha_i^1} \in H_1$. We have

$$h_1 = \sum_{i=1}^{d} h_{\alpha_i^1}, \quad (\epsilon, d-\epsilon, h_2) = \sum_{i=1}^{d} h_{\alpha_i^2}.$$

Let $\tau_1 = \{h_{\beta_1^1}, \ldots, h_{\beta_{m+1}^1}\}$ and $\tau_2 = \{h_{\beta_1^2}, \ldots, h_{\beta_{m+1}^2}\}$. Note that τ_1, τ_2 could be multisets.

Now we show that $\tau \in \Delta_h$ if and only if $\tau_2 \subseteq \{h_{\alpha_1^2}, \ldots, h_{\alpha_d^2}\}$. Observe that if $\tau \in \Delta_h$, then $(\epsilon, d-\epsilon, h_2) - \sum_{j=1}^{m+1} h_{\beta_j^2} \in H_{P_2'}$. Hence, $\tau_2 \subseteq \{h_{\alpha_1^2}, \ldots, h_{\alpha_d^2}\}$, by Lemma 3.15. On the other hand, if $\tau_2 \subseteq \{h_{\alpha_1^2}, \ldots, h_{\alpha_d^2}\}$, then $(\epsilon, d-\epsilon, h_2) - \sum_{j=1}^{m+1} h_{\beta_j^2} \in H_{P_2'}$. Since $\tau \in X_h$, there exists an $i_0 \in \{0, \ldots, \epsilon\}$ such that $\tau \in \Delta_{(\epsilon - i_0, d - \epsilon + i_0, h_2, h_1)}$. So $(\epsilon - i_0, d - \epsilon + i_0, h_2, h_1) - \sum_{j=1}^{m+1} h_{\beta_j} \in H$. Therefore, $h_1 - \sum_{j=1}^{m+1} h_{\beta_j^1} \in H_1$. We obtain $\tau \in \Delta_h$.

The proof of 'only if' part follows from the above claim. To prove 'if', it suffices to show that $\tau_2 \subseteq \{h_{\alpha_1^2}, \ldots, h_{\alpha_d^2}\}$. For $1 \leq i \leq \epsilon$, we have

$$(\epsilon - i, d - \epsilon + i, h_2) = \sum_{j=1}^{\epsilon - i} h_{P_2'} + \sum_{j=\epsilon-i+1}^{\epsilon} h_{P_2} + \sum_{j=\epsilon+1}^{\epsilon+r} h_{P_2} + \sum_{j=\epsilon-r+1}^{d} h_{\alpha_j^2} \in H_{P_2'}.$$

Let $i_0 \in \{0, \ldots, \epsilon\}$ be such that $\tau \in \Delta_{(\epsilon - i_0, d-\epsilon+i_0, h_2, h_1)}$. Since there are atmost r number of β_i's in τ with $\beta_i = \beta_i^1 \cup P_2$, where $\beta_i^1 \in \mathcal{I}(P_1)$, the multiplicity of h_{P_2} in τ_2 is atmost r. So by Lemma 3.15,

$$\tau_2 \subseteq \{h_{P_2'}, \ldots, h_{P_2'}, h_{P_2}, \ldots, h_{P_2}, h_{\alpha_{\epsilon-r+1}^2}, \ldots, h_{\alpha_d^2}\} \subseteq \{h_{\alpha_1^2}, \ldots, h_{\alpha_d^2}\},$$

where the multidegrees of $h_{P_2'}$ and h_{P_2} in the middle set are $\epsilon - i_0$ and r respectively. Hence, $\tau_2 \subseteq \{h_{\alpha_1^2}, \ldots, h_{\alpha_d^2}\}$. □

Remark 3.17 (1) Let $p \in \{2, 3\}$ and $h = \sum_{i=1}^{p+2} h_{\alpha_i} \in H$. Assume that there is an $\alpha_0^2 \in \mathcal{I}(P_2') \setminus \{\alpha_1^2, \ldots, \alpha_{p+2}^2\}$. Let $\widetilde{h} = \sum_{i=1}^{p+2} h_{\beta_i}$ be an element of H such that

$$\beta_i = \begin{cases} \alpha_i & \text{if } x \notin \alpha_i, \\ \alpha_i^1 \cup \alpha_0^2 & \text{if } x \in \alpha_i. \end{cases}$$

For $\tau = \{h_{\gamma_1}, \ldots, h_{\gamma_m}\}$, define $\tau' := \{h_{\nu_1}, \ldots, h_{\nu_m}\}$, where

$$\nu_j = \begin{cases} \gamma_j & \text{if } x \notin \gamma_j, \\ \gamma_j^1 \cup \alpha_0^2 & \text{if } x \in \gamma_i. \end{cases}$$

Then τ is a simplex of Δ_h if and only if τ' is a simplex of $\Delta_{\widetilde{h}}$. Therefore, $\Delta_h \cong \Delta_{\widetilde{h}}$.

(2) Let $p \in \{2, 3\}$ and $h = \sum_{i=1}^{p+2} h_{\alpha_i} \in H$. Let $\alpha^2, \widetilde{\alpha}^2 \in \{\alpha_1^2, \dots, \alpha_{p+2}^2\}$ with $\alpha^2 \neq \widetilde{\alpha}^2$. Let $\widetilde{h} = \sum_{i=1}^{p+2} h_{\beta_i}$ be an element of H such that

$$\beta_i = \begin{cases} \alpha_i & \text{if} \quad \alpha_i^2 \neq \alpha^2, \widetilde{\alpha}^2, \\ \alpha_i^1 \cup \widetilde{\alpha}^2 & \text{if} \quad \alpha_i = \alpha_i^1 \cup \alpha^2, \\ \alpha_i^1 \cup \alpha^2 & \text{if} \quad \alpha_i = \alpha_i^1 \cup \widetilde{\alpha}^2. \end{cases}$$

For $\tau = \{h_{\gamma_1}, \dots, h_{\gamma_m}\}$, define $\tau' := \{h_{\nu_1}, \dots, h_{\nu_m}\}$, where

$$\nu_j = \begin{cases} \gamma_j & \text{if} \quad \gamma_j^2 \neq \alpha^2, \widetilde{\alpha}^2, \\ \gamma_j^1 \cup \widetilde{\alpha}^2 & \text{if} \quad \gamma_j = \gamma_j^1 \cup \alpha^2, \\ \gamma_j^1 \cup \alpha^2 & \text{if} \quad \gamma_j = \gamma_j^1 \cup \widetilde{\alpha}^2. \end{cases}$$

Observe that τ is a simplex of Δ_h if and only if τ' is a simplex of $\Delta_{\widetilde{h}}$. Therefore, $\Delta_h \cong \Delta_{\widetilde{h}}$.

Proposition 3.18 *Let $h = (1, 3, h_2, h_1) = \sum_{i=1}^{4} h_{\alpha_i} \in H$. Assume that $\mathcal{I}(P_2') \subseteq \{\alpha_1^2, \dots, \alpha_4^2\}$ and there are exactly two i's with $\alpha_i^2 = P_2$. Let γ be an 1-cycle in Δ_h. If γ is homologous to 0 in X_h, then it is also homologous to 0 in Δ_h.*

Proof Let $\eta = \sum c_\sigma \sigma$, where $c_\sigma \in \mathbb{Z}$, be a 2-chain in X_h such that $\partial\eta = \gamma$. We construct an η' in Δ_h such that $\partial\eta' = \gamma$. Let $\{h_{\nu_1}, h_{\nu_2}, h_{\nu_3}\}$ be a simplex in η. By Lemma 3.16, it is not a simplex of Δ_h if and only if $\nu_j^2 = P_2$ for $j = 1, 2, 3$. Let $\sigma = \{h_{\nu_1}, h_{\nu_2}, h_{\nu_3}\}$ be a simplex in η such that it is not a simplex of Δ_h. Note that $(0, 4, h_2, h_1) - \sum_{i=1}^{3} h_{\nu_i} \in H$, call it h_{ν_4}. Observe that $\{h_{\nu_1}, \dots, h_{\nu_4}\}$ is a face of X_h. Define

$$\sigma' := \sum_{j=1}^{3} (-1)^{j-1} \{h_{\nu_4}, h_{\nu_1}, \dots, \widehat{h_{\nu_j}}, \dots, h_{\nu_3}\}.$$

By Lemma 3.16, $\nu_4^2 \neq P_2$. Therefore, σ' is a 2-chain in Δ_h. Observe that $\partial\sigma = \partial\sigma'$. Take $\eta' = \sum_{\sigma\in\eta} c_\sigma \sigma'$, where $\sigma' = \sigma$ if $\sigma \in \Delta_h$ otherwise σ' is as defined above for σ. Then η' is a 2-chain in Δ_h and $\partial\eta' = \gamma$. This completes the proof. □

Remark 3.19 Let Q be a poset such that it is a disjoint union of a poset P_1 and a chain $P_2' = \{a_1, a_2, x\}$, with $a_1 \lessdot a_2 \lessdot x$. Assume that $R[\mathcal{I}(P_1)]$ satisfies property N_2. Let H be the semigroup associated to $R[\mathcal{I}(Q)]$. Let $A = \{\beta_1^1, \dots, \beta_4^1\}$, $B = \{\beta_1^1, \beta_2^1, \delta_1^1, \delta_2^1\}$ where $\beta_j^1, \delta_i^1 \in \mathcal{I}(P_1)$, be two multisets with $\{\delta_1^1, \delta_2^1\} \cap \{\beta_3^1, \beta_4^1\} = \emptyset$, $\beta_1^1 \neq \beta_2^1$ and $\sum_{\beta\in A} h_\beta = \sum_{\beta\in B} h_\beta \in H_1$. Let

$$\mathcal{S} = \left\{\{\nu_1, \dots, \nu_4\} \subseteq \mathcal{I}(Q) : \left\{\nu_1^2, \dots, \nu_4^2\right\} = \mathcal{I}(P_2') \text{ and } \left\{\nu_1^1, \dots, \nu_4^1\right\} \in \{A, B\}\right\}.$$

Let Δ' be the simplicial complex whose facets are $\{h_{\nu_1}, \dots, h_{\nu_4}\}$, where $\{\nu_1, \dots, \nu_4\} \in \mathcal{S}$. The choices of A and B, up to isomorphism, are following:

(a) $\{\beta_1^1, \beta_2^1\} = \{\beta_3^1, \beta_4^1\}$, $\delta_3^1 \neq \delta_4^1$,
(b) $\{\beta_1^1, \beta_2^1\} = \{\beta_3^1, \beta_4^1\}$, $\delta_3^1 = \delta_4^1$,
(c) $\beta_1^1 = \beta_3^1$, $\beta_2^1 \neq \beta_4^1$, $\delta_3^1 \neq \delta_4^1$ and $\beta_2^1 \notin \{\delta_3^1, \delta_4^1\}$,
(d) $\beta_1^1 = \beta_3^1$, $\beta_2^1 \neq \beta_4^1$,δ_3^1 and $\delta_3^1 = \delta_4^1$,
(f) $\beta_1^1 = \beta_3^1$, $\beta_2^1 \neq \beta_4^1$, $\delta_3^1 \neq \delta_4^1$ and $\beta_2^1 = \delta_3^1$,
(g) $\beta_1^1 = \beta_3^1$, $\beta_2^1 \neq \beta_4^1$ and $\beta_2^1 = \delta_3^1 = \delta_4^1$,
(h) $\beta_1^1 = \beta_3^1 = \beta_4^1$, $\delta_3^1 \neq \delta_4^1$ and $\beta_2^1 \notin \{\delta_3^1, \delta_4^1\}$,
(i) $\beta_1^1 = \beta_3^1 = \beta_4^1$, $\beta_2^1 \neq \delta_3^1$ and $\delta_3^1 = \delta_4^1$,
(j) $\beta_1^1 = \beta_3^1 = \beta_4^1$, $\delta_3^1 \neq \delta_4^1$ and $\beta_2^1 = \delta_3^1$,
(k) $\beta_1^1 = \beta_3^1 = \beta_4^1$, $\beta_2^1 = \delta_3^1 = \delta_4^1$,

(l) $\{\beta_1^1, \beta_2^1\} \cap \{\beta_3^1, \beta_4^1\} = \emptyset$, $\beta_3^1 = \beta_4^1$, $\delta_3^1 = \delta_4^1$,
(m) each element of A and B appears with multiplicity 1 and $A \cap B = \{\beta_1^1, \beta_2^1\}$.

One can use a computer to check that $\widetilde{H}_1(\Delta') = 0$.

Proposition 3.20 *Let $h = (1, 3, h_2, h_1) = \sum_{i=1}^4 h_{\alpha_i} \in H$. Assume that $\{\alpha_1^2, \ldots, \alpha_4^2\} = \mathcal{I}(P_2')$ (as a multiset). Let γ be an 1-cycle in Δ_h. If γ is homologous to 0 in X_h, then it is also homologous to 0 in Δ_h.*

Proof Let $\eta = \sum c_\sigma \sigma$, where $c_\sigma \in \mathbb{Z}$ be a 2-chain in X_h such that $\partial\eta = \gamma$. Let $\{h_{\nu_1}, h_{\nu_2}, h_{\nu_3}\}$ be a simplex in η. By Lemma 3.16, it is not a simplex of Δ_h if and only if there exist exactly two $j's$ with $\nu_j^2 = P_2$. We prove the proposition by induction on $k^\eta := \sum |c_\sigma|$, where σ is a simplex of η but it is not a simplex of Δ_h.

Let $\sigma_1 = \{h_{\nu_1}, h_{\nu_2}, h_{\nu_3}\}$ be a simplex in η such that it is not a simplex of Δ_h and $\nu_{j_1}, \nu_{j_2} = P_2$. Let a be the sign of the coefficient of σ_1 in η. Observe that $\{h_{\nu_{j_1}}, h_{\nu_{j_2}}\}$ is a simplex in $\partial\sigma_1$ and it is not in Δ_h. Since $\partial\eta$ is in Δ_h, there is another $\{h_{\nu_{j_1}}, h_{\nu_{j_2}}\}$ in $\partial\eta$ with the opposite sign. So there is a simplex σ_2 of η but not a simplex of Δ_h such that $\sigma_1 \neq \sigma_2$ and $\partial(a\sigma_1 + b\sigma_2)$ is an 1-cycle in Δ_h, where b is the sign of the coefficient of σ_2 in η.

Let σ_1, σ_2 be as above. We will define an η_1 such that $k^{\eta_1} < k^\eta$. Suppose σ_1 and σ_2 are the faces of the same facet, say F. Let σ_3 and σ_4 be other two faces of F. Then $\sigma_3, \sigma_4 \in \Delta_h$, by Lemma 3.16 and there exist $c, d \in \{1, -1\}$ such that $\partial(c\sigma_3 + d\sigma_4) = \partial(a\sigma_1 + b\sigma_2)$. Define

$$\eta_1 := \eta - (a\sigma_1 + b\sigma_2) - (c\sigma_3 + d\sigma_4).$$

Observe that $\partial\eta_1 = \partial\eta$.

On the other hand, suppose σ_1 and σ_2 are not the faces of the same facet. For $i = 1, 2$, let F_i be the facet of X_h such that σ_i is a face of F_i. Write $F_1 = \{h_{\beta_1}, \ldots, h_{\beta_4}\}$ and $F_2 = \{h_{\beta_1}, h_{\beta_2}, h_{\delta_1}, h_{\delta_2}\}$. Let $A = \{\beta_1^1, \ldots, \beta_4^1\}$, $B = \{\beta_1^1, \beta_2^1, \delta_1^1, \delta_2^1\}$. For A and B, let Δ' be as defined in Remark 3.19. Observe that Δ' is a subsimplicial complex of Δ_h and $\partial(a\sigma_1 + b\sigma_2)$ is an 1-cycle in Δ'. Since $\widetilde{H}_1(\Delta') = 0$, there exists a 2-chain μ_{σ_1,σ_2} in Δ' such that $\partial\mu_{\sigma_1,\sigma_2} = \partial(a\sigma_1 + b\sigma_2)$. Define

$$\eta_1 := \eta - (a\sigma_1 + b\sigma_2) - \mu_{\sigma_1,\sigma_2}.$$

Observe that $\partial\eta_1 = \partial\eta$. Also, notice that in both cases, $k^{\eta_1} < k^\eta$. Hence the proof. □

Proof of Proposition 3.12 Let H and H' be the semigroup associated to $R[\mathcal{I}(Q)]$ and $R[\mathcal{I}(P)]$ respectively. To prove the theorem, by Proposition 2.2 and Lemma 3.3, it suffices to show that for all $\epsilon \in \{0, \ldots, 4\}$, if $h = (\epsilon, 4 - \epsilon, h_2, h_1) = \sum_{i=1}^4 h_{\alpha_i} \in H$ then $\widetilde{H}_1(\Delta_h) = 0$. We prove the theorem in the following two cases: $\mathcal{I}(P_2') \nsubseteq \{\alpha_1^2, \ldots, \alpha_4^2\}$ and $\mathcal{I}(P_2') \subseteq \{\alpha_1^2, \ldots, \alpha_4^2\}$. In particular, if $n \geq 3$, then we always have $\mathcal{I}(P_2') \nsubseteq \{\alpha_1^2, \ldots, \alpha_4^2\}$.

(a) Assume that $\mathcal{I}(P_2') \nsubseteq \{\alpha_1^2, \ldots, \alpha_4^2\}$. If $P_2' \in \{\alpha_1^2, \ldots, \alpha_4^2\}$, then by Remark 3.17(1), $\Delta_h \cong \Delta_{\widetilde{h}}$, where $\widetilde{h}$ is as defined in Remark 3.17(1). Observe that $\widetilde{h} = (0, 4, \widetilde{h'})$, where $\widetilde{h'} \in H'$. We know that $\Delta_{\widetilde{h}} \cong \Delta_{\widetilde{h'}}$. By hypothesis, $\widetilde{H}_1(\Delta_{\widetilde{h'}}) = 0$. Therefore, $\widetilde{H}_1(\Delta_{\widetilde{h}}) = 0$. If $P_2' \notin \{\alpha_1^2, \ldots, \alpha_4^2\}$, then $h = (0, 4, h')$, where $h' \in H'$. Thus, $\widetilde{H}_1(\Delta_h) = 0$.
(b) Now we assume that $\mathcal{I}(P_2') \subseteq \{\alpha_1^2, \ldots, \alpha_4^2\}$. We prove this case in two subcases $n = 1$ and $n = 2$. If $n = 1$, then by Remark 3.17(2), it is enough to consider the subcase $h = (1, 3, h_2, h_1)$ and there are exactly two α_i^2's with $\alpha_i^2 = P_2$. Let γ be an 1-cycle in Δ_h. By Proposition 3.14, there exists an 1-cycle γ' of $\Delta_{(0,4,h_2,h_1)}$ such that γ is homologous to γ' in X_h. By hypothesis, γ' is homologous to 0 in Δ_h. Thus, by Proposition 3.18, γ is homologous to 0 in Δ_h. This concludes the proof for $n = 1$. For $n = 2$, if $\mathcal{I}(P_2') \subseteq \{\alpha_1^2, \ldots, \alpha_4^2\}$, then $\mathcal{I}(P_2') = \{\alpha_1^2, \ldots, \alpha_4^2\}$. By the similar argument of the case $n = 1$ and Proposition 3.20, we are done in this subcase also. Hence the proof.

□

4 Minimal Koszul syzygies of Hibi rings

Let $S = K[x_1, \ldots, x_n]$ be a standard graded polynomial ring over a field K. Let $I = (f_1, \ldots, f_m)$ be a graded ideal in S. Let $\{e_1, \ldots, e_m\}$ be a basis of the free S-module S^m. Define a map $\varphi : S^m \to S$ by $\varphi(e_i) = f_i$. Then, $\ker\varphi$ is the second syzygy module of S/I, denoted by $Syz_2(S/I)$. Let f_i and f_j be two distinct generators of I.

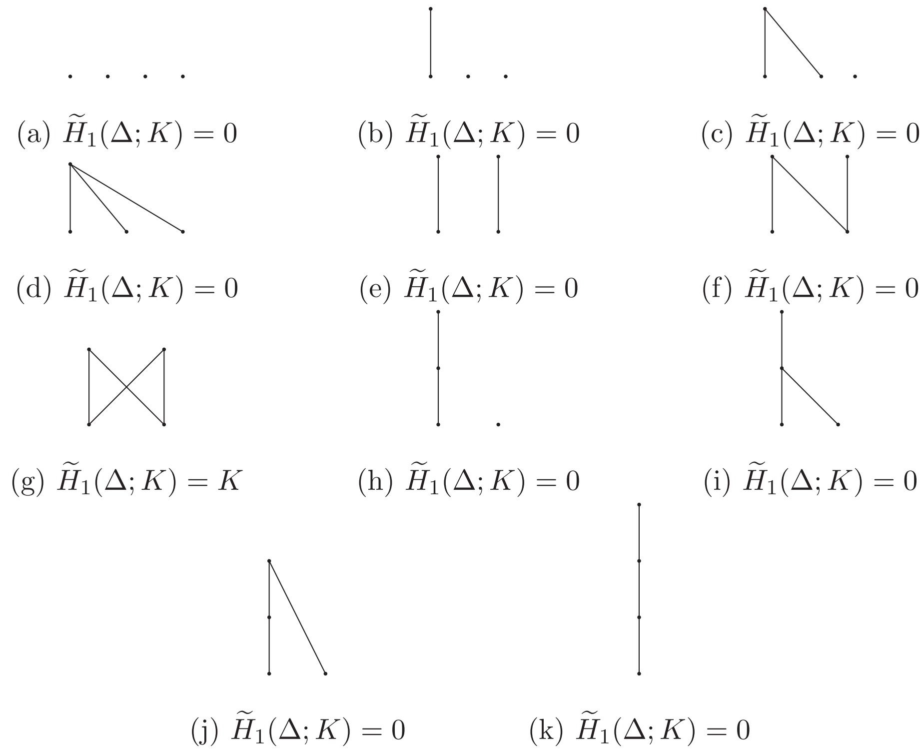

Fig. 1 All possible cardinality 4 subsets of L in Theorem 4.2

Then the Koszul relation $f_i e_j - f_j e_i$ belongs to $Syz_2(S/I)$. We say f_i, f_j a *Koszul relation pair* if $f_i e_j - f_j e_i$ is a minimal generator of $Syz_2(S/I)$.

Let $L = \mathcal{I}(P)$ be a distributive lattice. Let $R[L] = K[L]/I_L$ be the Hibi ring associated to L. Let $<$ be a total order on the variables of $K[L]$ with the property that $x_\alpha < x_\beta$ if $\alpha < \beta$ in L. Consider the reverse lexicographic order $<$ on $K[L]$ induced by this order of the variables. Recall from Sect. 2, we have

$$\mathrm{in}_<(I_L) = (x_\alpha x_\beta : \alpha, \beta \in L \text{ and } \alpha, \beta \text{ incomparable}).$$

Let us define $D_2 := \{(\alpha, \beta) : \alpha, \beta \in L \text{ and } \alpha, \beta \text{ incomparable}\}$.

4.1 Syzygies of initial Hibi ideals

Let K be a field and Δ be a simplicial complex on a vertex set $V = \{v_1, \ldots, v_n\}$. Let $K[\Delta]$ be the Stanley-Reisner ring of the simplicial complex Δ. We know that $K[\Delta] = S/I_\Delta$, where $S = K[x_1, \ldots, x_n]$ and $I_\Delta = \{x_{i_1} \cdots x_{i_r} : \{v_{i_1}, \ldots, v_{i_r}\} \notin \Delta\}$. Since $K[\Delta]$ is a $\mathbb{Z}^n$-graded S-module, it has a minimal $\mathbb{Z}^n$-graded free resolution. Let $W \subset V$; we set $\Delta_W = \{F \in \Delta : F \subset W\}$. It is clear that Δ_W is again a simplicial complex.

Let $L = \mathcal{I}(P)$ be a distributive lattice. We know that L is a poset under the order $\alpha \leq \beta$ if there is a chain from α to β. Let $\Delta(L)$ be the order complex of L. We have $K[\Delta(L)] = K[L]/I_{\Delta(L)}$, where $K[L] = K[\{x_\alpha : \alpha \in L\}]$ and $I_{\Delta(L)} = (x_{\alpha_1} \cdots x_{\alpha_r} : \{\alpha_1, \ldots, \alpha_r\} \notin \Delta(L))$.

Lemma 4.1 $I_{\Delta(L)} = \mathrm{in}_<(I_L)$.

Proof If $\alpha, \beta \in Q$ such that α and β are incomparable, then $\{\alpha, \beta\} \notin \Delta(L)$. Hence, $x_\alpha x_\beta \in I_{\Delta(L)}$.

On the other hand, if $x_{\alpha_1} \cdots x_{\alpha_r} \in I_{\Delta(L)}$, then $\{\alpha_1, \ldots, \alpha_r\}$ is not a chain. So there exist $\alpha, \beta \in \{\alpha_1, \ldots, \alpha_r\}$ such that α and β are incomparable. Hence, $x_{\alpha_1} \cdots x_{\alpha_r} \in (x_\alpha x_\beta) \subseteq \mathrm{in}_<(I_L)$. This concludes the proof. □

Theorem 4.2 *Let* $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in D_2$. *Then* $x_{\alpha_1} x_{\beta_1}, x_{\alpha_2} x_{\beta_2}$ *is a Koszul relation pair of* $K[L]/\mathrm{in}_<(I_L)$ *if and only if either* $\alpha_2 \vee \beta_2 \leq \alpha_1 \wedge \beta_1$ *or* $\alpha_1 \vee \beta_1 \leq \alpha_2 \wedge \beta_2$.

Proof Suppose $x_{\alpha_1}x_{\beta_1}$, $x_{\alpha_2}x_{\beta_2}$ is a Koszul relation pair of $K[\Delta(L)]$. Let $W = \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$. Then, by [2, Theorem 5.5.1], $\widetilde{H}_1(\Delta_W; K) \neq 0$. All possible subsets of L with cardinality 4 are listed in Fig. 1. For $W' \subset L$ with $\#W' = 4$, one can check that $\widetilde{H}_1(\Delta_{W'}; K) \neq 0$ only if W' is as in Fig. 1g. Hence, the forward part follows.

For the converse part, suppose $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in D_2$. Without loss of generality, assume that $\alpha_1 \vee \beta_1 \leq \alpha_2 \wedge \beta_2$. Let $W = \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$. It is easy to see that

$$\widetilde{H}_j(\Delta_W; K) = \begin{cases} K & \text{for} \quad j = 1, \\ 0 & \text{for} \quad j \neq 1. \end{cases}$$

So by [2, Theorem 5.5.1], $x_{\alpha_1}x_{\beta_1}$, $x_{\alpha_2}x_{\beta_2}$ is a Koszul relation pair of $K[\Delta(L)]$. Hence the proof. □

4.2 Syzygies of Hibi ideals

Let $L = \mathcal{I}(P)$ be a distributive lattice with $\#L = n$. Let $R[L] = K[L]/I_L$ be the Hibi ring associated to L. There exists a weight vector $w = (w_1, \ldots, w_n)$ with strictly positive integer coordinates such that $\text{in}_{<_w}(I_L) = \text{in}_<(I_L)$ [17, Theorem 22.3]. Consider the polynomial ring $\widetilde{K[L]} = K[L][t]$ and the integral weight vector $\widetilde{w} = (w_1, ..., w_n, 1)$. Let $f = \sum_i c_i l_i \in K[L]$, where $c_i \in K \setminus \{0\}$ and l_i is a monomial in $K[L]$. Let l be a monomial in f such that $w(l) = max_i\{w(l_i)\}$. Define $\tilde{f} = \sum_i t^{w(l)-w(l_i)} c_i l_i$. If we grade $\widetilde{K[L]}$ by $\deg(t) = 1$ and $\deg(x_i) = w_i$ for all i, then $\tilde{f}$ is homogeneous. Note that the image of $\tilde{f}$ in $\widetilde{K[L]}/(t-1)$ is f, and its image in $\widetilde{K[L]}/(t)$ is $\text{in}_{<_w}(f)$. Define $\widetilde{I}_L := (\tilde{f} \mid f \in I_L)$. We have

$$\widetilde{I}_L = (\widetilde{x_\alpha x_\beta - x_{\alpha\wedge\beta} x_{\alpha\vee\beta}} : \alpha, \beta \in L \text{ and } \alpha, \beta \text{ incomparable}).$$

Observation 4.3 *Let $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in D_2$. From the proof of* [17, Theorem 22.9]*, we have*

(a) *If $\widetilde{x_{\alpha_1}x_{\beta_1} - x_{\alpha_1\wedge\beta_1}x_{\alpha_1\vee\beta_1}}$, $\widetilde{x_{\alpha_2}x_{\beta_2} - x_{\alpha_2\wedge\beta_2}x_{\alpha_2\vee\beta_2}}$ is a Koszul relation pair of $\widetilde{K[L]}/\widetilde{I}_L$, then $x_{\alpha_1}x_{\beta_1}$, $x_{\alpha_2}x_{\beta_2}$ is a Koszul relation pair of $K[L]/\text{in}_<(I_L)$.*

(b) *If $\widetilde{x_{\alpha_1}x_{\beta_1} - x_{\alpha_1\wedge\beta_1}x_{\alpha_1\vee\beta_1}}$, $\widetilde{x_{\alpha_2}x_{\beta_2} - x_{\alpha_2\wedge\beta_2}x_{\alpha_2\vee\beta_2}}$ is not a Koszul relation pair of $\widetilde{K[L]}/\widetilde{I}_L$, then $x_{\alpha_1}x_{\beta_1} - x_{\alpha_1\wedge\beta_1}x_{\alpha_1\vee\beta_1}$, $x_{\alpha_2}x_{\beta_2} - x_{\alpha_2\wedge\beta_2}x_{\alpha_2\vee\beta_2}$ is not a Koszul relation pair of $R[L]$.*

(c) *From (a) and (b), we obtain that if $x_{\alpha_1}x_{\beta_1} - x_{\alpha_1\wedge\beta_1}x_{\alpha_1\vee\beta_1}$, $x_{\alpha_2}x_{\beta_2} - x_{\alpha_2\wedge\beta_2}x_{\alpha_2\vee\beta_2}$ is a Koszul relation pair of $R[L]$, then $x_{\alpha_1}x_{\beta_1}$, $x_{\alpha_2}x_{\beta_2}$ is a Koszul relation pair of $K[L]/\text{in}_<(I_L)$.*

Theorem 4.4 *Let $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in D_2$. If $x_{\alpha_1}x_{\beta_1} - x_{\alpha_1\wedge\beta_1}x_{\alpha_1\vee\beta_1}$, $x_{\alpha_2}x_{\beta_2} - x_{\alpha_2\wedge\beta_2}x_{\alpha_2\vee\beta_2}$ is a Koszul relation pair of $R[L]$, then either $\alpha_2 \vee \beta_2 \leq \alpha_1 \wedge \beta_1$ or $\alpha_1 \vee \beta_1 \leq \alpha_2 \wedge \beta_2$.*

Proof The proof follows from Observation 4.3(c) and Theorem 4.2. □

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