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ORIGINAL RESEARCH

Energy of graphs with respect to generalized distance matrix: Extremal results and bounds

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Abstract Let G be a simple connected graph of order n. Denote by $D(G)$ the distance matrix of G and by $Tr(G)$ the diagonal matrix of its vertex transmissions. For $0 \le \alpha \le 1$, the generalized distance matrix $D_\alpha(G)$ of G is defined as $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$. The generalized distance energy of a graph G (energy of G with respect to the generalized distance matrix) is defined as $E^{D_\alpha}(G) = \sum_{i=1}^{n} \left|\partial_i - \frac{2\alpha W(G)}{n}\right|$, where $W(G)$ is the transmission (also called the Wiener index) of a graph G and $\partial_1 \ge \partial_2 \ge \cdots \ge \partial_n$ are the eigenvalues of $D_\alpha(G)$. In this paper, we establish new upper and lower bounds for $E^{D_\alpha}(G)$ in terms of various graph invariants, and we characterize the extremal graphs for which these bounds are attained.

Keywords Generalized distance matrix · Generalized distance energy · Bound · Extremal graph

1 Introduction

Every graph G in this paper is connected, undirected, simple and has n vertices. The *distance matrix* $D(G)$ of a connected graph G is the (symmetric) matrix indexed by the vertices of G and with its (i, j)-entry d_{ij} equal to the *distance* between the vertices v_i and v_j, i.e., the length of a shortest path between v_i and v_j in G. Till now, the distance spectrum of a connected graph has been investigated extensively; for spectral properties of $D(G)$ see the survey [8]. The *diameter* of G is the maximum distance between any two vertices of G. We say that a graph is *k-transmission regular* (or *transmission regular*) if its distance matrix has a constant row sum equal to k. Naturally, just as regular graphs, transmission regular graphs are also of particular interest in spectral graph theory. The *Wiener index* (or *transmission*) of a graph G, denoted by $W(G)$, is a topological index used in theoretical chemistry to characterize the structure of organic molecules. It is calculated by summing the distances

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between all unordered pairs of vertices in the graph. Clearly, $W(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}$. The transmission of a vertex v_i, denoted by $Tr(v_i)$ (or Tr_i), is the sum of its distances to every other vertex in the graph.

In [9] Aouchiche and Hansen introduced the *distance Laplacian matrix* $D^L(G) = Tr(G) - D(G)$ and the *distance signless Laplacian matrix* $D^Q(G) = Tr(G) + D(G)$, where $Tr(G)$ is the diagonal matrix of vertex transmissions of G. The spectral properties of $D(G)$, $D^L(G)$ and $D^Q(G)$ have attracted much more attention of the researchers and a large number of papers have been published regarding their spectral properties, like spectral radius, second largest eigenvalue, smallest eigenvalue, etc. For some recent works we refer to [2,4,5,7–9,11,18] and the references therein.

For $0 \le \alpha \le 1$, Cui et al. [10] introduced the *generalized distance matrix* $D_\alpha(G)$ as a convex combinations of $Tr(G)$ and $D(G)$, defined as $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$ and also mentioned that the matrices $D_\alpha(G)$ can underpin a unified theory of the matrices $D(G)$, $D^L(G)$ and $D^Q(G)$. Since the matrix $D_\alpha(G)$ is real symmetric, all its eigenvalues are real. Therefore, we can arrange them as $\partial_1 \ge \partial_2 \ge \cdots \ge \partial_n$. The largest eigenvalue ∂_1 of the matrix $D_\alpha(G)$ is called the *generalized distance spectral radius* of G (from now onwards, we will denote ∂_1 by $\partial(G)$). The *generalized distance spectral spread* of a connected graph G is defined in [22] as $S_{D_\alpha}(G) = \partial_1 - \partial_n$. Various lower and upper bounds for the parameter $S_{D_\alpha}(G)$ can be found in [22].

The notion of energy of a graph [16] was introduced in 1978 by Ivan Gutman and has its origin in theoretical chemistry. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the adjacency eigenvalues of a graph G. The *energy* of a graph G, denoted by $E(G)$, is defined as $E(G) = \sum_{i=1}^{n}|\lambda_i|$ (see [17] and [20] for details and an exhaustive list of references). Let $\rho_1^D \ge \rho_2^D \ge \cdots \ge \rho_n^D$, $\rho_1^L \ge \rho_2^L \ge \cdots \ge \rho_n^L$ and $\rho_1^Q \ge \rho_2^Q \ge \cdots \ge \rho_n^Q$ be respectively the distance, distance Laplacian and distance signless Laplacian eigenvalues of the graph G. The distance energy of a graph G was introduced in [19] as the sum of the absolute values of the distance eigenvalues of G, that is $E^D(G) = \sum_{i=1}^{n}|\rho_i^D|$. It is easy to see that

$$\sum_{i=1}^{n}\rho_i^D = 0 \text{ and } \sum_{i=1}^{n}(\rho_i^D)^2 = 2\sum_{1\le i<j\le n}(d_{ij})^2. \tag{1}$$

For some recent results on the distance energy of a graph, we refer to [11,12,15,25] and the references therein.

Let us introduce auxiliary eigenvalues Θ_i, corresponding to the generalized distance eigenvalues as $\Theta_i = \partial_i - \frac{2\alpha W(G)}{n}$. Analogous to the definitions of distance Laplacian energy $E^L(G)$ (see [26]) and distance signless Laplacian energy $E^Q(G)$ (see [7]), Alhevaz et al. [3] defined the generalized distance energy of G as the mean deviation of the values of the generalized distance eigenvalues of G, that is

$$E^{D_\alpha}(G) = \sum_{i=1}^{n}\left|\partial_i - \frac{2\alpha W(G)}{n}\right| = \sum_{i=1}^{n}|\Theta_i|.$$

Let t be the largest positive integer such that $\partial_t \ge \frac{2\alpha W(G)}{n}$ and $M_k(G) = \sum_{i=1}^{k}\partial_i$ be the sum of k largest generalized distance eigenvalues of G. It was shown in [3] that

$$E^{D_\alpha}(G) = 2\left(M_t - \frac{2\alpha t W(G)}{n}\right) = 2\max_{1\le j\le n}\left(\sum_{i=1}^{j}\partial_i - \frac{2\alpha j W(G)}{n}\right).$$

It is easy to see that $\sum_{i=1}^{n}\Theta_i = 0$. Also, since $\sum_{i=1}^{n}\partial_i = 2\alpha W(G)$ and $\sum_{i=1}^{n}\partial_i^2 = \text{trace}[D_\alpha(G)]^2 = 2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2 + \alpha^2\sum_{i=1}^{n}Tr_i^2$, hence by the definition of Θ_i, we see that $\sum_{i=1}^{n}\Theta_i^2 = 2\zeta$, where $\zeta = (1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2 + \frac{\alpha^2}{2}\sum_{i=1}^{n}Tr_i^2 - \frac{2\alpha^2 W^2(G)}{n}$. It is clear from the definition that $E^{D_0}(G) = E^D(G)$ and

$2E^{D_{\frac{1}{2}}}(G) = E^{Q}(G)$. This shows that the concept of generalized distance energy of a graph G merges the theories of distance energy and the distance signless Laplacian energy of a graph G. Therefore, it will be interesting to study the quantity $E^{D_\alpha}(G)$ and explore some properties like the bounds, the dependence on the structure of a graph G and the dependence on the parameter α. For some recent results, we refer to [1,3,6,23] and the references therein.

In this paper, we give some new bounds for the generalized distance energy $E^{D_\alpha}(G)$ of a connected graph G. We obtain some upper and lower bounds for $E^{D_\alpha}(G)$ in terms of different graph invariants and characterize the extremal graphs attaining these bounds.

2 Preliminary results

We start this section with the following lemma, which characterizes the graphs with two distinct generalized distance eigenvalues.

Lemma 1 *A connected graph G has two distinct $D_\alpha(G)$ eigenvalues if and only if G is a complete graph.*

Proof The proof is similar to that of [18, Lemma 2], and is excluded. □

We will make use of the following lemma in our subsequent results.

Lemma 2 *[21] If A is an $n \times n$ non-negative matrix with the spectral radius $\lambda(A)$ and row sums $r_1, r_2, \ldots, r_n$, then $\min_{1\leq i\leq n} r_i \leq \lambda(A) \leq \max_{1\leq i\leq n} r_i$. Moreover, if A is irreducible, then one of the equalities holds if and only if the row sums of A are all equal.*

Note that the i-th row sum of $D_\alpha(G)$ is $r_i = \alpha Tr_i + (1-\alpha)\sum_{j=1}^{n} d_{ij} = Tr_i$. Hence, we get the following result.

Corollary 1 *Let G be a connected graph of order n. Let $Tr_{\max}$ and $Tr_{\min}$ be the largest and least transmissions of G, respectively. Then $Tr_{\min} \leq \partial(G) \leq Tr_{\max}$. Moreover, one of the equalities holds if and only if G is a transmission regular graph.*

Since for a graph with diameter at most two, we have $Tr_i = 2n-2-d_i$, for all $i = 1, \ldots, n$, it follows that a graph G with diameter at most two is transmission regular if and only if it is degree regular.

Corollary 2 *Let G be a connected graph of order n and with maximum degree Δ. Then*

$$\partial(G) \geq 2n - \Delta - 2,$$

with equality if and only if G is a regular graph with diameter at most 2.

Lemma 3 *[27] If A is a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then, for any non-zero vector $\mathbf{x} \in R^n$, we have*

$$\mathbf{x}^T A\mathbf{x} \geq \lambda_n \mathbf{x}^T\mathbf{x}.$$

Furthermore, the equality holds if and only if $\mathbf{x}$ is an eigenvector of A corresponding to the smallest eigenvalue λ_n.

Lemma 4 *[10] If G is a connected graph of order n, then*

$$\partial(G) \geq \frac{2W(G)}{n},$$

with equality if and only if G is transmission regular.

Lemma 5 *If the transmission degree sequence of G is $\{Tr_1, Tr_2, \ldots, Tr_n\}$, then*

$$\partial(G) \geq \sqrt{\frac{\sum_{i=1}^{n} Tr_i^2}{n}},$$

with equality if and only if G is transmission regular.

Proof The proof is analogous to that of [7, Theorem 2.2], and is excluded. □

Lemma 6 *[10] If the transmission degree sequence and the second transmission degree sequence of G are* $\{Tr_1, Tr_2, \ldots, Tr_n\}$ *and* $\{T_1, T_2, \ldots, T_n\}$*, respectively, then*

$$\partial(G) \geq \sqrt{\frac{\sum_{i=1}^{n}\left(\alpha Tr_i^2 + (1-\alpha)T_i\right)^2}{\sum_{i=1}^{n} Tr_i^2}}.$$

Moreover, if $\frac{1}{2} \leq \alpha \leq 1$*, the equality holds if and only if G is transmission regular.*

Remark 1 Keeping all of the notations from Lemma 6, we have

$$\partial(G) \geq \sqrt{\frac{\sum_{i=1}^{n}\left(\alpha Tr_i^2 + (1-\alpha)T_i\right)^2}{\sum_{i=1}^{n} Tr_i^2}} \geq \frac{2W(G)}{n}.$$

In fact, as we always have $\sum_{i=1}^{n} T_i = \sum_{i=1}^{n} Tr_i^2$, and also applying Cauchy-Schwarz inequality we have $\left(\sum_{i=1}^{n} T_i\right)^2 \leq n\sum_{i=1}^{n} T_i^2$ and $\left(\sum_{i=1}^{n} Tr_i\right)^2 \leq n\sum_{i=1}^{n} Tr_i^2$, hence we get

$$\begin{aligned}\partial(G) &\geq \sqrt{\frac{\sum_{i=1}^{n}\left(\alpha Tr_i^2 + (1-\alpha)T_i\right)^2}{\sum_{i=1}^{n} Tr_i^2}} \geq \sqrt{\frac{\left(\sum_{i=1}^{n}(\alpha Tr_i^2 + (1-\alpha)T_i)\right)^2}{n\sum_{i=1}^{n} Tr_i^2}} \\ &= \sqrt{\frac{\left(\sum_{i=1}^{n} Tr_i^2\right)^2}{n\sum_{i=1}^{n} Tr_i^2}} = \sqrt{\frac{\sum_{i=1}^{n} Tr_i^2}{n}} \geq \sqrt{\frac{\left(\sum_{i=1}^{n} Tr_i\right)^2}{n^2}} = \frac{2W(G)}{n}.\end{aligned}$$

Lemma 7 *[24] Let X and Y be Hermitian matrices of order n such that* $Z = X + Y$*. Then*

$$\begin{aligned}\lambda_k(Z) &\leq \lambda_j(X) + \lambda_{k-j+1}(Y), \quad n \geq k \geq j \geq 1,\\ \lambda_k(Z) &\geq \lambda_j(X) + \lambda_{k-j+n}(Y), \quad n \geq j \geq k \geq 1,\end{aligned}$$

where $\lambda_i(M)$ *is the i-th largest eigenvalue of the matrix M. In either of these inequalities, equality holds if and only if there exists a unit vector that is an eigenvector to each of the three eigenvalues involved.*

Lemma 8 *[14] Let B and C be two real symmetric matrices of size n. Then for any* $1 \leq k \leq n$*,*

$$\sum_{i=1}^{k}\lambda_i(B+C) \leq \sum_{i=1}^{k}\lambda_i(B) + \sum_{i=1}^{k}\lambda_i(C),$$

where $\lambda_i(M)$ *is the i-th largest eigenvalue of M.*

3 Bounds for the generalized distance energy

In this section, we obtain some sharp upper and lower bounds for the generalized distance energy $E^{D_\alpha}(G)$ of a connected graph G, in terms of the order n, the Wiener index $W(G)$, the transmission degrees, the generalized distance spectral spread $D_\alpha S(G)$ and the parameter $\alpha \in [0, 1]$. We also characterize the extremal graphs attaining these bounds.

The following results give the generalized distance energy of a graph of diameter 2 in terms of the sum of the adjacency eigenvalues of G.

Theorem 1 *Let G be a connected r-regular graph of order n and size m having diameter 2. Then*

$$E^{D_\alpha}(G) = 2(1-\alpha)\ (2n - 2t - r + U_{n-t+1}),$$

where $U_{n-t+1} = \sum_{i=1}^{n-t+1}\lambda_i$*, is the sum of the* $n-t+1$ *largest adjacency eigenvalues of G and t,* $1 \leq t \leq n$*, is the number of generalized distance eigenvalues of G which are greater than or equal to* $\frac{2\alpha W(G)}{n}$*.*

Proof Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the adjacency eigenvalues and $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$ be the generalized distance eigenvalues of G. Since G is a connected r-regular with diameter two, the transmission of each vertex $v \in V(G)$, $Tr(v) = r + 2(n-1-r) = 2n - r - 2$ and so $2W(G) = 2n(n-1) - 2m$. The generalized distance matrix can be written as

$$D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G) = \alpha(2n-r-2)I + (1-\alpha)(2J - 2I - A(G)),$$

hence $\partial_1 = 2n - 2 - r$ and $\partial_i = \alpha(2n - r + \lambda_{n-i+2}) - \lambda_{n-i+2} - 2$, for $i = 2, 3, \ldots, n$. Since t is the number of generalized distance eigenvalues of G which are greater than or equal to $\frac{2\alpha W(G)}{n}$, using the definition of generalized distance energy, we obtain

$$\begin{aligned} E^{D_\alpha}(G) &= 2\left(\sum_{i=1}^{t} \partial_i - \frac{2\alpha t W(G)}{n}\right) \\ &= 2\Bigg(2n - 2 - r + \sum_{i=2}^{t}\Big(\alpha(2n - r + \lambda_{n-i+2}) - \lambda_{n-i+2} - 2\Big) \\ &\quad -\alpha t\left(2n - 2 - \frac{2m}{n}\right)\Bigg) \\ &= 2\left(2n - 2t - r + \alpha\left(t\left(\frac{2m}{n} + 2 - r\right) - 2n + r\right) + (1-\alpha)\sum_{i=1}^{n-t+1}\lambda_i\right). \end{aligned}$$

Since $2m = nr$, from the above, we get the required result. □

The following theorem gives a lower bound for the generalized distance energy in terms of the order n, the Wiener index $W(G)$, the minimum transmission $Tr_{\min}$, and the parameter α of any graph G.

Theorem 2 *Let G be a connected graph of order n having minimum transmission degree $Tr_{\min}$ and Wiener index $W(G)$. Then*

$$E^{D_\alpha}(G) > \frac{2nTr_{\min} - 4\alpha W(G)}{n(n-1)}. \tag{2}$$

Proof Let $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$ be the generalized distance eigenvalues and t is the number of generalized distance eigenvalues of G which are greater than or equal to $\frac{2\alpha W(G)}{n}$. Note that $\sum_{i=1}^{n} \partial_i = 2\alpha W(G)$ and $2\alpha W(G) \geq \partial_1 + (n-1)\partial_n$, also by Corollary 1, $Tr_{\min} \leq \partial_1 \leq Tr_{\max}$. We have $\partial_n \leq \frac{2\alpha W(G) - \partial_1}{n-1} \leq \frac{2\alpha W(G) - Tr_{\min}}{n-1}$. Using the definition of generalized distance energy, we have

$$\begin{aligned} E^{D_\alpha}(G) &= 2\max_{1\leq j\leq n}\left(\sum_{i=1}^{j}\partial_i - \frac{2\alpha j W(G)}{n}\right) \geq 2\left(\sum_{i=1}^{n-1}\partial_i - \frac{2\alpha(n-1)W(G)}{n}\right) \\ &= 2\left(2\alpha W(G) - \partial_n - \frac{2\alpha(n-1)W(G)}{n}\right) = \frac{4\alpha W(G)}{n} - 2\partial_n \\ &\geq \frac{4\alpha W(G)}{n} - \frac{4\alpha W(G) - 2Tr_{\min}}{n-1} = \frac{2nTr_{\min} - 4\alpha W(G)}{n(n-1)}. \end{aligned}$$

By contradiction, we will prove that the inequality in (2) is strict. For this, we assume that equality holds in (2). Then all the inequalities in the above must be equalities. Thus we have

$$\max_{1\leq j\leq n}\left(\sum_{i=1}^{j}\partial_i - \frac{2\alpha j W(G)}{n}\right) = \sum_{i=1}^{n-1}\partial_i - \frac{2\alpha(n-1)W(G)}{n} \tag{3}$$

and

$$\partial_n = \frac{2\alpha W(G) - \partial_1}{n-1} = \frac{2\alpha W(G) - Tr_{\min}}{n-1}. \tag{4}$$

From (3), we have $t = n-1$. From (4), we have $\partial_2 = \cdots = \partial_n$ and G is a transmission regular graph (by Corollary 1). Since $t = n-1$, we have $\partial_n < \frac{2\alpha W(G)}{n} \leq \partial_{n-1} = \partial_n$, a contradiction. This completes the proof of the theorem. □

Since $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$, by taking $k = n$ and $j = 1$ in first inequality and $k = n$ and $j = n$ in second inequality of Lemma 7, we get

$$\begin{aligned}\lambda_n(\alpha Tr(G)) + \lambda_n((1-\alpha)D(G)) &\leq \lambda_n(D_\alpha(G)) \\ &\leq \lambda_1(\alpha Tr(G)) + \lambda_n((1-\alpha)D(G)),\end{aligned} \tag{5}$$

and hence we obtain the following upper bound for the smallest generalized distance eigenvalue ∂_n,

$$\partial_n(G) \leq \alpha Tr_{\max} + (1-\alpha)\rho_n^D, \tag{6}$$

where ρ_n^D is the smallest distance eigenvalue, with equality if and only if G is a transmission regular graph. Hence, we get the following result.

Theorem 3 *Let G be a connected graph of order n having maximum transmission degree $Tr_{\max}$ and smallest distance eigenvalue ρ_n^D. Then*

$$E^{D_\alpha}(G) \geq \frac{4\alpha W(G)}{n} - 2\alpha Tr_{\max} - 2(1-\alpha)\rho_n^D, \tag{7}$$

equality occurs if and only if G is a transmission regular graph and $t = n-1$.

Proof The first part of the proof is similar to that of Theorem 2.

Now, suppose that equality holds in (7). From equality in (3), we have $t = n-1$ and from equality in (6), we get G is a transmission regular graph.

Conversely, let G be a transmission regular graph and $t = n-1$. Then $Tr_{\max} = Tr_{\min}$ and $\partial_n = \alpha Tr_{\max} + (1-\alpha)\rho_n^D$. Since $t = n-1$, we have

$$\begin{aligned}E^{D_\alpha}(G) &= \sum_{i=1}^{n-1}\left(\partial_i - \frac{2\alpha W(G)}{n}\right) + \frac{2\alpha W(G)}{n} - \partial_n \\ &= 2\left(\frac{2\alpha W(G)}{n} - \partial_n\right) \\ &= -2(1-\alpha)\rho_n^D \\ &= \frac{4\alpha W(G)}{n} - 2\alpha Tr_{\max} - 2(1-\alpha)\rho_n^D.\end{aligned}$$

□

The following result gives a lower bound for the generalized distance energy $E^{D_\alpha}(G)$, in terms of the order n, the largest transmission of G, and also the parameter α.

Theorem 4 *For a graph G of order n, let $Tr_{\max}$ be the largest transmissions of G and*

$$\Gamma = \left|det\left(D_\alpha(G) - \frac{2\alpha W(G)}{n}I\right)\right|.$$

Then

$$\begin{aligned}E^{D_\alpha}(G) &\geq (1-\alpha)\frac{2W(G)}{n} \\ &+ \sqrt{2\zeta - \left(Tr_{\max} - \frac{2\alpha W(G)}{n}\right)^2 + (n-1)(n-2)\left(\frac{\Gamma}{Tr_{\max} - \frac{2\alpha W(G)}{n}}\right)^{\frac{2}{n-1}}},\end{aligned} \tag{8}$$

where $2\zeta = 2(1-\alpha)^2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \alpha^2\sum_{i=1}^n Tr_i^2 - \frac{4\alpha^2W^2(G)}{n}$. Equality holds if and only if either G is a complete graph or a k-transmission regular graph with three distinct D_α-eigenvalues

$$\left(k, \sqrt{\frac{M - k^2(1-\alpha)^2}{n-1}} + \alpha k, -\sqrt{\frac{M - k^2(1-\alpha)^2}{n-1}} + \alpha k\right),$$

where $M = 2(1-\alpha)^2\sum_{1\leq i<j\leq n}(d_{ij})^2$.

Proof We have

$$\left(E^{D_\alpha}(G)\right)^2 = \sum_{i=1}^{n}\left(\partial_i - \frac{2\alpha W(G)}{n}\right)^2 + 2\sum_{1\le i<j\le n}\left|\partial_i - \frac{2\alpha W(G)}{n}\right|\left|\partial_j - \frac{2\alpha W(G)}{n}\right|$$
$$= 2\zeta + 2\left(\partial_1 - \frac{2\alpha W(G)}{n}\right)\sum_{i=2}^{n}\left|\partial_i - \frac{2\alpha W(G)}{n}\right|$$
$$+2\sum_{2\le i<j\le n}\left|\partial_i - \frac{2\alpha W(G)}{n}\right|\left|\partial_j - \frac{2\alpha W(G)}{n}\right|, \tag{9}$$

where $2\zeta = 2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2 + \alpha^2\sum_{i=1}^{n} Tr_i^2 - \frac{4\alpha^2 W^2(G)}{n}$. By the Arithmetic-Geometric mean inequality, we get

$$\sum_{2\le i<j\le n}\left|\partial_i - \frac{2\alpha W(G)}{n}\right|\left|\partial_j - \frac{2\alpha W(G)}{n}\right|$$
$$\ge \frac{(n-1)(n-2)}{2}\left(\prod_{i=2}^{n}\left|\partial_i - \frac{2\alpha W(G)}{n}\right|\right)^{\frac{2}{n-1}} \tag{10}$$
$$= \frac{(n-1)(n-2)}{2}\left(\frac{\Gamma}{\partial_1 - \frac{2\alpha W(G)}{n}}\right)^{\frac{2}{n-1}}. \tag{11}$$

Using inequality (11) in (9), we get

$$\left(E^{D_\alpha}(G)\right)^2 \ge 2\zeta + 2\left(\partial_1 - \frac{2\alpha W(G)}{n}\right)\left(E^{D_\alpha}(G) - \left(\partial_1 - \frac{2\alpha W(G)}{n}\right)\right)$$
$$+(n-1)(n-2)\left(\frac{\Gamma}{\partial_1 - \frac{2\alpha W(G)}{n}}\right)^{\frac{2}{n-1}},$$

that is,

$$\left(E^{D_\alpha}(G)\right)^2 - 2\left(\partial_1 - \frac{2\alpha W(G)}{n}\right)E^{D_\alpha}(G) - \left[2\zeta - 2\left(\partial_1 - \frac{2\alpha W(G)}{n}\right)^2\right.$$
$$\left. +(n-1)(n-2)\left(\frac{\Gamma}{\partial_1 - \frac{2\alpha W(G)}{n}}\right)^{\frac{2}{n-1}}\right] \ge 0.$$

By solving the above inequality, we get

$$E^{D_\alpha}(G) \ge \partial_1 - \frac{2\alpha W(G)}{n}$$
$$+\sqrt{2\zeta - \left(\partial_1 - \frac{2\alpha W(G)}{n}\right)^2 + (n-1)(n-2)\left(\frac{\Gamma}{\partial_1 - \frac{2\alpha W(G)}{n}}\right)^{\frac{2}{n-1}}}.$$

Since by Corollary 1 and Lemma 4, $\frac{2W(G)}{n} \le \partial_1 \le Tr_{\max}$, we get

$$E^{D_\alpha}(G) \ge (1-\alpha)\frac{2W(G)}{n}.$$
$$+\sqrt{2\zeta - \left(Tr_{\max} - \frac{2\alpha W(G)}{n}\right)^2 + (n-1)(n-2)\left(\frac{\Gamma}{Tr_{\max} - \frac{2\alpha W(G)}{n}}\right)^{\frac{2}{n-1}}}. \tag{12}$$

Hence we get the required result in (8). Then the first part of the proof is done.

Suppose that equality holds in (8). Then all the inequalities in the above must be equalities. Then $\partial_1 = \frac{2W(G)}{n} = Tr_{\max}$. By Corollary 1 and Lemma 4, G is a transmission regular graph. From equality in (10), we get $\left|\partial_2 - \frac{2\alpha W(G)}{n}\right| = \left|\partial_3 - \frac{2\alpha W(G)}{n}\right| = \cdots = \left|\partial_n - \frac{2\alpha W(G)}{n}\right|$, and hence

$$\left(\sum_{i=2}^{n}\left|\partial_i - \frac{2\alpha W(G)}{n}\right|\right)^2 = (n-1)\left(2\zeta - \left(\partial_1 - \frac{2\alpha W(G)}{n}\right)^2\right),$$

then for each $i = 2, \ldots, n$, we have

$$\left|\partial_i - \frac{2\alpha W(G)}{n}\right| = \sqrt{\frac{2\zeta - \left(\partial_1 - \frac{2\alpha W(G)}{n}\right)^2}{n-1}}.$$

Hence $\left|\partial_i - \frac{2\alpha W(G)}{n}\right|$ can have at most two distinct values and we arrive at the following:
(i): G has exactly one distinct D_α-eigenvalue. Then $G = K_1$.
(ii): G has exactly two distinct D_α-eigenvalues. Then by Lemma 1, $G = K_n$.
(iii): G has exactly three distinct D_α-eigenvalues. Then $\partial_1 = \frac{2W(G)}{n}$, also $\left|\partial_i - \frac{2\alpha W(G)}{n}\right| = \sqrt{\frac{2\zeta - \left(\frac{2(1-\alpha)W(G)}{n}\right)^2}{n-1}}$, $i = 2, \ldots, n$. Also, G is k-transmission regular graph. Then we get G as a graph with three distinct D_α-eigenvalues

$$\left(k, \sqrt{\frac{M - k^2(1-\alpha)^2}{n-1}} + \alpha k, -\sqrt{\frac{M - k^2(1-\alpha)^2}{n-1}} + \alpha k\right),$$

where $M = 2(1-\alpha)^2 \sum_{1 \le i < j \le n}(d_{ij})^2$. Hence, the result follows. □

Theorem 5 *Let G be a connected graph of order n. If $0 \le \alpha \le \frac{1}{2}$, then*

$$E^{D_\alpha}(G) \ge (1-\alpha)\sqrt{\frac{\sum_{i=1}^{n} Tr_i^2}{n}} + (n-1)\left(\frac{\left|det\left(D_\alpha(G) - \frac{2\alpha W(G)}{n}\right)\right|}{(1-\alpha)\sqrt{\frac{\sum_{i=1}^{n} Tr_i^2}{n}}}\right)^{\frac{1}{n-1}}.$$

Equality holds if and only if either G is a complete graph or G is a graph with three distinct D_α-eigenvalues.

Proof Applying the Cauchy-Schwarz inequality for the vectors $\underbrace{(1, 1, \ldots, 1)}_{(n\text{-}1)\text{ times}}$ and $\left(\sqrt{\left|\partial_2 - \frac{2\alpha W(G)}{n}\right|}, \sqrt{\left|\partial_3 - \frac{2\alpha W(G)}{n}\right|}, \ldots, \sqrt{\left|\partial_n - \frac{2\alpha W(G)}{n}\right|}\right)$, we get

$$\begin{aligned} E^{D_\alpha}(G) &\ge \partial_1 - \frac{2\alpha W(G)}{n} + (n-1) \\ &\quad \cdot \left(\left|\partial_2 - \frac{2\alpha W(G)}{n}\right|\left|\partial_3 - \frac{2\alpha W(G)}{n}\right| \cdots \left|\partial_n - \frac{2\alpha W(G)}{n}\right|\right)^{\frac{1}{n-1}} \\ &= \partial_1 - \frac{2\alpha W(G)}{n} + (n-1) \cdot \left(\frac{\left|det\left(D_\alpha(G) - \frac{2\alpha W(G)}{n}\right)\right|}{\partial_1 - \frac{2\alpha W(G)}{n}}\right)^{\frac{1}{n-1}}. \end{aligned}$$

Define a function

$$f(x) = x + (n-1)\left(\frac{\left|det\left(D_\alpha(G) - \frac{2\alpha W(G)}{n}\right)\right|}{x}\right)^{\frac{1}{n-1}},$$

then

$$f'(x) = 1 - \frac{\left|det\left(D_\alpha(G) - \frac{2\alpha W(G)}{n}\right)\right|^{\frac{1}{n-1}}}{x^{\frac{n}{n-1}}},$$

and

$$f''(x) = \frac{n\left|det\left(D_\alpha(G) - \frac{2\alpha W(G)}{n}\right)\right|^{\frac{1}{n-1}}}{(n-1)x^{\frac{2n-1}{n-1}}}.$$

For maxima or minima, $f'(x) = 0$, which implies $x = \left|det\left(D_\alpha(G) - \frac{2\alpha W(G)}{n}\right)\right|^{\frac{1}{n}}$. At this point,

$$f''(x) = \frac{n}{n-1}\left|det\left(D_\alpha(G) - \frac{2\alpha W(G)}{n}\right)\right|^{\frac{-1}{n}} \geq 0 \text{ for all } n > 1.$$

Thus the function $f(x)$ attains its minimum at $x = \left|det\left(D_\alpha(G) - \frac{2\alpha W(G)}{n}\right)\right|^{\frac{1}{n}}$, and the minimum value is

$$f\left(\left|det\left(D_\alpha(G) - \frac{2\alpha W(G)}{n}\right)\right|^{\frac{1}{n}}\right) = n\left|det\left(D_\alpha(G) - \frac{2\alpha W(G)}{n}\right)\right|^{\frac{1}{n}}.$$

Since $2\sum_{1\leq i<j\leq n}(d_{ij})^2 < \frac{\left(\sum_{i=1}^n Tr_i\right)^2}{n}$ (see [13]), and by Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=1}^n\left|\partial_i - \frac{2\alpha W(G)}{n}\right|\right)^2 \leq n\sum_{i=1}^n\left(\partial_i - \frac{2\alpha W(G)}{n}\right)^2, \ \left(\sum_{i=1}^n T_i\right)^2 \leq n\sum_{i=1}^n T_i^2$$

and

$$\left(\sum_{i=1}^n Tr_i\right)^2 \leq n\sum_{i=1}^n Tr_i^2.$$

Then we obtain

$$\begin{aligned}
&\sum_{i=1}^n\left|\partial_i - \frac{2\alpha W(G)}{n}\right| \\
&\leq \sqrt{n\left(2(1-\alpha)^2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \alpha^2\sum_{i=1}^n Tr_i^2 - \frac{4\alpha^2 W^2(G)}{n}\right)} \\
&< \sqrt{n\left(\frac{(1-\alpha)^2\left(\sum_{i=1}^n Tr_i\right)^2}{n} + \alpha^2\sum_{i=1}^n Tr_i^2 - \frac{\alpha^2\left(\sum_{i=1}^n Tr_i\right)^2}{n}\right)} \\
&= \sqrt{n\left((1-2\alpha)\frac{\left(\sum_{i=1}^n Tr_i\right)^2}{n} + \alpha^2\sum_{i=1}^n Tr_i^2\right)}.
\end{aligned}$$

If $0 \leq \alpha \leq \frac{1}{2}$, then

$$\begin{aligned}
&\sqrt{n\left((1-2\alpha)\frac{\left(\sum_{i=1}^n Tr_i\right)^2}{n} + \alpha^2\sum_{i=1}^n Tr_i^2\right)} \\
&\leq \sqrt{n\left((1-2\alpha)\sum_{i=1}^n Tr_i^2 + \alpha^2\sum_{i=1}^n Tr_i^2\right)}
\end{aligned}$$

$$= (1-\alpha)\sqrt{n\sum_{i=1}^{n} Tr_i^2}.$$

Hence we have

$$(1-\alpha)\sqrt{\frac{\sum_{i=1}^{n} Tr_i^2}{n}} > \frac{\left|\partial_1 - \frac{2\alpha W(G)}{n}\right| + \cdots + \left|\partial_n - \frac{2\alpha W(G)}{n}\right|}{n}$$
$$\geq \left(\left|\partial_1 - \frac{2\alpha W(G)}{n}\right| \cdots \left|\partial_n - \frac{2\alpha W(G)}{n}\right|\right)^{\frac{1}{n}}.$$

For $0 \leq \alpha \leq \frac{1}{2}$, this implies

$$\left|det\left(D_\alpha(G) - \frac{2\alpha W(G)}{n}\right)\right|^{\frac{1}{n}} < (1-\alpha)\sqrt{\frac{\sum_{i=1}^{n} Tr_i^2}{n}}.$$

On the other hand, by Lemma 5 and inequality $\frac{2W(G)}{n} = \sqrt{\frac{(\sum_{i=1}^{n} Tr_i)^2}{n^2}} \leq \sqrt{\frac{\sum_{i=1}^{n} Tr_i^2}{n}}$, we have

$$\partial_1 \geq \sqrt{\frac{\sum_{i=1}^{n} Tr_i^2}{n}} \geq (1-\alpha)\sqrt{\frac{\sum_{i=1}^{n} Tr_i^2}{n}} + \frac{2\alpha W(G)}{n}$$
$$\geq \sqrt{\frac{2(1-\alpha)^2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \alpha^2\sum_{i=1}^{n} Tr_i^2 - \frac{4\alpha^2 W^2(G)}{n}}{n}}$$
$$+\frac{2\alpha W(G)}{n}.$$

Therefore, the function is increasing in the interval

$$\left|det\left(D_\alpha(G) - \frac{2\alpha W(G)}{n}\right)\right|^{\frac{1}{n}} < (1-\alpha)\sqrt{\frac{\sum_{i=1}^{n} Tr_i^2}{n}} \leq x,$$

then $f(x) \geq f\left((1-\alpha)\sqrt{\frac{\sum_{i=1}^{n} Tr_i^2}{n}}\right)$, and

$$E^{D_\alpha}(G) \geq (1-\alpha)\sqrt{\frac{\sum_{i=1}^{n} Tr_i^2}{n}} + (n-1)\left(\frac{\left|det\left(D_\alpha(G) - \frac{2\alpha W(G)}{n}\right)\right|}{(1-\alpha)\sqrt{\frac{\sum_{i=1}^{n} Tr_i^2}{n}}}\right)^{\frac{1}{n-1}}.$$

By this, the first part of the proof is done. The remaining of the proof is similar to that of Theorem 4, and hence is omitted. □

We now focus on establishing upper bounds for the generalized distance energy of graphs. We begin by relating the distance energy and generalized distance energy of graphs.

Theorem 6 *Let G be a connected graph with transmission degree sequence $\{Tr_1, Tr_2, \ldots, Tr_n\}$. Then*

$$E^{D_\alpha}(G) \leq (1-\alpha)E^D(G) + 2\alpha\sum_{i=1}^{t}\left(Tr_i - \frac{2W(G)}{n}\right), \tag{13}$$

where t, $1 \leq t \leq n$, is the number of generalized distance eigenvalues of G which are greater than or equal to $\frac{2\alpha W(G)}{n}$.

Proof Using Lemma 8, we obtain

$$\sum_{i=1}^{k} \partial_i \leq \alpha \sum_{i=1}^{k} Tr_i + (1-\alpha) \sum_{i=1}^{k} \rho_i^D,$$

where ρ_i^D is the i-th largest eigenvalue of distance matrix $D(G)$. From the definition of distance energy, we have

$$\begin{aligned} E^D(G) &= \sum_{i=1}^{n} |\rho_i^D| = 2 \sum_{\rho_i^D \geq 0} \rho_i^D \\ &= 2 \max \left\{ \sum_{i=1}^{k} \rho_i^D, \ \ 1 \leq k \leq n \right\} \geq 2 \sum_{i=1}^{k} \rho_i^D, \ \ 1 \leq k \leq n. \end{aligned}$$

From the above two results, for any k, we get

$$2 \sum_{i=1}^{k} \partial_i \leq 2\alpha \sum_{i=1}^{k} Tr_i + (1-\alpha) E^D(G).$$

From the above, we can write

$$2 \sum_{i=1}^{t} \partial_i - \frac{4\alpha t W(G)}{n} \leq 2\alpha \sum_{i=1}^{t} Tr_i + (1-\alpha) E^D(G) - \frac{4\alpha t W(G)}{n},$$

where t, $1 \leq t \leq n$, is the number of generalized distance eigenvalues of G which are greater than or equal to $\frac{2\alpha W(G)}{n}$. Hence, we get the required result. □

Recently, Alhevaz et al. [3] presented the following relation between distance energy and generalized distance energy of graphs:

$$E^{D_\alpha}(G) \leq (1-\alpha) E^D(G) + \alpha \sum_{i=1}^{n} \left| Tr_i - \frac{2W(G)}{n} \right|. \tag{14}$$

Remark 2 The given upper bound in (13) is better than the upper bound in (14).

Proof We can assume that $Tr_1 \geq Tr_2 \geq \cdots \geq Tr_n$. Suppose that z is the largest positive integer such that $Tr_z \geq \frac{2W(G)}{n}$. Thus, we have

$$\begin{aligned} \sum_{i=1}^{z} \left(Tr_i - \frac{2W(G)}{n} \right) &= \sum_{i=1}^{z} Tr_i - \frac{2zW(G)}{n} = 2W(G) - \sum_{i=z+1}^{n} Tr_i - \frac{2zW(G)}{n} \\ &= \frac{2W(G)(n-z)}{n} - \sum_{i=z+1}^{n} Tr_i = \sum_{i=z+1}^{n} \left(\frac{2W(G)}{n} - Tr_i \right). \end{aligned}$$

Also

$$\sum_{i=1}^{n} \left| Tr_i - \frac{2W(G)}{n} \right| = \sum_{i=1}^{z} \left(Tr_i - \frac{2W(G)}{n} \right) + \sum_{i=z+1}^{n} \left(\frac{2W(G)}{n} - Tr_i \right).$$

From the above two relations, we get

$$\sum_{i=1}^{n} \left| Tr_i - \frac{2W(G)}{n} \right| = 2 \sum_{i=1}^{z} \left(Tr_i - \frac{2W(G)}{n} \right).$$

Now we have to show that

$$\sum_{i=1}^{z}\left(Tr_i-\frac{2W(G)}{n}\right)=\max\left\{\sum_{i=1}^{k}\left(Tr_i-\frac{2W(G)}{n}\right);\ \ 1\leq k\leq n\right\},$$

that is,

$$\sum_{i=1}^{k}\left(Tr_i-\frac{2W(G)}{n}\right)\leq\sum_{i=1}^{z}\left(Tr_i-\frac{2W(G)}{n}\right),\ \ 1\leq k\leq n. \tag{15}$$

If $k=z$, then the equality holds in (15). Otherwise, $k\neq z$. Now, consider the following two cases:
(i): $k<z$. Since $\sum_{i=k+1}^{z}\left(Tr_i-\frac{2\alpha W(G)}{n}\right)\geq 0$ and

$$\sum_{i=1}^{z}\left(Tr_i-\frac{2W(G)}{n}\right)=\sum_{i=1}^{k}\left(Tr_i-\frac{2W(G)}{n}\right)+\sum_{i=k+1}^{z}\left(Tr_i-\frac{2W(G)}{n}\right),$$

we get the result in (15).
(ii): $k>z$. Since $\sum_{i=z+1}^{k}\left(Tr_i-\frac{2W(G)}{n}\right)<0$, we have

$$\begin{aligned}\sum_{i=1}^{k}\left(Tr_i-\frac{2W(G)}{n}\right)&=\sum_{i=1}^{z}\left(Tr_i-\frac{2W(G)}{n}\right)+\sum_{i=z+1}^{k}\left(Tr_i-\frac{2W(G)}{n}\right)\\&<\sum_{i=1}^{z}\left(Tr_i-\frac{2W(G)}{n}\right),\end{aligned}$$

which gives the result in (15). Therefore,

$$\begin{aligned}\sum_{i=1}^{n}\left|Tr_i-\frac{2W(G)}{n}\right|&=2\sum_{i=1}^{z}\left(Tr_i-\frac{2W(G)}{n}\right)\\&=2\max\left\{\sum_{i=1}^{k}\left(Tr_i-\frac{2W(G)}{n}\right),\ \ 1\leq k\leq n\right\}\\&\geq 2\sum_{i=1}^{t}\left(Tr_i-\frac{2W(G)}{n}\right).\end{aligned}$$

□

The following result gives an upper bound for the generalized distance energy $E^{D_\alpha}(G)$, in terms of the order n, the Wiener index $W(G)$, the transmission degrees, and the parameter α.

Theorem 7 *Let G be a connected graph of order n. If $0\leq\alpha\leq\frac{1}{2}$, then*

$$\begin{aligned}E^{D_\alpha}(G)\leq{}&(1-\alpha)\sqrt{\frac{\sum_{i=1}^{n}(\alpha Tr_i^2+(1-\alpha)T_i)^2}{\sum_{i=1}^{n}Tr_i^2}}\\&+\sqrt{(n-1)\left(2\zeta-(1-\alpha)^2\left(\frac{\sum_{i=1}^{n}(\alpha Tr_i^2+(1-\alpha)T_i)^2}{\sum_{i=1}^{n}Tr_i^2}\right)\right)},\end{aligned}\tag{16}$$

where $2\zeta=2(1-\alpha)^2\sum_{1\leq i<j\leq n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}$. Equality holds if and only if either G is a complete graph or G is a graph with three distinct D_α-eigenvalues.

Proof By applying the Cauchy-Schwarz inequality to the vectors $(\underbrace{1,1,\ldots,1}_{\text{(n-1) times}})$ and $\left(\left|\partial_2-\frac{2\alpha W(G)}{n}\right|,\right.$ $\left.\left|\partial_3-\frac{2\alpha W(G)}{n}\right|,\ldots,\left|\partial_n-\frac{2\alpha W(G)}{n}\right|\right)$, we get

$$\begin{aligned}&\left(E^{D_\alpha}(G)-\left(\partial_1-\frac{2\alpha W(G)}{n}\right)\right)^2\\&\leq(n-1)\Bigg(2(1-\alpha)^2\sum_{1\leq i<j\leq n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2\\&\quad-\frac{4\alpha^2W^2(G)}{n}-\left(\partial_1-\frac{2\alpha W(G)}{n}\right)^2\Bigg).\end{aligned}\tag{17}$$

Then

$$\begin{aligned}E^{D_\alpha}(G)&\leq\partial_1-\frac{2\alpha W(G)}{n}\\&+\sqrt{(n-1)\left(2(1-\alpha)^2\sum_{1\leq i<j\leq n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}-\left(\partial_1-\frac{2\alpha W(G)}{n}\right)^2\right)}.\end{aligned}$$

The function

$$f(x)=x+\sqrt{(n-1)\left(2(1-\alpha)^2\sum_{1\leq i<j\leq n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}-x^2\right)}$$

decreases if and only if $x\geq\sqrt{\frac{2(1-\alpha)^2\sum_{1\leq i<j\leq n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}}{n}}$. Since $2\sum_{1\leq i<j\leq n}(d_{ij})^2<\frac{(\sum_{i=1}^{n}Tr_i)^2}{n}$ (see [13]), and by Cauchy-Schwarz inequality we have $\left(\sum_{i=1}^{n}T_i\right)^2\leq n\sum_{i=1}^{n}T_i^2$ and $\left(\sum_{i=1}^{n}Tr_i\right)^2\leq n\sum_{i=1}^{n}Tr_i^2$, then we get

$$\begin{aligned}&\sqrt{\frac{2(1-\alpha)^2\sum_{1\leq i<j\leq n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}}{n}}\\&<\sqrt{\frac{\frac{(1-\alpha)^2(\sum_{i=1}^{n}Tr_i)^2}{n}+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{\alpha^2(\sum_{i=1}^{n}Tr_i)^2}{n}}{n}}\\&=\sqrt{\frac{(1-2\alpha)\frac{(\sum_{i=1}^{n}Tr_i)^2}{n}+\alpha^2\sum_{i=1}^{n}Tr_i^2}{n}}.\end{aligned}$$

If $0\leq\alpha\leq\frac{1}{2}$, then

$$\begin{aligned}\sqrt{\frac{(1-2\alpha)\frac{(\sum_{i=1}^{n}Tr_i)^2}{n}+\alpha^2\sum_{i=1}^{n}Tr_i^2}{n}}&\leq\sqrt{\frac{(1-2\alpha)\sum_{i=1}^{n}Tr_i^2+\alpha^2\sum_{i=1}^{n}Tr_i^2}{n}}\\&=(1-\alpha)\sqrt{\frac{\sum_{i=1}^{n}Tr_i^2}{n}}\\&\leq(1-\alpha)\sqrt{\frac{\left(\sum_{i=1}^{n}Tr_i^2\right)^2}{n\sum_{i=1}^{n}Tr_i^2}}\end{aligned}$$

$$\leq (1-\alpha)\sqrt{\frac{\left(\sum_{i=1}^{n}(\alpha Tr_i^2+(1-\alpha)T_i)\right)^2}{n\sum_{i=1}^{n}Tr_i^2}}$$
$$\leq (1-\alpha)\sqrt{\frac{\sum_{i=1}^{n}(\alpha Tr_i^2+(1-\alpha)T_i)^2}{\sum_{i=1}^{n}Tr_i^2}}.$$

Then for $0 \leq \alpha \leq \frac{1}{2}$, by Lemma 6 and Remark 1, we get

$$\begin{aligned}
\partial_1 &\geq \sqrt{\frac{\sum_{i=1}^{n}(\alpha Tr_i^2+(1-\alpha)T_i)^2}{\sum_{i=1}^{n}Tr_i^2}} \\
&\geq (1-\alpha)\sqrt{\frac{\sum_{i=1}^{n}(\alpha Tr_i^2+(1-\alpha)T_i)^2}{\sum_{i=1}^{n}Tr_i^2}}+\frac{2\alpha W(G)}{n} \\
&\geq \sqrt{\frac{2(1-\alpha)^2\sum_{1\leq i<j\leq n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}}{n}}+\frac{2\alpha W(G)}{n}.
\end{aligned}$$

Hence

$$f(x) \leq f\left((1-\alpha)\sqrt{\frac{\sum_{i=1}^{n}(\alpha Tr_i^2+(1-\alpha)T_i)^2}{\sum_{i=1}^{n}Tr_i^2}}\right),$$

and

$$\begin{aligned}
E^{D_\alpha}(G) \leq{}& (1-\alpha)\sqrt{\frac{\sum_{i=1}^{n}(\alpha Tr_i^2+(1-\alpha)T_i)^2}{\sum_{i=1}^{n}Tr_i^2}} \\
&+\sqrt{(n-1)\left(2\zeta-(1-\alpha)^2\left(\frac{\sum_{i=1}^{n}(\alpha Tr_i^2+(1-\alpha)T_i)^2}{\sum_{i=1}^{n}Tr_i^2}\right)\right)}.
\end{aligned}$$

Therefore, the first part of the proof is done. The remainder of the proof is similar to that of Theorem 4, and hence is omitted. □

Now, we give an upper bound for $E^{D_\alpha}(G)$ in terms of the order n, the Wiener index $W(G)$, the transmission degrees, the generalized distance spectral spread $S_{D_\alpha}(G)$ and the parameter α.

Theorem 8 *Let G be a connected graph of order $n \geq 2$. Then*

$$E^{D_\alpha}(G) \leq S_{D_\alpha}(G)+\sqrt{(n-2)\left(2(1-\alpha)^2\sum_{1\leq i<j\leq n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}\right)}. \tag{18}$$

Equality holds if and only if G is a graph with three or four distinct D_α-eigenvalues.

Proof Using the Cauchy-Schwarz inequality for the vectors $(\underbrace{1,1,\ldots,1}_{(n\text{-}2)\text{ times}})$ and $\left(\left|\partial_2-\frac{2\alpha W(G)}{n}\right|,\left|\partial_3-\frac{2\alpha W(G)}{n}\right|,\ldots,\left|\partial_{n-1}-\frac{2\alpha W(G)}{n}\right|\right)$, we have

$$\left(\sum_{i=2}^{n-1}\left|\partial_i-\frac{2\alpha W(G)}{n}\right|\right)^2 \leq \left(\sum_{i=2}^{n-1}1\right)\left(\sum_{i=2}^{n-1}\left|\partial_i-\frac{2\alpha W(G)}{n}\right|^2\right),$$

then

$$E^{D_\alpha}(G) \leq \left|\partial_1-\frac{2\alpha W(G)}{n}\right|+\left|\partial_n-\frac{2\alpha W(G)}{n}\right|$$

$$+\sqrt{(n-2)\left(2\zeta-\left(\left(\partial_1-\frac{2\alpha W(G)}{n}\right)^2+\left(\partial_n-\frac{2\alpha W(G)}{n}\right)^2\right)\right)},$$

where $2\zeta = 2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2+\alpha^2\sum_{i=1}^n Tr_i^2-\frac{4\alpha^2W^2(G)}{n}$. Let v_n be the vertex with $Tr_n = Tr_{\min}$. Taking $\mathbf{x}=(x_1,\ldots,x_n)^T$, where

$$x_i=\begin{cases}1 & \text{if } i=n,\\ 0 & \text{otherwise.}\end{cases}$$

Then by Lemma 3, we have

$$\partial_n(G)\le\frac{\mathbf{x}^TD_\alpha\mathbf{x}}{\mathbf{x}^T\mathbf{x}}=\alpha Tr_{\min}.$$

If $\mathbf{x}$ is the eigenvector of D_α corresponding to $\partial_n(G)$, then it must orthogonal to eigenvector of $\partial_1(G)$. It follows that $\mathbf{x}$ is not the eigenvector of D_α, then $\partial_n(G)<\alpha Tr_{\min}$. Also, since $Tr_{\min}=\frac{nTr_{\min}}{n}\le\frac{\sum_{i=1}^n Tr_i}{n}=\frac{2W(G)}{n}$, hence $\partial_n(G)<\alpha Tr_{\min}\le\frac{2\alpha W(G)}{n}$. Then

$$\begin{aligned}E^{D_\alpha}(G)&\le\partial_1-\partial_n\\&\quad+\sqrt{(n-2)\left(2\zeta-\left(\left(\partial_1-\frac{2\alpha W(G)}{n}\right)^2+\left(\partial_n-\frac{2\alpha W(G)}{n}\right)^2\right)\right)}.\end{aligned}$$

Define a function

$$f(x,y)=S_{D_\alpha}(G)+\sqrt{(n-2)\left(2\zeta-(x^2+y^2)\right)}.$$

Differentiating $f(x,y)$ with respect to x and y, we have

$$f_x=-\frac{x\sqrt{n-2}}{\sqrt{2\zeta-(x^2+y^2)}},\quad f_y=-\frac{y\sqrt{n-2}}{\sqrt{2\zeta-(x^2+y^2)}}$$

$$f_{xx}=-\frac{(2\zeta-y^2)\sqrt{n-2}}{\sqrt{(2\zeta-(x^2+y^2))^3}},\quad f_{yy}=-\frac{(2\zeta-x^2)\sqrt{n-2}}{\sqrt{(2\zeta-(x^2+y^2))^3}},$$

$$f_{xy}=-\frac{xy\sqrt{n-2}}{\sqrt{(2\zeta-(x^2+y^2))^3}}.$$

For maxima or minima, $f_x=0$ and $f_y=0$ which implies $x=y=0$. At this point the values of f_{xx}, f_{yy}, f_{xy} and $\Delta=f_{xx}f_{yy}-f_{xy}^2$ are $f_{xx}=-\sqrt{\frac{n-2}{2\zeta}}$, $f_{yy}=-\sqrt{\frac{n-2}{2\zeta}}$, $f_{xy}=0$ and $\Delta=f_{xx}f_{yy}-(f_{xy})^2=\frac{n-2}{2\zeta}\ge 0$. Then $f(x,y)$ attains maximum value at $x=y=0$, hence $f(0,0)=D_\alpha S(G)+\sqrt{2\zeta(n-2)}$. Thus

$$E^{D_\alpha}(G)\le S_{D_\alpha}(G)+\sqrt{2\zeta(n-2)}.$$

The remainder of the proof is similar to that of Theorem 4, and is omitted here. □

Since $D_\alpha(G)=\alpha Tr(G)+(1-\alpha)D(G)$, by taking $k=1$ and $j=1$ in first inequality of Lemma 7, we get

$$\lambda_1(D_\alpha(G))\le\lambda_1(\alpha Tr(G))+\lambda_1((1-\alpha)D(G)).\tag{19}$$

If $Tr_{\max}=\max_{1\le i\le n}Tr_i$, $Tr_{\min}=\min_{1\le i\le n}Tr_i$ and the distance spread $S_D(G)=\rho_1^D-\rho_n^D$, then from (5) and (19), we have

$$\begin{aligned}\lambda_1(D_\alpha(G))-\lambda_n(D_\alpha(G))&\le\alpha\left(\lambda_1(Tr(G))-\lambda_n(Tr(G))\right)\\&\quad+(1-\alpha)(\lambda_1(D(G))-\lambda_n(D(G)))\\&\le\alpha(Tr_{\max}-Tr_{\min})+(1-\alpha)S_D(G),\end{aligned}$$

hence $S_{D_\alpha}(G)\le\alpha(Tr_{\max}-Tr_{\min})+(1-\alpha)S_D(G)$. Then the following observation follows from Theorem 8.

Theorem 9 *Let G be a connected graph of order $n \geq 2$, having maximum transmission degree $Tr_{\max}$, minimum transmission degree $Tr_{\min}$ and the distance spread $S_D(G)$. Then*

$$E^{D_\alpha}(G) \leq \alpha(Tr_{\max} - Tr_{\min}) + (1-\alpha)S_D(G) + \sqrt{(n-2)\left(2(1-\alpha)^2 \sum_{1\leq i<j\leq n} (d_{ij})^2 + \alpha^2 \sum_{i=1}^{n} Tr_i^2 - \frac{4\alpha^2 W^2(G)}{n}\right)}.$$

Equality holds if and only if G is a graph with three or four distinct D_α-eigenvalues.

4 A concluding remark

The energy of a graph is one of the important topics in spectral graph theory. Graph energy is the energy of a symmetric matrix representing a network. In the literature, there are several different types of matrices that can be used to describe a network (e.g., an adjacency matrix, a Laplacian matrix, a distance matrix, a generalized distance matrix), each of these matrices produces a different energy. In this paper, our attention was focused on upper and lower bounds on the generalized distance energy of graphs in terms of different graph parameters. Moreover, we characterize the extremal graphs. Concluding this paper, we ask the following natural questions:

Problem 10 Which graph gives the maximal or minimal generalized distance energy in terms of n only?

Problem 11 To find some relations between the generalized distance energy and other graph energies for general graphs.

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Declarations

Conflicts of Interest The authors declare no conflict of financial or non-financial interest.

References

1. A. Alhevaz, M. Baghipur, K. C. Das and Y. Shang, Sharp bounds on (generalized) distance energy of graphs, *Mathematics*, 8(3) (2020), Article 426.
2. A. Alhevaz, M. Baghipur, H. A. Ganie and S. Pirzada, Brouwer type conjecture for the eigenvalues of distance signless Laplacian matrix of a graph, *Linear Multilinear Algebra*, 69 (2021), 2423–2440.
3. A. Alhevaz, M. Baghipur, H. A. Ganie and Y. Shang, On the generalized distance energy of graphs, *Mathematics*, 8(1) (2020), Article 17.
4. A. Alhevaz, M. Baghipur, H. A. Ganie and Y. Shang, Bounds for the generalized distance eigenvalues of a graph, *Symmetry*, 11(12) (2019), Article 1529.
5. A. Alhevaz, M. Baghipur, E. Hashemi and H. S. Ramane, On the distance signless Laplacian spectrum of graphs, *Bull. Malay. Math. Sci. Soc.*, 42(5) (2019), 2603–2621.
6. A. Alhevaz, M. Baghipur, E. Hashemi and Y. Shang, Some upper and lower bounds for D_α-energy of graphs, *J. Algebra Comb. Discrete Appl.*, 10(2) (2023), 73–86.
7. A. Alhevaz, M. Baghipur and S. Paul, On the distance signless Laplacian spectral radius and the distance signless Laplacian energy of graphs, *Discrete Math. Algorithm. Appl.*, 10(3) (2018), Article ID: 1850035 (19 pages).
8. M. Aouchiche and P. Hansen, Distance spectra of graphs: a survey, *Linear Algebra Appl.*, 458 (2014), 301–386.
9. M. Aouchiche and P. Hansen, Two Laplacians for the distance matrix of a graph, *Linear Algebra Appl.*, 439 (2013), 21–33.
10. S. Y. Cui, J. X. He and G. X. Tian, The generalized distance matrix, *Linear Algebra Appl.*, 563 (2019), 1–23.
11. K. C. Das, M. Aouchiche and P. Hansen, On (distance) Laplacian energy and (distance) signless Laplacian energy of graphs, *Discrete Appl. Math.*, 243 (2018), 172–185.

12. K. C. Das and I. Gutman, Upper bounds on distance energy, *MATCH Commun. Math. Comput. Chem.*, 86 (2021), 611–620.
13. R. C. Diaz and O. Rojo, Sharp upper bounds on the distance energies of a graph, *Linear Algebra Appl.*, 545 (2018), 55–75.
14. K. Fan. On a theorem of Weyl concerning eigenvalues of linear transformations I. *Proc. Nat. Acad. Sci.*, 35 (1949), 652–655.
15. H. A. Ganie, B. A. Rather, and Y. Shang. On comparison between the distance energies of a connected graph. *Heliyon*, 10(22) (2024), e40316.
16. I. Gutman, The energy of a graph, *Ber. Math. Statist. Sekt. Forschungsz. Graz*, 103 (1978), 1–22.
17. I. Gutman, The energy of a graph: Old and new results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), *Algebraic Combinatorics and Applications,* Springer-Verlag, Berlin, (2001), 196–211.
18. G. Indulal, Sharp bounds on the distance spectral radius and the distance energy of graphs, *Linear Algebra Appl.*, 430 (2009), 106–113.
19. G. Indulal, I. Gutman and A. Vijayakumar, On distance energy of graphs, *MATCH Commun. Math. Comput. Chem.*, 60 (2008), 461–472.
20. X. Li, Y. Shi and I. Gutman, *Graph Energy*, Springer, New York, 2012.
21. H. Minć, *Nonnegative Matrices,* John Wiley and Sons Inc., New York, 1988.
22. S. Pirzada, H. A. Ganie, A. Alhevaz and M. Baghipur, On spectral spread of generalized distance matrix of a graph, *Linear Multilinear Algebra*, 70(15) (2022), 2819–2835.
23. S. Pirzada, H. A. Ganie, B. A. Rather and R. U. Shaban, On generalized distance energy of graphs, *Linear Algebra Appl.*, 603 (2020), 1–19.
24. W. So, Commutativity and spectra of Hermitian matrices, *Linear Algebra Appl.* 212/213 (1994) 121–129.
25. S. Sun, K. C. Das, and Y. Shang, On maximal distance energy, *Mathematics*, 9(4) (2021), 360.
26. J. Yang, L. You and I. Gutman, Bounds on the distance Laplacian energy of graphs, *Kragujevac J. Math.*, 37 (2013), 245–255.
27. F. Zhang, *Matrix Theory: Basic Results and Techniques*, Springer-Verlag, New York, 1999.