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ORIGINAL RESEARCH

Some results on lattices with some maps

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Abstract In this paper, we introduce the notion of generalized permuting tri-derivations, g-derivations, and g-tri-derivations on lattices, and we investigate and generalize some properties discussed in [1] and [2]. We also provided several properties that characterize the g-derivations, g-tri-derivations, and the generalized permuting g-tri-derivations and their trace.

Keywords Lattice · g-derivations · Generalized permuting g-tri-derivations · Permuting tri-derivations

Mathematics Subject Classification 06B35 · 06B99

1 Introduction

In 1940, Birkhoff introduced the concept of lattice theory [3]. Following this, various researchers explored different aspects of the concept, such as the partially ordered sets (Posets) discussed by Hoffmann, and Balbes and Dwinger, who examined distributive lattices in [4]. Additionally, some scholars have investigated the analytic and algebraic properties of lattices [5–7].

In the application side, lattices have played an extremely important role in many disciplines, of information theory [8], information retrieval [9], information access controls [10] and in [11] Durfee applied tools form the geometry of numbers to solve several problems in cryptanalysis. They used algebraic techniques to cryptanalyze several public keycryptosystems and used tools form the theory of integer lattice to get some results.

The derivation is consequential topic to studying, Posner [12] defined the derivation on ring and many researchers studied the derivation theory in rings, near-rings and to BCI-algebras [13–16]. Multiderivations (e.g, bi-derivations, tri-derivations, or n-derivation in generals) have explored prime and semi-prime rings [17–20]. The concept of generalized derivation in rings is introduced by Braser [21] and Hvala [22]. This concept was studied by many researchers for example Agraç and Albas [23] of prime rings, Ozturk and Sapancy [24] on symmetric bi-derivation in prime rings and Jana et all [25] on KUS-algebras. In the context of lattice theory, the

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notion of lattice derivation was defined and developed in [16] by Szasz and was applied by Ferrari [26] and Xin [27] to investigate further properties.

Xin et al. [27] introduced the idea of derivation in lattices and studied various related properties. They also introduced the notion of generalized derivation in lattices and discussed its properties. Alshehri [28] further explored generalized derivation in lattices, building on these initial results.

Recently, in [29] Çeven defined symmetric bi-derivations and its trace for a lattice and proved some results and in [30] he has extended his definitions and theorems to n-derivation of lattices.

Öztürk et all [2] presented the idea of permuting tri-derivations in lattices and studied some related properties. In [1] Çeven also studied the notion of generalized symmetric bi-derivation for lattices and investigated various properties. Our research was primarily inspired by the work in [1] and [2].

Our article is split into two sections. The first half covers the concepts of g-derivations and g-tri-derivations on Lattices, as well as some results, while the second part covers the concept of generalized permuting tri-derivations on a Lattice, as well as its fundamental properties.

In the first part, we defined new notions called g-derivation and g-tri-derivation in Lattices, therefore we enriched this part with several important examples and theorems that characterize the g-derivations and g-tri-derivations.

In the second section, we present a new concept called generalized permuting tri-derivations for a Lattice, as well as examples that demonstrate the existence of this class of application. We also presented several properties that characterize in detail the generalized permutant tri-derivation and their trace, as well as we have generalized some results from the references [1] and [2].

2 Some preliminaries

In this section, we will give some definitions and outcomes that will be useful in the following.

Definition 1 [3, Lemma 1, p: 8] Let L be a nonempty set endowed with operations $\wedge$ and $\vee$. By a lattice $(L, \vee, \wedge)$, we mean a set L satisfying the following conditions:

1. $x \wedge x = x, x \vee x = x$,
2. $x \wedge y = y \wedge x, x \vee y = y \vee x$,
3. $(x \wedge y) \wedge z = x \wedge (y \wedge z), (x \vee y) \vee z = x \vee (y \vee z)$,
4. $(x \wedge y) \vee x = x, (x \vee y) \wedge x = x$

for all $x, y, z \in L$.

Definition 2 [3, Lemma 1, p: 8] Let $(L, \wedge, \vee)$ be a lattice. A binary relation $\leq$ is defined by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$.

Lemma 1 [27, Lemma 2.6, p: 308] *Let $(L, \wedge, \vee)$ be a lattice. Define the binary relation $\leq$ as the Definition 2.2. Then $(L, \leq)$ is a poset and for any $x, y \in L$, $x \wedge y$ is the greatest lower bound of $\{x, y\}$ and $x \vee y$ is the least upper bound of $\{x, y\}$.*

Definition 3 [3, Definition, p: 12] A lattice L is distributive if the identity (1) or (2) holds:

1. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
2. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

In any lattice, the conditions (1) and (2) are equivalent.

Definition 4 [3, Definition, p: 25] An ideal is a non-empty subset I of a lattice L with the properties:

1. $x \leq y, y \in I \Longrightarrow x \in I$,
2. $x, y \in I \Longrightarrow x \vee y \in I$.

If I_1 and I_2 are ideals of lattice L, so $I_1 \cap I_2$ is as well.

Definition 5 [16] Let L be a lattice. A mapping $D\colon L \longrightarrow L$ is a derivation if it satisfies the following condition:

1. $D(x \vee y) = D(x) \vee D(y)$ for all $x, y \in L$.
2. $D(x \wedge y) = (D(x) \wedge y) \vee (x \wedge D(y))$ for all $x, y \in L$.

Definition 6 [2, Definition 6, p: 416] Let L be a lattice. A mapping $D\colon L \times L \times L \longrightarrow L$ is called permuting if it satisfies the following condition:

$$D(x, y, z) = D(x, z, y) = D(y, x, z) = D(y, z, x) = D(z, x, y) = D(z, y, x)$$

for all $x, y, z \in L$.

Definition 7 [2, Definition 6, p: 416] A mapping $d\colon L \longrightarrow L$ defined by $d(x) = D(x, x, x)$ is called the trace of D, where D is a permuting mapping.

Definition 8 [2, Definition 7, p: 416] Let L be a lattice. The map $D\colon L^3 \to L$ will be called a permuting tri-derivation if D is a derivation according to all components; that is,

$$\begin{gathered} D(x \wedge w, y, z) = (D(x, y, z) \wedge w) \vee (x \wedge D(w, y, z)) \\ D(x, y \wedge w, z) = (D(x, y, z) \wedge w) \vee (y \wedge D(x, w, z)) \\ D(x, y, z \wedge w) = (D(x, y, z) \wedge w) \vee (z \wedge D(x, y, w)) \\ \text{for all } x, y, z, w \in L. \end{gathered}$$

Proposition 1 [2] *Let L be a lattice and D be a permuting tri-derivation on L with the trace d. Then the following assertions hold:*

(i) $D(x, y, z) \le x$, $D(x, y, z) \le y$ *and* $D(x, y, z) \le z$,
(ii) $D(x, y, z) \le (x \wedge y) \wedge z$,
(iii) $d(x) \le x$,
(iv) $d^2(x) = d(x)$,

for all $x, y, z \in L$.

Corollary 1 [2, Corollary 1, p: 418] *Let L be a lattice and D be a permuting tri-derivation on L. If* 1 *be the greatest element of L and* 0 *be the smallest element of L, then* $D(x, y, z) = 0$ *if least one the component is* 0*, and* $D(1, x, y) \le x$ *and* $D(1, x, y) \le y$*, for all* $x, y \in L$.

3 Some results on lattices involving g-derivations

This section introduces a new notion called generalized g-derivations and generalized g-tri-derivations for a lattice, followed by an example that demonstrates the existence of this type of application.

Definition 9 Let L_1 and L_2 be two lattices and $F, f, g : L_1 \to L_2$ be tree maps. We say F is generalized g-derivation from L_1 to L_2 associated with f, if $F(x \wedge y) = (F(x) \wedge g(y)) \vee (g(x) \wedge f(y))$ for all $x, y \in L_1$.

From the above definition, for each $x \in L_1$, we have $F(x) = g(x) \wedge (F(x) \vee f(x))$. This implies $F(x) \le g(x)$ for each $x \in L_1$.

Whenever $F = f$, then our definition is the old definition of g-derivation on a lattice L.

Example 1 Let R be a reduced SA-ring (i.e., a ring for which the sum of two annihilator ideals is an annihilator ideal). It is important to know that in this case R is an IN-ring, i.e., for each two ideals I, J of R $Ann(I \cap J) = Ann(I) + Ann(J)$, see [31]. Now consider two lattices $Ann(R)$ (i.e., the lattice of annihilator ideals of R) and $Id(R)$ (i.e., the lattice of ideals of R), see [32]. Put $f : Id(R) \to Ann(R)$, by $f(I) = Ann(I)$ and $g : Id(R) \to Ann(R)$, by $g(I) = R$, for each ideal I of R. Then f is a g-derivation (and f is a generalized g-derivation associated with f). For, $f(I \wedge J) = Ann(I \cap J) = Ann(I) + Ann(J)$. On the other hand, $(f(I) \wedge g(J)) \vee (g(I) \wedge f(J)) = (Ann(I) \cap R) \vee (R \cap Ann(J)) = Ann(I) \vee Ann(J) = Ann(Ann(Ann(I) + Ann(J))) = Ann(I) + Ann(J)$. Moreover, this is a g-derivation which is not an f-derivation. Consider two ideals $I = eR$ and $J = (1-e)R$, for some idempotent $e \in R$. Then $f(I \wedge J) = f(0) = R$, but $f(I) \wedge f(J) = 0$.

We may have a map $f : L_1 \to L_2$ which is not g-derivation for each $g \ge f$. See the next example.

Example 2 Let $L_1 = L_2 = \mathbb{R}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ by $f(1) = 2$ and $f(x) = 3$, for all $x \ne 1$. Then we have $f(-1) = f(1 \wedge -1) \ne f(1) \wedge f(-1)$. Thus f is not f-derivation. Now let $g > f$ be a function from $\mathbb{R}$ to $\mathbb{R}$. Then we have $(f(1) \wedge g(2)) \vee (g(1) \wedge f(2)) > 2 = f(1) = f(1 \wedge 2)$.

Definition 10 Let $f, g : L_1 \to L_2$ be two maps. Define $f \vee g$ and $f \wedge g$ from L_1 to L_2, by $(f \vee g)(x) = f(x) \vee g(x)$ and $(f \wedge g)(x) = f(x) \wedge g(x)$, respectively.

Proposition 2 *Let L_2 be a distributive lattice. The following statements hold.*

1. *If F_1 and F_2 are two generalized g-derivation from L_1 to L_2, then $F_1 \vee F_2$ is a generalized g-derivation.*
2. *If F is generalized g_1 and g_2-derivation from L_1 to L_2, then F is a generalized $g_1 \vee g_2$-derivation.*
3. *If F is generalized g_1 and g_2-derivation from L_1 to L_2, then F is a $g_1 \wedge g_2$-derivation.*

Proof (1) As F_1 and F_2 are two g-derivations from L_1 to L_2 and L_2 is a distributive lattice, we conclude that;

$$\begin{aligned}(F_1 \vee F_2)(x \wedge y) &= F_1(x \wedge y) \vee F_2(x \wedge y)\\ &= [(F_1(x) \wedge g(y)) \vee (g(x) \wedge f_1(y))] \vee [(F_2(x) \wedge g(y)) \vee\\ &\quad (g(x) \wedge f_2(y))]\\ &= ((F_1(x) \vee F_2(x)) \wedge g(y)) \vee (g(x) \wedge (f_1(y) \vee f_2(y))\\ &= ((F_1 \vee F_2)(x) \wedge g(y)) \vee (g(x) \wedge (f_1 \vee f_2)(y)).\end{aligned}$$

So we are done.

(2) By hypothesis, for $x, y \in L_1$, $F(x \wedge y) = (F(x) \wedge g_1(y)) \vee (g_1(x) \wedge f(y))$ and also $F(x \wedge y) = (F(x) \wedge g_2(y)) \vee (g_2(x) \wedge f(y))$. Thus we have;

$$\begin{aligned}F(x \wedge y) &= [(F(x) \wedge g_1(y)) \vee (g_1(x) \wedge f(y))] \vee [(F(x) \wedge g_2(y)) \vee (g_2(x) \wedge f(y))]\\ &= (F(x) \wedge (g_1(y) \vee g_2(y))) \vee ((g_1(x) \vee g_2(x)) \vee f(y))\\ &= (F(x) \wedge (g_1 \vee g_2)(x)) \vee ((g_1 \vee g_2)(y) \wedge f(y)).\end{aligned}$$

(3) By hypothesis, for $x, y \in L_1$, $F(x \wedge y) = (F(x) \wedge g_1(y)) \vee (g_1(x) \wedge f(y))$ and $F(x \wedge y) = (F(x) \wedge g_2(y)) \vee (g_2(x) \wedge f(y))$. This is also important to know that $F(x) \leq g_1(x)$ (resp., $F(y) \leq g_1(y)$) and $F(x) \leq g_2(x)$ (resp., $F(y) \leq g_2(y)$), for all $x \in L_1$ (resp., for all $y \in L_1$). This implies that;

$$\begin{aligned}F(x \wedge y) &= F(x \wedge y) \wedge F(x \wedge y)\\ &= [(F(x) \wedge g_1(y)) \vee (g_1(x) \wedge f(y))] \wedge [(F(x) \wedge g_2(y)) \vee (g_2(x) \wedge f(y))]\\ &= (F(x) \wedge g_1(y) \wedge g_2(y)) \vee (F(x) \wedge g_1(y) \wedge g_2(x) \wedge f(y)) \vee\\ &\quad (g_1(x) \wedge f(y)) \wedge F(x) \wedge g_2(y)) \vee (g_1(x) \wedge f(y) \wedge g_2(x) \wedge f(y))\\ &= (F(x) \wedge (g_1 \wedge g_2)(y)) \vee ((F(x) \wedge g_1(y) \wedge f(y)) \vee ((F(x) \wedge g_2(y) \wedge\\ &\quad f(y))((g_1 \wedge g_2)(x) \wedge f(y)).\end{aligned}$$

As $F(y) \leq (g_1 \wedge g_2)(y)$, the above equality coincides to the $(F(x) \wedge (g_1 \wedge g_2)(y)) \vee ((g_1 \wedge g_2)(x) \wedge f(y))$. So we are done. □

The above proposition gives the following result.

Corollary 2 *Let $F : L_1 \to L_2$ be a generalized g-derivation, $(L_2, \leq, \vee, \wedge)$ a distributive lattice and $C = \{h : L_1 \to L_2 |\ F$ is a h-derivation $\}$. Then $(C, \leq, \vee, \wedge)$ is a distributive lattice.*

A joinitive map is a concept in lattice theory. It refers to a type of function between lattices that preserves the join operation, which is the least upper bound or supremum of elements. More precisely, if F is a joinitive map between two lattices $(L_1, \leq_1)$ and $(L_2, \leq_1)$, then for any two elements $a, b \in L_1$, the following condition holds: $F(a \vee b) = F(a) \vee F(b)$, where $\vee$ denotes the join operation (supremum) in their respective lattices. This means that the image of the join of two elements under F is the join of the images of these elements under F. In other words, a joinitive map preserves the structure of the join operation, ensuring that the function respects the lattice's supremum relationships.

Proposition 3 *The following statements hold.*

1. *Let $F : L_1 \to L_2$. Then F is a generalized F-derivation and a joinitive mapping if and only if F is a homomorphism.*
2. *Let $F, g : L_1 \to L_2$ and F be a g-derivation. Then $F = g$ and is a joinitive mapping if and only if $F(x \vee y) = (F(x) \vee g(y)) \wedge (g(x) \vee F(y))$, for all $x, y \in L_1$.*

Proof (1) $\Rightarrow$). The generalized F-derivation of F implies $F(x \wedge y) = F(x) \wedge F(y)$. And since F is a joinitive mapping, $F(x \vee y) = F(x) \vee F(y)$. So F is a homomorphism.

$\Leftarrow$). By hypothesis, $F(x \vee y) = F(x) \vee F(y)$ and $F(x \wedge y) = F(x) \wedge F(y)$. Thus F is a joinitive mapping and a generalized F-derivation.

(2) $\Rightarrow$). F is g-derivation and $F = g$ is a joinitive map, hence

$$\begin{aligned}(F(x) \vee g(y)) \wedge (g(x) \vee F(y)) &= F(x) \vee F(y) \\ &= F(x \vee y).\end{aligned}$$

$\Leftarrow$). Put $x = y$. Then we have $F(x) = F(x \vee y) = F(x) \vee g(x)$. This implies $F \geq g$. As F is a generalized g-derivation, $F \leq g$. Thus $F = g$, and by hypothesis, $F(x \vee y) = F(x) \vee F(y)$, i.e., $F = g$ is a joinitive mapping. □

Theorem 4 *Let L_2 be a distributive lattice and F be a generalized g-derivation from L_1 to L_2. Then the following conditions are equivalent:*

1. *F is isotone,*
2. *$F(x \wedge y) = F(x) \wedge F(y)$,*
3. *F is a F-derivation,*
4. *$F(x \vee y) \wedge (g(x) \vee g(y)) = F(x) \vee F(y)$.*

Proof (1) $\Rightarrow$ (2). Assume that F is isotone. We can see that $F(x \wedge y) \leq F(x)$ and $F(x \wedge y) \leq F(y)$. Then $F(x \wedge y) \leq F(x) \wedge F(y)^{(*)}$. On the other hand,

$$\begin{aligned}F(x \wedge y) &= (F(x) \wedge g(y)) \vee (g(x) \wedge f(y)) \\ &\geq F(x) \wedge g(y)\end{aligned}$$

and

$$\begin{aligned}F(y \wedge x) &= (F(y) \wedge g(x)) \vee (g(y) \wedge f(x)) \\ &\geq F(y) \wedge g(x)\end{aligned}$$

From the two previous inequalities, we deduce that $F(y \wedge x) \geq F(x) \wedge F(y)^{(**)}$

Combining $(*)$ and $(**)$ we get $F(x \wedge y) = F(x) \wedge F(y)$.

(2) $\Rightarrow$ (1). Let $x \leq y$. Then $x = x \wedge y$, and so

$$\begin{aligned}F(x) &= F(x \wedge y) \\ &= F(x) \wedge F(y).\end{aligned}$$

It follows that $F(x) \leq F(y)$. This shows that F is isotone.

(2) $\Leftrightarrow$ (3). Obviously, F is generalized F-derivation if and only if

$$\begin{aligned}F(x \wedge y) &= (F(x) \wedge F(y)) \vee (F(y) \wedge F(x)) \\ &= F(x) \wedge F(y).\end{aligned}$$

(1) $\Rightarrow$ (4). Assume F is isotone. Since F is generalized g-derivation, we have

$$\begin{aligned}F(x) &= F((x \vee y) \wedge x) \\ &= (F(x \vee y) \wedge g(x)) \vee (g(x \vee y) \wedge F(x)) \\ &= (F(x \vee y) \wedge g(x)) \vee F(x) \\ &= F(x \vee y) \wedge g(x).\end{aligned}$$

Also

$$\begin{aligned}F(y) &= F((x \vee y) \wedge y) \\ &= (F(x \vee y) \wedge g(y)) \vee (g(x \vee y) \wedge F(y)) \\ &= (f(x \vee y) \wedge g(y)) \vee F(y) \\ &= f(x \vee y) \wedge g(y).\end{aligned}$$

Thus

$$F(x) \vee F(y) = (F(x \vee y) \wedge g(x)) \vee (F(x \vee y) \wedge g(y))$$
$$= F(x \vee y) \wedge (g(x) \vee g(y)).$$

(4) $\Rightarrow$ (1). Let $x \leq y$. Then $y = x \vee y$, and so $F(y) = F(x \vee y)$ and $F(y) \vee F(x) = F(y) \wedge (g(x) \vee g(y)) = F(y)$. It follows that $F(x) \leq F(y)$. This shows that F is isotone. □

Definition 11 Let $f, g : L^3 \to L$ be two permuting maps. We say f is a g-tri-derivation on a lattice L if for each $x, y, z, w \in L$, we have

$$f(x \wedge w, y, z) = (f(x, y, z) \wedge g(w, y, z)) \vee (g(x, y, z) \wedge f(w, y, z)),$$

$$f(x, y \wedge w, z) = (f(x, y, z) \wedge g(x, w, z)) \vee (g(x, y, z) \wedge f(x, w, z)),$$

$$f(x, y, z \wedge w) = (f(x, y, z) \wedge g(x, y, w)) \vee (g(x, y, z) \wedge f(x, y, w)).$$

Example 3 1. If $g(x) = x \wedge y \wedge z$ and f satisfies in the above definition, then f is a tri-derivation.
2. Let $a, b \in L, b \leq a, f, g : L^3 \to L$ be two functions by $f(x, y, z) = x \wedge y \wedge z \wedge b$ and $g(x, y, z) = x \wedge y \wedge z \wedge a$, respectively. Then we can see that f is a g-tri-derivation.

A mapping $d_f : L \to L$ defined by $d_f(x) = f(x, x, x)$ is called the trace of f, where f is a g-tri-derivation. Let y and z be two fixed elements of L, we define two maps $f_1, g_1 : L \to L$, by $f_1(x) = f(x, y, z)$ and $g_1(x) = g(x, y, z)$.

Proposition 5 *Let $f, g : L^3 \to L$ and f be a g-tri-derivation. The following statements hold.*

1. *For all $x, y, z \in L$, $f(x, y, z) \leq g(x, y, z)$,*
2. *For all $x \in L$, $d_f(x) \leq d_g(x)$,*
3. *f_1 is a g_1-derivation.*

Proof (1) f is g-derivation, so

$$f(x, y, z) = f(x \wedge x, y, z)$$
$$= (f(x, y, z) \wedge g(x, y, z)) \vee (g(x, y, z) \wedge f(x, y, z))$$
$$= f(x, y, z) \wedge g(x, y, z).$$

This shows $f(x, y, z) \leq g(x, y, z)$

(2) Put $x = y = z$ in Part (1).

(3) Since f is a g-tri-derivation, we have

$$f_1(x \wedge w) = f(x \wedge w, y, z)$$
$$= (f(x, y, z) \wedge g(w, y, z)) \vee (g(x, y, z) \wedge f(w, y, z))$$
$$= (f_1(x) \wedge g_1(w)) \vee (g_1(x) \wedge f_1(w))$$

which means that f_1 is a g_1-derivation. □

4 Generalized permuting g-tri-derivations on Lattice

Definition 12 Let L be a lattice. A mapping $D \colon L \times L \times L \to L$ be a permuting g-tri-derivation and $\Delta \colon L \times L \times L \to L$ be a permuting mapping. Δ is called generalized permuting g-tri-derivation associated with D, if it satisfies the following condition

$$\Delta(x \wedge w, y, z) = (\Delta(x, y, z) \wedge g(w, y, z) \vee (g(x, y, z) \wedge D(w, y, z))$$

For all $x, y, z, w \in L$.

It is obvious that $\Delta(x, y \wedge w, z) = (\Delta(x, y, z) \wedge g(x, w, z)) \vee (g(x, y, z) \wedge D(x, w, z))$ and $\Delta(x, y, z \wedge w) = (\Delta(x, y, z) \wedge g(x, y, w)) \vee (g(x, y, z) \wedge D(x, y, w))$.

Example 4 (i) Let D be a g-tri-derivation on a lattice L. Then D is a generalized permuting g-tri-derivation associated with D.

(ii) Let L be a lattice and $a, b \in L$ with $b < a$. Consider two maps $\Delta(x, y, z) = [(x \wedge y) \wedge z] \wedge b$ and $D(x, y, z) = [(x \wedge y) \wedge z] \wedge a$. Then, D is a permuting gt-ri-derivation on L and we can see that Δ is a generalized permuting tri-derivation associated with D on L, where $g(x, y, z) = x$.

(iii) Let L be a lattice with the least element 0. The mapping defined on L by $D(x, y, z) = 0$ is a permuting tri-derivation on L. Let $a \in L$ and define the mapping Δ on L by $\Delta(x, y, z) = [(x \wedge y) \wedge z] \wedge a$ for all $x, y, z \in L$. Then, we can see that Δ is a generalized permuting g-tri-derivation associated with D on L, where $g(x, y, z) = x$.

Example 5 Let L be a lattice. The mapping $\Delta(x, y, z) = (x \wedge y) \vee z$ is not a generalized permuting g-tri-derivation on L associated with $D(x, y, z) = 0$ and $g(x, y, z) = x$ for all $x \in L$.

Proposition 6 *Let Δ be a generalized permuting g-tri-derivation associated with a permuting g-tri-derivation D. Then the map $f_1 : L \to L$ defined by $f_1(x) = \Delta(x, y, z)$, where y, z are two fixed elements on L, is a generalized id_L-derivation on L.*

Proof We have

$$\begin{aligned} f_1(x \wedge a) &= \Delta(x \wedge a, y, z) \\ &= (\Delta(x, y, z) \wedge g(a, y, z)) \vee (g(x, y, z) \wedge D(a, y, z)) \\ &= (f_1(x) \wedge g_1(a)) \vee (g_1(x) \wedge d_1(a)). \end{aligned}$$

In this equation, the mapping $d_1 : L \to L, d_1(a) = D(a, y, z)$ is a g_1-derivation on L, where D is the permuting g-tri-derivation. Hence, the mapping f_1 is a generalized g_1-derivation on L. □

Theorem 7 *Let L be a lattice, Δ be a generalized permuting g-tri-derivation associated with permuting g-tri-derivation D, δ be the trace of Δ, d be the trace of D and g_1 be the trace of g. Then*

(i) $D(x, y, z) \leq \Delta(x, y, z)$ *for all* $x, y, z \in L$;
(ii) $\Delta(x, y, z) \leq g(x, y, z)$,
(iii) $d(x) \leq \delta(x) \leq g_1(x)$,
(iv) $d(x) = g_1(x) \Rightarrow \delta(x) = g_1(x)$

for all $x, y, z \in L$.

Proof (i) We have

$$\begin{aligned} \Delta(x, y, z) &= \Delta(x \wedge x, y, z) \\ &= (\Delta(x, y, z) \wedge g(x, y, z)) \vee (g(x, y, z) \wedge D(x, y, z)) \\ &= ((\Delta(x, y, z) \wedge g(x, y, z)) \vee D(x, y, z) \end{aligned}$$

It follows that $D(x, y, z) \leq \Delta(x, y, z)$ for all $x, y, z \in L$.

(ii) We have

$$\begin{aligned} \Delta(x, y, z) &= \Delta(x \wedge x, y, z) \\ &= (\Delta(x, y, z) \wedge g(x, y, z)) \vee (g(x, y, z) \wedge D(x, y, z)) \end{aligned}$$

And by (i), $D(x, y, z) \leq \Delta(x, y, z)$ for all $x, y, z \in L$, it follows that $g(x, y, z) \wedge D(x, y, z) \leq g(x, y, z) \wedge \Delta(x, y, z)$, it then $\Delta(x, y, z) = \Delta(x, y, z) \wedge g(x, y, z)$. Which implies that $\Delta(x, y, z) \leq g(x, y, z)$ for all $x, y, z \in L$.

(iii) Using (i) together with (ii), for $x = y = z$ we obtain $d(x) \leq \delta(x) \leq g_1(x)$ for all $x \in L$

(iv) The proof is clear by (iii). □

As an application of the previous theorem, we find the following corollary:

Corollary 3 *Let L be a lattice with a least element* 0*, Δ be a generalized permuting g-tri-derivation associated with a permuting g-tri-derivation D, then $\Delta(x, y, z) = 0$ implies that $D(x, y, z) = 0$.*

Proof The proof is clear. □

Theorem 8 *Let L be a lattice, Δ be a generalized permuting g-tri-derivation associated with a permuting g-tri-derivation D, δ be the trace of Δ and d be the trace of D. Then*

$$\delta(x \wedge y) = g(x, x \wedge y, x \wedge y) \wedge g(y, x \wedge y, x \wedge y).$$

for all $x, y \in L$.

Proof

$$\begin{aligned}
\delta(x \wedge y) &= \Delta(x \wedge y, x \wedge y, x \wedge y) \\
&= (\Delta(x, x \wedge y, x \wedge y) \wedge g(y, x \wedge y, x \wedge y)) \vee (g(x, x \wedge y, x \wedge y) \wedge \\
&\quad D(y, x \wedge y, x \wedge y)) \\
&= [(\Delta(x, x \wedge y, x \wedge y) \wedge g(y, x \wedge y, x \wedge y)) \vee (g(x, x \wedge y, x \wedge y)] \wedge \\
&\quad [(\Delta(x, x \wedge y, x \wedge y) \wedge g(y, x \wedge y, x \wedge y)) \vee D(y, x \wedge y, x \wedge y))] \\
&= \Big(\Delta(x, x \wedge y, x \wedge y) \vee (g(x, x \wedge y, x \wedge y)\Big) \wedge \Big(g(y, x \wedge y, x \wedge y) \vee \\
&\quad g(x, x \wedge y, x \wedge y)\Big) \wedge \Big(\Delta(x, x \wedge y, x \wedge y) \vee D(y, x \wedge y, x \wedge y)\Big) \wedge \\
&\quad \Big(g(y, x \wedge y, x \wedge y) \vee D(y, x \wedge y, x \wedge y)\Big) \\
&= g(x, x \wedge y, x \wedge y) \wedge \Big(g(y, x \wedge y, x \wedge y) \vee g(x, x \wedge y, x \wedge y)\Big) \wedge \\
&\quad \Big(\Delta(x, x \wedge y, x \wedge y) \vee D(y, x \wedge y, x \wedge y)\Big) \wedge g(y, x \wedge y, x \wedge y) \\
&= \Big(g(x, x \wedge y, x \wedge y) \wedge g(y, x \wedge y, x \wedge y)\Big) \wedge \\
&\quad \Big(\Delta(x, x \wedge y, x \wedge y) \vee D(y, x \wedge y, x \wedge y)\Big)
\end{aligned}$$

As $\Delta(x, x \wedge y, x \wedge y) \leq g(x, x \wedge y, x \wedge y)$ and $D(y, x \wedge y, x \wedge y) \leq g(y, x \wedge y, x \wedge y)$, the last equation shows that $\delta(x \wedge y) = g(x, x \wedge y, x \wedge y) \wedge g(y, x \wedge y, x \wedge y)$. And our proof is complete. □

Corollary 4 *Let L be a lattice, and Δ be a generalized permuting g-tri-derivation associated with a permuting g-tri-derivation D, δ be the trace of Δ and g_1 be the trace of g, then*

(i) $\delta(x) = g_1(x)$;
(ii) $\delta^2(x) = g_1^2(x)$;
(iii) $\delta^n(x) = g_1^n(x)$, *where* $n \in \mathbb{N}^*$;

for all $x \in L$.

Proof (i) Taking $x = y$ in theorem 4.2, we get the required result.
(ii) Putting $g_1(x)$ instead of x in (i) and applying it again.
(iii) Reasoning by recurrence on n.

□

Theorem 9 *Let L be a lattice and Δ be a generalized permuting g-tri-derivation associated with a permuting g-tri-derivation D, δ be the trace of Δ, d be the trace of D, and g_1 be the trace of g. Then*

$$d(x) \wedge d(y) \leq \delta(x) \wedge \delta(y) \leq g_1(x) \wedge g_1(y) \ \textit{for all}\ \ x, y \in L.$$

Proof According to theorem 4.1 (iii), we have $d(x) \leq \delta(x) \leq g_1(x)$ and $d(y) \leq \delta(y) \leq g_1(y)$ for all $x, y \in L$. Then according to theorem 4.1 (iii), we obtain $d(x) \wedge d(y) \leq \delta(x) \wedge \delta(y) \leq g_1(x) \wedge g_1(y)$ for all $x, y \in L$. □

Theorem 10 *Let L be a lattice and Δ be a generalized permuting g-tri-derivation associated with a permuting g-tri-derivation D, δ be the trace of Δ and $x \in L$. Let the greatest element of L be 1, then*

(i) *If* $g(1, x, x) \leq \delta(x)$, *then* $\delta(x) = g(1, x, x)$.
(ii) *If* $x \leq \delta(1)$, *then* $\delta(1) \leq g(\delta(1), x, x)$.

(iii) If $x \geq \delta(1)$, *then* $\delta(1) \leq g(x, \delta(1), \delta(1))$.

Proof (i) In view of theorem 4.2, we have $\delta(x \wedge y) = g(x, x \wedge y, x \wedge y) \wedge g(y, x \wedge y, x \wedge y)$ for all $x, y \in L$. In particular, for $y = 1$, we get $\delta(x \wedge 1) = g(x, x \wedge 1, x \wedge 1) \wedge g(1, x \wedge 1, x \wedge 1)$ for all $x \in L$. Since 1 is the greatest element of L, the previous expression gives $\delta(x) = g(x, x, x) \wedge g(1, x, x)$ for all $x \in L$. From corollary 4.2, we have $\delta(x) = g_1(x)$, then $\delta(x) \leq g(1, x, x)$ for all $x \in L$. On the other hand, by hypothesis we have the other inequality, that is $g(1, x, x) \leq \delta(x)$. So that, $\delta(x) = g(1, x, x)$.

(ii) Assume that $x \leq \delta(1)$. Once again, applying theorem 4.2 for $y = \delta(1)$, we infer that $\delta(x \wedge \delta(1)) = g(x, x, x) \wedge g(\delta(1), x, x)$, so that $\delta(x) = g_1(x) \wedge g(\delta(1), x, x)$. Invoking corollary 4.3, we get $\delta(x) \leq g(\delta(1), x, x)$.

(iii) Suppose that $x \geq \delta(1)$. Replacing y by $\delta(1)$ in theorem 4.2 and using our hypothesis, we obtain $\delta(x \wedge \delta(1)) = g(x, \delta(1), \delta(1)) \wedge g(\delta(1), \delta(1), \delta(1))$ which implies that $\delta^2(1) = g(x, \delta(1), \delta(1)) \wedge g_1(\delta(1))$. Accordingly, in virtue of corollary 4.2 (i) and (ii), we conclude that $\delta(1) \wedge g(x, \delta(1), \delta(1))$.

□

Corollary 5 *Let L be a lattice and* Δ *be a generalized permuting g-tri-derivation associated with a permuting g-tri-derivation D,* δ *be the trace of* δ *and the greatest element of L be* 1*, then* $\delta(1) = 1$ *if and only if* $g(1, x, x) = 1$ *for all* $x \in L$.

Proof $\Rightarrow$) By hypothesis, we have $x \leq 1$ for all $x \in L$ and $\delta(1) = 1$, using theorem 4.4 (ii), we obtain $1 \leq g(1, x, x)$ for all $x \in L$. Since 1 is the greatest element of L, thus $g(1, x, x) = 1$ for all $x \in L$.

$\Leftarrow$) Suppose that $g(1, x, x) = 1$ for all $x \in L$. From corollary 4.2 (i), we have $\delta(x) = g(x, x, x)$ for all $x \in L$. In particular, for $x = 1$ we get $\delta(1) = g(1, 1, 1) = 1$. □

Definition 13 Let L be a lattice, the map $\Delta \colon L \times L \times L \to L$ is called joinitive mapping ($\vee$-homomorphism) if it satisfies

$$\Delta(x \vee w, y, z) = \Delta(x, y, z) \vee \Delta(w, y, z)$$

$$\Delta(x, y \vee w, z) = \Delta(x, y, z) \vee \Delta(x, w, z)$$

$$\Delta(x, y, z \vee w) = \Delta(x, y, z) \vee \Delta(x, y, w)$$

For all $x, y, z, w \in L$.

Theorem 11 *Let L be a lattice and* Δ *be a joinitive and permuting g-tri-derivation with the trace* δ *on L, then*

(i) $\delta(x \vee y) = \delta(x) \vee \delta(y) \vee \Delta(x, x, y) \vee \Delta(x, y, y)$;
(ii) $\delta(x) \vee \delta(y) \leq \delta(x \vee y)$,

for all $x, y \in L$.

Proof (i) We have

$$\begin{aligned}
\delta(x \vee y) &= \Delta(x \vee y, x \vee y, x \vee y) \\
&= \Delta(x, x \vee y, x \vee y) \vee \Delta(y, x \vee y, x \vee y) \\
&= \Delta(x, x, x \vee y) \vee \Delta(x, y, x \vee y) \vee \Delta(y, x, x \vee y) \vee \Delta(y, y, x \vee y) \\
&= \Delta(x, x, x \vee y) \vee \Delta(x, y, x \vee y) \vee \Delta(y, y, x \vee y) \\
&= \Delta(x, x, x) \vee \Delta(x, x, y) \vee \Delta(x, y, x) \vee \Delta(x, y, y) \\
&\quad \vee \Delta(y, y, x) \vee \Delta(y, y, y) \\
&= \Delta(x, x, x) \vee \Delta(x, x, y) \vee \Delta(x, y, y) \vee \Delta(y, y, y).
\end{aligned}$$

Then $\delta(x \vee y) = \delta(x) \vee \Delta(x, x, y) \vee \Delta(x, y, y) \vee \delta(y)$ for all $x, y \in L$. (ii) It is obvious from (i). □

Remark 1 The result of the previous theorem remains valid for a generalized permuting g-tri-derivation Δ associated with a permuting g-tri-derivation D and the proof remains the same.

Notation 1 *Let L be a lattice, and* δ *be a mapping of L. We set*

$$Fix_\delta(L) = \{x \in L : \delta(x) = x\} \ \ and \ \ B_\delta(L) = \{x \in L : x \leq \delta(1)\}.$$

The following corollary is an immediate result of Theorem 4.4.

Corollary 6 *Let L be a lattice, and Δ be a generalized permuting g-tri-derivation associated with a permuting g-tri-derivation D with the trace δ, and 1 be the greatest element of L. Then the following properties hold:*

(i) $\delta(x) \in B_\delta(L)$ *for all* $x \in B_\delta(L) \cap Fix_\delta(L)$;
(ii) $g(\delta(1), x, x) \notin B_\delta(L)$ *for all* $x \in B_\delta(L)$;
(iii) If δ *is joinitive, then* $Fix_\delta(L)$ *and* $B_\delta(L)$ *are subgroups of* L;
(iv) $1 \in B_\delta$ *if and only if* $g(1, x, x) = 1$ *for all* $x \in L$;

For all $x, y \in L$.

Proposition 12 *Let L be a lattice, Δ_1 and Δ_2 are a two generalized permuting g-tri-derivation associated with a same permuting g-tri-derivation D, δ_1 be the trace of Δ_1, δ_2 be the trace of Δ_2 and d be the trace of D, then $\delta_1(\delta_2(x)) = \delta_2^2(x)$ and $\delta_2(\delta_1(x)) = \delta_1^2(x)$ for all $x \in L$.*

Proof From Theorem 4.1 (ii) and Theorem 4.2, we get

$$\begin{aligned}
\delta_1(\delta_2(x)) &= \delta_1\big(g_1(x) \wedge \delta_2(x)\big)\\
&= g\Big(g_1(x), g_1(x) \wedge \delta_2(x), g_1(x) \wedge \delta_2(x)\Big)\\
&\quad \wedge g\Big(\delta_2(x), g_1(x) \wedge \delta_2(x), g_1(x) \wedge \delta_2(x)\Big)\\
&= g\Big(g_1(x), \delta_2(x), \delta_2(x)\Big) \wedge g\Big(\delta_2(x), \delta_2(x), \delta_2(x)\Big)\\
&= \delta_2\big(g_1(x) \wedge \delta_2(x)\big)\\
&= \delta_2^2(x).
\end{aligned}$$

For the second equality, you just need to swap the role of δ_1 and δ_2. □

Proposition 13 *Let L be a distributive lattice. The following statements hold.*

1. *If Δ_1 and Δ_2 are generalized permuting g-tri-derivations associated with D, then $\Delta_1 \vee \Delta_2$ is too.*
2. *If Δ_1 and Δ_2 are generalized permuting g-tri-derivations associated with D, then $\Delta_1 \wedge \Delta_2$ is too.*
3. *If Δ is generalized permuting g-tri-derivation associated with D_1 and D_2, then Δ is generalized permuting gtri-derivation associated with $D_1 \vee D_2$.*

Proof (1) By hypothesis for $x, y, w, z \in L$, we have

$$\Delta_1(x \wedge w, y, z) = (\Delta_1(x, y, z) \wedge g(w, y, z)) \vee (g(x, y, z) \wedge D(w, y, z))$$

and

$$\Delta_2(x \wedge w, y, z) = (\Delta_2(x, y, z) \wedge g(w, y, z)) \vee (g(x, y, z) \wedge D(w, y, z)).$$

Thus

$$\begin{aligned}
(\Delta_1 \vee \Delta_2)(x \wedge w, y, z) &= \Delta_1(x \wedge w, y, z) \vee \Delta_2(x \wedge w, y, z)\\
&= (\Delta_1(x, y, z) \wedge g(w, y, z)) \vee (g(x, y, z) \wedge D(w, y, z)) \vee\\
&\quad (\Delta_2(x, y, z) \wedge g(w, y, z)) \vee (g(x, y, z) \wedge D(w, y, z))\\
&= (\Delta_1(x, y, z) \wedge g(w, y, z)) \vee (g(x, y, z) \wedge D(w, y, z)) \vee\\
&\quad (\Delta_2(x, y, z) \wedge g(w, y, z))\\
&= ((\Delta_1 \vee \Delta_2)(x, y, z) \wedge g(w, y, z)) \vee (g(w, y, z) \wedge D(w, y, z)).
\end{aligned}$$

(2) Similar to the Part (1), we have

$$\begin{aligned}
(\Delta_1 \wedge \Delta_2)(x \wedge w, y, z) &= \Delta_1(x \wedge w, y, z) \wedge \Delta_2(x \wedge w, y, z)\\
&= [(\Delta_1(x, y, z) \wedge g(w, y, z)) \vee (g(x, y, z) \wedge D(w, y, z))] \wedge\\
&\quad [(\Delta_2(x, y, z) \wedge g(w, y, z)) \vee (g(x, y, z) \wedge D(w, y, z))]\\
&= \Big(\Delta_1(x, y, z) \wedge g(w, y, z) \wedge \Delta_2(x, y, z)\Big) \vee\\
&\quad \Big(g(x, y, z) \wedge D(w, y, z)\Big)\\
&= \Big(\Delta_1 \wedge \Delta_2)(x, y, z) \wedge g(w, y, z)\Big) \vee \Big(g(w, y, z) \wedge D(x, y, z)\Big).
\end{aligned}$$

(3) By hypothesis for $x, y, w, z \in L$, we have

$$\Delta(x \wedge w, y, z) = (\Delta(x, y, z) \wedge g(w, y, z)) \vee (g(x, y, z) \wedge D_1(w, y, z))$$

and

$$\Delta(x \wedge w, y, z) = (\Delta(x, y, z) \wedge g(w, y, z)) \vee (g(x, y, z) \wedge D_2(w, y, z)).$$

Thus

$$\begin{aligned}\Delta(x \wedge w, y, z) &= (\Delta(x, y, z) \wedge g(w, y, z)) \vee (g(x, y, z) \wedge D_1(w, y, z)) \\ &\quad (\Delta(x, y, z) \wedge g(w, y, z)) \vee (g(x, y, z) \wedge D_2(w, y, z)) \\ &= ((\Delta(x, y, z) \wedge w) \vee (x \wedge (D_1(w, y, z) \vee D_2(w, y, z))) \\ &= (\Delta(x, y, z) \wedge g(w, y, z)) \vee (g(x, y, z) \wedge (D_1 \vee D_2)(w, y, z)).\end{aligned}$$

□

Corollary 7 *Let L be a distributive lattice, D be a g-tri-derivation on L and $\Gamma = \{\Delta : L^3 \to L| \ \ \Delta$ be a generalized permuting g-tri-derivation associated with $D\}$. Then $(\Gamma, \vee, \wedge)$ is a distributive lattice.*

Definition 14 Let L be a lattice, Δ be a generalized permuting g-tri-derivation associated with a permuting g-tri-derivation D and δ be the trace of Δ. δ is called an isotone mapping, if $x \leq y$ implies $\delta(x) \leq \delta(y)$.

Proposition 14 *Let L be a distributive lattice and Δ be a generalized permuting g-tri-derivation related to a permuting g-tri-derivation D, δ be the trace of Δ and d be the trace of D. If 1 is the greatest element of L, then the following conditions are equivalent:*

(i) δ *is an isotone mapping,*
(ii) $\delta(x) \vee \delta(y) \leq \delta(x \vee y)$ *for all* $x, y \in L$.

Proof $(i) \Rightarrow (ii)$. We have $x \leq x \vee y$ and $y \leq x \vee y$. Since δ is an isotone mapping, we obtain $\delta(x) \leq \delta(x \vee y)$ and $\delta(y) \leq \delta(x \vee y)$, so $\delta(x) \vee \delta(y) \leq \delta(x \vee y)$ for all $x, y \in L$.

(ii) $\Rightarrow$ (i). Suppose that $x \leq y$ and $\delta(x) \vee \delta(y) \leq \delta(x \vee y)$. Using the fact that $x \vee y = y$, then $\delta(x) \vee \delta(y) \leq \delta(y)$. Since $\delta(y) \leq \delta(x) \vee \delta(y)$, we find that $\delta(y) = \delta(x) \vee \delta(y)$, which assures that $\delta(x) \leq \delta(y)$ for all $x, y \in L$.
□

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