Indian J Pure Appl Math (2025) 56:1507–1522
https://doi.org/10.1007/s13226-024-00717-x

ORIGINAL RESEARCH

On eccentricity matrices of wheel graphs

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Received: 24 May 2022 / Accepted: 4 November 2024 / Published online: 25 November 2024
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Abstract The eccentricity matrix $E(G)$ of a simple connected graph G is obtained from the distance matrix $D(G)$ of G by retaining the largest distance in each row and column, and by defining the remaining entries to be zero. This paper focuses on the eccentricity matrix $E(W_n)$ of the wheel graph W_n with n vertices. By establishing a formula for the determinant of $E(W_n)$, we show that $E(W_n)$ is invertible if and only if $n \not\equiv 1 \pmod 3$. We determine the inertia of $E(W_n)$ by obtaining the determinant and inertia of the eccentricity matrix of the fan graph. Further, we derive a formula for the inverse of $E(W_n)$ by finding a vector $\mathbf{w} \in \mathbb{R}^n$ and an $n \times n$ symmetric Laplacian-like matrix $\widetilde{L}$ of rank $n-1$ such that

$$E(W_n)^{-1} = -\frac{1}{2}\widetilde{L} + \frac{6}{n-1}\mathbf{w}\mathbf{w}'.$$

Keywords Wheel graph · Fan graph · Eccentricity matrix · Distance matrix · Determinant · Inverse

Mathematics Subject Classification 05C12 · 05C50

1 Introduction

Let G be a simple connected graph on n vertices with the vertex set $\{v_1, v_2, \cdots, v_n\}$. Corresponding to G, several (graph) matrices have been introduced and studied, namely, the adjacency matrix, the Laplacian matrix, the distance matrix, the resistance matrix etc., see [3,7]. We recall the adjacency and distance matrices which are relevant to the discussion here. The *adjacency matrix* $A(G) := (a_{ij})$ of G is an $n \times n$ matrix with $a_{ij} = 1$ if v_i and v_j are adjacent and zero elsewhere. Let $d(v_i, v_j)$ be the length of a shortest path between v_i and v_j. The *distance matrix* of G, denoted by $D(G) := (d_{ij})$, is an $n \times n$ matrix with $d_{ij} = d(v_i, v_j)$ for all i and j. Then $A(G)$ and $D(G)$ are symmetric matrices with diagonal entries all equal to zero. These matrices have been well studied in the literature and have a wide range of applications in chemistry, physics, computer science, etc., see [3,6,7] and the references therein. Note that the adjacency matrix can be derived from the distance matrix by keeping the smallest non-zero distance (which is equal to one) in each row and column and by setting the remaining entries to be zero.

Communicated by Shariefuddin Pirzada.

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Now, let us turn our attention to the eccentricity matrix of a graph. Randić [17] introduced a new graph matrix known as D_{MAX} which is obtained from the distance matrix $D(G)$ by retaining the largest distance in each row and each column of $D(G)$, and setting the rest of the entries to zero. Wang et al. [18] introduced the D_{MAX} matrix in a slightly different form using the notion of the eccentricity of a vertex and called it the eccentricity matrix $E(G)$. The *eccentricity* of a vertex v_i, denoted by $e(v_i)$, is defined by $e(v_i)=\max\{d(v_i, v_j) : 1 \le j \le n\}$. Let us recall the equivalent formulation of D_{MAX} given in [18]. The *eccentricity matrix* of G is an $n \times n$ matrix with

$$(E(G))_{ij} = \begin{cases} d(v_i, v_j) & \text{if } d(v_i, v_j) = \min\{e(v_i), e(v_j)\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $(E(G))_{ij}$ is the (i, j)-th entry of $E(G)$. It is also referred to as the anti-adjacency matrix [18]. Note that $E(G)$ is a real symmetric matrix whose entries are non-negative. This new matrix $E(G)$ is applied to study the boiling point of hydrocarbons [20]. For more details on the applications of $E(G)$ in terms of molecular descriptor, we refer to [17,18,20]. Motivated by the concepts and results of other graph matrices, several spectral properties have been studied for the eccentricity matrix in [15–20]. The relation between the eigenvalues of $A(G)$ and $E(G)$ has been investigated for certain graphs in [18].

Unlike the adjacency and distance matrices of a connected graph, the eccentricity matrix need not be irreducible [18]. The problem of characterizing the irreducible eccentricity matrix posed by Wang et al. [18] remains open. They showed that the eccentricity matrix of a tree is irreducible. Mahato et al. [15] gave an alternative proof. In [19], this result was generalized by providing a class of graphs whose eccentricity matrices are irreducible.

An important problem in graph matrices is to find the determinants and inertias of these matrix classes. A famous result in the theory of distance matrix states that the determinant of a tree T with n vertices is $(-1)^{n-1}(n-1)2^{n-2}$ which depends only on the number of vertices [10]. The inertia of $D(T)$ was also found in [10]. These results were extended to the distance matrix of a weighted tree by Bapat et al. [4]. The inertias of the eccentricity matrices of certain graphs have been investigated in [15]. In this paper, we first study the determinant and inertia of the eccentricity matrix of the wheel graph W_n with n vertices which is irreducible.

The wheel graph has interesting applications in wireless sensor networks and pharmaceutical chemistry, for more details see [1,22]. Many remarkable studies on wheel graphs are documented in the literature. The determinants and inertias of the distance matrices of the wheel graph W_n and the fan graph $W_n - e$ (an edge-deleted subgraph of W_n where the edge e lies on the outer cycle) are studied in [24]. The inverse formulae for $D(W_n)$ and $D(W_n - e)$ are derived in [2] and [12] respectively. It is shown in [25] that the wheel graph W_n is determined by its Laplacian spectrum (the set of all eigenvalues of the Laplacian matrix, see definition below) except for $n = 7$. The number of spanning trees of a wheel graph (in fact, any graph generated by a wheel graph) is explicitly given in [8] without computing its Laplacian spectrum. In this paper, we show that the determinant of $E(W_n)$ is $2^{n-2}(1-n)$ if $n \not\equiv 1 \pmod 3$, which depends only on the number of vertices, and zero if $n \equiv 1 \pmod 3$, see Theorem 3.4. Further, we derive the inertia of $E(W_n)$ by computing the determinant and inertia of the eccentricity matrix of the fan graph.

In order to motivate our next result, we recall the Laplacian matrix L of G. The *Laplacian matrix* of G is given by $L := \bar{D} - A(G)$, where $\bar{D}$ is the diagonal matrix whose i-th diagonal entry is $\deg(v_i)$, the degree of the vertex v_i. Then L is an $n \times n$ symmetric matrix whose row sums are zero and rank of L is $n - 1$, see [3]. The problem of finding the inverses of different graph matrices has been extensively studied in the literature, see [2–5,9,11,12,26]. Graham and Lovász [11] gave an interesting formula for the inverse of the distance matrix $D(T)$ of a tree T which is given by

$$D(T)^{-1} = -\frac{1}{2}L + \frac{1}{2(n-1)}\boldsymbol{\tau}\boldsymbol{\tau}', \tag{1.1}$$

where $\boldsymbol{\tau}' = \big(2 - \deg(v_1), 2 - \deg(v_2), \cdots, 2 - \deg(v_n)\big)$. This result was extended to the distance matrix of a weighted tree in [4] and to the resistance matrix of G, see [3].

Let us recall that a real square matrix $\widetilde{L}$ is called a Laplacian-like matrix [26] if $\widetilde{L}\mathbf{e} = \mathbf{0}$ and $\mathbf{e}'\widetilde{L} = \mathbf{0}'$, where $\mathbf{e}$ denotes the column vector whose entries are all one. Note that all Laplacian matrices are Laplacian-like matrices. Inspired by the result of Graham and Lovász [11], similar inverse formulae have been obtained for the distance matrices of wheel graph W_n when n is even [2], block graphs [5], helm graphs [9], fan graphs [12] and complete graphs [26]. For all these matrices, the inverse formula is expressed as the sum of a Laplacian-like matrix and a rank one matrix.

Motivated by Graham and Lovász type inverse formulae for the distance matrices of several graphs and for the resistance matrices, it is natural to ask whether a similar formula can be obtained for the eccentricity matrices of some classes of graphs. A first attempt is to study the formula for trees. In [16], it is proved that the path on four vertices P_4 and the star graph $K_{1,n}$ are the only trees whose eccentricity matrices are non-singular. Since $D(K_{1,n}) = E(K_{1,n})$, the inverse formula for $E(K_{1,n})$ is given in (1.1). So, the problem of finding the inverse formulae for the eccentricity matrices of trees is completely settled. In order to take an initial step (besides trees) in this respect, we find an inverse formula for $E(W_n)$ when $n \not\equiv 1 \pmod 3$ which is of the form (1.1), see Theorem 5.11. Let us mention that our approach of deriving the formula for $E(W_n)^{-1}$ is quite different from those of $D(W_n)^{-1}$ in [2] and $D(W_n - e)^{-1}$ in [12].

The article is organized as follows. In Section 2, we fix the notations, collect some known results and study some properties of the eccentricity matrix of the wheel graph $E(W_n)$. In Section 3, we establish a formula to compute the determinant of $E(W_n)$. As a consequence, we show that $E(W_n)$ is invertible if and only if $n \not\equiv 1 \pmod 3$. Section 4 deals with the inertia of $E(W_n)$. We also obtain the determinant and the inertia of the eccentricity matrix of the fan graph. In Section 5, we obtain an explicit formula similar to (1.1) for the inverse of $E(W_n)$ when $n \not\equiv 1 \pmod 3$. The inverse formula is expressed as the sum of a symmetric Laplacian-like matrix $\widetilde{L}$ and a rank one matrix which is given by $E(W_n)^{-1} = -\frac{1}{2}\widetilde{L} + \frac{6}{n-1}\mathbf{w}\mathbf{w}'$, where $\mathbf{w} = \frac{1}{6}(7-n, 1, \cdots, 1)' \in \mathbb{R}^n$ and the rank of $\widetilde{L}$ is $n-1$.

2 Preliminaries and properties of $E(W_n)$

In this section, we first introduce a few notations and present some preliminary results on circulant matrices that are needed in the paper. Next, we compute the eccentricity matrix of the wheel graph and study some of its properties. We conclude this section by finding its spectral radius.

We assume that all the vectors are column vectors and are denoted by lowercase boldface letters. As usual I_n is the identity matrix of order n and $\mathbf{e_i}$ is the vector in $\mathbb{R}^n$ with the i-th coordinate 1 and 0 elsewhere. We write $\mathbf{e}_{n\times 1}$ ($\mathbf{0}_{n\times 1}$) to represent the vector in $\mathbb{R}^n$ whose coordinates are all one (respectively, zero). We use the notation J_n ($\mathbf{O_n}$) to denote the matrix with all elements equal to 1 (respectively, 0). We omit the subscript if the orders of the vector and the matrix are clear from the context. For the matrix A, we denote the transpose of A, the null space of A, i-th row of A and j-th column of A by A', $N(A)$, A_{i*} and A_{*j}, respectively. The notation $A > \mathbf{O}$ ($A \geq \mathbf{O}$) means that all the entries of A are positive (respectively, non-negative). The determinant of a square matrix A is denoted by $\det(A)$.

For a vector $\mathbf{c} = (c_1, \cdots, c_n)'$ in $\mathbb{R}^n$, the notation $\mathrm{Circ}(\mathbf{c}')$ stands for the circulant matrix of order n, and $T_n(a, b, c)$ denotes the tridiagonal matrix of order n which are given by

$$\mathrm{Circ}(\mathbf{c}') = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_n \\ c_n & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_n & c_1 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ c_3 & c_4 & c_5 & \cdots & c_1 & c_2 \\ c_2 & c_3 & c_4 & \cdots & c_n & c_1 \end{bmatrix} \text{ and } T_n(a,b,c) = \begin{bmatrix} a & b & 0 & \cdots & 0 & 0 & 0 \\ c & a & b & & 0 & 0 & 0 \\ 0 & c & a & & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \\ 0 & & & & a & b & 0 \\ 0 & \cdots & & & c & a & b \\ 0 & \cdots & & & 0 & c & a \end{bmatrix},$$

respectively where a, b and c are real numbers. The following properties of the circulant matrix will be used later in this paper.

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Suppose that $A = \mathrm{Circ}(\mathbf{x}')$ and $B = \mathrm{Circ}(\mathbf{y}')$. Then

$$\mathrm{Circ}(a\mathbf{x}' + b\mathbf{y}') = aA + bB \text{ and } AB = \mathrm{Circ}(\mathbf{x}'B). \tag{2.1}$$

If A is non-singular then A^{-1} is also circulant. We refer to the books [13,23] for more details.

Throughout this paper, we deal with a circulant matrix of order $n-1$, which is defined by some vector $\mathbf{x}$ in $\mathbb{R}^{n-1}$. On most occasions, the vector $\mathbf{x}$ follows symmetry in its last $n-2$ coordinates. The precise definition of symmetry is given below.

Definition 2.1 ([2,14]) Let n be a positive integer such that $n \geq 4$. A vector $\mathbf{x} = (x_1, x_2 \cdots, x_{n-1})'$ in $\mathbb{R}^{n-1}$ is said to follow symmetry in its last $n-2$ coordinates if

$$x_i = x_{n+1-i} \text{ for all } i = 2, 3, \cdots, n-1.$$

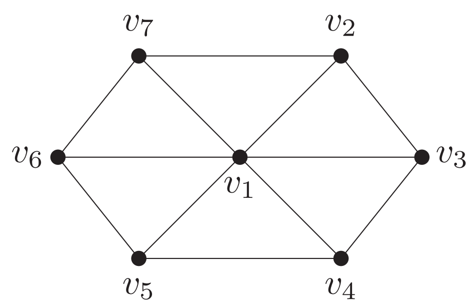

Fig. 1 Wheel graph on seven vertices

Equivalently, $\mathbf{x}$ follows symmetry in its last $n-2$ coordinates if it has the form

$$\mathbf{x} = \begin{cases} (x_1, x_2, x_3, \cdots, x_{\frac{n}{2}}, x_{\frac{n}{2}}, \cdots, x_3, x_2)' & \text{if } n \text{ is even}, \\ (x_1, x_2, x_3, \cdots, x_{\frac{n-1}{2}}, x_{\frac{n+1}{2}}, x_{\frac{n-1}{2}}, \cdots, x_3, x_2)' & \text{if } n \text{ is odd}. \end{cases}$$

In general, a circulant matrix C need not be symmetric, but if the defining vector of C follows symmetry in its last $n-2$ coordinates, then C is symmetric, which is observed in [14]. For the sake of completeness, we give a proof. In fact, the converse is also true.

Lemma 2.2 *Let $n \geq 4$ and $C = \mathrm{Circ}(\mathbf{c}')$ be the circulant matrix of order $n-1$ which is defined by $\mathbf{c} \in \mathbb{R}^{n-1}$. The vector $\mathbf{c}$ follows symmetry in its last $n-2$ coordinates if and only if C is a symmetric matrix.*

Proof Let $\mathbf{c} = (c_1, c_2, \cdots, c_{n-1})'$ and let $C = (a_{ij})$. Define the right shift operator $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T\big((f_1, f_2, \cdots, f_n)'\big) = (f_n, f_1, \cdots, f_{n-1})'$. Then $a_{ii} = c_1$ and i-th row of C is $T^{i-1}(\mathbf{c}') = \big(c_{n-(i-1)}, c_{n-(i-2)}, \cdots, c_{n-2}, c_{n-1}, c_1, c_2, \cdots, c_{n-i}\big)$. Assume that $i < j$. Then $j = i + k$ for some positive integer k. Now,

$$\begin{aligned} a_{ij} &= C_{i*}\mathbf{e_j} = T^{i-1}(\mathbf{c}')\mathbf{e_j} = T^{i-1}(\mathbf{c}')\mathbf{e_{i+k}} = c_{k+1}, \quad \text{and} \\ a_{ji} &= C_{j*}\mathbf{e_i} = T^{j-1}(\mathbf{c}')\mathbf{e_i} = c_{n-(j-i)} = c_{n-k}. \end{aligned}$$

Assume that $\mathbf{c}$ follows symmetry in its last $n-2$ coordinates. Then we have $c_{k+1} = c_{n-k}$ and hence C is symmetric. Conversely, if $a_{ij} = a_{ji}$ then $c_{k+1} = c_{n-k}$. This implies that $c_{n+1-i} = c_i$. Hence the proof. □

2.1 Eccentricity matrix of wheel graph

For $n \geq 4$, the wheel graph on n vertices W_n is a graph containing a cycle of length $n-1$ and a vertex (called the hub) not in the cycle which is adjacent to every other vertex in the cycle. Throughout this paper, we label the vertices of W_n by $v_1, v_2, \cdots v_n$, where the hub of W_n is labelled as v_1 and the vertices in the cycle are labelled as $v_2, \cdots v_n$ clockwise.

We observe that $D(W_4) = E(W_4)$. We assume that $n \geq 5$. Since v_1 is adjacent to all other v_i's in W_n, we have $e(v_1) = 1$. Note that all v_i's $(i \neq 1)$ are adjacent to exactly three vertices. Therefore there exists a vertex v_j such that $d(v_i, v_j) = 2$, where the existence of v_j follows from the assumption $n \geq 5$. Hence, $e(v_i) = 2$ for $i = 2, 3, \cdots, n$. Let $\mathbf{d} = (0, 1, 2, 2, \cdots, 2, 1)'$ and $\mathbf{u} = (0, 0, 2, 2, \cdots, 2, 0)'$ be the vectors fixed in $\mathbb{R}^{n-1}$. Then the distance matrix $D(W_n)$ and the eccentricity matrix $E(W_n)$ of the wheel graph are $n \times n$ symmetric matrices which can be written in the block form

$$D(W_n) = \begin{bmatrix} 0 & \mathbf{e}' \\ \mathbf{e} & \widetilde{D} \end{bmatrix} \text{ and } E(W_n) = \begin{bmatrix} 0 & \mathbf{e}' \\ \mathbf{e} & \widetilde{E} \end{bmatrix}, \tag{2.2}$$

respectively, where $\widetilde{D} = \mathrm{Circ}(\mathbf{d}')$ and $\widetilde{E} = \mathrm{Circ}(\mathbf{u}')$ are circulant matrices of order $n-1$.

2.2 The irreducible property of $E(W_n)$

Let us recall that a square matrix A of order n is said to be *irreducible* if there is no permutation matrix P of order n such that

$$P'AP = \begin{bmatrix} A_1 & A_2 \\ \mathbf{O} & A_3 \end{bmatrix},$$

where A_1 and A_3 are square matrices of order at least one. An equivalent condition for the irreducibility of a non-negative matrix is given below.

Theorem 2.3 *([13]) Let A be a square matrix of order n whose entries are non-negative. Then A is irreducible if and only if* $(I_n + A)^{n-1} > \mathbf{O}$.

The distance matrix of a connected graph is always irreducible but the eccentricity matrix need not be irreducible [18]. Characterizing the irreducible eccentricity matrices remains an open problem. It has been shown that the eccentricity matrix of a tree is irreducible [18] and this result was extended in [19].

In the following proposition, we show that the eccentricity matrix of the wheel graph is irreducible.

Proposition 2.4 *The eccentricity matrix of* W_n *is irreducible.*

Proof For $n = 4$, we have $E(W_n) = D(W_n)$ which is irreducible. If $n \geq 5$, then

$$(I_n + E(W_n))^2 = \begin{bmatrix} n & 2(n-3)\mathbf{e}' \\ 2(n-3)\mathbf{e} & J_{n-1} + \widetilde{E}^2 + 2\widetilde{E} + I_{n-1} \end{bmatrix}.$$

Since $\widetilde{E} \geq \mathbf{O}$, we have $J_{n-1} + \widetilde{E}^2 + 2\widetilde{E} + I_{n-1} > \mathbf{O}$. Now, the proof follows from Theorem 2.3. □

2.3 Euclidean distance matrix

We investigate the concept of the Euclidean distance matrix for $E(W_n)$. A real square matrix $M = (m_{ij})$ of order n is said to be an *Euclidean distance matrix* (EDM) if there exist n vectors $\mathbf{x_1}, \cdots, \mathbf{x_n}$ in $\mathbb{R}^r$ such that $m_{ij} = \|\mathbf{x_i} - \mathbf{x_j}\|_2^2$, where $\|.\|_2$ stands for the Euclidean norm on $\mathbb{R}^r$ [14]. It was shown that $D(W_n)$ is an EDM [14]. It is natural to ask whether $E(W_n)$ is an EDM. The answer to this question is negative. We show that $E(W_n)$ is not an EDM for all $n \geq 5$ by using the following equivalent condition of EDM. Note that $E(W_4)$ is an EDM because $E(W_4) = D(W_4)$.

Theorem 2.5 *(see [14]) Let M be a real symmetric matrix of order n whose diagonal entries are zero. Then M is an EDM if and only if* $\mathbf{x}'M\mathbf{x} \leq 0$ *for all* $\mathbf{x} \in \mathbb{R}^n$ *such that* $\mathbf{x}'\mathbf{e} = 0$.

Proposition 2.6 *For* $n \geq 5$, *the matrix* $E(W_n)$ *is not an Euclidean distance matrix.*

Proof For each n, we find a vector $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x}'\mathbf{e} = 0$ and $\mathbf{x}'E(W_n)\mathbf{x} > 0$. By a simple manipulation, the matrix $E(W_n)$ given in (2.2), can be written as

$$E(W_n) = \begin{bmatrix} 0 & \mathbf{e}' \\ \mathbf{e} & 2J_{n-1} \end{bmatrix} - 2\begin{bmatrix} 0 & \mathbf{0}' \\ \mathbf{0} & F \end{bmatrix}, \quad \text{where} \quad F = \text{Circ}\big((1, 1, \underbrace{0, \cdots, 0}_{\text{(n-4) times}}, 1)\big). \tag{2.3}$$

We first assume that n is odd. Consider the vector

$$\mathbf{y}' = (0, \tilde{\mathbf{y}}') \in \mathbb{R}^n, \text{ where } \tilde{\mathbf{y}} = (1, -1, 1, -1, \cdots, 1, -1)' \in \mathbb{R}^{n-1}.$$

Then $\mathbf{e}'\tilde{\mathbf{y}} = 0$ and $F\tilde{\mathbf{y}} = -\tilde{\mathbf{y}}$. This implies that

$$E(W_n)\mathbf{y} = \begin{bmatrix} \mathbf{e}'\tilde{\mathbf{y}} \\ 2J_{n-1}\tilde{\mathbf{y}} \end{bmatrix} - 2\begin{bmatrix} 0 \\ F\tilde{\mathbf{y}} \end{bmatrix} = 2\begin{bmatrix} 0 \\ \tilde{\mathbf{y}} \end{bmatrix} = 2\mathbf{y}.$$

Further, $\mathbf{y}'E(W_n)\mathbf{y} = 2\mathbf{y}'\mathbf{y} = 2(n-1) > 0$. We now consider the case n is even. Then $n = 2m$ where $m \geq 3$.

Case(i): Assume that m is odd. Consider the vector

$$\mathbf{x} = (0, 1, -1, 1, -1, \cdots, 1, -1, 0, 1, -1, \cdots, 1, -1)' \in \mathbb{R}^n,$$

where zero occurs in the first and $(m+1)$-th coordinates. Let $\mathbf{x}' = (0, \tilde{\mathbf{x}'})$. Then $\mathbf{e}'_{\mathbf{n}\times\mathbf{1}}\mathbf{x} = \mathbf{e}'_{(\mathbf{n-1})\times\mathbf{1}}\tilde{\mathbf{x}} = 0$. Therefore, $E(W_n)\mathbf{x} = -2\left[0\ (F\tilde{\mathbf{x}})'\right]'$. It can be verified that $E(W_n)\mathbf{x} = -2(0, -1, 1, -1, 1, \cdots, -1, 1, -1, 0, 0, 0, 1, -1, \cdots, 1, -1, 1)'$ where zero occurs in the first, m-th, $(m+1)$-th and $(m+2)$-th coordinates. Note that the signs of non-zero coordinates of $E(W_n)\mathbf{x}$ are the same as those of $\mathbf{x}$. Hence, $\mathbf{x}'E(W_n)\mathbf{x} = 2(n-4) > 0$.

Case(ii): Suppose that m is even. In this case, take

$$\mathbf{x} = (0, 1, -1, 1-1, \cdots, 1, -1, 1, 0, -1, 1, -1, 1, \cdots, -1, 1, -1)' \in \mathbb{R}^n,$$

where zero in the first and $(m+1)$-th coordinates. Proceeding similar to the proof of the previous case, we get $\mathbf{x}'E(W_n)\mathbf{x} = 2(n-4) > 0$. Therefore, by Theorem 2.5, the result follows. □

Remark 2.7 In fact, the proof of the above proposition shows that if n is odd, then $\mathbf{y}$ is an eigenvector of $E(W_n)$ corresponding to the eigenvalue 2.

2.4 Spectral radius of $E(W_n)$

The *spectral radius* of a square matrix A, denoted by $\rho(A)$, is the maximum of the moduli of the eigenvalues of A. We recall the fact that the matrix $E(W_n)$ is non-negative and irreducible. Hence, by Perron-Frobenius Theorem [13], the spectral radius $\rho(E(W_n))$ of $E(W_n)$ is an eigenvalue of $E(W_n)$ and there exists a non-negative eigenvector $\mathbf{x}$ corresponding to $\rho(E(W_n))$. That is, $E(W_n)\mathbf{x} = \rho(E(W_n))\mathbf{x}$ and $\mathbf{x} \geq 0$.

We end this section by computing the spectral radius $\rho(E(W_n))$ and a non-negative eigenvector corresponding to $\rho(E(W_n))$. Let us recall the concepts of equitable partition and the associated characteristic matrix [6].

Let A be a real symmetric matrix of order n whose rows and columns are indexed by $X = \{1, 2, \cdots, n\}$. Suppose that $\{X_1, X_2\}$ is a partition of X such that the number of elements in X_1 and X_2 are n_1 and n_2 respectively. Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where each A_{ij} denotes the block submatrix of A of order $n_i \times n_j$ formed by rows in X_i and the columns in X_j. If q_{ij} denotes the average row sum of A_{ij} (the sum of all entries in A_{ij} divided by the number of rows), then the matrix $Q = (q_{ij})$ is said to be a *quotient matrix* of A with respect to the given partition X. If each row sum of the block A_{ij} is constant (that is, $A_{ij}\mathbf{e}_{\mathbf{n_j}} = q_{ij}\mathbf{e}_{\mathbf{n_i}}$, for every i and j), then the partition is called *equitable*. The *characteristic matrix* $C = (c_{ij})$ with respect to the partition X is the $n \times 2$ matrix such that $c_{ij} = 1$, if $i \in X_j$ and 0 otherwise.

We state a result which gives the relation between $\rho(A)$ and $\rho(Q)$.

Theorem 2.8 *([21]) Let A be a non-negative matrix. If Q is the quotient matrix of A with respect to an equitable partition, then the spectral radius of A and Q are equal.*

Using the notion of equitable partition, we now determine the spectral radius of $E(W_n)$.

Theorem 2.9 *The spectral radius of $E(W_n)$ is $(n-4) + \sqrt{n^2 - 7n + 15}$.*

Proof We have $E(W_n) = \begin{bmatrix} 0 & \mathbf{e}' \\ \mathbf{e} & \widetilde{E} \end{bmatrix}$, where $\widetilde{E} = \text{Circ}\big((0, 0, \underbrace{2, \cdots, 2}_{\text{(n-4) times}}, 0)\big)$. Since $\widetilde{E}$ is a circulant matrix, it follows that all the row sums of $\widetilde{E}$ are the same and equal to $2(n-4)$. Now partition the vertex set of W_n as $X = \{\{v_1\}, \{v_2, \cdots, v_n\}\}$. Then X is an equitable partition. So, the quotient matrix of $E(W_n)$ with respect to X is $Q(W_n) = \begin{bmatrix} 0 & n-1 \\ 1 & 2(n-4) \end{bmatrix}$. As the eigenvalues of Q are $(n-4) \pm \sqrt{n^2 - 7n + 15}$, the proof follows by Theorem 2.8. □

From an eigenvector of the quotient matrix Q corresponding to $\rho(Q)$, we can obtain an eigenvector of $E(W_n)$ corresponding to $\rho(E(W_n))$ where the precise statement is given below.

Lemma 2.10 *([6]) Let C be a characteristic matrix of an equitable partition X of A. Let v be an eigenvector of the quotient matrix Q with respect to X for an eigenvalue λ. Then Cv is an eigenvector of A for the same eigenvalue λ.*

Remark 2.11 Let $\rho = (n-4) + \sqrt{n^2 - 7n + 15}$. The characteristic matrix of $E(W_n)$ with respect to X is of order $n \times 2$, whose first row is $\mathbf{e_1}$, and the remaining rows are $\mathbf{e_2}$, where $\mathbf{e_1}, \mathbf{e_2} \in \mathbb{R}^2$. Note that $(\frac{n-1}{\rho}, 1)'$ is an eigenvector of $Q(W_n)$ corresponding to $\rho(Q(W_n))$. By Lemma 2.10, $\big(\frac{1}{\rho}(n-1), \underbrace{1, \cdots, 1}_{\text{(n-1) times}}\big)'$ is an eigenvector of $E(W_n)$ corresponding to $\rho(E(W_n))$.

3 Determinant of $E(W_n)$

A result due to Graham and Pollak [10] showed that the determinant of $D(T)$ is equal to $(-1)^{n-1}(n-1)2^{n-2}$, where T is a tree on n vertices. It is clear that the formula depends only on the number of vertices of T. This result was generalized to the distance matrices of weighted trees in [4]. The determinant of the distance matrix of the wheel graph W_n was studied in [24]. More precisely, $\det(D(W_n)) = 1-n$ if n is even, and $\det(D(W_n)) = 0$ if n is odd.

The primary aim of this section is to establish a formula to find the determinant of $E(W_n)$. We show that

$$\det(E(W_n)) = \begin{cases} 0 & \text{if } n \equiv 1 \pmod 3, \\ 2^{n-2}(1-n) & \text{if } n \not\equiv 1 \pmod 3, \end{cases}$$

which is given in terms of the number of vertices alone. To prove the result, we obtain the recurrence relation (3.4) involving the determinant of the tridiagonal matrix $T_n(-2,-2,-2)$ and a bordered matrix B_n (defined in Lemma 3.3).

We need the following result, which gives the determinant of a tridiagonal matrix.

Theorem 3.1 *([23], Thm. 5.5) Let $T_n(a,b,c)$ be a tridiagonal matrix of order n, where a, b and c are real numbers. If $a^2 \neq 4bc$, then*

$$\det\big(T_n(a,b,c)\big) = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} \text{ where } \alpha = \frac{a+\sqrt{a^2-4bc}}{2} \text{ and } \beta = \frac{a-\sqrt{a^2-4bc}}{2}.$$

In the next two lemmas, we evaluate the determinants of two matrices which will be used to find the $\det(E(W_n))$.

Lemma 3.2 *Let $T_n = T_n(-2,-2,-2)$ be a tridiagonal matrix of order n. Then*

$$\det(T_n) = \begin{cases} 2^n & \text{if } n \equiv 0 \pmod 3, \\ -2^n & \text{if } n \equiv 1 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof We have $\det(T_n) = (-2)^n \det(T_n(1,1,1))$. Using Theorem 3.1, we get

$$\det(T_n) = (-2)^n\left(\frac{\left(\frac{1+i\sqrt{3}}{2}\right)^{n+1} - \left(\frac{1-i\sqrt{3}}{2}\right)^{n+1}}{i\sqrt{3}}\right).$$

Since $\left(\frac{1+i\sqrt{3}}{2}\right)^{n+1} = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{n+1} = \cos(n+1)\frac{\pi}{3} + i\sin(n+1)\frac{\pi}{3}$, we have

$$\det(T_n) = (-2)^n\left(\frac{2\sin\left((n+1)\frac{\pi}{3}\right)}{\sqrt{3}}\right). \tag{3.1}$$

Note that

$$\sin\left((n+1)\frac{\pi}{3}\right) = \begin{cases} (-1)^k\frac{\sqrt{3}}{2} & \text{if } n = 3k \text{ or } n = 3k+1, \\ 0 & \text{if } n = 3k+2. \end{cases} \tag{3.2}$$

The result follows by substituting (3.2) in (3.1). □

Let α be a non-zero number and $i \neq j$. The addition of α times row j (column j) of a matrix A to row i (column i) of A is denoted by $R_i \to R_i + \alpha R_j$ ($C_i \to C_i + \alpha C_j$). Note that the effect of this operation does not change the determinant of A.

Lemma 3.3 *Let T_n be defined as in Lemma 3.2. Let $B_n = \begin{bmatrix} 0 & \mathbf{e}' \\ \mathbf{e} & T_{n-1} \end{bmatrix}$ be a square matrix of order n. Then*

$$\det(B_n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3, \\ 2^{n-2}(\frac{n-1}{3}) & \text{if } n \equiv 1 \pmod 3, \\ -2^{n-2}(\frac{n+1}{3}) & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof We prove the result by induction on n. It is easy to see that the determinants of B_2, B_3 and B_4 are -1, 0 and 4 respectively.

Let $n \geq 5$. Suppose that the result is true for all positive integers less than n. By performing the row operation $R_3 \to R_3 - R_2$ first and then the column operation $C_3 \to C_3 - C_2$ on B_n, we get

$$\det(B_n) = \left| \begin{array}{ccc|cccc} 0 & 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & -2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -2 & 0 & \cdots & 0 \\ \hline 1 & 0 & -2 & & & & \\ \mathbf{e} & \mathbf{0} & \mathbf{0} & & & T_{n-3} & \end{array} \right|.$$

We first expand $\det(B_n)$ along the second row. Thus,

$$\det(B_n) = (-1) \left| \begin{array}{cc|cccc} 1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & -2 & 0 & \cdots & 0 \\ \hline 0 & -2 & & & & \\ \mathbf{0} & \mathbf{0} & & & T_{n-3} & \end{array} \right| + (-2) \left| \begin{array}{cc|cccc} 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & -2 & 0 & \cdots & 0 \\ \hline 1 & -2 & & & & \\ \mathbf{e} & \mathbf{0} & & & T_{n-3} & \end{array} \right|.$$

Next expanding the above determinants along the second column, and the resulting determinants along the second row, we have

$$\begin{aligned} \det(B_n) &= (-1)(2) \left| \begin{array}{cc|c} 1 & 1 & \mathbf{e}' \\ 0 & -2 & \mathbf{0}' \\ \hline 0 & -2 & \\ \mathbf{0} & \mathbf{0} & T_{n-4} \end{array} \right| + (-2)(2) \left| \begin{array}{cc|c} 0 & 1 & \mathbf{e}' \\ 0 & -2 & \mathbf{0}' \\ \hline 1 & -2 & \\ \mathbf{e} & \mathbf{0} & T_{n-4} \end{array} \right| \\ &= (-2)(-2) \begin{vmatrix} 1 & \mathbf{e}' \\ \mathbf{0} & T_{n-4} \end{vmatrix} + (-4)(-2) \ \det(B_{n-3}). \end{aligned}$$

Finally, expand the first determinant along the first column, we obtain the recursive formula

$$\det(B_n) = 4 \ \det(T_{n-4}) + 8 \ \det(B_{n-3}). \tag{3.3}$$

Case(i): Let $n \equiv 0 \pmod 3$. Then, $n-3 \equiv 0 \pmod 3$ and $n-4 \equiv 2 \pmod 3$. From Lemma 3.2 and by induction hypothesis, it follows that $\det(T_{n-4}) = \det(B_{n-3}) = 0$. Hence $\det(B_n) = 0$.

Case(ii): Suppose $n \equiv 1 \pmod 3$. Then, $n-3 \equiv 1 \pmod 3$ and $n-4 \equiv 0 \pmod 3$. By Lemma 3.2, we have $\det(T_{n-4}) = 2^{n-4}$. Using the recurrence relation (3.3) and the induction hypothesis, we get

$$\det(B_n) = 4(2^{n-4}) + 8\left(2^{n-5}\left(\frac{n-4}{3}\right)\right) = 2^{n-2}\left(\frac{n-1}{3}\right).$$

Case(iii): If $n \equiv 2 \pmod 3$, then the proof is similar to that of *Case(ii)*. □

Later in Theorem 4.3, we will show that $\det(B_n)$ is equal to the determinant of the eccentricity matrix of the fan graph.

The main result of this section is the following theorem, which gives the determinant of $E(W_n)$ in terms of the number of vertices of W_n.

Theorem 3.4 *Let $n \geq 5$. Then*

$$\det(E(W_n)) = \begin{cases} 2^{n-2}(1-n) & \text{if } n \not\equiv 1 \pmod 3, \\ 0 & \text{if } n \equiv 1 \pmod 3. \end{cases}$$

Proof From (2.2), we have $\det(E(W_n)) = \begin{vmatrix} 0 & \mathbf{e}' \\ \mathbf{e} & \widetilde{E} \end{vmatrix}$. For each $i = 2, 3, \cdots, n$, perform the row operations $R_i \to R_i - 2R_1$ on $E(W_n)$, we get

$$\det(E(W_n)) = \left|\begin{array}{cc|cccccc} 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -2 & -2 & 0 & \cdots & 0 & -2 \\ \hline 1 & -2 & & & & & \\ 1 & 0 & & & & & \\ \vdots & \vdots & & & T_{n-2} & & \\ 1 & 0 & & & & & \\ 1 & -2 & & & & & \end{array}\right|.$$

We first add the third, fourth, $\cdots$, n-th rows to the second row and then do the similar operations for columns, we get

$$\det(E(W_n)) = \left|\begin{array}{cc|c} 0 & n-1 & \mathbf{e}' \\ n-1 & -6(n-1) & -6\mathbf{e}' \\ \hline \mathbf{e} & -6\mathbf{e} & T_{n-2} \end{array}\right|.$$

By performing the row operation $R_2 \to R_2 + 6R_1$ first and then the column operation $C_2 \to C_2 + 6C_1$, we get

$$\det(E(W_n)) = \left|\begin{array}{cc|c} 0 & n-1 & \mathbf{e}' \\ n-1 & 6(n-1) & \mathbf{0}' \\ \hline \mathbf{e} & \mathbf{0} & T_{n-2} \end{array}\right|.$$

Expanding the above determinant along the second row, we have

$$\det(E(W_n)) = (-1)(n-1) \begin{vmatrix} n-1 & \mathbf{e}' \\ \mathbf{0} & T_{n-2} \end{vmatrix} + 6(n-1)\det(B_{n-1}).$$

Then expand the first determinant along the first column, we get the recursive formula

$$\det(E(W_n)) = (-1)(n-1)^2 \det(T_{n-2}) + 6(n-1)\det(B_{n-1}). \tag{3.4}$$

Case(i): Let $n \equiv 0 \pmod 3$. Then $n-1 \equiv 2 \pmod 3$ and $n-2 \equiv 1 \pmod 3$. Using Lemmas 3.2 and 3.3 in the above recurrence relation (3.4), we get

$$\det(E(W_n)) = (-1)(n-1)^2(-2^{n-2}) + 6(n-1)\Big(-2^{n-3}\,\frac{n}{3}\Big) = 2^{n-2}(1-n).$$

Case(ii): Suppose $n \equiv 1 \pmod 3$. Then, $n-1 \equiv 0 \pmod 3$ and $n-2 \equiv 2 \pmod 3$. Again, from Lemmas 3.2 and 3.3, we have $\det(T_{n-2}) = \det(B_{n-1}) = 0$. Hence, $\det(E(W_n)) = 0$.

Case(iii): If $n \equiv 2 \pmod 3$, then the proof is similar to that of *Case(i)*. □

As an immediate consequence of Theorem 3.4, we have the following result.

Theorem 3.5 *Let $n \geq 5$. Then $E(W_n)$ is invertible if and only if $n \not\equiv 1 \pmod 3$.*

4 Inertia of $E(W_n)$

Let us recall that for a real symmetric matrix A of order n, the *inertia* of A, denoted by $\mathrm{In}(A)$, is the ordered triple $(i_+(A), i_-(A), i_0(A))$, where $i_+(A)$, $i_-(A)$ and $i_0(A)$ respectively denote the number of positive, negative, and zero eigenvalues of A including the multiplicities. It is well known that $i_+(A) + i_-(A)$ is equal to the rank of A.

The inertias of the distance matrices of weighted trees and unicyclic graphs (i.e., connected graphs containing exactly one cycle) have been studied in [4]. It has been shown in [24] that

$$\mathrm{In}(D(W_n)) = \begin{cases} (1, n-1, 0) & \text{if } n \text{ is even,} \\ (1, n-2, 1) & \text{if } n \text{ is odd.} \end{cases}$$

From this result, it is evident that $D(W_n)$ has exactly one positive eigenvalue which is the spectral radius because $D(W_n) \geq 0$. The inertias of the eccentricity matrices of paths and lollipop graphs were investigated in [15]. In this section, we compute the inertia of the eccentricity matrix of the wheel graph using the notion of interlacing property. We also obtain the determinant and the inertia of $E(W_n - e)$ where $W_n - e$ is known as the fan graph.

We begin with an observation that the eccentricity matrix $E(W_n)$ of the wheel graph on n vertices is not a principal submatrix of $E(W_{n+1})$ as $\mathrm{Circ}(\mathbf{m_1}')$ is not a principal submatrix of $\mathrm{Circ}(\mathbf{m_2}')$, where $\mathbf{m_1} = (0, 0, 2, 2, \cdots, 2, 0)' \in \mathbb{R}^{n-1}$ and $\mathbf{m_2} = (0, 0, 2, 2, \cdots, 2, 0)' \in \mathbb{R}^n$. Interestingly, $E(W_n - e)$ is a leading principal submatrix of $E(W_{n+1})$. Without loss of generality, we assume that $e = v_2 v_n$. Then,

$$D(W_n - e) = \begin{bmatrix} 0 & \mathbf{e}' \\ \mathbf{e} & 2J_{n-1} \end{bmatrix} - \begin{bmatrix} 0 & \mathbf{0}' \\ \mathbf{0} & T_{n-1}(2, 1, 1) \end{bmatrix}.$$

Note that the eccentricities of the vertices with respect to the graphs W_n and $W_n - e$ are the same. So,

$$E(W_n - e) = \begin{bmatrix} 0 & \mathbf{e}' \\ \mathbf{e} & 2J_{n-1} \end{bmatrix} - \begin{bmatrix} 0 & \mathbf{0}' \\ \mathbf{0} & T_{n-1}(2, 2, 2) \end{bmatrix}.$$

As $T_{n-1}(2, 2, 2)$ is a leading principal submatrix of $T_n(2, 2, 2)$, it follows that $E(W_n - e)$ is a leading principal submatrix of $E(W_{n+1} - e)$.

Remark 4.1 We observe that

$$E(W_{n+1}) = \begin{bmatrix} 0 & \mathbf{e}' \\ \mathbf{e} & 2J_n \end{bmatrix} - \begin{bmatrix} 0 & \mathbf{0}' \\ \mathbf{0} & 2F \end{bmatrix}, \text{ where } F = \mathrm{Circ}\big((1, 1, \underbrace{0, \cdots, 0}_{\text{(n-3) times}}, 1)\big).$$

If we delete the last row and the last column of $2F$, the resulting matrix is $T_{n-1}(2, 2, 2)$. Hence $E(W_n - e)$ is a leading principal submatrix of $E(W_{n+1})$.

Remark 4.2 After performing the row operations $R_i \rightarrow R_i - 2R_1$ on the matrix $E(W_n - e)$ for each $i = 2, 3, \cdots, n$, it is evident that the resultant matrix is just B_n (defined in Theorem 3.3). Hence we have the following result which gives the determinant of the eccentricity matrix of the fan graph.

Theorem 4.3 *Let $n \geq 5$. Then*

$$\det\big(E(W_n - e)\big) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3, \\ 2^{n-2}(\frac{n-1}{3}) & \text{if } n \equiv 1 \pmod 3, \\ -2^{n-2}(\frac{n+1}{3}) & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

We need the following interlacing theorem to prove the inertia of $E(W_n)$.

Theorem 4.4 *([23]) Let $M = \begin{bmatrix} A & B \\ B' & C \end{bmatrix}$ be a symmetric matrix of order n, where A is a leading principal submatrix of M of order m ($1 \leq m \leq n$). If the eigenvalues of M and A are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_m$ respectively, then $\lambda_i \geq \beta_i \geq \lambda_{n-m+i}$, for all $i = 1, 2, \cdots, m$. In particular, when $m = n - 1$, we have*

$$\lambda_1 \geq \beta_1 \geq \lambda_2 \geq \beta_2 \geq \lambda_3 \geq \cdots \geq \lambda_{n-1} \geq \beta_{n-1} \geq \lambda_n.$$

Moreover, $i_+(M) \geq i_+(A)$ and $i_-(M) \geq i_-(A)$.

We first obtain the inertia of $E(W_n - e)$ which will be used to study the inertia of $E(W_n)$.

Lemma 4.5 *Let $n \geq 5$. Then*

$$\mathrm{In}\big(E(W_n - e)\big) = \begin{cases} (\frac{n}{3}, \frac{2n-3}{3}, 1) & \text{if } n \equiv 0 \pmod 3, \\ (\frac{n+2}{3}, \frac{2n-2}{3}, 0) & \text{if } n \equiv 1 \pmod 3, \\ (\frac{n+1}{3}, \frac{2n-1}{3}, 0) & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof The proof is by induction on the number of vertices n. If $n = 5$ then, by Theorem 4.3, $\det\big(E(W_5 - e)\big)$ is negative. Therefore, $i_0\big(E(W_5 - e)\big) = 0$. Consider the matrix $B = \begin{bmatrix} 0&1&1\\ 1&0&0\\ 1&0&0 \end{bmatrix}$. Then the eigenvalues of B are $\beta_1 = \sqrt{2}$, $\beta_2 = 0$ and $\beta_3 = -\sqrt{2}$. Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5$ be the eigenvalues of $E(W_5 - e)$. Note that B is a leading principal submatrix of $E(W_5 - e)$. By Theorem 4.4, we have $\lambda_1 \geq \beta_1$, $\lambda_2 \geq \beta_2 \geq \lambda_4$ and $\beta_3 \geq \lambda_5$. As λ_i's are non-zero, we have $\lambda_2 > 0$ and $\lambda_4 < 0$. Since $\det\big(E(W_5 - e)\big) < 0$, $i_-\big(E(W_5 - e)\big)$ must be 3. Thus, the inertia of $E(W_5 - e)$ is (2,3,0).

Assume that the result is true for $n - 1$. Since $E(W_{n-1} - e)$ is a leading principal submatrix of $E(W_n - e)$, we use interlacing theorem to the following three cases.

Case(i): Let $n \equiv 0 \pmod 3$. Then $n-1 \equiv 2 \pmod 3$. Since $\det\big(E(W_n - e)\big) = 0$, we have $i_0\big(E(W_n - e)\big) \geq 1$. By induction hypothesis, $\mathrm{In}\big(E(W_{n-1} - e)\big) = \big(\frac{n}{3}, \frac{2n-3}{3}, 0\big)$. Using Theorem 4.4, we get $\mathrm{In}\big(E(W_n - e)\big) = \big(\frac{n}{3}, \frac{2n-3}{3}, 1\big)$.

Case(ii): If $n \equiv 1 \pmod 3$, then $\det\big(E(W_n - e)\big) > 0$. Therefore $i_0\big(E(W_n - e)\big) = 0$. Notice that, in this case, $\mathrm{In}\big(E(W_{n-1} - e)\big) = \big(\frac{n-1}{3}, \frac{2n-5}{3}, 1\big)$. By Theorem 4.4 together with the fact that $i_0\big(E(W_n - e)\big) = 0$, we get $i_+\big(E(W_n - e)\big) \geq \big(\frac{n-1}{3}\big) + 1$ and $i_-\big(E(W_n - e)\big) \geq \big(\frac{2n-5}{3}\big) + 1$. Hence, $\mathrm{In}\big(E(W_n - e)\big) = \big(\frac{n+2}{3}, \frac{2n-2}{3}, 0\big)$.

Case(iii): Suppose $n \equiv 2 \pmod 3$. Then $\det\big(E(W_n - e)\big) < 0$. This implies that $i_0\big(E(W_n - e\big) = 0$. It follows from induction hypothesis that $\mathrm{In}\big(E(W_{n-1} - e)\big) = \big(\frac{n+1}{3}, \frac{2n-4}{3}, 0\big)$. Again, by Theorem 4.4, we have $i_+\big(E(W_n - e)\big) \geq \frac{n+1}{3}$ and $i_-\big(E(W_n - e)\big) \geq \frac{2n-4}{3}$. Note that $\big(\frac{2n-4}{3}\big)$ is an even integer and $\frac{n+1}{3} + \frac{2n-4}{3} = n - 1$. Since $\det\big(E(W_n - e)\big) < 0$, $i_-\big(E(W_n - e)\big)$ must be odd. Thus, $\mathrm{In}\big(E(W_n - e)\big) = \big(\frac{n+1}{3}, \frac{2n-1}{3}, 0\big)$. □

From Theorem 3.5, it follows that the rank of $E(W_n)$ is n if and only if $n \not\equiv 1 \pmod 3$. The next result gives the rank of $E(W_n)$ if $n \equiv 1 \pmod 3$ which will be used to derive the inertia of $E(W_n)$.

Theorem 4.6 *If $n \equiv 1 \pmod 3$, then the rank of $E(W_n)$ is $n - 2$.*

Proof Since $n \equiv 1 \pmod 3$, we have $n = 3k + 1$ for some $k \geq 1$. Consider the vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ which are given by

$$\mathbf{x} = (0, \underbrace{1, 0, -1}, \underbrace{1, 0, -1}, \cdots, \underbrace{1, 0, -1})' \tag{4.1}$$

and

$$\mathbf{y} = (0, \underbrace{0, 1, -1}, \underbrace{0, 1, -1}, \cdots, \underbrace{0, 1, -1})'. \tag{4.2}$$

By simple verification, we can see that $E(W_n)\mathbf{x} = E(W_n)\mathbf{y} = 0$. Moreover, $\mathbf{x}$ and $\mathbf{y}$ are linearly independent. Therefore, the dimension of the null space of $E(W_n)$ is at least two and hence the rank of $E(W_n)$ is at most $n - 2$. To prove the result, it is enough to find an $(n-2) \times (n-2)$ submatrix of $E(W_n)$ whose determinant is non-zero. As $n \equiv 1 \pmod 3$, $n - 2 \equiv 2 \pmod 3$. From Theorem 4.3, we have $\det\big(E(W_{n-2} - e)\big) \neq 0$. Since $E(W_{n-2} - e)$ is a leading principal submatrix of $E(W_n)$, the result follows. □

We now proceed to compute the inertia of the eccentricity matrix of the wheel graph.

Theorem 4.7 *Let $n \geq 5$. The inertia of $E(W_n)$ is*

$$\mathrm{In}\big(E(W_n)\big) = \begin{cases} (\frac{n+3}{3}, \frac{2n-3}{3}, 0) & \textit{if } n \equiv 0 \pmod 3, \\ (\frac{n-1}{3}, \frac{2n-5}{3}, 2) & \textit{if } n \equiv 1 \pmod 3, \\ (\frac{n+1}{3}, \frac{2n-1}{3}, 0) & \textit{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof If $n = 5$, then the proof is similar to that of Lemma 4.5. Let $n \geq 6$. We consider the following three cases.

Case(i): Let $n \equiv 0 \pmod 3$. Then from Theorem 3.4, $\det\big(E(W_n)\big) < 0$. Note that $E(W_{n-1} - e)$ is a leading principal submatrix of $E(W_n)$. By Theorem 4.4 and Lemma 4.5, we have $i_+\big(E(W_n)\big) \geq i_+\big(E(W_{n-1} - e)\big) = \frac{n}{3}$, and $i_-\big(E(W_n)\big) \geq i_-\big(E(W_{n-1} - e)\big) = \frac{2n-3}{3}$. As $\big(\frac{2n-3}{3}\big)$ is an odd number and $\det\big(E(W_n)\big) < 0$, we get $i_+(E(W_n)) = \frac{n}{3} + 1$, the result follows in this case.

Case(ii): Suppose $n \equiv 1 \pmod 3$. By Theorem 4.6, rank of $E(W_n)$ is $n - 2$ and hence $i_0\big(E(W_n)\big) = 2$. Again, using Theorem 4.4 and Lemma 4.5, we get $i_+\big(E(W_n)\big) \geq i_+\big(E(W_{n-1} - e)\big) \geq \frac{n-1}{3}$, and $i_-\big(E(W_n)\big) \geq i_-\big(E(W_{n-1} - e)\big) \geq \frac{2n-5}{3}$. Since the sum of $i_+\big((E(W_n)\big)$, $i_-\big((E(W_n)\big)$ and $i_0\big((E(W_n)\big)$ is equal to n, we have $\mathrm{In}\big(E(W_n)\big) = \big(\frac{n-1}{3}, \frac{2n-5}{3}, 2\big)$.

Case(iii): If $n \equiv 2 \pmod 3$, then the proof goes similar to that of *Case(i)*. □

Remark 4.8 It has been proved that the distance matrix of W_n has a unique positive eigenvalue [24]. But this property does not hold for $E(W_n)$ because it has at least two positive eigenvalues for all $n \geq 5$. From the above result also, we can conclude that $E(W_n)$ is not an EDM (see Theorem 2 in [14]).

5 Inverse formula for $E(W_n)$

Graham and Lovász [11] obtained the formula (1.1) for the inverse of the distance matrix $D(T)$ of a tree T which is expressed as the sum of the Laplacian matrix and a rank one matrix. Motivated by this result, a similar inverse formula has been derived for the distance matrices of several graphs, see [3–5,9,12,26].

Balaji et al. [2] obtained an inverse formula similar to (1.1) for the distance matrix of W_n when n is even. The formula for $D(W_n)^{-1}$ is given as the sum of a Laplacian-like matrix and a rank one matrix. The objective of this section is to find a similar inverse formula for $E(W_n)$ when $n \not\equiv 1 \pmod 3$. We derive this formula by finding a suitable Laplacian-like matrix $\widetilde{L}$ and a vector $\mathbf{w} \in \mathbb{R}^n$ such that

$$E(W_n)\mathbf{w} = (1/\alpha)\mathbf{e} \text{ and } \widetilde{L}E(W_n) + 2I = 2\mathbf{w}\mathbf{e}', \tag{5.1}$$

where α is a real number. If (5.1) holds then $E(W_n)^{-1} = -\frac{1}{2}\widetilde{L} + \alpha\mathbf{w}\mathbf{w}'$ which is the formula we want to obtain. We remark that the proof technique for obtaining $E(W_n)^{-1}$ differs significantly from those of $D(W_n)^{-1}$ in [2] and $D(W_n - e)^{-1}$ in [12].

The next result shows the uniqueness of the Laplacian-like matrix if the inverse of a real symmetric matrix is of the form (1.1).

Theorem 5.1 *Let A be a symmetric non-singular matrix of order n. Suppose there exist $\alpha \in \mathbb{R}$ and $\mathbf{w} \in \mathbb{R}^n$ such that $A^{-1} = -\frac{1}{2}\widetilde{L} + \alpha\mathbf{w}\mathbf{w}'$ for some $n \times n$ matrix $\widetilde{L}$ with $\widetilde{L}\mathbf{e} = \mathbf{0}$. Then the following statements hold.*

(i) The vector $\mathbf{w}$ is unique up to a scalar multiplication.
(ii) The scalar α and the matrix $\widetilde{L}$ are unique.

Proof The non-singularity of A and $\widetilde{L}\mathbf{e} = \mathbf{0}$ guarantee that $\mathbf{w} \neq \mathbf{0}$, $\alpha \neq 0$ and $\mathbf{w}'\mathbf{e} \neq 0$.

(i) Suppose that $A^{-1} = -\frac{1}{2}N + \beta\mathbf{u}\mathbf{u}'$ where $\beta \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{R}^n$ and N is a square matrix of order n such that $N\mathbf{e} = \mathbf{0}$. Then $\alpha\mathbf{w}\mathbf{w}'\mathbf{e} = \beta\mathbf{u}\mathbf{u}'\mathbf{e}$ because $\widetilde{L}\mathbf{e} = N\mathbf{e} = \mathbf{0}$. This implies that $\mathbf{u} = \left(\alpha\mathbf{w}'\mathbf{e}/\beta\mathbf{u}'\mathbf{e}\right)\mathbf{w}$.
(ii) If $A^{-1} = -\frac{1}{2}N + \beta\mathbf{w}\mathbf{w}'$, then $\alpha = \beta$ is obtained by finding $A^{-1}\mathbf{e}$ together with the facts $\mathbf{w} \neq \mathbf{0}$ and $\mathbf{w}'\mathbf{e} \neq 0$. Now it is trivial to verify the uniqueness of $\widetilde{L}$. □

Remark 5.2 If we impose a condition $\mathbf{w}'\mathbf{e} = 1$ on the vector $\mathbf{w}$ in the hypothesis of Theorem 5.1, then $\mathbf{w}$ is unique for A^{-1}.

As mentioned earlier, the primary aim of this section is to find a formula similar to (1.1) for $E(W_n)^{-1}$. To achieve this, it is sufficient to find a vector $\mathbf{w} \in \mathbb{R}^n$ and a matrix $\widetilde{L}$ satisfying the identities given in (5.1). We search for such an $\mathbf{w}$ with an additional condition $\mathbf{w}'\mathbf{e} = 1$. By Remark 5.2, such a vector is unique (if it exists) and hence by Theorem 5.1, the scalar α and the matrix $\widetilde{L}$ in (5.1) are unique. If we choose $\mathbf{w} = \frac{1}{6}(7-n, 1, 1, \cdots, 1)' \in \mathbb{R}^n$ then the first identity $E(W_n)\mathbf{w} = (1/\alpha)\mathbf{e}$ is satisfied with $\mathbf{w}'\mathbf{e} = 1$ and $\alpha = \frac{6}{n-1}$.

Next, we derive a necessary condition for the Laplacian-like matrix $\widetilde{L}$ in Graham and Lovász type formula for $E(W_n)^{-1}$. To do this, let us recall the following two results.

Let $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$. Then $1, \omega, \omega^2, \cdots \omega^{n-1}$ are the n-th roots of unity.

Theorem 5.3 *([23]) Let $\mathbf{c} = (c_0, c_1, \cdots, c_{n-1})' \in \mathbb{R}^n$ and $C = \mathrm{Circ}(\mathbf{c}')$. If $f(\lambda) = c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_{n-1}\lambda^{n-1}$, then the eigenvalues of C are $f(\omega^j)$, $j = 0, 1, \cdots, n-1$.*

Theorem 5.4 *([23]) Let P and S be square matrices. Suppose $A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ and S is non-singular. Then A is non-singular if and only if $H := P - QS^{-1}R$ is non-singular. Moreover, $A^{-1} = \begin{bmatrix} H^{-1} & -H^{-1}QS^{-1} \\ -S^{-1}RH^{-1} & S^{-1} + S^{-1}RH^{-1}QS^{-1} \end{bmatrix}$.*

Theorem 5.5 *Let $n \not\equiv 1 \pmod 3$ and $\mathbf{w} = \frac{1}{6}(7-n, 1, 1, \cdots, 1)' \in \mathbb{R}^n$. If $E(W_n)^{-1} = -\frac{1}{2}\widetilde{L} + \frac{6}{n-1}\mathbf{w}\mathbf{w}'$ then $\widetilde{L} = \frac{1}{3}\begin{bmatrix} n-1 & -\mathbf{e}' \\ -\mathbf{e} & M \end{bmatrix}$, where M is a circulant matrix of order $n-1$.*

Proof We have $E(W_n) = \begin{bmatrix} 0 & \mathbf{e}' \\ \mathbf{e} & \widetilde{E} \end{bmatrix}$ where $\widetilde{E} = \mathrm{Circ}(\mathbf{u}')$ with $\mathbf{u} = (0, 0, 2, 2, \cdots, 2, 0)'$ in $\mathbb{R}^{n-1}$. For $j = 0, 1, 2, \cdots, n-2$, let $z_j = \omega^j$ where $\omega = \cos(\frac{2\pi}{n-1}) + i\sin(\frac{2\pi}{n-1})$, an $(n-1)$-th root of unity. From Theorem 5.3, the eigenvalues of $\widetilde{E}$ are

$$\lambda_1 = 2(n-4) \text{ and } \lambda_{j+1} = 2(z_j^2 + z_j^3 + \cdots + z_j^{n-3}),\ \ j = 1, 2, \ldots, n-2.$$

Since $\sum_{k=0}^{n-2} z_j^k = 0$, λ_{j+1} can be written as

$$\lambda_{j+1} = -2(1 + z_j + z_j^{n-2}) = -2\left(1 + z_j + \frac{1}{z_j}\right) = -2\left(1 + 2\cos\left(\frac{2\pi j}{n-1}\right)\right).$$

Thus, $\det(\widetilde{E}) = (-1)^{n-2}2^{n-1}(n-4)\prod_{j=1}^{n-2}\left(1 + 2\cos\left(\frac{2\pi j}{n-1}\right)\right)$. Note that $\frac{2\pi j}{n-1}$ lies in the interval $(0, 2\pi)$ for $1 \le j \le n-2$. Since $n \not\equiv 1 \pmod 3$, it is easy to see that there is no $j \in \{1, 2, \cdots, n-2\}$ such that $\cos\left(\frac{2\pi j}{n-1}\right) = -\frac{1}{2}$. That is, $\det(\widetilde{E}) \neq 0$. As $\widetilde{E}\mathbf{e} = 2(n-4)\mathbf{e}$, we have $\widetilde{E}^{-1}\mathbf{e} = \frac{1}{2(n-4)}\mathbf{e}$. By Theorem 5.4,

$$E(W_n)^{-1} = \frac{1}{n-1}\begin{bmatrix} 2(4-n) & \mathbf{e}' \\ \mathbf{e} & (n-1)\widetilde{E}^{-1} - \dfrac{1}{2(n-4)} J_{n-1} \end{bmatrix}.$$

Since $\widetilde{L} = 2\left(\frac{6}{n-1}\mathbf{w}\mathbf{w}' - E(W_n)^{-1}\right)$, the desired result can be obtained by using (2.1) and the fact that the inverse of a circulant matrix is circulant. □

The following theorem gives a sufficient condition on $\widetilde{L}$ such that the second identity in (5.1) is satisfied.

Theorem 5.6 *Let* $\widetilde{E} = \mathrm{Circ}(\mathbf{u}')$ *where* $\mathbf{u} = (0, 0, 2, 2, \cdots, 2, 0)' \in \mathbb{R}^{n-1}$ *and* $\mathbf{w} = \frac{1}{6}(7-n, 1, 1, \cdots, 1)' \in \mathbb{R}^n$. *Let M be a circulant symmetric matrix of order $n-1$ such that* $M\mathbf{e} = \mathbf{e}$ *and* $M\widetilde{E} + 6I_{n-1} = 2J_{n-1}$. *If* $\widetilde{L} = \frac{1}{3}\begin{bmatrix} n-1 & -\mathbf{e}' \\ -\mathbf{e} & M \end{bmatrix}$ *then $\widetilde{L}$ is a symmetric Laplacian-like matrix satisfying* $\widetilde{L}E(W_n) + 2I_n = 2\mathbf{w}\mathbf{e}'$.

Proof Since $M\mathbf{e} = \mathbf{e}$, it is easy to see that $\widetilde{L}\mathbf{e} = \mathbf{0}$. Postmultiplying the matrix $\widetilde{L}$ by $E(W_n)$, we get

$$\widetilde{L}E(W_n) = \frac{1}{3}\begin{bmatrix} 1-n & (n-1)\mathbf{e}' - \mathbf{e}'\widetilde{E} \\ M\mathbf{e} & M\widetilde{E} - J_{n-1} \end{bmatrix}.$$

Using the fact $\widetilde{E}\mathbf{e} = 2(n-4)\mathbf{e}$ and the conditions $M\mathbf{e} = \mathbf{e}$ and $M\widetilde{E} + 6I_{n-1} = 2J_{n-1}$, we have

$$\widetilde{L}E(W_n) + 2I_n = \frac{1}{3}\begin{bmatrix} 7-n & (7-n)\mathbf{e}' \\ \mathbf{e} & J_{n-1} \end{bmatrix} = 2\mathbf{w}\mathbf{e}'.$$

Hence the proof. □

To construct $\widetilde{L}$, we need to find the matrix M explicitly. That is, to find the vector $\mathbf{x} \in \mathbb{R}^{n-1}$ such that $M = \mathrm{Circ}(\mathbf{x}')$. We start with computing the inverse of the eccentricity matrix of the wheel graph on six vertices to find $\mathbf{x}$ in this case.

Example 5.7 Consider the eccentricity matrix of W_6 which is given by

$$E(W_6) = \begin{bmatrix} 0 & \mathbf{e}' \\ \mathbf{e}_{5\times 1} & \mathrm{Circ}((0, 0, 2, 2, 0)) \end{bmatrix}.$$

Note that $\mathbf{w} = \frac{1}{6}(1, 1, 1, 1, 1, 1)'$ and $\alpha = \frac{6}{5}$. Then by direct verification, we see that

$$E(W_6)^{-1} = \frac{1}{5}\left[\begin{array}{c|ccccc} -4 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & \frac{-3}{2} & 1 & 1 & \frac{-3}{2} \\ 1 & \frac{-3}{2} & 1 & \frac{-3}{2} & 1 & 1 \\ 1 & 1 & \frac{-3}{2} & 1 & \frac{-3}{2} & 1 \\ 1 & 1 & 1 & \frac{-3}{2} & 1 & \frac{-3}{2} \\ 1 & \frac{-3}{2} & 1 & 1 & \frac{-3}{2} & 1 \end{array}\right] = -\frac{1}{2}\widetilde{L}(W_6) + \frac{6}{5}\mathbf{w}\mathbf{w}',$$

where the Laplacian-like matrix $\widetilde{L}(W_6)$ is given by

$$\widetilde{L}(W_6) = \frac{1}{3}\left[\begin{array}{c|ccccc} 5 & -1 & -1 & -1 & -1 & -1 \\ \hline -1 & -1 & 2 & -1 & -1 & 2 \\ -1 & 2 & -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 & -1 & 2 \\ -1 & 2 & -1 & -1 & 2 & -1 \end{array}\right].$$

That is, $\widetilde{L}(W_6) = \frac{1}{3}\begin{bmatrix} 5 & -\mathbf{e}' \\ -\mathbf{e} & M \end{bmatrix}$ where $M = \mathrm{Circ}((-1, 2, -1, -1, 2))$.

The above example illustrates the vector $\mathbf{x} = (-1, 2, -1, -1, 2)' \in \mathbb{R}^5$ for the case $n = 6$. Note that the vector $\mathbf{x}$ follows symmetry in its last four coordinates. That is, the second coordinate is equal to the last coordinate of $\mathbf{x}$, and the third and fourth coordinates of $\mathbf{x}$ are the same. In fact, it is observed that the same kind of pattern is retained for the matrices of higher orders.

The following significant vectors $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ in $\mathbb{R}^{n-1}$ are identified from the observations made from numerical examples which play a central role in finding a formula for $E(W_n)^{-1}$.
Let $n \equiv 2 \pmod 3$. Define

$$\bar{\mathbf{x}} = (1,\ \underbrace{1, -2, 1},\ \cdots,\ \underbrace{1, -2, 1})'. \tag{5.2}$$

If $n \equiv 0 \pmod 3$, we define

$$\bar{\mathbf{y}} = (-1,\ \underbrace{2, -1, -1},\ \cdots,\ \underbrace{2, -1, -1},\ 2)'. \tag{5.3}$$

Remark 5.8 Balaji et al. [2] derived an inverse formula similar to (1.1) for $D(W_n)$ when n is even. They had to deal only with the case when n is even and introduced only one special vector. But in the case of $E(W_n)^{-1}$, we have to deal with two cases $n \equiv 2 \pmod 3$ and $n \equiv 0 \pmod 3$.

Now we construct a symmetric Laplacian-like matrix $\widetilde{L}$ of order n for W_n using Theorems 5.5 and 5.6 and the above defined vectors $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$. Later we will show that $\widetilde{L}$ is a symmetric matrix of rank $n-1$ (see Theorem 5.12).

Definition 5.9 Let $n \not\equiv 1 \pmod 3$ and $n \geq 5$. Define

$$\widetilde{L} := \frac{1}{3}\begin{bmatrix} n-1 & \mathbf{e}' \\ \mathbf{e} & M \end{bmatrix}, \tag{5.4}$$

where

$$M = \begin{cases} \mathrm{Circ}(\bar{\mathbf{x}}') & \text{if } n \equiv 2 \pmod 3, \\ \mathrm{Circ}(\bar{\mathbf{y}}') & \text{if } n \equiv 0 \pmod 3. \end{cases} \tag{5.5}$$

Remark 5.10 The matrix M can also be introduced as a sum of circulant matrices along the lines of that given for $D(W_n)$ in [2] and similar analysis can be carried out. However, our approach is different from that of [2].

We are now ready to give a formula for the inverse of the eccentricity matrix of W_n which is similar to the form of (1.1). That is, the inverse of $E(W_n)$ is expressed as a sum of a symmetric Laplacian-like matrix of rank $n-1$ and a rank one matrix.

Theorem 5.11 *Let $n \geq 5$ and $n \not\equiv 1 \pmod 3$. Consider the matrix $\widetilde{L}$ given in (5.4). Suppose that $\mathbf{w} = \frac{1}{6}(7-n, 1, \cdots, 1)' \in \mathbb{R}^n$. Then $\widetilde{L}$ is a symmetric Laplacian-like matrix such that*

$$E(W_n)^{-1} = -\frac{1}{2}\widetilde{L} + \frac{6}{n-1}\mathbf{w}\mathbf{w}'.$$

Proof Since each of $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ follows symmetry in its last $n-2$ coordinates, the matrix M defined in (5.5) is symmetric by Lemma 2.2. Therefore, $\widetilde{L}$ is symmetric. Also, we have $E(W_n)\mathbf{w} = \frac{n-1}{6}\mathbf{e}$. In order to get the desired formula, we need to obtain the second identity in (5.1). By Theorem 5.6, it is enough to prove that $M\mathbf{e} = \mathbf{e}$ and $M\widetilde{E} + 6I_{n-1} = 2J_{n-1}$. It is easy to see that $\mathbf{e}'\bar{\mathbf{x}} = \mathbf{e}'\bar{\mathbf{y}} = 1$. Since M is a circulant matrix, all the row sums of M are equal to $\mathbf{e}'\bar{\mathbf{x}}$ if $n \equiv 2 \pmod 3$ and $\mathbf{e}'\bar{\mathbf{y}}$ if $n \equiv 0 \pmod 3$. Therefore, $M\mathbf{e} = \mathbf{e}$. Now we claim that $M\widetilde{E} + 6I_{n-1} = 2J_{n-1}$. Since $\widetilde{E} = \mathrm{Circ}(\mathbf{u}')$ where $\mathbf{u}' = (0, 0, 2, \cdots, 2, 0) \in \mathbb{R}^{n-1}$, the columns of $\widetilde{E}$ can be written as

$$\widetilde{E}_{*i} = \begin{cases} 2(\mathbf{e} - \mathbf{e_1} - \mathbf{e_2} - \mathbf{e_{n-1}}) & \text{if } i = 1, \\ 2(\mathbf{e} - \mathbf{e_{i-1}} - \mathbf{e_i} - \mathbf{e_{i+1}}) & \text{if } 2 \le i \le n-2, \\ 2(\mathbf{e} - \mathbf{e_1} - \mathbf{e_{n-2}} - \mathbf{e_{n-1}}) & \text{if } i = n-1. \end{cases}$$

First, assume that $n \equiv 2 \pmod 3$. Note that M and $\widetilde{E}$ are circulant. Using (2.1), we write

$$M\widetilde{E} = \mathrm{Circ}(\bar{\mathbf{x}}'\widetilde{E}), \text{ where } \bar{\mathbf{x}} = \big(1, \underbrace{1, -2, 1}, \cdots, \underbrace{1, -2, 1}\big)' \in \mathbb{R}^{n-1}.$$

Let $\mathbf{z}' = \bar{\mathbf{x}}'\widetilde{E} = (z_1, \cdots, z_{n-1})$. Since $\bar{\mathbf{x}}'\mathbf{e} = 1$, we have $z_1 = \bar{\mathbf{x}}'\widetilde{E}_{*1} = 2\bar{\mathbf{x}}'(\mathbf{e} - \mathbf{e_1} - \mathbf{e_2} - \mathbf{e_{n-1}}) = -4$, and $z_{n-1} = \bar{\mathbf{x}}'\widetilde{E}_{*(n-1)} = 2\bar{\mathbf{x}}'(\mathbf{e} - \mathbf{e_1} - \mathbf{e_{n-2}} - \mathbf{e_{n-1}}) = 2$. Note that the sum of any three consecutive coordinates of $\bar{\mathbf{x}}$ is zero. So, we get $z_i = \bar{\mathbf{x}}'\widetilde{E}_{*i} = 2\bar{\mathbf{x}}'\mathbf{e} = 2$ for $i = 2, 3, \cdots, n-2$. Thus, $\mathbf{z} = (-4, 2, 2, \cdots, 2)'$. Suppose $n \equiv 0 \pmod 3$. Then $M\widetilde{E} = \mathrm{Circ}(\bar{\mathbf{y}}'\widetilde{E})$. By the same arguments as above, we have $\bar{\mathbf{y}}'\widetilde{E} = \mathbf{z}'$. Therefore $M\widetilde{E} + 6I_{n-1} = \mathrm{Circ}(\mathbf{z}') + \mathrm{Circ}(6\mathbf{e_1'}) = \mathrm{Circ}(2\mathbf{e}') = 2J_{n-1}$. Hence, $\widetilde{L}$ is a Laplacian-like matrix such that $\widetilde{L}E(W_n) + 2I_n = 2\mathbf{w}\mathbf{e}'$. This completes the proof. □

In the following, we show that the Laplacian-like matrix $\widetilde{L}$, given in (5.9), is of rank $n-1$. Hence $\widetilde{L}$ has some of the properties of the Laplacian matrix. The proof of the following theorem is similar to that of Theorem 2 in [2], which is given for the sake of completeness.

Theorem 5.12 *Consider the $n \times n$ matrix $\widetilde{L}$ given in (5.4). Then the rank of $\widetilde{L}$ is $n-1$.*

Proof Suppose $\widetilde{L}\mathbf{x} = 0$ where $\mathbf{x}$ is a non-zero vector in $\mathbb{R}^n$. Then $\mathbf{x}'\widetilde{L}E(W_n) = 0$. Using the relation $\widetilde{L}E(W_n) + 2I = 2\mathbf{w}\mathbf{e}'$, we get $\mathbf{x}' = (\mathbf{x}'\mathbf{w})\mathbf{e}'$ where the vector $\mathbf{w} = \frac{1}{6}(7-n, 1, 1, \cdots, 1)'$. Thus, $\mathbf{x}$ is a scalar multiple of $\mathbf{e}$. This gives that the dimension of nullspace of $\widetilde{L}$ is at most one. Since $\widetilde{L}\mathbf{e} = 0$, we have nullity of $\widetilde{L}$ is one. Hence rank of $\widetilde{L}$ is $n-1$. □

Concluding remarks

In this article, we studied the eccentricity matrix $E(W_n)$ of the wheel graph W_n. We computed the determinants and the inertias of $E(W_n)$ and $E(W_n - e)$. For the non-singular case, we established a formula for the inverse of $E(W_n)$ which was expressed as the sum of a symmetric Laplacian-like matrix and a rank one matrix. It would be interesting to study similar inverse formulae for the eccentricity matrices of other graphs.

Acknowledgements We thank the anonymous reviewer for his/her comments and suggestions. The first author wishes to thank NIT Tiruchirappalli for financial support under seed grant project. The second author acknowledges financial support from the Council of Scientific and Industrial Research, (CSIR), India.

Conflict of interest No potential conflict of interest was reported by the authors.

References

1. H. M. Afzal Siddiqui and M. Imran, Computing the metric dimension of wheel related graphs, *Appl. Math. Comput.* 242 (2014), 624–632.
2. R. Balaji, R. B. Bapat and S. Goel, An inverse formula for the distance matrix of a wheel graph with an even number of vertices, *Linear Algebra Appl.* 610 (2021), 274–292.
3. R. B. Bapat, *Graphs and matrices*, Springer, London, 2014.
4. R.B. Bapat, S. J. Kirkland and M. Neumann, On distance matrices and Laplacians, *Linear Algebra Appl.* 401 (2005), 193–209.

5. R. B. Bapat and S. Sivasubramanian, Inverse of the distance matrix of a block graph, *Linear Multilinear Algebra* 59 (2011), no. 12, 1393–1397.
6. A. E. Brouwer and W. H. Haemers, *Spectra of graphs*, Springer, New York, 2012.
7. D. Cvetković, P. Rowlinson and S. Simić, *An introduction to the theory of graph spectra*, Cambridge University Press, Cambridge, 2010.
8. S. N. Daoud, Complexity of graphs generated by wheel graph and their asymptotic limits, *J. Egyptian Math. Soc.* 25 (2017), no. 4, 424–433.
9. S. Goel, On distance matrices of helm graphs obtained from wheel graphs with an even number of vertices, *Linear Algebra Appl.* 621 (2021), 86–104.
10. R. L. Graham and H. O. Pollak, On the addressing problem for loop switching, *Bell System Tech. J.* 50 (1971), 2495–2519.
11. R. L. Graham and L. Lovász, Distance matrix polynomials of trees, *Adv. Math.* 29 (1978), 60–88.
12. C. Hao, S. Li and L. Zhang, An inverse formula for the distance matrix of a fan graph, *Linear Multilinear Algebra* (2021). https://doi.org/10.1080/03081087.2021.2011827
13. R. A. Horn and C. R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 2013.
14. G. Jaklič and J. Modic, Euclidean graph distance matrices of generalizations of the star graph, *Appl. Math. Comput.* 230 (2014), 650–663.
15. I. Mahato, R. Gurusamy, M. Rajesh Kannan and S. Arockiaraj, Spectra of eccentricity matrices of graphs, *Discrete Appl. Math.* 285 (2020), 252–260.
16. I. Mahato, R. Gurusamy, M. Rajesh Kannan and S. Arockiaraj, On the spectral radius and the energy of eccentricity matrix of a graph, *Linear Multilinear Algebra.* (2021). https://doi.org/10.1080/03081087.2021.2015274
17. M. Randić, D_{MAX}—matrix of dominant distances in a graph, *MATCH Commun. Math. Comput. Chem.* 70 (2013), 221–238.
18. J. Wang, M. Lu, F. Belardo and M. Randić, The anti-adjacency matrix of a graph: eccentricity matrix, *Discrete Appl. Math.* 251 (2018), 299–309.
19. J. Wang, M. Lu, L. Lu and F. Belardo, Spectral properties of the eccentricity matrix of graphs, *Discrete Appl. Math.* 279 (2020), 168–177.
20. J. Wang, X. Lei, W. Wei, X. Luo and S. Li, On the eccentricity matrix of graphs and its applications to the boiling point of hydrocarbons. *Chemometr Intell Lab.* 207 (2020), 104173.
21. L. You, M. Yang, W. So and W. Xi, On the spectrum of an equitable quotient matrix and its application, *Linear Algebra Appl.* 577 (2019), 21–40.
22. X. Yu, L. Zhou and X. Li, A novel hybrid localization scheme for deep mine based on wheel graph and chicken swarm optimization, *Computer Networks* 154 (2019), 73–78.
23. F. Zhang, *Matrix Theory*, Springer, New York, 2011.
24. X. Zhang and C. Song, The distance matrices of some graphs related to wheel graphs, *J. Appl. Math.* 2013, Art. ID 707954, 5 pp.
25. Y. Zhang, X. Liu and X. Yong, Which wheel graphs are determined by their Laplacian spectra?, *Comput. Math. Appl.* 58 (2009), no. 10, 1887–1890.
26. H. Zhou, The inverse of the distance matrix of a distance well-defined graph, *Linear Algebra Appl.* 517 (2017), 11–29.

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