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ORIGINAL RESEARCH

Joint Functional Independence of the Riemann Zeta-Function

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Abstract By the Ostrowski theorem, the Riemann zeta-function $\zeta(s)$ does not satisfy any algebraic-differential equation. Voronin proved that the function $\zeta(s)$ does not satisfy algebraic-differential equation with continuous coefficients. In the paper, a joint generalization of the Voronin theorem is given, i. e., that a collection $(\zeta(s_1), \ldots, \zeta(s_r))$ does not satisfy a certain algebraic-differential equation with continuous coefficients.

Keywords Algebraic-differential independence · Functional independence · Gram function · Riemann zeta-function · Universality

Mathematics Subject Classification 11M06

1 Introduction

Let $\mathcal{T}$ be the class of certain type functions. The function f is called independent with respect to $\mathcal{T}$ if it does not satisfy the equation

$$F(f) = 0$$

with any $F \in \mathcal{T}$. The first result on the function independence was obtained by O. Hölder for the Euler gamma-function $\Gamma(s)$, $s = \sigma + it$, with respect to the class of polynomials [4]. In other words, he proved the algebraic-differential independence of $\Gamma(s)$, i. e., that there is no any polynomial $p(s_1, \ldots, s_r) \not\equiv 0$ such that

$$p\left(\Gamma(s), \Gamma'(s), \ldots, \Gamma^{(r-1)}(s)\right) \equiv 0.$$

The aim of this note is a certain independence of the Riemann zeta-function $\zeta(s)$. Recall that the function $\zeta(s)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

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and is analytically continued to the whole complex plane, except for the point $s = 1$ which is a simple pole with residue 1. D. Hilbert, in the description of his 18th problem mentioned [15] that the function $\zeta(s)$ is algebraically-differentially independent, and this is a result of the Hölder theorem on the independence of the function $\Gamma(s)$ and the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Moreover, he conjectured that the function

$$\zeta(s, x) = \sum_{m=1}^{\infty} \frac{x^m}{m^s}$$

also is algebraically-differentially independent. These Hilbert's conjectures were proved independently by D. D. Mordukhai-Boltovskoi [9] and A. Ostrowski [11]. A. G. Postnikov generalized [13] the latter results for the function

$$L(s, x, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)x^m}{m^s},$$

where $\chi(m)$ is a Dirichlet character.

The most general theorem on the independence of the function $\zeta(s)$ belongs to S. M. Voronin. In [15], he proved the so-called functional independence of $\zeta(s)$. More precisely, the following theorem is valid.

Theorem 1 *Suppose that* $F_0, \dots, F_m : \mathbb{C}^N \to \mathbb{C}$ *are continuous functions, and identically for* $s \in \mathbb{C}$

$$\sum_{k=0}^{m} s^k F_k\left(\zeta(s), \zeta'(s), \dots, \zeta^{(N-1)}(s)\right) = 0.$$

Then $F_k \equiv 0$ *for all* $k = 0, 1, \dots, m$.

In [17], S. M. Voronin generalized Theorem 1 for a collection of Dirichlet L-functions

$$L(s, \chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m)}{m^s}, \quad \mathrm{Re}s > 1, \ \ j = 1, \dots, r,$$

with pairwise non-equivalent Dirichlet characters $\chi_1, \dots, \chi_r$.

Our aim is a joint extension of Theorem 1. For non-negative integers $k_1, \dots, k_r$, denote by $\mathcal{A}$ the set of complex numbers $A_{k_1,\dots,k_r}$ such that $A_{k_1,\dots,k_r} \neq 0$ for the tuples $(k_1, 0, \dots, 0), \dots, (0, \dots, 0, k_r)$.

Theorem 2 *Suppose that* $A_{k_1,\dots,k_r} \in \mathcal{A}$*, and* $F_0, \dots, F_m : \mathbb{C}^{N_1+\cdots+N_r} \to \mathbb{C}$ *are continuous functions, and identically for* $s_1, \dots, s_r \in \mathbb{C}$

$$\begin{aligned}&\sum_{k=0}^{m} F_k\left(\zeta(s_1), \zeta'(s_1), \dots, \zeta^{(N_1-1)}(s_1), \dots, \zeta(s_r), \zeta'(s_r), \dots, \zeta^{(N_r-1)}(s_r)\right)\\ &\quad\times \sum_{k_1+\cdots+k_r=k} A_{k_1,\dots,k_r} s_1^{k_1} \cdots s_r^{k_r} = 0. \end{aligned} \tag{1}$$

Then $F_k \equiv 0$ *for all* $k = 0, 1, \dots, m$.

Equation (1) is new in the theory of the function $\zeta(s)$.

A proof of Theorem 2 is based on a joint universality theorem for the function $\zeta(s)$, and an implied denseness of its set of values.

2 Universality

By a Voronin's work [16], it is well known that the function $\zeta(s)$ is universal in the sense that its shifts $\zeta(s+i\tau)$, $\tau \in \mathbb{R}$, approximate a wide class of analytic functions. We recall a modern version of the Voronin theorem. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous non-vanishing functions on K that are analytic in the interior of K. Let meas$A$ be the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the Voronin theorem, see, for example, [8], asserts that, for every $K \in \mathcal{K}$ and $f(s) \in H_0(K)$ and every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon \right\} > 0. \tag{2}$$

The latter statement has a generalization for generalized shifts $\zeta(s + i\varphi(\tau))$ with a certain function $\varphi(\tau)$, moreover, a joint version on the simultaneous approximation of a collection of analytic functions by $(\zeta(s + i\varphi_1(\tau)), \dots, \zeta(s + i\varphi_r(\tau)))$ is known. We will use the function $\varphi(\tau)$ involving the Gram functions.

Denote by $\theta(t)$, $t > 0$, the increment of the argument of the function $\pi^{-s/2}\Gamma(s/2)$ along the segment connecting the points $s = 1/2$ and $s = 1/2 + it$. The function $\theta(t)$ is monotonically increasing and unbounded from above for $t > t^* = 6.2898\dots$, therefore the equation

$$\theta(t) = (\tau - 1)\pi, \quad \tau \geqslant 0,$$

has the unique solution t_τ for $t \geqslant t^*$, see [6]. The numbers t_n, $n \in \mathbb{N}$, are called the Gram points, and are used for the investigation of non-trivial zeros of the function $\zeta(s)$. By analogy, t_τ with arbitrary positive τ is called the Gram function.

In [7], the following joint universality theorem for $\zeta(s)$ has been obtained.

Theorem 3 *Suppose that $\alpha_1, \dots, \alpha_r$ are fixed different positive numbers. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left| \zeta(s + it_\tau^{\alpha_j}) - f_j(s) \right| < \varepsilon \right\} > 0.$$

Moreover "lim inf" *can be replaced by* "lim" *for all but at most countably many $\varepsilon > 0$.*

In Theorem 3, it is important that the numbers $\alpha_1, \dots, \alpha_r$ are different. The proof is probabilistic based on a limit theorem for weakly convergent probability measures in the space of analytic functions.

Note that joint universality theorems obtained in [10] and [12] also can be applied for the proof of Theorem 3.

3 Denseness

It is well known that the set of values of the Riemann zeta-function is dense in a certain sense. The first result in this direction belongs to H. Bohr. In [1], he proved that the function $\zeta(s)$ takes every non-zero value infinitely many times in the strip

$$\{s \in \mathbb{C} : 1 < \sigma < 1 + \delta\}$$

with arbitrary positive δ. Further, H. Bohr and R. Courant proved [2] that, for every fixed σ, $1/2 < \sigma \leqslant 1$, the set

$$\{\zeta(\sigma + it) : t \in \mathbb{R}\}$$

is everywhere dense in $\mathbb{C}$. The latter result was generalized by S. M. Voronin. He obtained in [14] that the set

$$\{(\zeta(s_1 + i\tau), \dots, \zeta(s_r + i\tau)) : \tau \in \mathbb{R}\}$$

with arbitrary fixed complex numbers $s_1, \dots, s_r$, $1/2 < \operatorname{Re} s_j < 1$, $j = 1, \dots, r$, and $s_j \neq s_k$ for $j \neq k$, and the set

$$\left\{ \left(\zeta(s + i\tau), \zeta'(s + i\tau), \dots, \zeta^{(r-1)}(s + i\tau) \right) : \tau \in \mathbb{R} \right\}$$

with arbitrary s, $1/2 < \sigma < 1$, are everywhere dense in $\mathbb{C}^r$.

Denote by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta. The space $H(D)$ is infinite-dimensional. Therefore the Voronin universality theorem (2) is a infinite-dimensional generalization of the mentioned above Bohr-Courant theorem.

In this section we will prove the following denseness result.

Theorem 4 *Suppose that $\alpha_1, \ldots, \alpha_r$ are fixed different positive numbers, and $N_1, \ldots, N_r \in \mathbb{N}$. Then, for every fixed σ, $1/2 < \sigma < 1$, the set*

$$\Big\{\Big(\zeta(\sigma + it_\tau^{\alpha_1}), \zeta'(\sigma + it_\tau^{\alpha_1}), \ldots, \zeta^{(N_1-1)}(\sigma + it_\tau^{\alpha_1}), \ldots,$$
$$\zeta(\sigma + it_\tau^{\alpha_r}), \zeta'(\sigma + it_\tau^{\alpha_r}), \ldots, \zeta^{(N_r-1)}(\sigma + it_\tau^{\alpha_r})\Big) : \tau \in \mathbb{R}\Big\}$$

is everywhere dense in $\mathbb{C}^{N_1+\cdots+N_r}$.

For the proof of Theorem 4, the following property of polynomials [5] is useful.

Lemma 5 *Suppose that $s_0 \neq 0, s_1, \ldots, s_m$ are complex numbers. Then there exists a polynomial $p(s)$ such that*

$$s_k = \left(\mathrm{e}^{p(s)}\right)^{(k)}\Big|_{s=0}, \quad k = 0, 1, \ldots, m.$$

Proof of Theorem 4 Let $s_{jk} \in \mathbb{C}$, $j = 1, \ldots, r$, $k = 0, 1, \ldots, N_j$, $s_{j0} \neq 0$, $j = 1, \ldots, r$. Fix $1/2 < \widehat{\sigma} < 1$. In view of topology of $\mathbb{C}^{N_1+\ldots+N_r}$, it suffices to show that, for every $\varepsilon > 0$, there exists $\tau \in \mathbb{R}$ such that

$$\left|\zeta^{(k)}(\widehat{\sigma} + it_r^{\alpha_j}) - s_{jk}\right| < \varepsilon. \tag{3}$$

By Lemma 5, there exists the polynomials $p_j(s)$, $j = 1, \ldots, r$, such that

$$s_{jk} = \left(\mathrm{e}^{p_j(s-\widehat{\sigma})}\right)^{(k)}\Big|_{s=\widehat{\sigma}}, \quad k = 0, 1, \ldots, N_j, \ j = 1, \ldots, r. \tag{4}$$

Further, let $\varrho = (1/2)\min(\widehat{\sigma} - 1/2, 1 - \widehat{\sigma})$, and consider the disk $K = \{s : |s - \widehat{\sigma}| \leqslant \varrho\}$. Suppose finally that $\varepsilon_1 > 0$. By Theorem 2, there exists $\tau \in \mathbb{R}$ (actually, there exists a sequence $\{\tau_n\} \subset \mathbb{R}$, $\tau_n \to \infty$ as $n \to \infty$) such that

$$\sup_{s\in K}\left|\zeta(s + it_\tau^{\alpha_j}) - \mathrm{e}^{p_j(s-\widehat{\sigma})}\right| < \varepsilon_1 \tag{5}$$

for all $j = 1, \ldots, r$. Let L be the circle $|s - \widehat{\sigma}| = \varrho/2$. Then the integral Cauchy formula implies

$$\zeta^{(k)}(\widehat{\sigma} + it_\tau^{\alpha_j}) - \left(\mathrm{e}^{p_j(s-\widehat{\sigma})}\right)^{(k)}\Big|_{s=\widehat{\sigma}} = \frac{k!}{2\pi i}\int_L \frac{\zeta(z + it_\tau^{\alpha_j}) - \mathrm{e}^{p_j(z-\widehat{\sigma})}}{(z - \widehat{\sigma})^{k+1}}\mathrm{d}z.$$

Hence, in view of (4) and (5),

$$\left|\zeta^{(k)}(\widehat{\sigma} + it_\tau^{\alpha_j}) - s_{jk}\right| \leqslant \frac{k!\varepsilon_1}{2\pi}\int_L \frac{|\mathrm{d}z|}{|z - \widehat{\sigma}|^{k+1}} = \frac{k!\varepsilon_1}{2\pi}\left(\frac{2}{\varrho}\right)^k.$$

Given ε, taking

$$\varepsilon_1 < 2\pi\varepsilon \min_{1\leqslant k\leqslant \max(N_1,\ldots,N_r)} \frac{1}{k!}\left(\frac{\varrho}{2}\right)^k,$$

we arrive at (3). □

4 Proof of the main theorem

Proof of Theorem 2 Let, for brevity, $N = N_1 + \cdots + N_r$. Denote by $\underline{s}$ the points of $\mathbb{C}^N$, and by d_N the distance in $\mathbb{C}^N$. Moreover, let

$$Z_k(s_1, \ldots, s_r) = \sum_{k_1+\cdots+k_r=k} A_{k_1,\ldots,k_r}\, s_1^{k_1} \cdots s_r^{k_r}.$$

Without loss of generality, we may suppose that $\alpha_1 = \max\{\alpha_1, \ldots, \alpha_r\}$.

We will prove that $F_m \equiv 0$. Suppose, on the contrairy, that $F_m \not\equiv 0$. Then there exists $\underline{s}_0 \in \mathbb{C}^N$ such that $F_m(\underline{s}_0) \neq 0$. The continuity of F_m implies that, for a certain fixed $\delta > 0$, there exists $\varepsilon > 0$ such that

$$|F_m(\underline{s})| > \delta \tag{6}$$

for all $\underline{s} \in G_\varepsilon = \{\underline{s} \in \mathbb{C}^N : d_N(\underline{s}, \underline{s}_0) < \varepsilon\}$. Take $0 < \varepsilon_1 < \varepsilon$, and define $G_{\varepsilon_1} = \{\underline{s} \in \mathbb{C}^N : d_N(\underline{s}, \underline{s}_0) \leqslant \varepsilon_1\}$. Then the functions $F_0, F_1, \ldots, F_{m-1}$ are bounded in the disc G_{ε_1}. Define one more disc $G_{\varepsilon_2} = \{\underline{s} \in \mathbb{C}^N : d_N(\underline{s}, \underline{s}_0) \leqslant \varepsilon_2\}$ with $0 < \varepsilon_2 < \varepsilon_1$. By the proof of Theorem 4, there exists a sequence $\{\tau_n\} \subset \mathbb{R}$, $\tau_n \to \infty$ as $n \to \infty$, such that, for a fixed $1/2 < \sigma < 1$,

$$\underline{\zeta}(\sigma, \underline{t}_{\tau_n}^{\underline{\alpha}}) \in G_{\varepsilon_2},$$

where $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$, $\underline{t}_{\tau_n}^{\underline{\alpha}} = (t_{\tau_n}^{\alpha_1}, \ldots, t_{\tau_n}^{\alpha_r})$, and

$$\underline{\zeta}(\sigma, \underline{t}_{\tau_n}^{\underline{\alpha}}) = \Big(\zeta(\sigma + it_{\tau_n}^{\alpha_1}), \zeta'(\sigma + it_{\tau_n}^{\alpha_1}), \ldots, \zeta^{(N_1)}(\sigma + it_{\tau_n}^{\alpha_1}),$$
$$\ldots, \zeta(\sigma + it_{\tau_n}^{\alpha_r}), \zeta'(\sigma + it_{\tau_n}^{\alpha_r}), \ldots, \zeta^{(N_r)}(\sigma + it_{\tau_n}^{\alpha_r})\Big).$$

Therefore, in view of (6),

$$\left|F_m\left(\underline{\zeta}(\sigma, \underline{t}_{\tau_n}^{\underline{\alpha}})\right)\right| > \delta. \tag{7}$$

It is known [6] that, for $\tau \to \infty$,

$$t_\tau = \frac{2\pi\tau}{\log\tau}\left(1 + \frac{\log\log\tau}{\log\tau}(1 + o(1))\right).$$

Thus, by a remark on the number α_1, as $n \to \infty$,

$$Z_m(\sigma + it_{\tau_n}^{\alpha_1}, \ldots, \sigma + it_{\tau_n}^{\alpha_r}) = A_{m,0,\ldots,0}(\sigma + it_{\tau_n}^{\alpha_1})^m + o(|\sigma + it_{\tau_n}^{\alpha_1}|^m),$$

and

$$Z_k(\sigma + it_{\tau_n}^{\alpha_1}, \ldots, \sigma + it_{\tau_n}^{\alpha_r}) = o(|\sigma + it_{\tau_n}^{\alpha_1}|^m)$$

for $k < m$. This and (7) show that

$$\left|\sum_{k=0}^{m} F_k\left(\underline{\zeta}(\sigma, \underline{t}_{\tau_n}^{\underline{\alpha}})\right) Z_k\left(\sigma + i\underline{t}_{\tau_n}^{\underline{\alpha}}\right)\right| \to \infty$$

as $n \to \infty$. However, this contradicts the hypothesis (1) of the theorem. Thus, $F_m \equiv 0$.

Now, similarly as above, we obtain that $F_k \equiv 0$ for $k = m-1, \ldots, 1, 0$. The theorem is proved. □

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