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ORIGINAL RESEARCH

Applications of Nörlund's formula and linear forms for continued fractions

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Abstract We provide a method for determining explicit symbolic evaluations for infinite families of continued fractions with quadratic partial numerators and constant partial denominators. In this manner, we obtain continued fraction identities generalizing results due to Ramanujan and Stieltjes. Our method relies on Nörlund's formula together with generalizations of classical hypergeometric series identities due to Rakha and Rathie. We also introduce and apply a related technique concerning linear forms for continued fractions to build on the work of Dougherty-Bliss and Zeilberger related to "The Ramanujan Machine."

Keywords Continued fraction · Nörlund's formula · Hypergeometric series

Mathematics Subject Classification 11A55 · 33C05

1 Introduction

For sequences $(a_n : n \in \mathbb{N}_0)$ and $(b_n : n \in \mathbb{N}_0)$, we adopt the convention such that the continued fraction

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} \tag{1}$$

is denoted as

$$b_0 + \mathop{\mathrm{K}}_{m=1}^{\infty} \frac{a_m}{b_m} = b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots$$

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and we express its n-th convergent of the continued fraction in (1) as A_n/B_n, where $A_{n+2} = b_{n+2}A_{n+1} + a_{n+2}A_n$ and $B_{n+2} = b_{n+2}B_{n+1} + a_{n+2}B_n$, with $A_0 = b_0$, $B_0 = 1$, $A_1 = b_0b_1 + a_1$, and $B_1 = b_1$. These recursions are often referred to as *Euler–Wallis formulas*, and date back to the discoveries of Euler in 1748 and Wallis in 1656 (see [9, Section 1.2]). In this article, we consider an application of Nörlund's formula [13] together with generalizations of classical hypergeometric identities due to Rakha and Rathie [15] in order to generalize continued fraction identities due to Ramanujan and Stieltjes. We also apply Nörlund's formula to build on the work of Dougherty-Bliss and Zeilberger [6].

Our article is, in large part, inspired by the following result included in Ramanujan's first letter to Hardy and originally proved by Bauer in 1872, with reference to Part II of Ramanujan's Notebooks [2, p. 145]:
Theorem (Ramanujan): For $s > 0$,

$$s + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{(2n-1)^2}{2s} = 4\left(\frac{\Gamma\left(\frac{s+3}{4}\right)}{\Gamma\left(\frac{s+1}{4}\right)}\right)^2. \tag{2}$$

The following 1890 identity due to Stieltjes [16] is another source of inspiration for this article:

$$\frac{1}{s + \mathrm{K}_{n=1}^{\infty} n(n+1)/s} = \int_0^{\infty} \frac{e^{-sx}}{\cosh^2 x}\,dx = s\int_0^{\infty} e^{-sx}\tanh x\,dx. \tag{3}$$

This is related to Laplace transforms of elliptic functions [9, §86]. We see that both (2) and (3) provide evaluations for infinite families of continued fractions with a free real variable involved, and such that the partial numerators form a quadratic polynomial sequence and the partial denominators are constant. We introduce a method for determining further identities for families of continued fractions of this form. We also investigate the evaluation of infinite continued fractions with linear partial denominators and numerators, building on the recent work by Dougherty-Bliss and Zeilberger [6]. We confirm all conjectures related to expressions for continued fractions concerning the constant e in "The Ramanujan Machine" (See https://www.ramanujanmachine.com [14]).

1.1 Nörlund's formula

Lorentzen and Waadeland [10] provided identities for expressing, in terms of hypergeometric functions, continued fractions such that the a-sequence and the b-sequence in (1) are, respectively, a polynomial of degree two and a polynomial of degree at most one. The classical hypergeometric function, which is often called the Gauss hypergeometric function, is defined by [1]

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad \Re(c-a-b) > 0, |z| \leq 1, c \notin \mathbb{Z}_{\leq 0},$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$. As in [17], we refer to the identity

$$1 - \frac{z(a+b+1)}{c} + \frac{1}{c}\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{(a+n)(b+n)(z-z^2)}{c+n-z(a+b+2n+1)} = \frac{{}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, z\right]}{{}_2F_1\left[\begin{matrix} a+1, b+1 \\ c+1 \end{matrix}\,\middle|\, z\right]} \tag{4}$$

for suitably bounded a, b, c, and z as Nörlund's formula (see [11]). Explicitly, following [10], the above identity holds on the domain

$$D_f := \left\{(a, b, c, z) \subset \mathbb{C}^4 : \mathrm{Re}(z) < \frac{1}{2} \text{ or } z = \frac{1}{2}, \mid \Im(a+b) \mid < \Re(2c-a-b-1)\right\}.$$

As indicated in [17], Nörlund's formula may be used to express infinite continued fractions of the form

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{c_1n^2 + c_2n + c_3}{d_1n + d_2} \tag{5}$$

in terms of hypergeometric functions of the following form:

$$ {}_2F_1\left[\begin{matrix} *,* \\ * \end{matrix}\,\middle|\, \frac{1}{2}\left(1-\frac{d_1}{\sqrt{d_1^2+4c_1}}\right)\right]. $$

More explicitly, the infinite continued fraction (5) is equal to $-d_2$ plus the quotient indicated as follows. The numerator of this quotient is the product of

$$ \left(\sqrt{d_1^2+4c_1}(c_1+c_2)-d_1(c_1+c_2)+2d_2c_1\right) $$

and

$$ {}_2F_1\left[\begin{matrix} \frac{c_2}{2c_1}-\frac{\sqrt{c_2^2-4c_1c_3}}{2c_1},\ \frac{c_2}{2c_1}+\frac{\sqrt{c_2^2-4c_1c_3}}{2c_1} \\ -\frac{c_2d_1}{2c_1\sqrt{d_1^2+4c_1}}-\frac{d_1}{2\sqrt{d_1^2+4c_1}}+\frac{c_2}{2c_1}+\frac{d_2}{\sqrt{d_1^2+4c_1}}+\frac{1}{2} \end{matrix}\,\middle|\, \frac{1}{2}\left(1-\frac{d_1}{\sqrt{d_1^2+4c_1}}\right)\right]. $$

The denominator of the aforementioned quotient is $2c_1$ times:

$$ {}_2F_1\left[\begin{matrix} \frac{c_2}{2c_1}-\frac{\sqrt{c_2^2-4c_1c_3}}{2c_1}+1,\ \frac{c_2}{2c_1}+\frac{\sqrt{c_2^2-4c_1c_3}}{2c_1}+1 \\ -\frac{c_2d_1}{2c_1\sqrt{d_1^2+4c_1}}-\frac{d_1}{2\sqrt{d_1^2+4c_1}}+\frac{c_2}{2c_1}+\frac{d_2}{\sqrt{d_1^2+4c_1}}+\frac{3}{2} \end{matrix}\,\middle|\, \frac{1}{2}\left(1-\frac{d_1}{\sqrt{d_1^2+4c_1}}\right)\right]. $$

2 Infinite families

Setting $d_1 = 0$ and $\Re(d_2/\sqrt{c_1}) > | \Im(c_2/c_1) |$ in the latter formulation of Nörlund's formula given above (5), we obtain that

$$ d_2 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{c_1n^2+c_2n+c_3}{d_2} \tag{6} $$

evaluates as below:

$$ \frac{\left(2d_2c_1+2\sqrt{c_1}(c_1+c_2)\right)\, {}_2F_1\left[\begin{matrix} \frac{c_2}{2c_1}-\frac{\sqrt{c_2^2-4c_1c_3}}{2c_1},\ \frac{c_2}{2c_1}+\frac{\sqrt{c_2^2-4c_1c_3}}{2c_1} \\ \frac{d_2}{2\sqrt{c_1}}+\frac{c_2}{2c_1}+\frac{1}{2} \end{matrix}\,\middle|\, \frac{1}{2}\right]}{2c_1\cdot {}_2F_1\left[\begin{matrix} 1+\frac{c_2}{2c_1}-\frac{\sqrt{c_2^2-4c_1c_3}}{2c_1},\ 1+\frac{c_2}{2c_1}+\frac{\sqrt{c_2^2-4c_1c_3}}{2c_1} \\ \frac{d_2}{2\sqrt{c_1}}+\frac{c_2}{2c_1}+\frac{3}{2} \end{matrix}\,\middle|\, \frac{1}{2}\right]}. \tag{7} $$

Our evaluation method is given by the application of a generalization of Gauss's second summation theorem to evaluate these ${}_2F_1\left(\frac{1}{2}\right)$-series, specifically when $\frac{d_2}{2\sqrt{c_1}}+\frac{1}{2}$ is a non-negative integer. To this end, the following generalization of Gauss's second summation theorem, as presented by Rakha and Rathie [15], can be utilized:

$$ \begin{aligned} {}_2F_1\left[\begin{matrix} a,b \\ \frac{1}{2}(a+b+j+1) \end{matrix}\,\middle|\, \frac{1}{2}\right] &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}j+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} \frac{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}-\frac{1}{2}j\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}j+\frac{1}{2}\right)} \\ &\quad\times \sum_{r=0}^{j} \binom{j}{r} \frac{(-1)^r\Gamma\left(\frac{1}{2}b+\frac{1}{2}r\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}j+\frac{1}{2}r+\frac{1}{2}\right)} \end{aligned} \tag{8} $$

for $j \in \mathbb{N}_0$. Similarly, by expressing the ${}_2F_1\left(\frac{1}{2}\right)$-series in (7) in an equivalent way as ${}_2F_1(-1)$-series according to Pfaff's hypergeometric transform

$$ {}_2F_1\left[\begin{matrix} a,b \\ c \end{matrix}\,\middle|\, x\right] = (1-x)^{-a}\, {}_2F_1\left[\begin{matrix} a,c-b \\ c \end{matrix}\,\middle|\, \frac{x}{x-1}\right], $$

then we may employ the following generalization of Kummer's hypergeometric identity given by Rakha and Rathie [15]. For $j \in \mathbb{N}_0$:

$$
{}_2F_1\left[\begin{matrix} A, B \\ 1+A-B+j \end{matrix} \middle| -1\right] = \frac{2^{-A}\Gamma\left(\frac{1}{2}\right)\Gamma(B-j)\Gamma(1+A-B+j)}{\Gamma(B)\Gamma\left(\frac{1}{2}A-B+\frac{1}{2}j+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}A-B+\frac{1}{2}j+1\right)} \times \sum_{r=0}^{j}(-1)^r\binom{j}{r}\frac{\Gamma\left(\frac{1}{2}A-B+\frac{1}{2}j+\frac{1}{2}r+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}A-\frac{1}{2}j+\frac{1}{2}r+\frac{1}{2}\right)}. \tag{9}
$$

For a given continued fraction K as in (6), computer algebra systems cannot evaluate K or the partial quotients of K or the Euler–Wallis recursions for K and also cannot evaluate the ${}_2F_1$-series associated with the Nörlund form of K. Our evaluation method drawing upon the work of Rakha and Rathie [15] may be used to produce new evaluations for families of continued fractions of the form indicated in (6) and involving a free real parameter. The classical formulas (2) and (3) may also be proved using our evaluation method.

In a recent article [6], Dougherty-Bliss and Zeilberger introduced a process for automatically determining and proving evaluations for infinite families of infinite continued fractions, building on the work on "The Ramanujan Machine" introduced in [14]. A main component of the evaluation technique from [6] concerns the identity

$$
1+k+\mathop{\mathrm{K}}_{n=1}^{\infty}\frac{an}{n+k+1} = \frac{a^D}{(D-1)!\left(e^a-\sum_{s=0}^{D-1}\frac{a^s}{a!}\right)} \tag{10}
$$

for $D = a+k+1 \geq 1$. Nörlund's formula, which is not involved in [6,14], can be applied to prove and generalize (10), as we consider in Section 3.

2.1 Applications

For all of the continued fraction identities introduced in this section, current computer algebra system software can evaluate neither the continued fractions therein nor the corresponding Euler–Wallis recursions. To begin with, the identity below may be seen as a natural variant of both Ramanujan's identity (2) and Stieltjes' identity (3).

Example 1 The identity

$$
\frac{b}{2}+\mathop{\mathrm{K}}_{n=0}^{\infty}\frac{n^2+n+1}{b} = 2\cdot\frac{\Gamma\left(\frac{b-i\sqrt{3}+3}{4}\right)\Gamma\left(\frac{b+i\sqrt{3}+3}{4}\right)}{\Gamma\left(\frac{b-i\sqrt{3}+1}{4}\right)\Gamma\left(\frac{b+i\sqrt{3}+1}{4}\right)}
$$

holds true for $\Re(b) > 0$. This is obtained by setting $c_1 = c_2 = c_3 = 1$, $d_1 = 0$, and $d_2 = b$ in Nörlund's formula and twofold application of the identity for the ${}_2F_1$-family indicated (9), with $j = 0$ and $j = 2$.

Example 2 The identity

$$
\frac{b}{2}+\mathop{\mathrm{K}}_{n=0}^{\infty}\frac{n^2+n-1}{b} = 2\cdot\frac{\Gamma\left(\frac{b-\sqrt{5}+3}{4}\right)\Gamma\left(\frac{b+\sqrt{5}+3}{4}\right)}{\Gamma\left(\frac{b-\sqrt{5}+1}{4}\right)\Gamma\left(\frac{b+\sqrt{5}+1}{4}\right)}
$$

holds true for $\Re(b) > 0$. This can be obtained by setting $c_1 = c_3 = 1$ and $c_2 = 3$ and $d_1 = 0$ and $d_2 = b$ in Nörlund's formula, and by applying a reindexing argument and a twofold application of the identity for the ${}_2F_1$-family indicated (9).

Example 3 The identity

$$
\frac{b}{2}+\mathop{\mathrm{K}}_{n=0}^{\infty}\frac{4n^2+4n+3}{b} = 4\cdot\frac{\Gamma\left(\frac{b-2i\sqrt{2}+6}{8}\right)\Gamma\left(\frac{b+2i\sqrt{2}+6}{8}\right)}{\Gamma\left(\frac{b-2i\sqrt{2}+2}{8}\right)\Gamma\left(\frac{b+2i\sqrt{2}+2}{8}\right)}
$$

holds true for $\Re(b) > 0$. This can be obtained by setting $c_1 = c_2 = 4$ and $c_3 = 3$ and $d_1 = 0$ and $d_2 = b$ in Nörlund's formula, and by applying a reindexing argument and a twofold application of the identity for the ${}_2F_1$-family indicated (9).

Example 4 The identity

$$3+\mathop{\mathrm{K}}_{n=1}^{\infty}\frac{9n^2-6n+b}{3}=$$
$$\frac{b\,\Gamma\left(\frac{2-\sqrt{1-b}}{6}\right)\Gamma\left(\frac{\sqrt{1-b}-1}{6}\right)-6\left(\sqrt{1-b}-1\right)\Gamma\left(\frac{5-\sqrt{1-b}}{6}\right)\Gamma\left(\frac{\sqrt{1-b}+2}{6}\right)}{6\Gamma\left(\frac{5-\sqrt{1-b}}{6}\right)\Gamma\left(\frac{\sqrt{1-b}+2}{6}\right)-6\Gamma\left(\frac{2-\sqrt{1-b}}{6}\right)\Gamma\left(\frac{\sqrt{1-b}+5}{6}\right)}$$

holds true for a suitably bounded variable b. This follows from Nörlund's formula, by setting $c_1=9$, $c_2=-6$, $c_3=b$, $d_1=0$, and $d_2=3$, and by then using Rakha and Rathie's hypergeometric identity indicated in (8).

Example 5 The identity

$$6+\mathop{\mathrm{K}}_{n=1}^{\infty}\frac{4n^2-8n+b}{6}=$$
$$\frac{\Gamma\left(2-\frac{\sqrt{4-b}}{2}\right)\Gamma\left(\frac{\sqrt{4-b}+2}{4}\right)\left(\frac{\sqrt{4-b}+6}{\Gamma\left(1-\frac{\sqrt{4-b}}{4}\right)\Gamma\left(\frac{\sqrt{4-b}+6}{4}\right)}+\frac{4\left(b+4\sqrt{4-b}+8\right)}{b\Gamma\left(\frac{1}{2}-\frac{\sqrt{4-b}}{4}\right)\Gamma\left(\frac{\sqrt{4-b}+4}{4}\right)}\right)\Gamma\left(\frac{\sqrt{4-b}}{4}\right)}{4\Gamma\left(-\frac{\sqrt{4-b}}{2}-1\right)\left(-\frac{3\Gamma\left(\frac{\sqrt{4-b}+2}{4}\right)}{\Gamma\left(-\frac{\sqrt{4-b}}{4}-\frac{1}{2}\right)}-\frac{\Gamma\left(\frac{\sqrt{4-b}+6}{4}\right)}{\Gamma\left(\frac{1}{2}-\frac{\sqrt{4-b}}{4}\right)}+\frac{\Gamma\left(\frac{\sqrt{4-b}}{4}\right)}{\Gamma\left(-\frac{\sqrt{4-b}}{4}-1\right)}+\frac{3\Gamma\left(\frac{\sqrt{4-b}+4}{4}\right)}{\Gamma\left(-\frac{\sqrt{4-b}}{4}\right)}\right)}$$

holds true for a suitably bounded variable b. This follows from Nörlund's formula and the Rakha–Rathie hypergeometric series shown in (8), which together give us that

$$6+\mathop{\mathrm{K}}_{n=1}^{\infty}\frac{4n^2-8n+b}{6}=\frac{4\,{}_2F_1\left[\left.\begin{matrix}-\frac{1}{8}\sqrt{64-16b}-1,\ \frac{1}{8}\sqrt{64-16b}-1\\ 1\end{matrix}\,\right|\,\frac{1}{2}\right]}{{}_2F_1\left[\left.\begin{matrix}-\frac{1}{8}\sqrt{64-16b},\ \frac{1}{8}\sqrt{64-16b}\\ 2\end{matrix}\,\right|\,\frac{1}{2}\right]}.$$

3 Linear Forms

The principal result of Dougherty-Bliss and Zeilberger in [6] is the identity in (10) involving continued fractions with linear partial numerators and linear partial denominators. With regard to Nörlund's formula, we are to derive a related continued fraction formula, again in the case where the a-sequence and the b-sequence are both linear polynomials.

Let α, β, and z be any complex numbers with $\beta\notin\mathbb{Z}_{\leq 0}$. It is stated [8] that

$$\mathop{\mathrm{K}}_{n=1}^{\infty}\frac{(\alpha+n)z}{\beta+n-z}=\frac{\beta\,{}_1F_1\left[\left.\begin{matrix}\alpha\\ \beta\end{matrix}\,\right|\,z\right]}{{}_1F_1\left[\left.\begin{matrix}\alpha+1\\ \beta+1\end{matrix}\,\right|\,z\right]}+z-\beta, \tag{11}$$

where ${}_1F_1\left[\left.\begin{matrix}\alpha\\ \beta\end{matrix}\,\right|\,z\right]$ is the confluent hypergeometric function. We substitute $\alpha=c_2/c_1$, $\beta=(c_1+d_1d_2)/d_1^2$, and $z=c_1/d_1^2$ in the above identity, then the continued fraction with linear forms can be determined by the following proposition [17, pp. 6].

Proposition 1 *Let d_1, c_1, and $\beta=(c_1+d_1d_2)/d_1^2$ be nonzero complex numbers with $\beta\notin\mathbb{Z}_{\leq 0}$. We have*

$$\mathop{\mathrm{K}}_{n=1}^{\infty}\frac{c_1n+c_2}{d_1n+d_2}=\frac{d_1\beta\,{}_1F_1\left[\left.\begin{matrix}c_2/c_1\\ \beta\end{matrix}\,\right|\,c_1/d_1^2\right]}{{}_1F_1\left[\left.\begin{matrix}c_2/c_1+1\\ \beta+1\end{matrix}\,\right|\,c_1/d_1^2\right]}-d_2. \tag{12}$$

More specifically, if we set $c_2 = 0$*, and* $\beta \in \mathbb{N}$*, then*

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{c_1 n}{d_1 n + d_2} = \frac{c_1^{\beta}}{d_1^{2\beta-1}(\beta-1)!\left[e^{c_1/d_1^2} - \sum_{j=0}^{\beta-1} \frac{c_1^j}{j! d_1^{2j}}\right]} - d_2. \tag{13}$$

Proof It is known for $a \notin \mathbb{Z}_{\leq 0}$ [12, (8.5.1)]

$$ {}_1F_1\left[\begin{matrix} 1 \\ a+1 \end{matrix} \middle| z\right] = a z^{-a} e^z \gamma(a, z),$$

where $\gamma(a, z)$ is the incomplete Gamma functions defined for $\Re a > 0$. If $a \in \mathbb{N}$, then

$$\gamma(a, z) = \int_0^z t^{a-1} e^{-t}\, dt = \frac{(a-1)!}{e^z} \cdot \left[e^z - \sum_{j=0}^{a-1} \frac{z^j}{j!}\right].$$

Following these two facts, we have (13). □

The identity in (10) given by Dougherty-Bliss and Zeilberger [6] can be obtained by setting $c_1 = a$, $d_1 = 1$, $d_2 = k + 1$, and $\beta = k + 1 + a \in \mathbb{N}$ in (13). Moreover, we highlight the special case obtain by setting $c_1 = d_1 = d_2 + 1 = a$ in (13):

$$a - 1 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{an}{a(n+1) - 1} = \frac{1}{\sqrt[a]{e} - 1}. \tag{14}$$

This gives the following two formulas, recalling the work of Euler indicated in [7, pp.12, 14]:

$$\frac{1}{1+}\frac{2}{2+}\frac{3}{3+}\cdots = \frac{1}{e-1}, \qquad 1 + \frac{2}{3+}\frac{4}{5+}\frac{6}{7+}\cdots = \frac{1}{\sqrt{e}-1}. \tag{15}$$

Moreover, for any nonzero complex numbers a and $\frac{b+1}{a} \notin \mathbb{Z}_{\leq 0}$, we obtain that

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{an + b + 1}{an + b} = 1. \tag{16}$$

This gives the following formulas:

$$1 = \frac{2}{1+}\frac{3}{2+}\frac{4}{3+}\cdots = \frac{3}{2+}\frac{5}{4+}\frac{7}{6+}\cdots = \frac{4}{3+}\frac{6}{5+}\frac{8}{7+}\cdots. \tag{17}$$

The 13 conjectures for e in PDF#1 that were generated by the MITM-RF algorithm in "The Ramanujan Machine" https://www.ramanujanmachine.com (ref. [14]) are easily confirmed by using (13).

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Declarations

Conflict of interest No potential conflict of interest was reported by the authors.

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