Indian J Pure Appl Math (2025) 56:1549–1562
https://doi.org/10.1007/s13226-024-00699-w

ORIGINAL RESEARCH

The generalized distance spectra of the $\mathcal{M}$-join of graphs, II

G. Kalaivani · R. Rajkumar

Received: 19 June 2022 / Accepted: 6 September 2024 / Published online: 24 October 2024

Abstract In this paper, we obtain the generalized distance spectra and the distance spectra of the graphs constructed by 2736 and 1588 binary graph operations of the $\mathcal{M}$-join type, respectively when the constituting graphs satisfy some conditions. Using these results, we deduce some existing results on the distance spectra of graphs constructed by some graph operations in the literature. As applications, we construct infinite pairs of generalized distance cospectral graphs.

Keywords $\mathcal{M}$-join of graphs · Generalized distance spectrum · Distance cospectral graphs

Mathematics Subject Classification 05C50 · 05C76

1 Introduction

Spectral graph theory investigates the relationship between the spectrum of the different matrices associated with graphs and their structural properties. The adjacency matrix, the Laplacian matrix, the signless Laplacian matrix, the normalized Laplacian matrix, the distance matrix are some important matrices associated with a graph; see [7]. Aouchiche and Hansen [6] defined the Laplacian distance matrix and the signless Laplacian distance matrix of a graph. Later, Cui et al. [11] introduced the generalized distance matrix of a graph.

Describing the spectra of a given graph in terms of the spectra of some other graphs to a possible extent is a natural problem of interest in spectral graph theory. To achieve this, researchers used several graph operations as tools for constructing graphs from the given graphs. The adjacency (Laplacian, signless Laplacian, normalized Laplacian) spectra of graphs constructed by such graph operations have been extensively studied in the literature by several researchers. For instance, see [8,13,22,23] and the references therein.

The distance (distance Laplacian, distance signless Laplacian, generalized distance) spectra of graphs constructed by various graph operations have been obtained by several researchers; for instance, see [3–5,9,10,12,16,17,20,21].

In this direction, determining the generalized distance spectra of the graphs constructed by the $\mathcal{M}$-join [13] type operations is a problem of interest. In [19] authors obtained the generalized distance spectra of graphs constructed by several unary and binary graph operations of $\mathcal{M}$-join type. As a squeal to this, in this paper we aim to obtain the generalized distance spectra of the graphs constructed by several binary graph operations of

Communicated by S Sivaramakrishnan.

G. Kalaivani · R. Rajkumar (✉)
Department of Mathematics, The Gandhigram Rural Institute (Deemed to be University), Gandhigram, Tamil Nadu 624 302, India
E-mail: rrajmaths@yahoo.co.in

G. Kalaivani
E-mail: kalaivani.slg@gmail.com

$\mathcal{M}$-join type and to construct distance (distance Laplacian, distance signless Laplacian) cospectral graphs. A part of this paper is a part of the dissertation [18].

The rest of this paper is arranged as follows: In Sect. 2, we give some basic definitions, notations and results which are used in the subsequent sections. In Sect. 3, we determine the spectra with corresponding eigenvectors of three families of block matrices of specific form. In Sect. 4, using some unary graph operations defined in [13], we define two families of binary graph operations and determine the generalized distance spectra of the graphs constructed by them when the constituting graphs satisfy some conditions, by using the results of Sect. 3. These results enable us to construct infinite pairs of generalized distance cospectral graphs. In Sect. 5, we define three families of binary graph operations using the unary graph operations considered in Sect. 4 and determine the distance spectra and the generalized distance spectra of graphs constructed by these binary graph operations when the constituting graphs satisfy some conditions, by using the results of Sect. 3. Using these results, we deduce the main results of the distance spectra of graphs obtained in [1,16]. The results of this section enable us to construct infinite pairs of generalized distance cospectral graphs.

2 Preliminaries

Throughout this paper, we consider only finite and simple graphs. Let G be a graph with vertex set $V(G)$. We denote $u \sim v$ ($u \nsim v$) if the vertices u and v are adjacent (non-adjacent) in G. For a vertex v in G, $N_G(v)$ denotes the set of all vertices in G which are adjacent with v. The *degree* of v in G is the cardinality of the set $N_G(v)$. If each vertex of G has the same degree r, then G is said to be *r-regular*. The complement of G, denoted by $\overline{G}$ is defined as the graph with vertex set $V(G)$ and two distinct vertices are adjacent in $\overline{G}$ if and only if they are non-adjacent in G. The union of two graphs G_1 and G_2, denoted by $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. For $u, v \in V(G)$, $u \neq v$, the distance from u to v in G, denoted by $d_G(u, v)$ is defined as the length of a shortest path from u to v (if it exists); otherwise it is defined to be ∞. G is said to be connected if there exists a path joining any two distinct vertices of G. The diameter of a connected graph G, denoted by $diam(G)$ is the maximum distance between all pairs of distinct vertices of G. I_n denotes the identity matrix of order n and $J_{n\times m}$ denotes the all one matrix of order $n \times m$.

Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of G, denoted by $A(G)$, is the $0-1$ matrix of size $n \times n$ whose rows and columns are indexed by $V(G)$ and for $i, j = 1, 2, \ldots, n$, the (i, j)-th entry of $A(G)$ is 1 if and only if $i \neq j$ and $v_i \sim v_j$.

The *distance matrix* of a connected graph G, denoted by $D(G)$, is the square matrix whose rows and columns are indexed by $V(G)$ and for $i, j = 1, 2, \ldots, n$, the (i, j)-th entry of $D(G)$ is $d_G(v_i, v_j)$ if $i \neq j$; 0 otherwise.

The following are introduced in [6]: The *transmission* of a vertex v of a connected graph G, denoted by $Tr_G(v)$, is defined as the sum of the distances from v to all other vertices of G. G is said to be t-transmission regular, if each vertex of G has the same transmission t. A graph is transmission regular if it is t-transmission regular for some t. The *transmission matrix* of G, denoted by $Tr(G)$, is defined as the diagonal matrix $diag(Tr_G(v_1), Tr_G(v_2), \ldots, Tr_G(v_n))$. Notice that if G is a t-transmission regular graph, then $Tr(G) = tI_n$. The *distance Laplacian matrix* $D^L(G)$ of G is defined as $Tr(G) - D(G)$. The *distance signless Laplacian matrix* $D^Q(G)$ of G is defined as $Tr(G) + D(G)$.

The *generalized distance matrix* $D_\alpha(G)$ of G is defined by $D_\alpha(G) = \alpha(Tr(G)) + (1-\alpha)D(G)$ for $0 \leq \alpha \leq 1$ [11]. Notice that $D_0(G) = D(G)$, $D_{\frac{1}{2}}(G) = \frac{1}{2}D^Q(G)$, $D_1(G) = Tr(G)$ and $D_\alpha(G) - D_\beta(G) = (\alpha - \beta)D^L(G)$, for $0 \leq \alpha, \beta \leq 1$ with $\alpha \neq \beta$.

The spectrum of $A(G)$ (resp. $D(G)$, $D^L(G)$, $D^Q(G)$, $D_\alpha(G)$) is called the *adjacency (resp. distance, distance Laplacian, distance signless Laplacian, generalized distance) spectrum* of G. Two graphs are said to be *adjacency (resp. distance, distance Laplacian, distance signless Laplacian, generalized distance) cospectral* if they have same adjacency (resp. distance, distance Laplacian, distance signless Laplacian, generalized distance) spectrum.

Let G and H be graphs with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$ and let M be a $0-1$ matrix of size $n \times m$. The *M-join* of G and H, denoted by $G \vee_M H$, is the graph obtained by taking one copy of G and H, and joining the vertices u_i and v_j if and only if the (i, j)-th entry of M is 1 for $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$ [13,15].

The definition of M-join of two graphs can be extended to a sequence of k graphs as follows [13]: Let $\mathcal{H}_k = (H_1, H_2, \ldots, H_k)$ be a sequence of graphs with $|V(H_i)| = n_i$ for $i = 1, 2, \ldots, k$ and let

$\mathcal{M} = (M_{12}, M_{13}, \ldots, M_{1k}, M_{23}, M_{24}, \ldots, M_{2k}, \ldots, M_{(k-1)k})$, where M_{ij} is a $0-1$ matrix of size $n_i \times n_j$. The *$\mathcal{M}$-join of the graphs in $\mathcal{H}_k$*, denoted by $\bigvee_{\mathcal{M}} \mathcal{H}_k$, is the graph $\bigcup\limits_{\substack{i,j=1,\\ i<j}}^{k} \left(H_i \vee_{M_{ij}} H_j\right)$.

Two graphs with the same number of vertices are said to *commute* if their adjacency matrices commute with each other. For the existence and properties of commuting graphs, we refer the reader to [2].

The following result will be used to prove the results of the subsequent sections.

Theorem 1 ([14], Corollary 2) *Let A be a $k \times k$ block matrix whose blocks A_{ij}, $1 \le i, j \le k$ are $n \times n$ real symmetric matrices commute with each other, then the spectrum of A consists of the spectra of the matrices E_h, $h = 1, 2, \ldots, n$ whose ij-th entry is $a_{ij}^{(h)}$, where $a_{ij}^{(h)}$ is an eigenvalue of A_{ij} corresponding to the same eigenvector X_h for each $i, j = 1, 2, \ldots, k$.*

3 Spectra of three families of 4 × 4 block matrices

In this section, we determine the spectra with corresponding eigenvectors of three families of block matrices of specific form.

For $i = 1, 2$, let A_i be a real symmetric matrix of order n_i and let it has equal row sums r_i. Then A_i has n_i real eigenvalues; let them be denoted by $(r_i =)\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in_i}$ with corresponding orthogonal eigenvectors $(J_{n_i \times 1} =)X_{i1}, X_{i2}, \ldots, X_{in_i}$ for $i = 1, 2$. Now we consider the matrix

$$D = \begin{bmatrix} a_1 J_{n_1} + a_2 A_1 + a_3 I_{n_1} & a'_1 J_{n_1} + a'_2 A_1 + a'_3 I_{n_1} & c_1 J_{n_1 \times n_2} & c_2 J_{n_1 \times n_2} \\ a'_1 J_{n_1} + a'_2 A_1 + a'_3 I_{n_1} & a_1 J_{n_1} + a_2 A_1 + a_3 I_{n_1} & c_3 J_{n_1 \times n_2} & c_4 J_{n_1 \times n_2} \\ c_1 J_{n_2 \times n_1} & c_3 J_{n_2 \times n_1} & b_1 J_{n_2} + b_2 A_2 + b_3 I_{n_2} & b'_1 J_{n_2} + b'_2 A_2 + b'_3 I_{n_2} \\ c_2 J_{n_2 \times n_1} & c_4 J_{n_2 \times n_1} & b'_1 J_{n_2} + b'_2 A_2 + b'_3 I_{n_2} & b_1 J_{n_2} + b_2 A_2 + b_3 I_{n_2} \end{bmatrix}, \quad (1)$$

where $a_i, a'_i, b_i, b'_i, c_j \in \mathbb{R}$ for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$.

Theorem 2 *The spectrum of the matrix D is*

(i) $(a_2 + a'_2)\lambda_{1j} + a_3 + a'_3$, $(a_2 - a'_2)\lambda_{1j} + a_3 - a'_3$ *for* $j = 2, 3, \ldots, n_1$;
(ii) $(b_2 + b'_2)\lambda_{2k} + b_3 + b'_3$, $(b_2 - b'_2)\lambda_{2k} + b_3 - b'_3$ *for* $k = 2, 3, \ldots, n_2$;
(iii) the roots of the equation $x^4 - 2(\sigma_1 + \chi_1)x^3 + [\sigma_1^2 - \sigma_2^2 + \chi_1^2 - \chi_2^2 + 4\sigma_1\chi_1 - cn_1n_2]x^2 - \{2[\sigma_1(\chi_1^2 - \chi_2^2) + \chi_1(\sigma_1^2 - \sigma_2^2)] - cn_1n_2(\sigma_1 + \chi_1) + 2n_1n_2[\sigma_2(c_1c_3 + c_2c_4) + \chi_2(c_1c_2 + c_3c_4)]\}x + (\sigma_1^2 - \sigma_2^2)(\chi_1^2 - \chi_2^2) - n_1n_2[c\sigma_1\chi_1 - 2(\sigma_1\chi_2(c_1c_2 + c_3c_4) - \sigma_2\chi_2(c_1c_4 + c_2c_3) + \chi_1\sigma_2(c_1c_3 + c_2c_4))] + n_1^2n_2^2(c_1c_4 - c_2c_3)^2 = 0$,

where $\sigma_1 = a_1n_1 + a_2r_1 + a_3$, $\sigma_2 = a'_1n_1 + a'_2r_1 + a'_3$, $\chi_1 = b_1n_2 + b_2r_2 + b_3$, $\chi_2 = b'_1n_2 + b'_2r_2 + b'_3$ *and* $c = c_1^2 + c_2^2 + c_3^2 + c_4^2$.

Proof Since X_{1j} is orthogonal to $J_{n_1 \times 1}$ for $j = 2, 3, \ldots, n_1$ and X_{2k} is orthogonal to $J_{n_2 \times 1}$ for $k = 2, 3, \ldots, n_2$, it can be easily verified that, the vectors $\left[X_{1j}{}^T \ X_{1j}{}^T \ \mathbf{0}_{1\times 2n_2}\right]^T$, $\left[X_{1j}{}^T \ -X_{1j}{}^T \ \mathbf{0}_{1\times 2n_2}\right]^T$, $\left[\mathbf{0}_{1\times 2n_1} \ X_{2k}{}^T \ X_{2k}{}^T\right]^T$ and $\left[\mathbf{0}_{1\times 2n_1} \ X_{2k}{}^T \ -X_{2k}{}^T\right]^T$ are eigenvectors of D corresponding to the eigenvalues $(a_2 + a'_2)\lambda_{1j} + a_3 + a'_3$, $(a_2 - a'_2)\lambda_{1j} + a_3 - a'_3$, $(b_2 + b'_2)\lambda_{2k} + b_3 + b'_3$ and $(b_2 - b'_2)\lambda_{2k} + b_3 - b'_3$ respectively. Thus we have obtained $2(n_1 + n_2) - 4$ eigenvalues of D with corresponding eigenvectors.

Notice that all these eigenvectors are orthogonal to the four vectors $\left[J_{1\times n_1} \ \mathbf{0}_{1\times(n_1+2n_2)}\right]^T$, $\left[\mathbf{0}_{1\times n_1} \ J_{1\times n_1} \ \mathbf{0}_{1\times 2n_2}\right]^T$, $\left[\mathbf{0}_{1\times 2n_1} \ J_{1\times n_2} \ \mathbf{0}_{1\times n_2}\right]^T$ and $\left[\mathbf{0}_{1\times(2n_1+n_2)} \ J_{1\times n_2}\right]^T$. This implies that the space spanned by these four vectors and the space spanned by the remaining four eigenvectors of D are same. So the remaining four eigenvectors of D are of the form $\left[\alpha_1 J_{1\times n_1} \ \alpha_2 J_{1\times n_1} \ \alpha_3 J_{1\times n_2} \ \alpha_4 J_{1\times n_2}\right]^T$ for some $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \neq (0, 0, 0, 0)$.

To find the remaining four eigenvalues, assume that ν is an eigenvalue of D with corresponding eigenvector $\left[\alpha_1 J_{1\times n_1} \ \alpha_2 J_{1\times n_1} \ \alpha_3 J_{1\times n_2} \ \alpha_4 J_{1\times n_2}\right]^T$. Then we have

$$D \begin{bmatrix} \alpha_1 J_{n_1\times 1} \\ \alpha_2 J_{n_1\times 1} \\ \alpha_3 J_{n_2\times 1} \\ \alpha_4 J_{n_2\times 1} \end{bmatrix} = \nu \begin{bmatrix} \alpha_1 J_{n_1\times 1} \\ \alpha_2 J_{n_1\times 1} \\ \alpha_3 J_{n_2\times 1} \\ \alpha_4 J_{n_2\times 1} \end{bmatrix}.$$

The above equation is equivalent to

$$[\mathcal{P} - \nu I_{2(n_1+n_2)}] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2}$$

where

$$\mathcal{P} = \begin{bmatrix} a_1n_1 + a_2r_1 + a_3 & a_1'n_1 + a_2'r_1 + a_3' & c_1n_2 & c_2n_2 \\ a_1'n_1 + a_2'r_1 + a_3' & a_1n_1 + a_2r_1 + a_3 & c_3n_2 & c_4n_2 \\ c_1n_1 & c_3n_1 & b_1n_2 + b_2r_2 + b_3 & b_1'n_2 + b_2r_2' + b_3' \\ c_2n_1 & c_4n_1 & b_1'n_2 + b_2'r_2 + b_3' & b_1n_2 + b_2r_2 + b_3 \end{bmatrix}.$$

Notice that (2) has a non-trivial solution only if $\mathcal{P} - \nu I_{2(n_1+n_2)}$ is singular. It follows that the remaining four eigenvalues of D are the eigenvalues of the matrix $\mathcal{P}$ whose characteristic equation is stated in this theorem. This completes the proof. □

Now, we consider the matrix D' which is obtained by taking $c_1 = c_4$ and $c_2 = c_3$ in D. Let R' denote the diagonal matrix of order $2(n_1 + n_2)$ in which the i-th diagonal entry is the i-th row sum of the matrix D', i.e.,

$$R' = \begin{bmatrix} w_1 I_{2n_1} & \mathbf{0}_{2n_1\times 2n_2} \\ \mathbf{0}_{2n_2\times 2n_1} & w_2 I_{2n_2} \end{bmatrix},$$

where $w_1 = (a_1 + a_1')n_1 + (c_1 + c_2)n_2 + (a_2 + a_2')r_1 + a_3 + a_3'$ and $w_2 = (c_1 + c_2)n_1 + (b_1 + b_1')n_2 + (b_2 + b_2')r_2 + b_3 + b_3'$.

For $0 \le \alpha \le 1$, we define the matrix D_α' as follows:

$$D_\alpha' = \alpha R' + (1 - \alpha)D'. \tag{3}$$

Theorem 3 *The spectrum of the matrix D_α' is*

(i) $\alpha[(a_1 + a_1')n_1 + (c_1 + c_2)n_2 + (a_2 + a_2')r_1 - (a_2 + a_2')\lambda_{1j}] + (a_2 + a_2')\lambda_{1j} + a_3 + a_3'$ *for* $j = 2, 3, \ldots, n_1$,
(ii) $\alpha[(a_1 + a_1')n_1 + (c_1 + c_2)n_2 + (a_2 + a_2')r_1 - (a_2 - a_2')\lambda_{1j} + 2a_3'] + (a_2 - a_2')\lambda_{1j} + a_3 - a_3'$ *for* $j = 2, 3, \ldots, n_1$,
(iii) $\alpha[(c_1 + c_2)n_1 + (b_1 + b_1')n_2 + (b_2 + b_2')r_2 - (b_2 + b_2')\lambda_{2k}] + (b_2 + b_2')\lambda_{2k} + b_3 + b_3'$ *for* $k = 2, 3, \ldots, n_2$,
(iv) $\alpha[(c_1 + c_2)n_1 + (b_1 + b_1')n_2 + (b_2 + b_2')r_2 - (b_2 - b_2')\lambda_{2k} + 2b_3'] + (b_2 - b_2')\lambda_{2k} + b_3 - b_3'$ *for* $k = 2, 3, \ldots, n_2$,
(v) $\frac{1}{2}(\sigma_1 + \sigma_2 + \chi_1 + \chi_2) \pm \frac{1}{2}[(\sigma_1 + \sigma_2 + \chi_1 + \chi_2)^2 - 4(\sigma_1 + \sigma_2)(\chi_1 + \chi_2) + 4(1 - \alpha)^2(c_1 + c_2)^2 n_1n_2]^{\frac{1}{2}}$,
$\frac{1}{2}(\sigma_1 - \sigma_2 + \chi_1 - \chi_2) \pm \frac{1}{2}[(\sigma_1 - \sigma_2 + \chi_1 - \chi_2)^2 - 4(\sigma_1 - \sigma_2)(\chi_1 - \chi_2) + 4(1 - \alpha)^2(c_1 - c_2)^2 n_1n_2]^{\frac{1}{2}}$,

where $\sigma_1 = \alpha[a_1'n_1 + (c_1 + c_2)n_2 + a_2'r_1 + a_3'] + a_1n_1 + a_2r_1 + a_3$, $\sigma_2 = (1 - \alpha)(a_1'n_1 + a_2'r_1 + a_3')$, $\chi_1 = \alpha[(c_1 + c_2)n_1 + b_1'n_2 + b_2'r_2 + b_3'] + b_1n_2 + b_2r_2 + b_3$ *and* $\chi_2 = (1 - \alpha)(b_1'n_2 + b_2'r_2 + b_3')$.

Proof For $j = 2, 3, \ldots, n_1$ and $k = 2, 3, \ldots, n_2$, it is easy to verify that $\left[X_{1j}{}^T \; X_{1j}{}^T \; \mathbf{0}_{1\times 2n_2}\right]^T$, $\left[X_{1j}{}^T \; -X_{1j}{}^T \; \mathbf{0}_{1\times 2n_2}\right]^T$, $\left[\mathbf{0}_{1\times 2n_1} \; X_{2k}{}^T \; X_{2k}{}^T\right]^T$ and $\left[\mathbf{0}_{1\times 2n_1} \; X_{2k}{}^T \; -X_{2k}{}^T\right]^T$ are eigenvectors of D_α' corresponding to the eigenvalues respectively given in (i), (ii), (iii) and (iv) of this theorem.

As in the proof of Theorem 2, the remaining four eigenvectors of D_α' are of the form $\left[\alpha_1 J_{1\times n_1} \alpha_2 J_{1\times n_1} \alpha_3 J_{1\times n_2} \alpha_4 J_{1\times n_2}\right]^T$ for some $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \neq (0, 0, 0, 0)$. Let ν be an eigenvalue of D_α' with corresponding one such eigenvector. Then we get

$$[\mathcal{P} - \nu I_{2(n_1+n_2)}]\left[\alpha_1 \; \alpha_2 \; \alpha_3 \; \alpha_4\right]^T = \left[0\;0\;0\;0\right]^T, \tag{4}$$

where

$$\mathcal{P} = \begin{bmatrix} \sigma_1 & \sigma_2 & (1-\alpha)c_1n_2 & (1-\alpha)c_2n_2 \\ \sigma_2 & \sigma_1 & (1-\alpha)c_2n_2 & (1-\alpha)c_1n_2 \\ (1-\alpha)c_1n_1 & (1-\alpha)c_2n_1 & \chi_1 & \chi_2 \\ (1-\alpha)c_2n_1 & (1-\alpha)c_1n_1 & \chi_2 & \chi_1 \end{bmatrix},$$

$\sigma_1 = \alpha[a_1'n_1 + (c_1 + c_2)n_2 + a_2'r_1 + a_3'] + a_1n_1 + a_2r_1 + a_3$, $\sigma_2 = (1 - \alpha)(a_1'n_1 + a_2'r_1 + a_3')$, $\chi_1 = \alpha[(c_1 + c_2)n_1 + b_1'n_2 + b_2'r_2 + b_3'] + b_1n_2 + b_2r_2 + b_3$ and $\chi_2 = (1 - \alpha)(b_1'n_2 + b_2'r_2 + b_3')$.

The Eq. (4) has a non-trivial solution only if $\mathcal{P} - \nu I_{2(n_1+n_2)}$ is singular. So we have that the remaining four eigenvalues of D_α' are the eigenvalues of $\mathcal{P}$ which are given in (v) of this theorem. This completes the proof. □

Next, we consider the matrix D'' which is obtained by taking $n_1 = n_2 = n$ in D. Let R'' denote the diagonal matrix of order $4n$ in which the i-th diagonal entry is the i-th row sum of the matrix D''. For $0 \le \alpha \le 1$, we define the matrix D''_α as follows:

$$D''_\alpha = \alpha R'' + (1-\alpha)D''. \tag{5}$$

Theorem 4 *If A_1 and A_2 commute, then the spectrum of the matrix D''_α is*

(i) $\frac{1}{2}\alpha(s_1+s_2)+(1-\alpha)(a_2\lambda_{1i}+a_3) \pm \frac{1}{2}\Big\{[\alpha(s_1+s_2)+2(1-\alpha)(a_2\lambda_{1i}+a_3)]^2 - 4[\alpha s_1+(1-\alpha)(a_2\lambda_{1i}+a_3)][\alpha s_2+(1-\alpha)(a_2\lambda_{1i}+a_3)] + 4(1-\alpha)^2(a'_2\lambda_{1i}+a'_3)^2\Big\}^{\frac{1}{2}}$ *for $i = 2, 3, \ldots, n$;*

(ii) $\frac{1}{2}\alpha(t_1+t_2)+(1-\alpha)(b_2\lambda_{2i}+b_3) \pm \frac{1}{2}\Big\{[\alpha(t_1+t_2)+2(1-\alpha)(b_2\lambda_{2i}+b_3)]^2 - 4[\alpha t_1+(1-\alpha)(b_2\lambda_{2i}+b_3)][\alpha t_2+(1-\alpha)(b_2\lambda_{2i}+b_3)] + 4(1-\alpha)^2(b'_2\lambda_{2i}+b'_3)^2\Big\}^{\frac{1}{2}}$ *for $i = 2, 3, \ldots, n$;*

(iii) the roots of the equation
$x^4 - (\sigma_1+\sigma_2+\chi_1+\chi_2)x^3 + [(\sigma_1+\sigma_2)(\chi_1+\chi_2)+\sigma_1\sigma_2+\chi_1\chi_2 - n^2(1-\alpha)^2(c_1^2+c_2^2+c_3^2+c_4^2)-\tau_1^2-\tau_2^2]x^2 + \{(\sigma_1+\sigma_2)(\tau_2^2-\chi_1\chi_2)+(\chi_1+\chi_2)(\tau_1^2-\sigma_1\sigma_2)+n^2(1-\alpha)^2[\sigma_1(c_3^2+c_4^2)+\sigma_2(c_1^2+c_2^2)+\chi_1(c_2^2+c_4^2)+\chi_2(c_1^2+c_3^2)-2\tau_1(c_1c_3+c_2c_4)-2\tau_2(c_1c_2+c_3c_4)]\}x + \tau_1^2\tau_2^2+\sigma_1\sigma_2\chi_1\chi_2-\sigma_1\sigma_2\tau_2^2-\chi_1\chi_2\tau_1^2 - 2n^2(1-\alpha)^2[\tau_1\tau_2(c_1c_4+c_2c_3)-\tau_1(\chi_2c_1c_3+\chi_1c_2c_4)-\tau_2(\sigma_2c_1c_2+\sigma_1c_3c_4)] - n^2(1-\alpha)^2[\sigma_2(\chi_2c_1^2+\chi_1c_2^2)+\sigma_1(\chi_2c_3^2+\chi_1c_4^2)] + n^4(1-\alpha)^4(c_1c_4-c_2c_3)^2 = 0,$

where $s_1 = (a_1+a'_1+c_1+c_2)n+(a_2+a'_2)r_1+a_3+a'_3$, $s_2 = (a_1+a'_1+c_3+c_4)n+(a_2+a'_2)r_1+a_3+a'_3$, $t_1 = (b_1+b'_1+c_1+c_3)n+(b_2+b'_2)r_2+b_3+b'_3$, $t_2 = (b_1+b'_1+c_2+c_4)n+(b_2+b'_2)r_2+b_3+b'_3$, $\tau_1 = (1-\alpha)(a'_1n+a'_2r_1+a'_3)$, $\tau_2 = (1-\alpha)(b'_1n+b'_2r_2+b'_3)$, $\sigma_i = \alpha s_i+(1-\alpha)(a_1n+a_2r_1+a_3)$ *and* $\chi_i = \alpha t_i+(1-\alpha)(b_1n+b_2r_2+b_3)$ *for $i = 1, 2$.*

Proof Since A_1, A_2, J_n and I_n commute with each other, we have that all the blocks of the 4×4 block matrix D''_α commute with each other. From Theorem 1, the spectrum of D''_α consists of the spectra of the following matrices:

$$\begin{bmatrix} \sigma_1 & \tau_1 & (1-\alpha)c_1n & (1-\alpha)c_2n \\ \tau_1 & \sigma_2 & (1-\alpha)c_3n & (1-\alpha)c_4n \\ (1-\alpha)c_1n & (1-\alpha)c_3n & \chi_1 & \tau_2 \\ (1-\alpha)c_2n & (1-\alpha)c_4n & \tau_2 & \chi_2 \end{bmatrix}$$

and

$$\begin{bmatrix} \alpha s_1+(1-\alpha)(a_2\lambda_{1i}+a_3) & (1-\alpha)(a'_2\lambda_{1i}+a'_3) & 0 & 0 \\ (1-\alpha)(a'_2\lambda_{1i}+a'_3) & \alpha s_2+(1-\alpha)(a_2\lambda_{1i}+a_3) & 0 & 0 \\ 0 & 0 & \alpha t_1+(1-\alpha)(a_2\lambda_{1i}+a_3) & (1-\alpha)(b'_2\lambda_{2i}+b'_3) \\ 0 & 0 & (1-\alpha)(b'_2\lambda_{2i}+b'_3) & \alpha t_2+(1-\alpha)(a_2\lambda_{1i}+a_3) \end{bmatrix}$$

for $i = 2, 3, \ldots, n$. The characteristic equation of the first matrix is given in (iii) of this theorem. The eigenvalues of the remaining $n-1$ matrices are given in (i) and (ii) of this theorem. This completes the proof. □

4 Families of M_1-join and M_2-join of graphs and their generalized distance spectra

Let G be a graph on n vertices. In Table 1, we give some unary graph operations of G defined in [13].

For $i = 1, 2$, let G_i be a graph on n_i vertices. Let

$$M_1 = J_{2n_1\times 2n_2} \text{ and } M_2 = \begin{bmatrix} J_{n_1\times n_2} & \mathbf{0}_{n_1\times n_2} \\ \mathbf{0}_{n_1\times n_2} & J_{n_1\times n_2} \end{bmatrix}.$$

For each $i, j \in \{1, 2, \ldots, 24\}$, we consider the following two graphs:

(i) M_1-join of $U_i(G_1)$ and $U_j(G_2)$;
(ii) M_2-join of $U_i(G_1)$ and $U_j(G_2)$,

Table 1 Some unary graph operations on a graph G

S. no.	Description	Name of the unary graph operation	Notation
1.	$G \vee_{I_n} G$	Mirror graph of G	$\mathcal{U}_1(G)$
2.	$G \vee_{J_n} G$	Join neighbourhood graph of G	$\mathcal{U}_2(G)$
3.	$G \vee_{J_n - I_n} G$	VC-neighbourhood graph of G	$\mathcal{U}_3(G)$
4.	$G \vee_{A(G)} G$	N-neighbourhood graph of G	$\mathcal{U}_4(G)$
5.	$G \vee_{A(G)+I_n} G$	$\overline{N}$-neighbourhood graph of G	$\mathcal{U}_5(G)$
6.	$G \vee_{A(\overline{G})} G$	NC-neighbourhood graph of G	$\mathcal{U}_6(G)$
7.	$G \vee_{A(\overline{G})+I_n} G$	$\overline{N}C$-neighbourhood graph of G	$\mathcal{U}_7(G)$
8.	$\overline{G} \vee_{I_n} \overline{G}$	Mirror complement graph of G	$\mathcal{U}_8(G)$
9.	$\overline{G} \vee_{J_n} \overline{G}$	Join-neighbourhood-complement graph of G	$\mathcal{U}_9(G)$
10.	$\overline{G} \vee_{J_n - I_n} \overline{G}$	VC-neighbourhood-complement graph of G	$\mathcal{U}_{10}(G)$
11.	$\overline{G} \vee_{A(G)} \overline{G}$	N-neighbourhood complement graph of G	$\mathcal{U}_{11}(G)$
12.	$\overline{G} \vee_{A(G)+I_n} \overline{G}$	$\overline{N}$-neighbourhood-complement graph of G	$\mathcal{U}_{12}(G)$
13.	$\overline{G} \vee_{A(\overline{G})} \overline{G}$	NC-neighbourhood complement graph of G	$\mathcal{U}_{13}(G)$
14.	$\overline{G} \vee_{A(\overline{G})+I_n} \overline{G}$	$\overline{N}C$-neighbourhood-complement graph of G	$\mathcal{U}_{14}(G)$
15.	$K_n \vee_{A(G)} K_n$	Fully complete duplicate graph of G	$\mathcal{U}_{15}(G)$
16.	$K_n \vee_{A(G)+I_n} K_n$	Fully complete $D\overline{N}$-graph of G	$\mathcal{U}_{16}(G)$
17.	$K_n \vee_{A(\overline{G})} K_n$	Fully complete complemented duplicate graph of G	$\mathcal{U}_{17}(G)$
18.	$K_n \vee_{A(\overline{G})+I_n} K_n$	Fully complete closed duplicate graph of G	$\mathcal{U}_{18}(G)$
19.	$\overline{K}_n \vee_{A(G)} \overline{K}_n$	Duplicate graph of G	$\mathcal{U}_{19}(G)$
20.	$\overline{K}_n \vee_{A(G)+I_n} \overline{K}_n$	$D\overline{N}$-graph of G	$\mathcal{U}_{20}(G)$
21.	$\overline{K}_n \vee_{A(\overline{G})} \overline{K}_n$	Complemented duplicate graph of G	$\mathcal{U}_{21}(G)$
22.	$\overline{K}_n \vee_{A(\overline{G})+I_n} \overline{K}_n$	Closed duplicate graph of G	$\mathcal{U}_{22}(G)$
23.	$G \vee_{\mathbf{0}_n} G$	Union of two copies of G	$\mathcal{U}_{23}(G)$
24.	$\overline{G} \vee_{\mathbf{0}_n} \overline{G}$	Union of two copies of $\overline{G}$	$\mathcal{U}_{24}(G)$

where the graphs $U_i(G_1)$ and $U_j(G_2)$ are given in Table 1.

Now, we proceed to obtain the generalized distance spectra of the two families of graphs mentioned above under some conditions on the constituting graphs and to construct generalized distance cospectral graphs by using these results.

Theorem 5 *For $i = 1, 2$, let G_i be an r_i-regular graph on n_i vertices and let the eigenvalues of $A(G_i)$ be $(r_i =)\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in_i}$. For each $p, q \in \{1, 2, \ldots, 24\}$, the generalized distance spectrum of $\mathcal{U}_p(G_1) \vee_{M_1} \mathcal{U}_q(G_2)$ is*

(i) $\alpha[(a_1 + a_1')n_1 + 2n_2 + (a_2 + a_2')r_1 - (a_2 + a_2')\lambda_{1j}] + (a_2 + a_2')\lambda_{1j} + a_3 + a_3'$,
$\alpha[(a_1 + a_1')n_1 + 2n_2 + (a_2 + a_2')r_1 - (a_2 - a_2')\lambda_{1j} + 2a_3'] + (a_2 - a_2')\lambda_{1j} + a_3 - a_3'$ *for* $j = 2, 3, \ldots, n_1$;

(ii) $\alpha[2n_1 + (b_1 + b_1')n_2 + (b_2 + b_2')r_2 - (b_2 + b_2')\lambda_{2k}] + (b_2 + b_2')\lambda_{2k} + b_3 + b_3'$,
$\alpha[2n_1 + (b_1 + b_1')n_2 + (b_2 + b_2')r_2 - (b_2 - b_2')\lambda_{2k} + 2b_3'] + (b_2 - b_2')\lambda_{2k} + b_3 - b_3'$ *for* $k = 2, 3, \ldots, n_2$;

(iii) $\sigma_1 - \sigma_2$, $\chi_1 - \chi_2$ *and* $\frac{1}{2}(\sigma_1+\sigma_2+\chi_1+\chi_2) \pm \frac{1}{2}[(\sigma_1+\sigma_2+\chi_1+\chi_2)^2 - 4(\sigma_1+\sigma_2)(\chi_1+\chi_2) + 16(1-\alpha)^2 n_1 n_2]^{\frac{1}{2}}$,

where the values $(a_1, a_2, a_3, a_1', a_2', a_3')$ and $(b_1, b_2, b_3, b_1', b_2', b_3')$ are given in the rows against the graphs $\mathcal{U}_p(G_1)$ and $\mathcal{U}_q(G_2)$, respectively in Table 2, $\sigma_1 = \alpha[a_1'n_1 + 2n_2 + a_2'r_1 + a_3'] + a_1n_1 + a_2r_1 + a_3$, $\sigma_2 = (1-\alpha)(a_1'n_1 + a_2'r_1 + a_3')$, $\chi_1 = \alpha[2n_1 + b_1'n_2 + b_2'r_2 + b_3'] + b_1n_2 + b_2r_2 + b_3$ and $\chi_2 = (1-\alpha)(b_1'n_2 + b_2'r_2 + b_3')$.

Proof Fix $p, q \in \{1, 2, \ldots, 24\}$. Taking $A_1 = A(G_1)$; $A_2 = A(G_2)$; $(c_1, c_2) = (1, 1)$; the values $(a_1, a_2, a_3, a_1', a_2', a_3')$ and $(b_1, b_2, b_3, b_1', b_2', b_3')$ as given in the rows against the graphs $\mathcal{U}_p(G_1)$ and $\mathcal{U}_q(G_2)$, respectively in Table 2, and substituting these in (3), we get the generalized distance matrix of $\mathcal{U}_p(G_1) \vee_{M_1} \mathcal{U}_q(G_2)$. Therefore, by substituting these values in Theorem 3, we get the generalized distance spectrum of $\mathcal{U}_p(G_1) \vee_{M_1} \mathcal{U}_q(G_2)$ as stated in this theorem. □

Notice that we have obtained the generalized distance spectra of the graphs constructed by 576 binary graph operations from Theorem 5.

In the following result we construct infinite pairs of generalized distance cospectral graphs by using Theorem 5.

Corollary 1 *For $i = 1, 2$, let G_i and H_i be two r_i-regular adjacency cospectral graphs on n_i vertices. For each $p, q \in \{1, 2, \ldots, 24\}$, $\mathcal{U}_p(G_1) \vee_{M_1} \mathcal{U}_q(G_2)$ and $\mathcal{U}_p(H_1) \vee_{M_1} \mathcal{U}_q(H_2)$ are generalized distance cospectral. In particular, they are distance (distance Laplacian, distance signless Laplacian) cospectral.*

Theorem 6 *For $i = 1, 2$, let G_i be an r_i-regular graph on n_i vertices satisfying one of the following:*

(a) For $p, q \in \{1, 2, \ldots, 22\}$, the graphs G_1 and G_2 satisfy the additional conditions given in the rows against the graphs $\mathcal{U}_p(G_1)$ and $\mathcal{U}_q(G_2)$, respectively in Table 3;

(b) For $p \in \{1, 2, \ldots, 22\}$, the graph G_1 satisfies the additional conditions given in the row against the graph $\mathcal{U}_p(G_1)$ in Table 3; for $q \in \{3, 4, 6, 10, 11, 13, 15, 17, 19, 21, 23, 24\}$, the graph G_2 satisfies the additional conditions given in the row against the graph $\mathcal{U}_q(G_2)$ in Table 4;

(c) For $p \in \{3, 4, 6, 10, 11, 13, 15, 17, 19, 21, 23, 24\}$, the graph G_1 satisfies the additional conditions given in the row against the graph $\mathcal{U}_p(G_1)$ in Table 4; for $q \in \{1, 2, \ldots, 22\}$, the graph G_2 satisfies the additional conditions given in the row against the graph $\mathcal{U}_q(G_2)$ in Table 3.

For $i = 1, 2$, if the eigenvalues of $A(G_i)$ are $(r_i =)\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in_i}$, then the generalized distance spectrum of $\mathcal{U}_p(G_1) \vee_{M_2} \mathcal{U}_q(G_2)$ is

(i) $\alpha[(a_1 + a_1')n_1 + 3n_2 + (a_2 + a_2')r_1 - (a_2 + a_2')\lambda_{1j}] + (a_2 + a_2')\lambda_{1j} + a_3 + a_3'$, $\alpha[(a_1 + a_1')n_1 + 3n_2 + (a_2 + a_2')r_1 - (a_2 - a_2')\lambda_{1j} + 2a_3'] + (a_2 - a_2')\lambda_{1j} + a_3 - a_3'$ *for* $j = 2, 3, \ldots, n_1$;

(ii) $\alpha[3n_1 + (b_1 + b_1')n_2 + (b_2 + b_2')r_2 - (b_2 + b_2')\lambda_{2k}] + (b_2 + b_2')\lambda_{2k} + b_3 + b_3'$, $\alpha[3n_1 + (b_1 + b_1')n_2 + (b_2 + b_2')r_2 - (b_2 - b_2')\lambda_{2k} + 2b_3'] + (b_2 - b_2')\lambda_{2k} + b_3 - b_3'$ *for* $k = 2, 3, \ldots, n_2$;

(iii) $\frac{1}{2}(\sigma_1 + \sigma_2 + \chi_1 + \chi_2) \pm \frac{1}{2}[(\sigma_1 + \sigma_2 + \chi_1 + \chi_2)^2 - 4(\sigma_1 + \sigma_2)(\chi_1 + \chi_2) + 36(1-\alpha)^2 n_1 n_2]^{\frac{1}{2}}$ *and* $\frac{1}{2}(\sigma_1 - \sigma_2 + \chi_1 - \chi_2) \pm \frac{1}{2}[(\sigma_1 - \sigma_2 + \chi_1 - \chi_2)^2 - 4(\sigma_1 - \sigma_2)(\chi_1 - \chi_2) + 4(1-\alpha)^2 n_1 n_2]^{\frac{1}{2}}$,

where the values $(a_1, a_2, a_3, a_1', a_2', a_3')$ and $(b_1, b_2, b_3, b_1', b_2', b_3')$ corresponding to the graphs $\mathcal{U}_p(G_1)$ and $\mathcal{U}_q(G_2)$ are given in the respective rows of the corresponding tables, $\sigma_1 = \alpha[a_1'n_1 + 3n_2 + a_2'r_1 + a_3'] + a_1n_1 + a_2r_1 + a_3$, $\sigma_2 = (1-\alpha)(a_1'n_1 + a_2'r_1 + a_3')$, $\chi_1 = \alpha[3n_1 + b_1'n_2 + b_2'r_2 + b_3'] + b_1n_2 + b_2r_2 + b_3$ and $\chi_2 = (1-\alpha)(b_1'n_2 + b_2'r_2 + b_3')$.

Proof Fix $p, q \in \{1, 2, \ldots, 24\}$. Consider the graphs G_1 and G_2 with particular assumptions as given in the theorem. Taking $A_1 = A(G_1)$; $A_2 = A(G_2)$; $(c_1, c_2) = (2, 1)$; the values $(a_1, a_2, a_3, a_1', a_2', a_3')$ and $(b_1, b_2, b_3, b_1', b_2', b_3')$ as given in the rows against the graphs $\mathcal{U}_p(G_1)$ and $\mathcal{U}_q(G_2)$ of the corresponding tables, and substituting these in (3), we get the generalized distance matrix of $\mathcal{U}_p(G_1) \vee_{M_2} \mathcal{U}_q(G_2)$. Therefore, by substituting these values in Theorem 3, we get the generalized distance spectrum of $\mathcal{U}_p(G_1) \vee_{M_2} \mathcal{U}_q(G_2)$ as stated in this theorem. □

Notice that we have obtained the generalized distance spectra of graphs constructed by 572 binary graph operations from Theorem 6.

For $i = 1, 2$, let G_i be a graph on n_i vertices. The graphs $\Theta_1(G_1, G_2)$, $\Theta_2(G_1, G_2)$ and $\Theta_3(G_1, G_2)$ defined in [1] can be viewed as $\mathcal{U}_2(G_1) \vee_{M_2} \mathcal{U}_1(G_2)$, $\mathcal{U}_2(G_1) \vee_{M_2} \mathcal{U}_6(G_2)$ and $\mathcal{U}_1(G_1) \vee_{M_2} \mathcal{U}_6(G_2)$, respectively. The Indu–Bala product [16] of G_1 and G_2 can be viewed as $\mathcal{U}_{23}(G_1) \vee_{M_2} \mathcal{U}_1(G_2)$.

In the following three results we deduce the results on the distance spectra of the graphs constructed by these four operations.

Corollary 2 (Theorem 2.2 [1]) *For $i = 1, 2$, let G_i be an r_i-regular graph on n_i vertices. Then the distance spectrum of $\Theta_1(G_1, G_2)$ is obtained by taking $\alpha = 0$ in the generalized distance spectrum of $\mathcal{U}_2(G_1) \vee_{M_2} \mathcal{U}_1(G_2)$ determined in Theorem 6.*

Corollary 3 (Theorems 2.3, 2.4 [1]) *For $i = 1, 2$, let G_i be an r_i-regular graph on n_i vertices. Suppose $N_{G_2}(x) - N_{G_2}(y) \neq \{y\}$ for all adjacent vertices x and y in G_2. Then the distance spectra of $\Theta_2(G_1, G_2)$ and $\Theta_3(G_1, G_2)$ are obtained by taking $\alpha = 0$ in the generalized distance spectra of $\mathcal{U}_2(G_1) \vee_{M_2} \mathcal{U}_6(G_2)$ and $\mathcal{U}_1(G_1) \vee_{M_2} \mathcal{U}_6(G_2)$, respectively which are determined in Theorem 6.*

Corollary 4 (Theorem 2.1 [16]) *For $i = 1, 2$, let G_i be an r_i-regular graph on n_i vertices. Then the distance spectrum of Indu-Bala product of G_1 and G_2 is obtained by taking $\alpha = 0$ in the generalized distance spectrum of $\mathcal{U}_{23}(G_1) \vee_{M_2} \mathcal{U}_1(G_2)$ determined in Theorem 6.*

In the following result we construct infinite pairs of generalized distance cospectral graphs by using Theorem 6.

Corollary 5 *Let G_1 and G_2 (resp. H_1 and H_2) be graphs satisfying the conditions of Theorem 6. If G_1 and H_1 (resp. G_2 and H_2) are adjacency cospectral, then $\mathcal{U}_p(G_1) \vee_{M_2} \mathcal{U}_q(G_2)$ and $\mathcal{U}_p(H_1) \vee_{M_2} \mathcal{U}_q(H_2)$ are generalized distance cospectral for the respective values of p and q. In particular, they are distance (distance Laplacian, distance signless Laplacian) cospectral.*

In Tables 2, 3 and 4, G denotes a graph on n vertices.

Table 2 The necessary values for proving Theorems 5, 8, 9, 11 and 12

S. no.	Unary operation on the graph G	Values
1.	$\mathcal{U}_1(G)$	$(2, -1, -2, 2, 0, -1)$
2.	$\mathcal{U}_2(G)$	$(2, -1, -2, 1, 0, 0)$
3.	$\mathcal{U}_3(G)$	$(2, -1, -2, 1, 0, 1)$
4.	$\mathcal{U}_4(G)$	$(2, -1, -2, 2, -1, 0)$
5.	$\mathcal{U}_5(G)$	$(2, -1, -2, 2, -1, -1)$
6.	$\mathcal{U}_6(G)$	$(2, -1, -2, 1, 1, 1)$
7.	$\mathcal{U}_7(G)$	$(2, -1, -2, 1, 1, 0)$
8.	$\mathcal{U}_8(G)$	$(1, 1, -1, 2, 0, -1)$
9.	$\mathcal{U}_9(G)$	$(1, 1, -1, 1, 0, 0)$
10.	$\mathcal{U}_{10}(G)$	$(1, 1, -1, 1, 0, 1)$
11.	$\mathcal{U}_{11}(G)$	$(1, 1, -1, 2, -1, 0)$
12.	$\mathcal{U}_{12}(G)$	$(1, 1, -1, 2, -1, -1)$
13.	$\mathcal{U}_{13}(G)$	$(1, 1, -1, 1, 1, 1)$
14.	$\mathcal{U}_{14}(G)$	$(1, 1, -1, 1, 1, 0)$
15.	$\mathcal{U}_{15}(G)$	$(1, 0, -1, 2, -1, 0)$
16.	$\mathcal{U}_{16}(G)$	$(1, 0, -1, 2, -1, -1)$
17.	$\mathcal{U}_{17}(G)$	$(1, 0, -1, 1, 1, 1)$
18.	$\mathcal{U}_{18}(G)$	$(1, 0, -1, 1, 1, 0)$
19.	$\mathcal{U}_{19}(G)$	$(2, 0, -2, 2, -1, 0)$
20.	$\mathcal{U}_{20}(G)$	$(2, 0, -2, 2, -1, -1)$
21.	$\mathcal{U}_{21}(G)$	$(2, 0, -2, 1, 1, 1)$
22.	$\mathcal{U}_{22}(G)$	$(2, 0, -2, 1, 1, 0)$
23.	$\mathcal{U}_{23}(G)$	$(2, -1, -2, 2, 0, 0)$
24.	$\mathcal{U}_{24}(G)$	$(1, 1, -1, 2, 0, 0)$

Table 3 The necessary values for proving Theorem 6

S. no.	Unary operation on the graph G	Additional conditions	Values
1.	$\mathcal{U}_1(G)$	–	$(2, -1, -2, 3, -1, -2)$
2.	$\mathcal{U}_2(G)$	–	$(2, -1, -2, 1, 0, 0)$
3.	$\mathcal{U}_3(G)$	$n \geq 2;\ G = \overline{K}_n$	$(2, 0, -2, 1, 0, 2)$
4.	$\mathcal{U}_3(G)$	$G \neq \overline{K}_n$	$(2, -1, -2, 1, 0, 1)$
5.	$\mathcal{U}_4(G)$	$n \geq 2$; G is connected; $diam(G) \leq 2$	$(2, -1, -2, 2, -1, 0)$
6.	$\mathcal{U}_5(G)$	$n \geq 2$; G is connected; $diam(G) \leq 2$	$(2, -1, -2, 2, -1, -1)$
7.	$\mathcal{U}_6(G)$	$n \geq 2$; for any two $u \sim v$ in G, $N_G(u) \backslash N_G(v) \neq \{v\}$	$(2, -1, -2, 1, 1, 2)$
8.	$\mathcal{U}_7(G)$	–	$(2, -1, -2, 1, 1, 0)$
9.	$\mathcal{U}_8(G)$	–	$(1, 1, -1, 2, 1, -1)$
10.	$\mathcal{U}_9(G)$	–	$(1, 1, -1, 1, 0, 0)$
11.	$\mathcal{U}_{10}(G)$	$n \geq 2;\ G = K_n$	$(2, 0, -2, 1, 0, 2)$
12.	$\mathcal{U}_{10}(G)$	$G \neq K_n$	$(1, 1, -1, 1, 0, 1)$
13.	$\mathcal{U}_{11}(G)$	$n \geq 2$; for any two $u \nsim v$ in G, $N_{\overline{G}}(u) \backslash N_{\overline{G}}(v) \neq \{v\}$	$(1, 1, -1, 2, -1, 1)$
14.	$\mathcal{U}_{12}(G)$	–	$(1, 1, -1, 2, -1, -1)$
15.	$\mathcal{U}_{13}(G)$	$n \geq 2$; for any two $u \sim v$ in G, $N_{\overline{G}}(u) \cap N_{\overline{G}}(v) \neq \emptyset$	$(1, 1, -1, 1, 1, 1)$
16.	$\mathcal{U}_{14}(G)$	$G \neq K_n$; for any two $u \sim v$ in G, $N_{\overline{G}}(u) \cap N_{\overline{G}}(v) \neq \emptyset$	$(1, 1, -1, 1, 1, 0)$
17.	$\mathcal{U}_{15}(G)$	$G \neq \overline{K}_n$	$(1, 0, -1, 2, -1, 0)$
18.	$\mathcal{U}_{16}(G)$	–	$(1, 0, -1, 2, -1, -1)$
19.	$\mathcal{U}_{17}(G)$	$G \neq K_n$	$(1, 0, -1, 1, 1, 1)$
20.	$\mathcal{U}_{18}(G)$	–	$(1, 0, -1, 1, 1, 0)$
21.	$\mathcal{U}_{19}(G)$	$G \neq \overline{K}_n$	$(2, 0, -2, 2, -1, 0)$
22.	$\mathcal{U}_{20}(G)$	–	$(2, 0, -2, 2, -1, -1)$
23.	$\mathcal{U}_{21}(G)$	$G \neq \overline{K}_n$	$(2, 0, -2, 1, 1, 1)$
24.	$\mathcal{U}_{22}(G)$	–	$(2, 0, -2, 1, 1, 0)$

Table 4 The necessary values for proving Theorem 6

S. no.	Unary operation on the graph G	Additional conditions	Values
1.	$\mathcal{U}_3(G)$	$n=1$	$(2,-1,-2,3,0,0)$
2.	$\mathcal{U}_4(G)$	$\overline{K}_n$	$(2,-1,-2,3,0,0)$
3.	$\mathcal{U}_6(G)$	K_n	$(2,-1,-2,3,0,0)$
4.	$\mathcal{U}_{10}(G)$	$n=1$	$(1,1,-1,3,0,0)$
5.	$\mathcal{U}_{11}(G)$	$\overline{K}_n$	$(1,1,-1,3,0,0)$
6.	$\mathcal{U}_{13}(G)$	K_n	$(1,1,-1,3,0,0)$
7.	$\mathcal{U}_{15}(G)$	$\overline{K}_n$	$(1,0,-1,3,0,0)$
8.	$\mathcal{U}_{17}(G)$	K_n	$(1,0,-1,3,0,0)$
9.	$\mathcal{U}_{19}(G)$	$\overline{K}_n$	$(2,0,-2,3,0,0)$
10.	$\mathcal{U}_{21}(G)$	K_n	$(2,0,-2,3,0,0)$
11.	$\mathcal{U}_{23}(G)$	–	$(2,-1,-2,3,0,0)$
12.	$\mathcal{U}_{24}(G)$	–	$(1,1,-1,3,0,0)$

5 Families of M_3-join, M_4-join and M_5-join of graphs and their generalized distance spectra

For $i=1,2$, let G_i be a graph on n_i vertices. Let

$$M_3=\begin{bmatrix} J_{n_1\times n_2} & \mathbf{0}_{n_1\times n_2}\\ \mathbf{0}_{n_1\times n_2} & \mathbf{0}_{n_1\times n_2}\end{bmatrix},\ M_4=\begin{bmatrix} J_{n_1\times n_2} & J_{n_1\times n_2}\\ \mathbf{0}_{n_1\times n_2} & \mathbf{0}_{n_1\times n_2}\end{bmatrix} \text{ and } M_5=\begin{bmatrix} J_{n_1\times n_2} & J_{n_1\times n_2}\\ \mathbf{0}_{n_1\times n_2} & J_{n_1\times n_2}\end{bmatrix}.$$

For each $i, j\in\{1,2,\ldots,24\}$, we consider the following three graphs:

(ii) M_3-join of $U_i(G_1)$ and $U_j(G_2)$;
(ii) M_4-join of $U_i(G_1)$ and $U_j(G_2)$;
(iii) M_5-join of $U_i(G_1)$ and $U_j(G_2)$,

where the graphs $U_i(G_1)$ and $U_j(G_2)$ are given in Table 1.

Now we proceed to obtain the distance spectra of the three families of graphs mentioned above under some conditions on the constituting graphs and to construct distance cospectral graphs by using these results.

Theorem 7 *Let G_1 be an r_1-regular graph on n_1 vertices with additional conditions given in the row against the graph $\mathcal{U}_p(G_1)$ in Table 5. Let G_2 be an r_2-regular graph on n_2 vertices with additional conditions given in the row against the graph $\mathcal{U}_q(G_2)$ in Table 5. For $i=1,2$, let the eigenvalues of $A(G_i)$ be $(r_i=)\lambda_{i1},\lambda_{i2},\ldots,\lambda_{in_i}$. Then for each $p,q\in\{1,2,\ldots,22\}$, the distance spectrum of $\mathcal{U}_p(G_1)\vee_{M_3}\mathcal{U}_q(G_2)$ is*

(i) $(a_2+a_2')\lambda_{1j}+a_3+a_3'$, $(a_2-a_2')\lambda_{1j}+a_3-a_3'$ *for* $j=2,3,\ldots,n_1$;
(ii) $(b_2+b_2')\lambda_{2k}+b_3+b_3'$, $(b_2-b_2')\lambda_{2k}+b_3-b_3'$ *for* $k=2,3,\ldots,n_2$;
(iii) the roots of the equation $x^4-2(\sigma_1+\chi_1)x^3+[\sigma_1^2-\sigma_2^2+\chi_1^2-\chi_2^2+4\sigma_1\chi_1-14n_1n_2]x^2-\{2[\sigma_1(\chi_1^2-\chi_2^2)+\chi_1(\sigma_1^2-\sigma_2^2)]-14n_1n_2(\sigma_1+\chi_1)+2n_1n_2[8\sigma_2+8\chi_2]\}x+(\sigma_1^2-\sigma_2^2)(\chi_1^2-\chi_2^2)-n_1n_2[14\sigma_1\chi_1-2(8\sigma_1\chi_2-7\sigma_2\chi_2+8\chi_1\sigma_2)]+n_1^2n_2^2=0$,

where the values $(a_1,a_2,a_3,a_1',a_2',a_3')$ and $(b_1,b_2,b_3,b_1',b_2',b_3')$ are given in the rows against the graphs $\mathcal{U}_p(G_1)$ and $\mathcal{U}_q(G_2)$, respectively in Table 5, $\sigma_1=a_1n_1+a_2r_1+a_3$, $\sigma_2=a_1'n_1+a_2'r_1+a_3'$, $\chi_1=b_1n_2+b_2r_2+b_3$ and $\chi_2=b_1'n_2+b_2'r_2+b_3'$.

Proof Fix $p,q\in\{1,2,\ldots,22\}$. Taking $A_1=A(G_1)$; $A_2=A(G_2)$; $(c_1,c_2,c_3,c_4)=(2,3,1,2)$; the values $(a_1,a_2,a_3,a_1',a_2',a_3')$ and $(b_1,b_2,b_3,b_1',b_2',b_3')$ as given in the rows against the graphs $\mathcal{U}_p(G_1)$ and $\mathcal{U}_q(G_2)$, respectively in Table 5, and substituting these in (1), we get the distance matrix of $\mathcal{U}_p(G_1)\vee_{M_3}\mathcal{U}_q(G_2)$. Therefore, by substituting these values in Theorem 2, we get the distance spectrum of $\mathcal{U}_p(G_1)\vee_{M_3}\mathcal{U}_q(G_2)$ as stated in this theorem. □

Notice that we have obtained the distance spectra of graphs constructed by 484 binary graph operations from Theorem 7.

In the following result we construct infinite pairs of distance cospectral graphs by using Theorem 7.

Corollary 6 *Let G_1 and G_2 (resp. H_1 and H_2) be graphs satisfying the conditions of Theorem 7. If G_1 and H_1 (resp. G_2 and H_2) are adjacency cospectral, then $\mathcal{U}_p(G_1)\vee_{M_3}\mathcal{U}_q(G_2)$ and $\mathcal{U}_p(H_1)\vee_{M_3}\mathcal{U}_q(H_2)$ are distance cospectral for the respective values of p and q.*

Theorem 8 *Let $p \in \{1, 2, \dots, 22\}$ and $q \in \{1, 2, \dots, 24\}$. Let G_1 be an r_1-regular graph on n_1 vertices with additional conditions given in the row against the graph $\mathcal{U}_p(G_1)$ in Table 5. Let G_2 be an r_2-regular graph on n_2 vertices. For $i = 1, 2$, let the eigenvalues of $A(G_i)$ be $(r_i =)\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{in_i}$. Then the distance spectrum of $\mathcal{U}_p(G_1) \vee_{M_4} \mathcal{U}_q(G_2)$ is*

(i) $(a_2 + a'_2)\lambda_{1j} + a_3 + a'_3$, $(a_2 - a'_2)\lambda_{1j} + a_3 - a'_3$ *for* $j = 2, 3, \dots, n_1$;
(ii) $(b_2 + b'_2)\lambda_{2k} + b_3 + b'_3$, $(b_2 - b'_2)\lambda_{2k} + b_3 - b'_3$ *for* $k = 2, 3, \dots, n_2$ *and*
(iii) the roots of the equation
$$x^4 - 2(\sigma_1 + \chi_1)x^3 + [\sigma_1^2 - \sigma_2^2 + \chi_1^2 - \chi_2^2 + 4\sigma_1\chi_1 - 10n_1n_2]x^2 - \{2[\sigma_1(\chi_1^2 - \chi_2^2) + \chi_1(\sigma_1^2 - \sigma_2^2)] - 10n_1n_2(\sigma_1 + \chi_1) + 2n_1n_2[4\sigma_2 + 5\chi_2]\}x + (\sigma_1^2 - \sigma_2^2)(\chi_1^2 - \chi_2^2) - n_1n_2[10\sigma_1\chi_1 - 2(5\sigma_1\chi_2 - 4\sigma_2\chi_2 + 4\chi_1\sigma_2)] = 0,$$

where the values $(a_1, a_2, a_3, a'_1, a'_2, a'_3)$ and $(b_1, b_2, b_3, b'_1, b'_2, b'_3)$ are given in the rows against the graphs $\mathcal{U}_p(G_1)$ and $\mathcal{U}_q(G_2)$, respectively in Tables 2 and 5, $\sigma_1 = a_1n_1 + a_2r_1 + a_3$, $\sigma_2 = a'_1n_1 + a'_2r_1 + a'_3$, $\chi_1 = b_1n_2 + b_2r_2 + b_3$ and $\chi_2 = b'_1n_2 + b'_2r_2 + b'_3$.

Proof Fix $p \in \{1, 2, \dots, 22\}$ and $q \in \{1, 2, \dots, 24\}$. Taking $A_1 = A(G_1)$; $A_2 = A(G_2)$; $(c_1, c_2, c_3, c_4) = (2, 2, 1, 1)$; the values $(a_1, a_2, a_3, a'_1, a'_2, a'_3)$ and $(b_1, b_2, b_3, b'_1, b'_2, b'_3)$ as given in the rows against the graphs $\mathcal{U}_p(G_1)$ and $\mathcal{U}_q(G_2)$, respectively in Tables 2 and 5 and substituting these in (1), we get the distance matrix of $\mathcal{U}_p(G_1) \vee_{M_4} \mathcal{U}_q(G_2)$. Therefore, by substituting these values in Theorem 2, we get the distance spectrum of $\mathcal{U}_p(G_1) \vee_{M_4} \mathcal{U}_q(G_2)$ as stated in this theorem. □

Notice that we have obtained the distance spectra of graphs constructed by 528 binary graph operations from Theorem 8.

In the following result we construct infinite pairs of distance cospectral graphs by using Theorem 8.

Corollary 7 *Let G_1 and G_2 (resp. H_1 and H_2) be graphs satisfying the conditions of Theorem 8. If G_1 and H_1 (resp. G_2 and H_2) are adjacency cospectral, then $\mathcal{U}_p(G_1) \vee_{M_4} \mathcal{U}_q(G_2)$ and $\mathcal{U}_p(H_1) \vee_{M_4} \mathcal{U}_q(H_2)$ are distance cospectral for the respective values of p and q.*

Theorem 9 *For $i = 1, 2$, let G_i be an r_i-regular graph on n_i vertices and let the eigenvalues of $A(G_i)$ be $(r_i =)\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{in_i}$. For each $p, q \in \{1, 2, \dots, 24\}$, the distance spectrum of $\mathcal{U}_p(G_1) \vee_{M_5} \mathcal{U}_q(G_2)$ is*

(i) $(a_2 + a'_2)\lambda_{1j} + a_3 + a'_3$, $(a_2 - a'_2)\lambda_{1j} + a_3 - a'_3$ *for* $j = 2, 3, \dots, n_1$;
(ii) $(b_2 + b'_2)\lambda_{2k} + b_3 + b'_3$, $(b_2 - b'_2)\lambda_{2k} + b_3 - b'_3$ *for* $k = 2, 3, \dots, n_2$ *and*
(iii) the roots of the equation
$$x^4 - 2(\sigma_1 + \chi_1)x^3 + [\sigma_1^2 - \sigma_2^2 + \chi_1^2 - \chi_2^2 + 4\sigma_1\chi_1 - 7n_1n_2]x^2 - \{2[\sigma_1(\chi_1^2 - \chi_2^2) + \chi_1(\sigma_1^2 - \sigma_2^2)] - 7n_1n_2(\sigma_1 + \chi_1) + 2n_1n_2[3\sigma_2 + 3\chi_2]\}x + (\sigma_1^2 - \sigma_2^2)(\chi_1^2 - \chi_2^2) - n_1n_2[7\sigma_1\chi_1 - 2(3\sigma_1\chi_2 - 3\sigma_2\chi_2 + 3\chi_1\sigma_2)] + n_1^2n_2^2 = 0,$$

where the values $(a_1, a_2, a_3, a'_1, a'_2, a'_3)$ and $(b_1, b_2, b_3, b'_1, b'_2, b'_3)$ are given in the rows against the graphs $\mathcal{U}_p(G_1)$ and $\mathcal{U}_q(G_2)$, respectively in Table 2; $\sigma_1 = a_1n_1 + a_2r_1 + a_3$, $\sigma_2 = a'_1n_1 + a'_2r_1 + a'_3$, $\chi_1 = b_1n_2 + b_2r_2 + b_3$ and $\chi_2 = b'_1n_2 + b'_2r_2 + b'_3$.

Proof Fix $p, q \in \{1, 2, \dots, 24\}$. Taking $A_1 = A(G_1)$; $A_2 = A(G_2)$; $(c_1, c_2, c_3, c_4) = (2, 1, 1, 1)$; the values $(a_1, a_2, a_3, a'_1, a'_2, a'_3)$ and $(b_1, b_2, b_3, b'_1, b'_2, b'_3)$ as given in the rows against the graphs $\mathcal{U}_p(G_1)$ and $\mathcal{U}_q(G_2)$, respectively in Table 2 and substituting these in (1), we get the distance matrix of $\mathcal{U}_p(G_1) \vee_{M_5} \mathcal{U}_q(G_2)$. Therefore, by substituting these values in Theorem 2, we get the distance spectrum of $\mathcal{U}_p(G_1) \vee_{M_5} \mathcal{U}_q(G_2)$ as stated in this theorem. □

Notice that we have obtained the distance spectra of graphs constructed by 576 binary graph operations from Theorem 9.

In the following result we construct infinite pairs of distance cospectral graphs by using Theorem 9.

Corollary 8 *For $i = 1, 2$, let G_i and H_i be r_i-regular adjacency cospectral graphs on n_i vertices. Then for each $p, q \in \{1, 2, \dots, 24\}$, $\mathcal{U}_p(G_1) \vee_{M_5} \mathcal{U}_q(G_2)$ and $\mathcal{U}_p(H_1) \vee_{M_5} \mathcal{U}_q(H_2)$ are distance cospectral.*

In Table 5, G denotes a graph on n vertices.

Next we proceed to obtain the generalized distance spectra of the three families of graphs mentioned in this section under some conditions on the constituting graphs and to construct generalized distance cospectral graphs by using these results.

Table 5 The necessary values for proving Theorems 7, 8, 10 and 11

S. no.	Unary operation on the graph G	Additional assumptions	Values
1.	$\mathcal{U}_1(G)$	G is connected; $diam(G) \leq 2$	$(2, -1, -2, 3, -1, -2)$
2.	$\mathcal{U}_2(G)$	-	$(2, -1, -2, 1, 0, 0)$
3.	$\mathcal{U}_3(G)$	$n \geq 3$; $G = \overline{K}_n$	$(2, 0, -2, 1, 0, 2)$
4.	$\mathcal{U}_3(G)$	$n \geq 2$; $G \neq \overline{K}_n$	$(2, -1, -2, 1, 0, 1)$
5.	$\mathcal{U}_4(G)$	$n \geq 2$; G is connected; $diam(G) \leq 2$	$(2, -1, -2, 2, -1, 0)$
6.	$\mathcal{U}_5(G)$	G is connected; $diam(G) \leq 2$	$(2, -1, -2, 2, -1, -1)$
7.	$\mathcal{U}_6(G)$	$G \neq \overline{K}_n$; for any two $u \sim v$ in G, $N_G(u)\backslash N_G(v) \neq \{v\}$; for any two $u \nsim v$ in G, $N_{\overline{G}}(u) \cap N_{\overline{G}}(v) \neq \emptyset$	$(2, -1, -2, 1, 1, 2)$
8.	$\mathcal{U}_7(G)$	$-$	$(2, -1, -2, 1, 1, 0)$
9.	$\mathcal{U}_8(G)$	for any two $u \sim v$ in G, $N_{\overline{G}}(u) \cap N_{\overline{G}}(v) \neq \emptyset$	$(1, 1, -1, 2, 1, -1)$
10.	$\mathcal{U}_9(G)$	$-$	$(1, 1, -1, 1, 0, 0)$
11.	$\mathcal{U}_{10}(G)$	$n \geq 3$; $G = K_n$	$(2, 0, -2, 1, 0, 2)$
12.	$\mathcal{U}_{10}(G)$	$n \geq 2$; $G \neq K_n$	$(1, 1, -1, 1, 0, 1)$
13.	$\mathcal{U}_{11}(G)$	$G \neq K_n$; for any two $u \sim v$ in G, $N_G(u) \cap N_G(v) \neq \emptyset$; for any two $u \nsim v$ in G, $N_{\overline{G}}(u)\backslash N_{\overline{G}}(v) \neq \{v\}$	$(1, 1, -1, 2, -1, 1)$
14.	$\mathcal{U}_{12}(G)$	$-$	$(1, 1, -1, 2, -1, -1)$
15.	$\mathcal{U}_{13}(G)$	$n \geq 2$; for any two $u \sim v$ in G, $N_{\overline{G}}(u) \cap N_{\overline{G}}(v) \neq \emptyset$	$(1, 1, -1, 1, 1, 1)$
16.	$\mathcal{U}_{14}(G)$	for any two $u \sim v$ in G, $N_{\overline{G}}(u) \cap N_{\overline{G}}(v) \neq \emptyset$	$(1, 1, -1, 1, 1, 0)$
17.	$\mathcal{U}_{15}(G)$	$G \neq \overline{K}_n$	$(1, 0, -1, 2, -1, 0)$
18.	$\mathcal{U}_{16}(G)$	$-$	$(1, 0, -1, 2, -1, -1)$
19.	$\mathcal{U}_{17}(G)$	$G \neq K_n$	$(1, 0, -1, 1, 1, 1)$
20.	$\mathcal{U}_{18}(G)$	$-$	$(1, 0, -1, 1, 1, 0)$
21.	$\mathcal{U}_{19}(G)$	for any two $u, v \in V(G)$, $N_G(u) \cap N_G(v) \neq \emptyset$	$(2, 0, -2, 3, -2, 0)$
22.	$\mathcal{U}_{20}(G)$	G is connected; $diam(G) \leq 2$	$(2, 0, -2, 3, -2, -2)$
23.	$\mathcal{U}_{21}(G)$	for any two $u, v \in V(G)$, $N_{\overline{G}}(u) \cap N_{\overline{G}}(v) \neq \emptyset$	$(2, 0, -2, 1, 2, 2)$
24.	$\mathcal{U}_{22}(G)$	for any two $u \sim v$ in G, $N_{\overline{G}}(u) \cap N_{\overline{G}}(v) \neq \emptyset$	$(2, 0, -2, 1, 2, 0)$

Theorem 10 *Let $p, q \in \{1, 2, \ldots, 22\}$. Let G_1 be an r_1-regular graph on n vertices with additional conditions given in the row against the graph $\mathcal{U}_p(G_1)$ in Table 5. Let G_2 be an r_2-regular graph on n vertices with additional conditions given in the row against the graph $\mathcal{U}_q(G_2)$ in Table 5. For $i = 1, 2$, let the eigenvalues of $A(G_i)$ be $(r_i =)\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in_i}$. If G_1 and G_2 commute, then the generalized distance spectrum of $\mathcal{U}_p(G_1) \vee_{M_3} \mathcal{U}_q(G_2)$ is*

(i) $\alpha[(a_1 + a'_1 + 4)n + (a_2 + a'_2)r_1 + a_3 + a'_3] + (1-\alpha)(a_2\lambda_{1i} + a_3) \pm \big\{[\alpha((a_1 + a'_1 + 4)n + (a_2 + a'_2)r_1 + a_3 + a'_3) + (1-\alpha)(a_2\lambda_{1i} + a_3)]^2 - [\alpha((a_1 + a'_1 + 5)n + (a_2 + a'_2)r_1 + a_3 + a'_3) + (1-\alpha)(a_2\lambda_{1i} + a_3)][\alpha((a_1 + a'_1 + 3)n + (a_2 + a'_2)r_1 + a_3 + a'_3) + (1-\alpha)(a_2\lambda_{1i} + a_3)] + (1-\alpha)^2(a'_2\lambda_{1i} + a'_3)^2\big\}^{\frac{1}{2}}$ *for $i = 2, 3, \ldots, n$;*

(ii) $\alpha[(b_1 + b'_1 + 4)n + (b_2 + b'_2)r_2 + b_3 + b'_3] + (1-\alpha)(b_2\lambda_{2i} + b_3) \pm \big\{[\alpha((b_1 + b'_1 + 4)n + (b_2 + b'_2)r_2 + b_3 + b'_3) + (1-\alpha)(b_2\lambda_{2i} + b_3)]^2 - [\alpha((b_1 + b'_1 + 3)n + (b_2 + b'_2)r_2 + b_3 + b'_3) + (1-\alpha)(b_2\lambda_{2i} + b_3)][\alpha((b_1 + b'_1 + 5)n + (b_2 + b'_2)r_2 + b_3 + b'_3) + (1-\alpha)(b_2\lambda_{2i} + b_3)] + (1-\alpha)^2(b'_2\lambda_{2i} + b'_3)^2\big\}^{\frac{1}{2}}$ *for $i = 2, 3, \ldots, n$ and*

(iii) the roots of the equation
$$x^4 - (\sigma_1 + \sigma_2 + \chi_1 + \chi_2)x^3 + [(\sigma_1 + \sigma_2)(\chi_1 + \chi_2) + \sigma_1\sigma_2 + \chi_1\chi_2 - 18n^2(1-\alpha)^2 - \tau_1^2 - \tau_2^2]x^2 + \{(\sigma_1 + \sigma_2)(\tau_2^2 - \chi_1\chi_2) + (\chi_1 + \chi_2)(\tau_1^2 - \sigma_1\sigma_2) + n^2(1-\alpha)^2[5\sigma_1 + 13\sigma_2 + 13\chi_1 + 5\chi_2 - 16\tau_1 - 16\tau_2]\}x + \tau_1^2\tau_2^2 + \sigma_1\sigma_2\chi_1\chi_2 - \sigma_1\sigma_2\tau_2^2 - \chi_1\chi_2\tau_1^2 - 2n^2(1-\alpha)^2[7\tau_1\tau_2 - \tau_1(2\chi_2 + 6\chi_1) - \tau_2(6\sigma_2 + 2\sigma_1)] - n^2(1-\alpha)^2[\sigma_2(4\chi_2 + 9\chi_1) + \sigma_1(\chi_2 + 4\chi_1)] + n^4(1-\alpha)^4 = 0,$$

where the values $(a_1, a_2, a_3, a'_1, a'_2, a'_3)$ and $(b_1, b_2, b_3, b'_1, b'_2, b'_3)$ are given in the rows against the graphs $\mathcal{U}_p(G_1)$ and $\mathcal{U}_q(G_2)$, respectively in Table 5, $\tau_1 = (1-\alpha)(a'_1 n + a'_2 r_1 + a'_3)$, $\tau_2 = (1-\alpha)(b'_1 n + b'_2 r_2 + b'_3)$,

$\sigma_1 = \alpha[(a_1+a_1'+5)n+(a_2+a_2')r_1+a_3+a_3']+(1-\alpha)(a_1n+a_2r_1+a_3)$, $\sigma_2 = \alpha[(a_1+a_1'+3)n+(a_2+a_2')r_1+a_3+a_3']+(1-\alpha)(a_1n+a_2r_1+a_3)$, $\chi_1 = \alpha[(b_1+b_1'+3)n+(b_2+b_2')r_2+b_3+b_3']+(1-\alpha)(b_1n+b_2r_2+b_3)$ *and* $\chi_2 = \alpha[(b_1+b_1'+5)n+(b_2+b_2')r_2+b_3+b_3']+(1-\alpha)(b_1n+b_2r_2+b_3)$.

Proof Since G_1 and G_2 are commuting regular graphs, $A(G_1)$ and $A(G_2)$ commute with each other. Fix $p, q \in \{1, 2, \ldots, 22\}$. Taking $A_1 = A(G_1)$; $A_2 = A(G_2)$; $(c_1, c_2, c_3, c_4) = (2, 3, 1, 2)$; the values $(a_1, a_2, a_3, a_1', a_2', a_3')$ and $(b_1, b_2, b_3, b_1', b_2', b_3')$ as given in the rows against the graphs $\mathcal{U}_p(G_1)$ and $\mathcal{U}_q(G_2)$, respectively in Table 5, and substituting these in (5), we get the generalized distance matrix of $\mathcal{U}_p(G_1) \vee_{M_3} \mathcal{U}_q(G_2)$. Therefore, by substituting these values in Theorem 4, we get the generalized distance spectrum of $\mathcal{U}_p(G_1) \vee_{M_3} \mathcal{U}_q(G_2)$ as stated in this theorem. □

Notice that we have obtained the generalized distance spectra of graphs constructed by 484 binary graph operations from Theorem 10.

In the following result we construct infinite pairs of generalized distance cospectral graphs by using Theorem 10.

Corollary 9 *Let G_1 and G_2 (resp. H_1 and H_2) be graphs satisfying the conditions of Theorem 10. Suppose G_1 and H_1 (resp. G_2 and H_2) are adjacency cospectral, then $\mathcal{U}_p(G_1) \vee_{M_3} \mathcal{U}_q(G_2)$ and $\mathcal{U}_p(H_1) \vee_{M_3} \mathcal{U}_q(H_2)$ are generalized distance cospectral for the respective values of p and q. In particular, they are distance (distance Laplacian, distance signless Laplacian) cospectral.*

Theorem 11 *Let $p \in \{1, 2, \ldots, 22\}$ and $q \in \{1, 2, \ldots, 24\}$. Let G_1 be an r_1-regular graph on n vertices with additional conditions given in the row against the graph $\mathcal{U}_p(G_1)$ in Table 5. Let G_2 be an r_2-regular graph on n vertices. For $i = 1, 2$, let the eigenvalues of $A(G_i)$ be $(r_i =)\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in_i}$. If G_1 and G_2 commute, then the generalized distance spectrum of $\mathcal{U}_p(G_1) \vee_{M_4} \mathcal{U}_q(G_2)$ is*

(i) $\alpha[(a_1+a_1'+3)n+(a_2+a_2')r_1+a_3+a_3']+(1-\alpha)(a_2\lambda_{1i}+a_3) \pm \big\{[\alpha((a_1+a_1'+3)n+(a_2+a_2')r_1+a_3+a_3')+(1-\alpha)(a_2\lambda_{1i}+a_3)]^2 - [\alpha((a_1+a_1'+4)n+(a_2+a_2')r_1+a_3+a_3')+(1-\alpha)(a_2\lambda_{1i}+a_3)][\alpha((a_1+a_1'+2)n+(a_2+a_2')r_1+a_3+a_3')+(1-\alpha)(a_2\lambda_{1i}+a_3)]+(1-\alpha)^2(a_2'\lambda_{1i}+a_3')^2\big\}^{\frac{1}{2}}$ *for* $i = 2, 3, \ldots, n$;

(ii) $\alpha[(b_1+b_1'+3)n+(b_2+b_2')r_2+b_3+b_3']+(1-\alpha)(b_2\lambda_{2i}+b_3) \pm \big\{[\alpha((b_1+b_1'+3)n+(b_2+b_2')r_2+b_3+b_3')+(1-\alpha)(b_2\lambda_{2i}+b_3)]^2 - [\alpha((b_1+b_1'+3)n+(b_2+b_2')r_2+b_3+b_3')+(1-\alpha)(b_2\lambda_{2i}+b_3)][\alpha((b_1+b_1'+3)n+(b_2+b_2')r_2+b_3+b_3')+(1-\alpha)(b_2\lambda_{2i}+b_3)]+(1-\alpha)^2(b_2'\lambda_{2i}+b_3')^2\big\}^{\frac{1}{2}}$ *for* $i = 2, 3, \ldots, n$ *and*

(iii) the roots of the equation
$$x^4-(\sigma_1+\sigma_2+\chi_1+\chi_2)x^3+[(\sigma_1+\sigma_2)(\chi_1+\chi_2)+\sigma_1\sigma_2+\chi_1\chi_2-10n^2(1-\alpha)^2-\tau_1^2-\tau_2^2]x^2+\{(\sigma_1+\sigma_2)(\tau_2^2-\chi_1\chi_2)+(\chi_1+\chi_2)(\tau_1^2-\sigma_1\sigma_2)+n^2(1-\alpha)^2[2\sigma_1+8\sigma_2+5\chi_1+5\chi_2-8\tau_1-10\tau_2]\}x+\tau_1^2\tau_2^2+\sigma_1\sigma_2\chi_1\chi_2-\sigma_1\sigma_2\tau_2^2-\chi_1\chi_2\tau_1^2-2n^2(1-\alpha)^2[4\tau_1\tau_2-\tau_1(2\chi_2+2\chi_1)-\tau_2(4\sigma_2+\sigma_1)]-n^2(1-\alpha)^2[\sigma_2(4\chi_2+4\chi_1)+\sigma_1(\chi_2+\chi_1)]=0,$$

where the values $(a_1, a_2, a_3, a_1', a_2', a_3')$ and $(b_1, b_2, b_3, b_1', b_2', b_3')$ are given in the rows against the graphs $\mathcal{U}_p(G_1)$ and $\mathcal{U}_q(G_2)$, respectively in Tables 2 and 5; $\tau_1 = (1-\alpha)(a_1'n+a_2'r_1+a_3')$, $\tau_2 = (1-\alpha)(b_1'n+b_2'r_2+b_3')$, $\sigma_1 = \alpha[(a_1+a_1'+4)n+(a_2+a_2')r_1+a_3+a_3']+(1-\alpha)(a_1n+a_2r_1+a_3)$, $\sigma_2 = \alpha[(a_1+a_1'+2)n+(a_2+a_2')r_1+a_3+a_3']+(1-\alpha)(a_1n+a_2r_1+a_3)$, $\chi_1 = \alpha[(b_1+b_1'+3)n+(b_2+b_2')r_2+b_3+b_3']+(1-\alpha)(b_1n+b_2r_2+b_3)$ *and* $\chi_2 = \alpha[(b_1+b_1'+3)n+(b_2+b_2')r_2+b_3+b_3']+(1-\alpha)(b_1n+b_2r_2+b_3)$.

Proof Since G_1 and G_2 are commuting graphs, $A(G_1)$ and $A(G_2)$ commute with each other. Fix $p \in \{1, 2, \ldots, 22\}$ and $q \in \{1, 2, \ldots, 24\}$. Taking $A_1 = A(G_1)$; $A_2 = A(G_2)$; $(c_1, c_2, c_3, c_4) = (2, 2, 1, 1)$; the values $(a_1, a_2, a_3, a_1', a_2', a_3')$ and $(b_1, b_2, b_3, b_1', b_2', b_3')$ as given in the rows against the graphs $\mathcal{U}_p(G_1)$ and $\mathcal{U}_q(G_2)$, respectively in Table 2 and 5 and substituting these in (5), we get the generalized distance matrix of $\mathcal{U}_p(G_1) \vee_{M_4} \mathcal{U}_q(G_2)$. Therefore, by substituting these values in Theorem 4, we get the generalized distance spectrum of $\mathcal{U}_p(G_1) \vee_{M_4} \mathcal{U}_q(G_2)$ as stated in this theorem. □

Notice that we have obtained the generalized distance spectra of graphs constructed by 528 binary graph operations from Theorem 11.

In the following result we construct infinite pairs of generalized distance cospectral graphs by using Theorem 11.

Corollary 10 *Let G_1 and G_2 (resp. H_1 and H_2) be graphs satisfying the conditions of Theorem 11. If G_1 and H_1 (resp. G_2 and H_2) are adjacency cospectral, then $\mathcal{U}_p(G_1) \vee_{M_4} \mathcal{U}_q(G_2)$ and $\mathcal{U}_p(H_1) \vee_{M_4} \mathcal{U}_q(H_2)$ are generalized distance cospectral for the respective values of p and q. In particular, they are distance (distance Laplacian, distance signless Laplacian) cospectral.*

Theorem 12 *For $i = 1, 2$, let G_i be an r_i-regular graph on n vertices and let the eigenvalues of $A(G_i)$ be $(r_i =)\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in_i}$. If G_1 and G_2 commute, then for each $p, q \in \{1, 2, \ldots, 24\}$, the generalized distance spectrum of $\mathcal{U}_p(G_1) \vee_{M_5} \mathcal{U}_q(G_2)$ is*

(i) $\alpha[(a_1+a_1'+\frac{5}{2})n+(a_2+a_2')r_1+a_3+a_3']+(1-\alpha)(a_2\lambda_{1i}+a_3) \pm \Big\{[\alpha((a_1+a_1'+\frac{5}{2})n+(a_2+a_2')r_1+a_3+a_3')+(1-\alpha)(a_2\lambda_{1i}+a_3)]^2-[\alpha((a_1+a_1'+3)n+(a_2+a_2')r_1+a_3+a_3')+(1-\alpha)(a_2\lambda_{1i}+a_3)][\alpha((a_1+a_1'+2)n+(a_2+a_2')r_1+a_3+a_3')+(1-\alpha)(a_2\lambda_{1i}+a_3)]+(1-\alpha)^2(a_2'\lambda_{1i}+a_3')^2\Big\}^{\frac{1}{2}}$ *for $i = 2, 3, \ldots, n$;*

(ii) $\alpha[(b_1+b_1'+\frac{5}{2})n+(b_2+b_2')r_2+b_3+b_3']+(1-\alpha)(b_2\lambda_{2i}+b_3) \pm \Big\{[\alpha((b_1+b_1'+\frac{5}{2})n+(b_2+b_2')r_2+b_3+b_3')+(1-\alpha)(b_2\lambda_{2i}+b_3)]^2-[\alpha((b_1+b_1'+3)n+(b_2+b_2')r_2+b_3+b_3')+(1-\alpha)(b_2\lambda_{2i}+b_3)][\alpha((b_1+b_1'+2)n+(b_2+b_2')r_2+b_3+b_3')+(1-\alpha)(b_2\lambda_{2i}+b_3)]+(1-\alpha)^2(b_2'\lambda_{2i}+b_3')^2\Big\}^{\frac{1}{2}}$ *for $i = 2, 3, \ldots, n$;*

(iii) *the roots of the equation*
$x^4-(\sigma_1+\sigma_2+\chi_1+\chi_2)x^3+[(\sigma_1+\sigma_2)(\chi_1+\chi_2)+\sigma_1\sigma_2+\chi_1\chi_2-7n^2(1-\alpha)^2-\tau_1^2-\tau_2^2]x^2+\{(\sigma_1+\sigma_2)(\tau_2^2-\chi_1\chi_2)+(\chi_1+\chi_2)(\tau_1^2-\sigma_1\sigma_2)+n^2(1-\alpha)^2[2\sigma_1+5\sigma_2+2\chi_1+5\chi_2-6\tau_1-6\tau_2]\}x+\tau_1^2\tau_2^2+\sigma_1\sigma_2\chi_1\chi_2-\sigma_1\sigma_2\tau_2^2-\chi_1\chi_2\tau_1^2-2n^2(1-\alpha)^2[3\tau_1\tau_2-\tau_1(2\chi_2+\chi_1)-\tau_2(2\sigma_2+\sigma_1)]-n^2(1-\alpha)^2[\sigma_2(4\chi_2+\chi_1)+\sigma_1(\chi_2+\chi_1)]+n^4(1-\alpha)^4=0,$

where the values $(a_1, a_2, a_3, a_1', a_2', a_3')$ and $(b_1, b_2, b_3, b_1', b_2', b_3')$ are given in the rows against the graphs $\mathcal{U}_p(G_1)$ and $\mathcal{U}_q(G_2)$, respectively in Table 2, $\tau_1 = (1-\alpha)(a_1'n + a_2'r_1 + a_3')$, $\tau_2 = (1-\alpha)(b_1'n + b_2'r_2 + b_3')$, $\sigma_1 = \alpha[(a_1+a_1'+3)n+(a_2+a_2')r_1+a_3+a_3']+(1-\alpha)(a_1n+a_2r_1+a_3)$, $\sigma_2 = \alpha[(a_1+a_1'+2)n+(a_2+a_2')r_1+a_3+a_3']+(1-\alpha)(a_1n+a_2r_1+a_3)$, $\chi_1 = \alpha[(b_1+b_1'+3)n+(b_2+b_2')r_2+b_3+b_3']+(1-\alpha)(b_1n+b_2r_2+b_3)$ and $\chi_2 = \alpha[(b_1+b_1'+2)n+(b_2+b_2')r_2+b_3+b_3']+(1-\alpha)(b_1n+b_2r_2+b_3)$.

Proof Since G_1 and G_2 are commuting regular graphs, $A(G_1)$ and $A(G_2)$ commute with each other. Fix $p, q \in \{1, 2, \ldots, 24\}$. Taking $A_1 = A(G_1)$; $A_2 = A(G_2)$; $(c_1, c_2, c_3, c_4) = (2, 1, 1, 1)$; the values $(a_1, a_2, a_3, a_1', a_2', a_3')$ and $(b_1, b_2, b_3, b_1', b_2', b_3')$ as given in the rows against the graphs $\mathcal{U}_p(G_1)$ and $\mathcal{U}_q(G_2)$, respectively in Table 2 and substituting these in (5), we get the generalized distance matrix of $\mathcal{U}_p(G_1) \vee_{M_5} \mathcal{U}_q(G_2)$. Therefore, by substituting these values in Theorem 4, we get the generalized distance spectrum of $\mathcal{U}_p(G_1) \vee_{M_5} \mathcal{U}_q(G_2)$ as stated in this theorem. □

Notice that we have obtained the generalized distance spectra of graphs constructed by 576 binary graph operations from Theorem 12.

In the following result we construct infinite pairs of generalized distance cospectral graphs by using Theorem 12.

Corollary 11 *For $i = 1, 2$, let G_i and H_i be r_i-regular adjacency cospectral graphs on n vertices. If (G_1, G_2) and (H_1, H_2) are pairs of commuting graphs, then for each $p, q \in \{1, 2, \ldots, 24\}$, $\mathcal{U}_p(G_1) \vee_{M_5} \mathcal{U}_q(G_2)$ and $\mathcal{U}_p(H_1) \vee_{M_5} \mathcal{U}_q(H_2)$ are generalized distance cospectral. In particular, they are distance (distance Laplacian, distance signless Laplacian) cospectral.*

Acknowledgements The first author would like to acknowledge the support of University Grants Commission, India for SRF (NTA Ref. No.: 211610155238).

Author contributions Both authors contributed equally.

Data availability Not applicable.

Code availability Not applicable.

Funding The first was supported by the University Grants Commission, India, NTA Ref. No.: 211610155238.

Declarations

Conflict of interest Both authors declare that they have no conflicts of interest

Ethical approval Not applicable.

Consent to participate Not applicable.

Consent for publication Authors give their consent for publication of this article in Indian Journal of Pure and Applied Mathematics.

References

1. Adiga, C., Rakshith, B.R.: Distance spectra of some graph operations and some new distance equienergetic graphs of diameter 3. TWMS J. of Apl. & Eng. Math. **9**(3), 581–588 (2019). http://belgelik.isikun.edu.tr/xmlui/handle/iubelgelik/2746
2. Akbari, S., Moazami, F., Mohammadian, A.: Commutativity of the adjacency matrices of graphs. Discrete Math. **309**, 595–600 (2009). https://doi.org/10.1016/j.disc.2008.09.006
3. Alhevaz, A., Baghipur, M., Hashemi, E., Ramane, H.S.: On the distance signless Laplacian spectrum of graphs. Bull. Malays. Math. Sci. Soc. **42**(5), 2603–2621 (2019). https://doi.org/10.1007/s40840-018-0619-8
4. Alhevaz, A., Baghipur, M., Ganie, H. A., Shang, Y.: The generalized distance spectrum of the join of graphs. Symmetry. **12**(1), 169 (2020). https://doi.org/10.3390/sym12010169
5. Aouchiche, M., Hansen, P.: Distance spectra of graphs: A survey. Linear Algebra Appl. **458**, 201–386 (2014). https://doi.org/10.1016/j.laa.2014.06.010
6. Aouchiche, M., Hansen, P.: Two Laplacians for the distance matrix of a graph. Linear Algebra Appl. **439**(1), 21–33 (2013). https://doi.org/10.1016/j.laa.2013.02.030
7. Bapat, R.B.: Graphs and Matrices, Springer, London, 2010.
8. Barik, S., Kalita, D., Pati, S., Sahoo, G.: Spectra of graphs resulting from various graph operations and products: a survey. Spec. Matrices. **6**(1), 323–342 (2018). https://doi.org/10.1515/spma-2018-0027
9. Barik, S., Sahoo, G.: On the distance spectra of coronas. Linear Multilinear Algebra. **65**(8), 1617–1628 (2017). https://doi.org/10.1080/03081087.2016.1249448
10. Cardoso, D.M., Díaz, R. C., Rojo, O.: Distance matrices on the H-join of graphs: a general result and applications. Linear Algebra Appl. **559**, 34–53 (2018). https://doi.org/10.1016/j.laa.2018.08.024
11. Cui, S.Y., He, J.X. Tian, G.X.: The generalized distance matrix. Linear Algebra Appl. **563**, 1–23 (2019). https://doi.org/10.1016/j.laa.2018.10.014
12. Ganie, H.A.: On distance Laplacian spectrum (energy) of graphs. Discrete Math. Algorithms Appl. **12**(05), Article No. 2050061 (2020). https://doi.org/10.1142/S1793830920500615
13. Gayathri, M., Rajkumar, R.: Spectra of partitioned matrices and the $\mathcal{M}$-join of graphs. Ric. di Mat. **73**, 213–260 (2024). https://doi.org/10.1007/s11587-021-00589-x
14. Haynsworth, E.V.: A reduction formula for partitioned matrices. J. Res. Nat. Bureau Stand. **64**(3), 171–174 (1960).
15. Hedetniemi, S.: On classes of graphs defined by special cutsets of lines. Many Facets of Graph Theory: Proc. Conf. Western MiGammagan Univ., Kalamazoo/Mi. 1968, Lect. Notes Math. 110, 171–189, (2006). https://doi.org/10.1007/BFb0060115.
16. Indulal, G., Balakrishnan, R.: Distance spectrum of Indu–Bala product of graphs. AKCE Int. J. Graphs Comb. **13**(3), 230–234 (2016). https://doi.org/10.1016/j.akcej.2016.06.012
17. Indulal, G., Scaria, D.C., Liu, X.: The distance spectrum of the subdivision vertex join and subdivision edge join of two regular graphs. Discrete Math. Lett. **1**, 36–41 (2019). https://doi.org/10.1142/S1793830919500538
18. Kalaivani, G.: Distance spectra of graphs constructed by several unary and binary graph operations, M.Phil Dissertation, The Gandhigram Rural Institute (Deemed to be University), 2020.
19. Kalaivani, G., Rajkumar, R.: The generalized distance spectra of the $\mathcal{M}$-join of graphs. Le Matematiche. (2024) (Accepted for publication)
20. Paul, S.: On distance and distance Laplacian spectra of corona of two graphs. Discrete Math. Algorithms Appl. **8**(01), Article No. 1650007 (2016). https://doi.org/10.1142/S1793830916500075
21. Paul, S.: Distance Laplacian spectra of joined union of graph. Asian-Eur. J. Math. **15**(2), Article No. 2250039 (2021). https://doi.org/10.1142/S1793557122500395
22. Pavithra, R., Rajkumar, R.: Spectra of M-edge rooted product of graphs. Indian J. Pure Appl. Math. **52**, 1235–1255 (2021). https://doi.org/10.1007/s13226-021-00027-6
23. Rajkumar, R., Gayathri, M.: Spectra of generalized corona of graphs constrained by vertex subsets. Le Matematiche. **761**(6), 175–205 (2021). https://doi.org/10.4418/2021.76.1.12