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ORIGINAL RESEARCH

# The dualities between variable anisotropic Hardy and Campanato spaces with variable exponents

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**Abstract** Let $\Theta$ be a continuous multi-level ellipsoid cover of $\mathbb{R}^n$, $p(\cdot):\mathbb{R}^n \to (0,\ \infty]$ be a variable exponent function satisfying the globally log-Hölder continuous condition, and let $\mathcal{H}^{p(\cdot)}(\Theta)$ be the variable anisotropic Hardy space with variable exponent introduced by Yang et al. (Anal Geom Metr Spaces 9:65–89, 2021). In this article, the authors obtain the dual spaces of $\mathcal{H}^{p(\cdot)}(\Theta)$. In addition, the authors also establish a boundedness criterion of a class of linear operators from $\mathcal{H}^{p(\cdot)}(\Theta)$ to $L^{p(\cdot)}(\mathbb{R}^n)$.

**Keywords** Hardy space · Continuous ellipsoid cover · Atomic decomposition · Campanato space · Linear operator

**Mathematics Subject Classification** 42B35 · 42B30 · 42B25 · 42B20

## 1 Introduction

The real-variable theory of Hardy space on the Euclidean space $\mathbb{R}^n$, initiated by Stein et al. [21], plays an essential role in the harmonic analysis and **PDE**s; see, for examples, [14,20–22]. As is known to all, classical Hardy space $\mathcal{H}^p(\mathbb{R}^n)$ is a good substitution of $L^p(\mathbb{R}^n)$ when $p \in (0, 1]$. For example, the Riesz transforms are bounded on $\mathcal{H}^p(\mathbb{R}^n)$, but not on $L^p(\mathbb{R}^n)$ when $p \in (0, 1]$. Moreover, extending classical Hardy spaces arising in harmonic analysis of Euclidean spaces $\mathbb{R}^n$ to other domains and non-isotropic settings is an essential topic. In 2003, Bownik [3] introduced anisotropic Hardy spaces $\mathcal{H}^p_A(\mathbb{R}^n)$, where the Euclidean balls are replaced by images of the unit ball by powers of a fixed expansion matrix $A$. In 2011, Dekel et al. [12] generalized Bownik's spaces by constructing highly geometric Hardy spaces with pointwise variable anisotropy $\mathcal{H}^p(\Theta)$, $0 < p \le 1$, where the Euclidean balls are replaced by continuous ellipsoid cover $\Theta$. Different from Bownik's spaces, the anisotropy in $\mathcal{H}^p(\Theta)$ can change rapidly from point to point in $\mathbb{R}^n$ and from level to level in depth. Moreover, the ellipsoid cover $\Theta$ is a very general setting which includes the classical isotropic setting, non-isotropic setting of Calderón and Torchinsky [6] and the anisotropic setting of Bownik [3] as special cases, see more details in [3,10].

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On the other hand, the variable function spaces have found their applications in fluid dynamics [1,2], image processing [7,15], **PDE**s, variational calculus [5,13] and harmonic analysis [8]. In 2012, Nakai and Sawano [18] introduced the variable Hardy space $\mathcal{H}^{p(\cdot)}(\mathbb{R}^n)$, via the radial grand maximal function, and then obtained some real-variable characterizations of the space, such as the characterizations in terms of the atomic and the molecular decompositions. Then, Sawano [19], Yang et al. [28] and Zhuo et al. [29] further contributed to the theory. In 2018, Liu et al. [17] introduced the variable anisotropic Hardy space $\mathcal{H}_A^{p(\cdot)}(\mathbb{R}^n)$ defined over $\mathbb{R}^n$, where $A$ is a general expansive matrix on $\mathbb{R}^n$, it extends the theory of Hardy spaces on $\mathbb{R}^n$ of Bownik [3]. Later, Wang et al. [26] established molecular decomposition of $\mathcal{H}_A^{p(\cdot)}(\mathbb{R}^n)$. Very recently, Yang et al. [27] further generalized Dekel's spaces $\mathcal{H}^p(\Theta)$ and Liu's space $\mathcal{H}_A^{p(\cdot)}(\mathbb{R}^n)$ by constructing the variable anisotropic Hardy spaces with variable exponents $\mathcal{H}^{p(\cdot)}(\Theta)$ associated with an continuous ellipsoid cover $\Theta$, and then established its atomic characterization and the boundedness of the singular integral operators from $\mathcal{H}^{p(\cdot)}(\Theta)$ to $L^{p(\cdot)}(\mathbb{R}^n)$ and from $\mathcal{H}^{p(\cdot)}(\Theta)$ to itself.

To complete the theory of the variable anisotropic Hardy spaces with variable exponents $\mathcal{H}^{p(\cdot)}(\Theta)$, motivated by Huang et al.'s work [16], in this article, we obtain the dual spaces of $\mathcal{H}^{p(\cdot)}(\Theta)$. In addition, we also establish a boundedness criterion of a class of linear operators from $\mathcal{H}^{p(\cdot)}(\Theta)$ to $L^{p(\cdot)}(\mathbb{R}^n)$.

For the convenience of the readers, we summarize briefly our main results as follows. For more concrete statement, please see Theorems 3.12 and 4.2 in Sects. 3 and 4, respectively.

**Theorem 1.1** *Let $\Theta$ be a continuous ellipsoid cover, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in (\max\{p_+, 1\}, \infty)$ with $p_+$ as in* (2.3) *and $s \in [N_{p(\cdot)}, \infty) \cap \mathbb{N}_0$. Then the dual space of $\mathcal{H}^{p(\cdot)}(\Theta)$, denoted by $(\mathcal{H}^{p(\cdot)}(\Theta))^*$ is $\mathcal{L}_{p(\cdot),\, q',\, s,\, \underline{p}}(\Theta)$. Here, the definitions of $\Theta$, $C^{\log}(\mathbb{R}^n)$, $N_{p(\cdot)}$ and $\mathcal{H}^{p(\cdot)}(\Theta)$ are as in Sect.* 2, *and the definitions of $\mathcal{L}_{p(\cdot),\, q',\, s,\, \underline{p}}(\Theta)$ and $\mathcal{H}_{q,\, s}^{p(\cdot)}(\Theta)$ are as in Sect.* 3.

**Theorem 1.2** *Let $\Theta$ be a continuous ellipsoid cover, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in [1, \infty] \cap (p_+, \infty]$ with $p_+$ as in* (2.3), *$s \in [N_{p(\cdot)}, \infty) \cap \mathbb{N}_0$ and $\beta > 1/p_-$ with $p_-$ as in* (2.3). *Suppose $T$ is a linear operator on $L^q(\mathbb{R}^n)$ and satisfies that, for any $(p(\cdot),\, \infty,\, s)$-atom $b$, $\operatorname{supp} b \subset \theta_{z,\, t} \in \Theta$ with $z \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $x \in (\theta_{z,\, t-J})^{\complement}$,*

$$|T(b)(x)| \le C \|\chi_{\theta_{z,\, t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} \left[\mathcal{M}_\Theta(\chi_{\theta_{z,\, t}})(x)\right]^\beta,$$

*where $\mathcal{M}_\Theta$ is as in* (4.2) *and $J$ is as in Lemma* 2.2. *Then $T$ is bounded from $\mathcal{H}^{p(\cdot)}(\Theta)$ to $L^{p(\cdot)}(\mathbb{R}^n)$. Here, the definitions of $\theta_{z,\, t}$ and $\theta_{z,\, t-J}$ are as in Sect.* 2, *and the definition of $(p(\cdot),\, \infty,\, s)$-atom is as in Sect.* 3.

To be precise, this article is organized as follows.

In Sect. 2, we recall some definitions and properties of continuous ellipsoid cover $\Theta$, the variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ and the variable anisotropic Hardy spaces with variable exponents $\mathcal{H}^{p(\cdot)}(\Theta)$. In Sect. 3, we first introduce variable anisotropic Campanato-type spaces $\mathcal{L}_{p(\cdot),\, q,\, s,\, \eta}(\Theta)$ (see Definition 3.1) and give some basic properties. Then, we prove that the dual space of $\mathcal{H}^{p(\cdot)}(\Theta)$ is $\mathcal{L}_{p(\cdot),q',s,\underline{p}}(\Theta)$ (see Theorem 3.12). In Sect. 4, via borrowing the atomic characterization of $\mathcal{H}^{p(\cdot)}(\Theta)$, we establish a boundedness criterion of a class of linear operators from $\mathcal{H}^{p(\cdot)}(\Theta)$ to $L^{p(\cdot)}(\mathbb{R}^n)$ (see Theorem 4.2).

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1,\, 2,\, \ldots\}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. For any $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$, and $\partial^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$. Throughout the whole paper, we denote by $C$ a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $D \lesssim F$ means that $D \le CF$. If $D \lesssim F$ and $F \lesssim D$, we then write $D \sim F$. For any sets $E \subset \mathbb{R}^n$, we use $E^{\complement}$ to denote the set $\mathbb{R}^n \setminus E$. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of Schwartz functions, $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions.

## 2 Preliminaries

In this section, we recall some notations. Firstly, let us begin with the definition of continuous ellipsoid covers, which is from [10, Definition 2.4]. An *ellipsoid* $\xi$ in $\mathbb{R}^n$ is an image of the Euclidean unit ball $\mathbb{B}^n := \{x \in \mathbb{R}^n : |x| < 1\}$ under an affine transform, i.e.

$$\xi := M_\xi(\mathbb{B}^n) + c_\xi,$$

where $M_\xi$ is a nonsingular matrix and $c_\xi \in \mathbb{R}^n$ is the center.

**Definition 2.1** Say that

$$\Theta := \{\theta_{x,\,t} : x \in \mathbb{R}^n,\, t \in \mathbb{R}\}$$

is a *continuous ellipsoid cover* of $\mathbb{R}^n$, or shortly an *ellipsoid cover*, if there exist positive constants $p(\Theta) := \{a_1, \ldots, a_6\}$, such that:

(i) For every $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, there exists an ellipsoid $\theta_{x,\,t} := M_{x,\,t}(\mathbb{B}^n) + x$ satisfying

$$a_1 2^{-t} \le |\theta_{x,\,t}| \le a_2 2^{-t}. \tag{2.1}$$

(ii) Intersecting ellipsoids from $\Theta$ satisfy a "shape condition", i.e., for any $x,\ y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $s \ge 0$, if $\theta(x,\ t) \cap \theta(y,\ t+s) \ne \emptyset$, then

$$a_3 2^{-a_4 s} \le \frac{1}{\|(M_{y,\,t+s})^{-1} M_{x,\,t}\|} \le \|(M_{x,\,t})^{-1} M_{y,\,t+s}\| \le a_5 2^{-a_6 s}. \tag{2.2}$$

Here, $\|\cdot\|$ is the matrix norm given by $\|M\| := \max_{|x|=1} |Mx|$ for an $n \times n$ real matrix $M$.

**Lemma 2.2** [12, Lemma 2.3] *Let* $\Theta$ *be a continuous ellipsoid cover. Then there exists* $J := J(\mathbf{p}(\Theta)) \ge 1$ *such that, for any* $x \in \mathbb{R}^n$ *and* $t \in \mathbb{R}$,

$$\theta_{x,\,t} \subset \frac{1}{2}\theta_{x,\,t-J}.$$

More properties about ellipsoid covers, see [4,10,12]. Now we recall the notion of variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$. For any measurable function $p(\cdot) : \mathbb{R}^n \to (0,\ \infty]$, let

$$p_- := \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad p_+ := \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x) \quad \text{and} \quad \underline{p} := \min\{p_-, 1\}. \tag{2.3}$$

Define $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot)$ satisfying $0 < p_- \le p_+ < \infty$.

For any measurable function $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the *variable Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^n)$ denotes the set of measurable functions $f$ on $\mathbb{R}^n$ such that, for some $\lambda > 0$,

$$\varrho_{p(\cdot)}(f/\lambda) := \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg–Nakano norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\ \infty) : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

Next, we recall the definition of variable anisotropic Hardy space with variable exponent [27] via the grand radial maximal function.

Let $C^{\log}(\mathbb{R}^n)$ be the set of all functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the *globally log-Hölder continuous condition*, namely, there exist $C_{\log}(p), C_\infty \in (0, \infty)$ and $p_\infty \in \mathbb{R}$ such that, for any $x, y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \le \frac{C_{\log}(p)}{\log\left(e + \frac{1}{|x-y|}\right)}$$

and

$$|p(x) - p_\infty| \le \frac{C_\infty}{\log(e + |x|)}.$$

For any $N,\ \widetilde{N} \in \mathbb{N}_0$ with $N \le \widetilde{N}$, let

$$\mathcal{S}_{N,\,\widetilde{N}} := \left\{\psi \in \mathcal{S} :\ \|\psi\|_{\mathcal{S}_{N,\,\widetilde{N}}} := \max_{\alpha\in\mathbb{N}_0^n,\,|\alpha|\le N} \sup_{y\in\mathbb{R}^n} (1+|y|)^{\widetilde{N}} |\partial^\alpha \psi(y)| \le 1\right\}.$$

For any $\varphi \in \mathcal{S}$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $\theta(x,\ t) = M_{x,\,t}(\mathbb{B}^n) + x \in \Theta$, denote

$$\varphi_{x,\,t}(y) := \left|\det(M_{x,t}^{-1})\right| \varphi(M_{x,\,t}^{-1} y), \quad y \in \mathbb{R}^n.$$

**Definition 2.3** Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The *radial maximal function* of $f$ is defined by

$$M_\varphi^\circ f(x) := \sup_{t\in\mathbb{R}} |f * \varphi_{x,t}(x)| \quad \text{for all } x \in \mathbb{R}^n.$$

For any $N,\ \widetilde{N} \in \mathbb{N}_0$ with $N \le \widetilde{N}$, the *radial grand maximal function* of $f$ is defined by

$$M_{N,\widetilde{N}}^\circ f(x) := \sup_{\varphi \in \mathcal{S}_{N,\widetilde{N}}(\mathbb{R}^n)} M_\varphi^\circ f(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Let $\Theta$ be a continuous ellipsoid cover of $\mathbb{R}^n$ with parameters $\mathbf{p}(\Theta) = \{a_1, \ldots, a_6\}$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and let $\underline{p}$ be as in (2.3). We define $N_{p(\cdot)} := N_{p(\cdot)}(\Theta)$ as the minimal integer satisfying

$$N_{p(\cdot)}(\Theta) > \frac{\max(1, a_4)n + 1}{a_6 \underline{p}}, \tag{2.4}$$

and then $\widetilde{N}_{p(\cdot)} := \widetilde{N}_{p(\cdot)}(\Theta)$ as the minimal integer satisfying

$$\widetilde{N}_{p(\cdot)}(\Theta) > \frac{a_4 N_{p(\cdot)}(\Theta) + 1}{a_6}.$$

**Definition 2.4** [27] Let $\Theta$ be a continuous ellipsoid cover, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $M^\circ := M^\circ_{N_{p(\cdot)},\widetilde{N}_{p(\cdot)}}$. The *variable anisotropic Hardy space with variable exponent* is defined as

$$\mathcal{H}^{p(\cdot)}(\Theta) := \left\{ f \in \mathcal{S}' :\ M^\circ f \in L^{p(\cdot)}(\mathbb{R}^n) \right\}$$

with the quasi-norm $\|f\|_{\mathcal{H}^{p(\cdot)}(\Theta)} := \|M^\circ f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

## 3 Variable anisotropic Campanato spaces

In this section, we mainly establish the duality theory of $\mathcal{H}^{p(\cdot)}(\Theta)$. In what follows, for any $s \in \mathbb{Z}_+$, $\mathbb{P}_s(\mathbb{R}^n)$ denotes the set of all polynomials on $\mathbb{R}^n$ with degree not greater than $s$; for any $y \in \mathbb{R}^n$, $t \in \mathbb{R}$, $\theta := \theta_{y,t} \in \Theta$ with $\Theta$ as in Definition 2.1 and any locally integrable function $g$ on $\mathbb{R}^n$, we use $P_\theta^s g$ to denote the minimizing polynomial of $g$ with degree not greater than $s$, which means that $P_\theta^s g$ is the unique polynomial $f \in \mathbb{P}_s(\mathbb{R}^n)$ such that, for any $h \in \mathbb{P}_s(\mathbb{R}^n)$,

$$\int_\theta [g(x) - f(x)]h(x)dx = 0.$$

Now we introduce the following variable anisotropic Campanato-type spaces. For any given $q \in [1, \infty)$, we use $L^q_{\mathrm{loc}}(\mathbb{R}^n)$ to denote the set of all $q$-order locally integrable functions on $\mathbb{R}^n$.

**Definition 3.1** Let $\Theta$ be a continuous ellipsoid cover, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $q \in [1, \infty)$, $\eta \in (0, \infty)$ and $s \in \mathbb{Z}_+$. The *variable anisotropic Campanato-type space* $\mathcal{L}_{p(\cdot),q,s,\eta}(\Theta)$ is defined to be the set of all $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{L}_{p(\cdot),q,s,\eta}(\Theta)} := \sup \left\| \left\{ \sum_{i\in\mathbb{N}} \left[ \frac{|\lambda_i| \chi_{\theta_i}}{\|\chi_{\theta_i}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^\eta \right\}^{1/\eta} \right\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\times \sum_{j\in\mathbb{N}} \left\{ \frac{|\lambda_j|\,|\theta_j|}{\|\chi_{\theta_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[ \frac{1}{|\theta_j|} \int_{\theta_j} \left| f(x) - P^s_{\theta_j} f(x) \right|^q dx \right]^{1/q} \right\} < \infty,$$

where the supremum is taken over all $\{\theta_i\}_{i\in\mathbb{N}} \subset \Theta$ and $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ with $\sum_i \lambda_i \neq 0$. Here and hereafter, for any $i \in \mathbb{N}$, $x_i \in \mathbb{R}^n$ and $t_i \in \mathbb{R}$, always denote $\theta_i := \theta_{x_i,t_i} \in \Theta$.

*Remark 3.2* We point out that $\mathbb{P}_s(\mathbb{R}^n) \subset \mathcal{L}_{p(\cdot),\,q,\,s,\,\eta}(\Theta)$. Indeed,

$$\|f\|_{\mathcal{L}_{p(\cdot),\,q,\,s,\,\eta}(\Theta)} = 0$$

if and only if $f \in \mathbb{P}_s(\mathbb{R}^n)$. Throughout this article, we always identify $f \in \mathcal{L}_{p(\cdot),\,q,\,s,\,\eta}(\Theta)$ with $\{f + P : P \in \mathbb{P}_s(\mathbb{R}^n)\}$.

**Proposition 3.3** *Let $\Theta$ be a continuous ellipsoid cover, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $q \in [1,\infty)$, $\eta \in (0,\infty)$ and $s \in \mathbb{Z}_+$. For any $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, define*

$$|||f|||_{\mathcal{L}_{p(\cdot),\,q,\,s,\,\eta}(\Theta)} := \sup\inf \left\| \left\{ \sum_{i\in\mathbb{N}} \left[ \frac{|\lambda_i|\chi_{\theta_i}}{\|\chi_{\theta_i}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\eta} \right\}^{1/\eta} \right\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)} \times \sum_{j\in\mathbb{N}} \left\{ \frac{|\lambda_j|\,|\theta_j|}{\|\chi_{\theta_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[ \frac{1}{|\theta_j|} \int_{\theta_j} |f(x) - P(x)|^q \, dx \right]^{1/q} \right\},$$

*where the supremum is the same as in Definition* 3.1 *and the infimum is taken over all $P \in \mathbb{P}_s(\mathbb{R}^n)$. Then, for any $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$|||f|||_{\mathcal{L}_{p(\cdot),\,q,\,s,\,\eta}(\Theta)} = \|f\|_{\mathcal{L}_{p(\cdot),\,q,\,s,\,\eta}(\Theta)}.$$

To prove Proposition 3.3, we need the following Lemma 3.4, whose proof is similar to that of [23], the details being omitted.

**Lemma 3.4** *Let $\Theta$ be a continuous ellipsoid cover, $q \in [1,\infty)$, $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ and $s \in \mathbb{Z}_+$. Then there exists a positive constant $C$, independent of $f$ and $\theta$, such that*

$$\sup_{x\in\theta} P^s_\theta(f)(x) \le C \frac{\int_\theta |f(x)|dx}{|\theta|},$$

*where $\theta := \theta_{y,t} \in \Theta$ with $y \in \mathbb{R}^n$ and $t \in \mathbb{R}$.*

*Proof of Proposition 3.3* By the definitions of $\|\cdot\|_{\mathcal{L}_{p(\cdot),\,q,\,s,\,\eta}(\Theta)}$ and $|||\cdot|||_{\mathcal{L}_{p(\cdot),\,q,\,s,\,\eta}(\Theta)}$, it is easy to see that, for any $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$,

$$|||f|||_{\mathcal{L}_{p(\cdot),\,q,\,s,\,\eta}(\Theta)} \le \|f\|_{\mathcal{L}_{p(\cdot),\,q,\,s,\,\eta}(\Theta)}.$$

Conversely, from Lemma 3.4 and the Hölder inequality with $q \in [1,\infty)$, we conclude that, for any $\theta \in \Theta$ and $g \in \mathbb{P}_s(\mathbb{R}^n)$,

$$\begin{aligned} \left[ \frac{1}{|\theta|} \int_\theta \left| P^s_\theta(g-f)(x) \right|^q dx \right]^{1/q} &\lesssim \frac{1}{|\theta|} \int_\theta |g(x) - f(x)| dx \\ &\lesssim \left[ \frac{1}{|\theta|} \int_\theta |g(x) - f(x)|^q \, dx \right]^{1/q}. \end{aligned}$$

Thus, for any $\{\theta_j\}_{j\in\mathbb{N}} \subset \Theta$ and $\{\lambda\}_{j\in\mathbb{N}} \subset \mathbb{C}$, from the Minkowski inequality and the fact that, for any $g \in \mathbb{P}_s(\mathbb{R}^n)$, $P^s_\theta g = g$, we deduce that

$$\begin{aligned} &\sum_{j\in\mathbb{N}} \left\{ \frac{|\lambda_j|\,|\theta_j|}{\|\chi_{\theta_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[ \frac{1}{|\theta_j|} \int_{\theta_j} \left| f(x) - P^s_{\theta_j} f(x) \right|^q dx \right]^{1/q} \right\} \\ &\lesssim \sum_{j\in\mathbb{N}} \left\{ \frac{|\lambda_j|\,|\theta_j|}{\|\chi_{\theta_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[ \frac{1}{|\theta_j|} \int_{\theta_j} \left| P^s_{\theta_j}(g-f)(x) + f(x) - g(x) \right|^q dx \right]^{1/q} \right\} \\ &\lesssim \sum_{j\in\mathbb{N}} \left\{ \frac{|\lambda_j|\,|\theta_j|}{\|\chi_{\theta_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[ \frac{1}{|\theta_j|} \int_{\theta_j} |f(x) - g(x)|^q dx \right]^{1/q} \right\}, \end{aligned}$$

which, combined with, the definitions of $\|\cdot\|_{\mathcal{L}_{p(\cdot),q,s,\eta}(\Theta)}$ and $|||\cdot|||_{\mathcal{L}_{p(\cdot),q,s,\eta}(\Theta)}$, further implies that

$$\|f\|_{\mathcal{L}_{p(\cdot),q,s,\eta}(\Theta)} \lesssim |||f|||_{\mathcal{L}_{p(\cdot),q,s,\eta}(\Theta)}.$$

This finishes the proof of Proposition 3.3. □

**Proposition 3.5** *Let $\Theta$ be a continuous ellipsoid cover, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $q \in [1,\infty)$, $\eta \in (0,\infty)$ and $s \in \mathbb{Z}_+$. For any $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, define*

$$\widetilde{\|f\|}_{\mathcal{L}_{p(\cdot),q,s,\eta}(\Theta)} := \sup \left\| \left\{ \sum_{i=1}^{m} \left[ \frac{|\lambda_i|\chi_{\theta_i}}{\|\chi_{\theta_i}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\eta} \right\}^{1/\eta} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} \times \sum_{j=1}^{m} \left\{ \frac{|\lambda_j|\,|\theta_j|}{\|\chi_{\theta_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[ \frac{1}{|\theta_j|} \int_{\theta_j} \left| f(x) - P^s_{\theta_j} f(x) \right|^q dx \right]^{1/q} \right\},$$

*where the supremum is taken over all $m \in \mathbb{N}$, $\{\theta_i\}_{i=1}^m \subset \Theta$ and $\{\lambda\}_{i=1}^m \subset \mathbb{C}$, and satisfying*

$$\left\| \left\{ \sum_{i\in\mathbb{N}} \left[ \frac{|\lambda_i|\chi_{\theta_i}}{\|\chi_{\theta_i}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\eta} \right\}^{1/\eta} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty. \tag{3.1}$$

*Then, for any $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$\widetilde{\|f\|}_{\mathcal{L}_{p(\cdot),q,s,\eta}(\Theta)} = \|f\|_{\mathcal{L}_{p(\cdot),q,s,\eta}(\Theta)}.$$

*Proof* Let $p(\cdot)$, $q$, $\eta$ and $s$ be as in the present proposition and $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$. By the definitions of $\|\cdot\|_{\mathcal{L}_{p(\cdot),q,s,\eta}(\Theta)}$ and $\widetilde{\|\cdot\|}_{\mathcal{L}_{p(\cdot),q,s,\eta}(\Theta)}$, it is easy to see that

$$\widetilde{\|f\|}_{\mathcal{L}_{p(\cdot),q,s,\eta}(\Theta)} \le \|f\|_{\mathcal{L}_{p(\cdot),q,s,\eta}(\Theta)}.$$

Conversely, let $\{\theta_i\}_{i=1}^m \subset \Theta$ and $\{\lambda_i\}_{i\in 1}^m \subset \mathbb{C}$ satisfy (3.1). We have

$$\begin{aligned}
&\lim_{m\to\infty} \left\| \left\{ \sum_{i=1}^{m} \left[ \frac{|\lambda_i|\chi_{\theta_i}}{\|\chi_{\theta_i}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\eta} \right\}^{1/\eta} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} \\
&\quad \times \sum_{j=1}^{m} \left\{ \frac{|\lambda_j|\,|\theta_j|}{\|\chi_{\theta_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[ \frac{1}{|\theta_j|} \int_{\theta_j} \left| f(x) - P^s_{\theta_j} f(x) \right|^q dx \right]^{1/q} \right\} \\
&= \left\| \left\{ \sum_{i\in\mathbb{N}} \left[ \frac{|\lambda_i|\chi_{\theta_i}}{\|\chi_{\theta_i}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\eta} \right\}^{1/\eta} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} \\
&\quad \times \sum_{j\in\mathbb{N}} \left\{ \frac{|\lambda_j|\,|\theta_j|}{\|\chi_{\theta_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[ \frac{1}{|\theta_j|} \int_{\theta_j} \left| f(x) - P^s_{\theta_j} f(x) \right|^q dx \right]^{1/q} \right\}.
\end{aligned}$$

Thus, for any given $\epsilon \in (0,+\infty)$, there exists an $m_0 \in \mathbb{N}$ such that

$$\begin{aligned}
&\left\| \left\{ \sum_{i\in\mathbb{N}} \left[ \frac{|\lambda_i|\chi_{\theta_i}}{\|\chi_{\theta_i}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\eta} \right\}^{1/\eta} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} \\
&\quad \times \sum_{j\in\mathbb{N}} \left\{ \frac{|\lambda_j|\,|\theta_j|}{\|\chi_{\theta_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[ \frac{1}{|\theta_j|} \int_{\theta_j} \left| f(x) - P^s_{\theta_j} f(x) \right|^q dx \right]^{1/q} \right\}
\end{aligned}$$

$$
\begin{aligned}
&< \left\| \left\{ \sum_{i=1}^{m_0} \left[ \frac{|\lambda_i| \chi_{\theta_i}}{\|\chi_{\theta_i}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^\eta \right\}^{1/\eta} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} \\
&\quad \times \sum_{j=1}^{m_0} \left\{ \frac{|\lambda_j| \, |\theta_j|}{\|\chi_{\theta_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[ \frac{1}{|\theta_j|} \int_{\theta_j} \left| f(x) - P_{\theta_j}^s f(x) \right|^q dx \right]^{1/q} \right\} + \epsilon \\
&\le \widetilde{\|f\|}_{\mathcal{L}_{p(\cdot),\, q,\, s,\, \eta}(\Theta)} + \epsilon,
\end{aligned}
$$

which, together with the arbitrariness of $\epsilon \in (0,\ +\infty)$, and $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ satisfying (3.1), further implies that

$$
\|f\|_{\mathcal{L}_{p(\cdot),\, q,\, s,\, \eta}(\Theta)} \le \widetilde{\|f\|}_{\mathcal{L}_{p(\cdot),\, q,\, s,\, \eta}(\Theta)}.
$$

This finishes the proof Proposition 3.5. □

Motivated by [24,25], now we introduce another anisotropic variable Campanato space $\mathcal{L}_{p(\cdot),q,s}(\Theta)$, and show that $\mathcal{L}_{p(\cdot),\, q,\, s,\, \eta}(\Theta) = \mathcal{L}_{p(\cdot),\, q,\, s}(\Theta)$.

**Definition 3.6** Let $\Theta$ be a continuous ellipsoid cover, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $q \in [1, \infty)$ and $s \in \mathbb{Z}_+$. The *variable anisotropic Campanato space* $\mathcal{L}_{p(\cdot),\, q,\, s}(\Theta)$ is defined to be the set of all $f \in L^q_{\text{loc}}(\mathbb{R}^n)$ such that

$$
\|f\|_{\mathcal{L}_{p(\cdot),\, q,\, s}(\Theta)} := \sup_{y\in\mathbb{R}^n, t\in\mathbb{R}} \inf_{P\in\mathbb{P}_s(\mathbb{R}^n)} \frac{|\theta_{y,\,t}|}{\|\chi_{\theta_{y,\,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[ \frac{1}{|\theta_{y,\,t}|} \int_{\theta_{y,\,t}} |f(x) - P(x)|^q \, dx \right]^{1/q} < \infty,
$$

where $\theta_{y,\,t} \in \Theta$.

*Remark 3.7* (i) For any $f \in L^q_{\text{loc}}(\mathbb{R}^n)$, we define

$$
|||f|||_{\mathcal{L}_{p(\cdot),\, q,\, s}(\Theta)} := \sup_{y\in\mathbb{R}^n, t\in\mathbb{R}} \frac{|\theta_{y,\,t}|}{\|\chi_{\theta_{y,\,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[ \frac{1}{|\theta_{y,\,t}|} \int_{\theta_{y,\,t}} \left| f(x) - P_\theta^s f(x) \right|^q dx \right]^{1/q}.
$$

By an argument similar to that used in the proof of Proposition 3.3, we obtain that $|||f|||_{\mathcal{L}_{p(\cdot),\, q,\, s}(\Theta)}$ is an equivalent quasi-norm of the Campanato space $\mathcal{L}_{p(\cdot),\, q,\, s}(\Theta)$.

(ii) From Proposition 3.3 and Definition 3.6, we deduce that

$$
\mathcal{L}_{p(\cdot),\, q,\, s,\, \eta}(\Theta) \subset \mathcal{L}_{p(\cdot),\, q,\, s}(\Theta)
$$

for all indices as in Definition 3.1, and this inclusion is continuous.

**Proposition 3.8** *Let $\Theta$ be a continuous ellipsoid cover, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $p_+ \in (0, 1]$ with $p_+$ as in* (2.3), *$q \in [1, \infty)$ and $s \in \mathbb{Z}_+$. Then*

$$
\mathcal{L}_{p(\cdot),\, q,\, s,\, \eta}(\Theta) = \mathcal{L}_{p(\cdot),\, q,\, s}(\Theta)
$$

*with equivalent quasi-norms.*

To prove Proposition 3.8, we need the following Lemma 3.9, which is a variable anisotropic extension of [24, Lemma 4.2].

**Lemma 3.9** *Let $\Theta$ be a continuous ellipsoid cover and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then, for any $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and $\{\theta_i\}_{i\in\mathbb{N}} \subset \Theta$,*

$$
\sum_{i\in\mathbb{N}} |\lambda_i| \le \left\| \left\{ \sum_{i\in\mathbb{N}} \left[ \frac{|\lambda_i| \chi_{\theta_i}}{\|\chi_{\theta_i}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)},
$$

*where $\underline{p}$ is as in* (2.3).

*Proof of Proposition 3.8* Let $p(\cdot)$, $p_+$, $q$, $\eta$ and $s$ be as in the present proposition. By Proposition 3.3, it suffices to prove that $\mathcal{L}_{p(\cdot),q,s}(\Theta) \subset \mathcal{L}_{p(\cdot),q,s,\eta}(\Theta)$. For any $f \in \mathcal{L}_{p(\cdot),q,s}(\Theta)$, by Lemma 3.9 and Remark 3.7(i), we obtain that

$$\begin{aligned}\|f\|_{\mathcal{L}_{p(\cdot),q,s,\eta}(\Theta)} &\lesssim \sup\left(\sum_{i\in\mathbb{N}}|\lambda_i|\right)^{-1}\sum_{j\in\mathbb{N}}\left\{\frac{|\lambda_j|\,|\theta_j|}{\|\chi_{\theta_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\left[\frac{1}{|\theta_j|}\int_{\theta_j}\left|f(x)-P^s_{\theta_j}f(x)\right|^q dx\right]^{1/q}\right\}\\ &\lesssim \sup\left(\sum_{i\in\mathbb{N}}|\lambda_i|\right)^{-1}\sum_{j\in\mathbb{N}}|\lambda_j|\,|||f|||_{\mathcal{L}_{p(\cdot),q,s}(\Theta)} \sim \|f\|_{\mathcal{L}_{p(\cdot),q,s,\eta}(\Theta)},\end{aligned}$$

where the supremum is taken over all $\{\theta_i\}_{i\in\mathbb{N}} \subset \Theta$ and $\{\lambda\}_{i\in\mathbb{N}} \subset \mathbb{C}$ with $\sum_i \lambda_i \neq 0$. This finishes the proof of Proposition 3.8. □

Next, we establish the dual space of $\mathcal{H}^{p(\cdot)}(\Theta)$. More precisely, using both the atomic characterizations of $\mathcal{H}^{p(\cdot)}(\Theta)$ established in [27] and some basic tools from functional analysis, we prove that the variable anisotropic Campanato-type space $\mathcal{L}_{p(\cdot),q',s,\underline{p}}(\Theta)$ is the dual space of the Hardy space $\mathcal{H}^{p(\cdot)}(\Theta)$ for the full range $p(\cdot) \in C^{\log}(\mathbb{R}^n)$.

**Definition 3.10** [27, Definition 4.1] For a continuous ellipsoid cover $\Theta$, we say that $(p(\cdot), q, s)$ is *admissible* if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $1 < q \le \infty$ and $s \in [N_{p(\cdot)}, \infty) \cap \mathbb{N}_0$. An *anisotropic* $(p(\cdot), q, s)$*-atom* is a function $a : \mathbb{R}^n \to \mathbb{R}$ such that

(i) $\operatorname{supp} a := \overline{\{x \in \mathbb{R}^n : a(x) \neq 0\}} \subset \theta_{x,t} \in \Theta$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$;

(ii) $\|a\|_{L^q(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n}|a(x)|^q\,dx\right)^{1/q} \le \frac{|\theta_{x,t}|^{1/q}}{\|\chi_{\theta_{x,t}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}$;

(iii) $\int_{\mathbb{R}^n} a(y)y^\alpha\,dy = 0$ for all $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \le s$.

**Definition 3.11** [27, Definition 4.2] Let $\Theta$ be a continuous ellipsoid cover, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $(p(\cdot), q, s)$ an admissible triple as in Definition 3.10. The variable anisotropic atomic Hardy space $\mathcal{H}^{p(\cdot)}_{q,s}(\Theta)$ associated with $\Theta$ is defined to be the set of all tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ of the form $f = \sum_{i=1}^{\infty}\lambda_i a_i$, where the series converges in $\mathcal{S}'(\mathbb{R}^n)$, $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and $\{a_i\}_i$ are $(p(\cdot), q, s)$-atoms respectively, supported, on $\{\theta_{x_i,t_i}\}_{i\in\mathbb{N}} \subset \Theta$. Moreover, the quasi-norm of $f \in \mathcal{H}^{p(\cdot)}_{q,s}(\Theta)$ is defined by

$$\|f\|_{\mathcal{H}^{p(\cdot)}_{q,s}(\Theta)} := \inf\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\chi_{\theta_{x_i,t_i}}}{\|\chi_{\theta_{x_i,t_i}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where the infimum is taken over all admissible decompositions of $f$ as above.

Now we state the main result as follows.

**Theorem 3.12** *Let $\Theta$ be a continuous ellipsoid cover, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in (\max\{p_+, 1\}, \infty)$ with $p_+$ as in (2.3) and $s \in [N_{p(\cdot)}, \infty) \cap \mathbb{N}_0$. Then the dual space of $\mathcal{H}^{p(\cdot)}(\Theta)$, denoted by $(\mathcal{H}^{p(\cdot)}(\Theta))^*$ is $\mathcal{L}_{p(\cdot),q',s,\underline{p}}(\Theta)$ in the following sense*:

(i) *Let $g \in \mathcal{L}_{p(\cdot),q',s,\underline{p}}(\Theta)$. Then*

$$\Lambda_g : f \to \Lambda_g(f) := \int_{\mathbb{R}^n} f(x)g(x)dx, \tag{3.2}$$

*initially defined for any $f \in \mathcal{H}^{p(\cdot)}_{q,s}(\Theta)$, has a bounded extension to $\mathcal{H}^{p(\cdot)}(\Theta)$.*

(ii) *Conversely, any continuous linear functional on $\mathcal{H}^{p(\cdot)}(\Theta)$, arises as in* (3.2) *with a unique $g \in \mathcal{L}_{p(\cdot),q',s,\underline{p}}(\Theta)$. Moreover, $\|g\|_{\mathcal{L}_{p(\cdot),q',s,\underline{p}}(\Theta)} \sim \|\Lambda_g\|_{(\mathcal{H}^{p(\cdot)}(\Theta))^*}$, where the positive equivalence constants are independent of $g$.*

The framework to prove Theorem 3.12 is motivated by Huang et al. [16]. The following Lemma 3.13 show the atomic characterizations of $\mathcal{H}^{p(\cdot)}(\Theta)$, which were established in [27, Theorem 4.3].

**Lemma 3.13** *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $q \in (\max\{p_+, 1\}, \infty]$ with $p_+$ as in* (2.3), *and let $\Theta$ be a continuous ellipsoid cover and $(p(\cdot),\ q,\ s)$ an admissible triple as in Definition* 3.10, *then*

$$\mathcal{H}^{p(\cdot)}_{q,\,s}(\Theta) = \mathcal{H}^{p(\cdot)}(\Theta)$$

*with equivalent quasi-norms.*

*Proof of Theorem 3.12* We first prove Theorem 3.12(i) by considering the range of $q \in (\max\{p_+, 1\}, \infty)$.

Let $g \in \mathcal{L}_{p(\cdot),\,q',\,s,\,\underline{p}}(\Theta)$. For any $f \in \mathcal{H}^{p(\cdot)}_{q,\,s}(\Theta)$, by Definition 4.2, we obtain that there exist $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and a sequence $\{a_i\}_{i\in\mathbb{N}}$ of $(p(\cdot),\ q,\ s)$-atoms supported, respectively, on $\theta_{x_i,\,t_i} \in \Theta$ with some $x_i \in \mathbb{R}^n$ and $t_i \in \mathbb{R}$ such that $f = \sum_{i\in\mathbb{N}} \lambda_i a_i \in \mathcal{S}'$ and

$$\|f\|_{\mathcal{H}^{p(\cdot)}_{q,\,s}(\Theta)} \sim \left\| \left\{ \sum_i \left[ \frac{|\lambda_i|\chi_{\theta_{x_i,\,t_i}}}{\|\chi_{\theta_{x_i,\,t_i}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

By this, the vanishing moments of $a_i$, the Hölder inequality, the size condition of $a_i$, Proposition 3.3 and Lemma 3.13, we obtain that

$$\begin{aligned}
\left|\Lambda_g(f)\right| &\le \sum_{i\in\mathbb{N}} |\lambda_i| \left| \int_{\mathbb{R}^n} a_i(x) g(x) dx \right| \\
&= \sum_{i\in\mathbb{N}} |\lambda_i| \inf_{P\in\mathbb{P}_s(\mathbb{R}^n)} \left| \int_{\theta_i} a_i(x)[g(x) - P(x)] dx \right| \\
&\le \sum_{i\in\mathbb{N}} \frac{|\lambda_i||\theta_{x_i,\,t_i}|}{\|\chi_{\theta_i}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \inf_{P\in\mathbb{P}_s(\mathbb{R}^n)} \left[ \frac{1}{|\theta_{x_i,\,t_i}|} \int_{\theta_{x_i,\,t_i}} |g(x) - P(x)|^{q'} dx \right]^{1/q'} \\
&\lesssim \|g\|_{\mathcal{L}_{p(\cdot),\,q',\,s,\,\underline{p}}(\Theta)} \left\| \left\{ \sum_{i\in\mathbb{N}} \left[ \frac{|\lambda_i|\chi_{\theta_{x_i,\,t_i}}}{\|\chi_{\theta_{x_i,\,t_i}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\sim \|g\|_{\mathcal{L}_{p(\cdot),\,q',\,s,\,\underline{p}}(\Theta)} \|f\|_{\mathcal{H}^{p(\cdot)}_{q,\,s}(\Theta)} \sim \|g\|_{\mathcal{L}_{p(\cdot),\,q',\,s,\,\underline{p}}(\Theta)} \|f\|_{\mathcal{H}^{p(\cdot)}(\Theta)},
\end{aligned}$$

which further implies that (i) holds true.

We now show (ii). For any $\Lambda \in (\mathcal{H}^{p(\cdot)}(\Theta))^*$, By an argument similar to that used in the proof of [24, Theorem 4.4], we easily obtain that there exists a unique $g \in \mathcal{L}_{p(\cdot),\,q,\,s}(\Theta)$ such that for any $f \in \mathcal{H}^{p(\cdot)}_{q,\,s}(\Theta)$,

$$\Lambda_g(f) = \int_{\mathbb{R}^n} f(x)g(x)dx.$$

Therefore, to finish the proof of (ii), it suffices to prove that $g \in \mathcal{L}_{p(\cdot),\,q',\,s,\,\underline{p}}(\Theta)$. Indeed, for any $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ with $\sum_i \lambda_i \neq 0$, and $\theta_i := \theta_{x_i,\,t_i} \in \Theta$ with any $x_i \in \mathbb{R}^n$ and $t_i \in \mathbb{R}$, let $\eta_i \in L^q(\theta_i)$ with $\|\eta_i\|_{L^q(\theta_i)} = 1$ satisfy that

$$\left[ \int_{\theta_i} \left| g(x) - P^s_{\theta_i} f(x) \right|^{q'} dx \right]^{1/q'} = \int_{\theta_i} \left[ g(x) - P^s_{\theta_i} f(x) \right] \eta_i(x) dx \tag{3.3}$$

and, for any $x \in \mathbb{R}^n$, define

$$a_i(x) := \frac{|\theta_i|^{1/q}[\eta_i(x) - P^s_{\theta_i}\eta_i(x)]\chi_{\theta_i}}{\|\chi_{\theta_i}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\eta_i(x) - P^s_{\theta_i}\eta_i(x)\|_{L^q(\theta_i)}}.$$

By this, we know that, for any $i \in \mathbb{N}$, $a_i$ is an $(p(\cdot), q, s)$-atom. Therefore, by Lemma 3.13, we obtain that $\sum_{i\in\mathbb{N}} \lambda_i a_i \in \mathcal{H}^{p(\cdot)}(\Theta)$. From this, combined with (3.3) and the facts that $\Lambda \in (\mathcal{H}^{p(\cdot)}(\Theta))^*$ and

$$\|\eta_i - P^s_{\theta_i}\eta_i\|_{L^q(\theta_i)} \lesssim 1,$$

we deduce that

$$\begin{aligned}
&\sum_{i\in\mathbb{N}} \frac{|\lambda_i|\,|\theta_i|}{\|\chi_{\theta_i}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[\frac{1}{|\theta_i|}\int_{\theta_i} \left|g(x)-P_\theta^s g(x)\right|^{q'}dx\right]^{1/q'}\\
&\quad= \sum_{i\in\mathbb{N}} \frac{|\lambda_i|\,|\theta_i|^{1/q}}{\|\chi_{\theta_i}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \int_{\theta_i}\left[g(x)-P_\theta^s g(x)\right]\eta_i(x)dx\\
&\quad= \sum_{i\in\mathbb{N}} \frac{|\lambda_i|\,|\theta_i|^{1/q}}{\|\chi_{\theta_i}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \int_{\theta_i}\left[\eta_i(x)-P_\theta^s \eta_i(x)\right]g(x)\chi_{\theta_i}(x)dx\\
&\quad\lesssim \sum_{i\in\mathbb{N}}|\lambda_i|\int_{\theta_i} a_i(x)g(x)dx \sim \sum_{i\in\mathbb{N}}|\lambda_i|\Lambda_g(a_i)\sim \Lambda_g\left(\sum_{i\in\mathbb{N}}|\lambda_i|a_i\right)\\
&\quad\lesssim \left\|\sum_{i\in\mathbb{N}}|\lambda_i|a_i\right\|_{\mathcal{H}^{p(\cdot)}(\Theta)} \lesssim \left\|\left\{\sum_i\left[\frac{|\lambda_i|\chi_{\theta_{x_i,t_i}}}{\|\chi_{\theta_{x_i,t_i}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

By this, we obtain that $g\in\mathcal{L}_{p(\cdot),q',s,\underline{p}}(\Theta)$. This finishes the proof of (ii) and hence the proof of Theorem 3.12. □

## 4 A class of linear operators on $\mathcal{H}^{p(\cdot)}(\Theta)$

In this section, we obtain a boundedness criterion for a class of linear operators from $\mathcal{H}^{p(\cdot)}(\Theta)$ to $L^{p(\cdot)}(\mathbb{R}^n)$. We first recall the definitions of quasi-distance $\rho_\Theta$ and *Hardy–Littlewood maximal operators* associated with a continuous ellipsoid covers $\Theta$.

Dekel et al. have shown that an ellipsoid cover $\Theta$ induces a quasi-distance $\rho_\Theta$ on $\mathbb{R}^n$, see [10, Proposition 2.7]. Moreover, $(\mathbb{R}^n,\ \rho_\Theta,\ dx)$ and the Lebesgue measure is a space of homogeneous type, [10, Proposition 2.10], which implies in the following Proposition 4.1.

**Proposition 4.1** *Let $\Theta$ be a continuous ellipsoid cover.*

(i) [10, Proposition 2.7] *The function $\rho:\ \mathbb{R}^n\times\mathbb{R}^n\to[0,\infty)$ defined by*

$$\rho_\Theta(x,y):=\inf_{\theta\in\Theta}\{|\theta|:x,y\in\theta\}$$

*is a quasi-distance on $\mathbb{R}^n$.*

(ii) [10, Proposition 2.10] *Let*

$$B_{\rho_\Theta}(x,r):=\{y\in\mathbb{R}^n:\rho_\Theta(x,y)<r\}. \tag{4.1}$$

*Then*

$$|B_{\rho_\Theta}(x,r)|\sim r\quad \textit{for all } x\in\mathbb{R}^n,\quad r>0,$$

*where the constants of equivalence depend only on* $\mathbf{p}(\Theta)$.

Let $\Theta$ be a continuous ellipsoid cover. For any locally integrable function $f$ on $\mathbb{R}^n$, the *Hardy–Littlewood maximal operators* $\mathcal{M}_{B_{\rho_\Theta}}$ and $\mathcal{M}_\Theta$ are defined, respectively, to be

$$\mathcal{M}_{B_{\rho_\Theta}}f(x):=\sup_{r>0}\sup_{B_{\rho_\Theta}(y,r)\ni x}\frac{1}{|B_{\rho_\Theta}(y,r)|}\int_{B_{\rho_\Theta}(y,r)}|f(z)|\,dz$$

and

$$\mathcal{M}_\Theta f(x):=\sup_{t\in\mathbb{R}}\sup_{\theta_{y,t}\ni x}\frac{1}{|\theta_{y,t}|}\int_{\theta_{y,t}}|f(z)|\,dz, \tag{4.2}$$

where $B_{\rho_\Theta}(y,r)$ is as in (4.1). By [12, Lemma 3.2], we know that

$$\mathcal{M}_{B_{\rho_\Theta}}f(x)\sim\mathcal{M}_\Theta f(x)\quad\text{for all } f\in L_{\mathrm{loc}}(\mathbb{R}^n),\quad x\in\mathbb{R}^n.$$

**Theorem 4.2** *Let $\Theta$ be a continuous ellipsoid cover, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in [1, \infty] \cap (p_+, \infty]$ with $p_+$ as in* (2.3), *$s \in [N_{p(\cdot)}, \infty) \cap \mathbb{N}_0$ and $\beta > 1/p_-$ with $p_-$ as in* (2.3). *Suppose $T$ is a linear operator on $L^q(\mathbb{R}^n)$ and satisfies that, for any $(p(\cdot),\ \infty,\ s)$-atom $b$, $\operatorname{supp} b \subset \theta_{z,\,t} \in \Theta$ with $z \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $x \in (\theta_{z,\,t-J})^{\complement}$,*

$$|T(b)(x)| \le C\|\chi_{\theta_{z,\,t}}\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)} \left[\mathcal{M}_\Theta(\chi_{\theta_{z,\,t}})(x)\right]^{\beta}, \tag{4.3}$$

*where $\mathcal{M}_\Theta$ is as in* (4.2) *and $J$ is as in Lemma* 2.2. *Then $T$ is bounded from $\mathcal{H}^{p(\cdot)}(\Theta)$ to $L^{p(\cdot)}(\mathbb{R}^n)$. Moreover, there exists a positive constant $C$ such that, for all $f \in \mathcal{H}^{p(\cdot)}(\Theta)$,*

$$\|T(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{\mathcal{H}^{p(\cdot)}(\Theta)}. \tag{4.4}$$

To prove Theorem 4.2, we need several technical lemmas as follows.

**Lemma 4.3** [27, Lemma 5.10] *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $1 < q < \infty$, and $s \ge N_{p(\cdot)}$. Then, for any $f \in L^q(\mathbb{R}^n) \cap \mathcal{H}^{p(\cdot)}(\Theta)$, there exist a sequence of $(p(\cdot), \infty, s)$-atoms $\{a_i\}_{i\in\mathbb{N}}$, a sequence $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$, and a positive constant $C$ independent of $f$ such that*

$$\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\chi_{\theta_{x_i,\,t_i}}}{\|\chi_{\theta_{x_i,\,t_i}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{\mathcal{H}^{p(\cdot)}(\Theta)}$$

*and*

$$f = \sum_{i\in\mathbb{N}} \lambda_i\, a_i \ \text{ converges in } L^q(\mathbb{R}^n) \text{ and almost everywhere,}$$

*where $\underline{p}$ is as in* (2.3).

**Lemma 4.4** (i) [9, Lemma 2.3] *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then, for any $s \in (0, \infty)$ and $f \in L^{p(\cdot)}(\mathbb{R}^n)$,*

$$\||f|^s\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|f\|^s_{L^{sp(\cdot)}(\mathbb{R}^n)}.$$

(ii) [9, Lemma 2.3] *In addition, for any $\lambda \in \mathbb{C}$ and $f,\ g \in L^{p(\cdot)}(\mathbb{R}^n)$, it holds true that*

$$\|\lambda f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = |\lambda|\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \text{ and } \|f+g\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} \le \|f\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)(\mathbb{R}^n)} + \|g\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)},$$

*where $\underline{p}$ is as in* (2.3).

**Lemma 4.5** [27, Lemma 4.5] *Let $r(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $q \in [1, \infty] \cap (r_+, \infty]$ with $r_+$ as in* (2.3). *Assume that $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$, $\{\theta_{x_i,\,t_i}\}_{i\in\mathbb{N}} \subset \Theta$ and $\{a_i\}_{i\in\mathbb{N}} \subset L^q(\mathbb{R}^n)$ satisfy, for any $i \in \mathbb{N}$, $\operatorname{supp} a_i \subset \theta_{x_i,\,t_i}$,*

$$\|a_i\|_{L^q(\mathbb{R}^n)} \le \frac{|\theta_{x_i,\,t_i}|^{1/q}}{\|\chi_{\theta_{x_i,\,t_i}}\|_{L^{r(\cdot)}(\mathbb{R}^n)}}$$

*and*

$$\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\chi_{\theta_{x_i,\,t_i}}}{\|\chi_{\theta_{x_i,\,t_i}}\|_{L^{r(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{r}}\right\}^{1/\underline{r}}\right\|_{L^{r(\cdot)}(\mathbb{R}^n)} < \infty.$$

*Then*

$$\left\|\left[\sum_{i\in\mathbb{N}}|\lambda_i a_i|^{\underline{r}}\right]^{\frac{1}{\underline{r}}}\right\|_{L^{r(\cdot)}(\mathbb{R}^n)} \le C\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\chi_{\theta_{x_i,\,t_i}}}{\|\chi_{\theta_{x_i,\,t_i}}\|_{L^{r(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{r}}\right\}^{1/\underline{r}}\right\|_{L^{r(\cdot)}(\mathbb{R}^n)},$$

*where $C$ is a positive constant independent of $\lambda_i$, $a_i$ and $\theta_{x_i,\,t_i}$.*

The following Lemma 4.6 gives Fefferman–Stein vector-valued inequality of the maximal operator $\mathcal{M}_\Theta$ on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$.

**Lemma 4.6** [27, Lemma 3.1] *Let $r \in (1, \infty]$. Assume that $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ satisfies $1 < p_- \le p_+ < \infty$. Then there exists a positive constant $C := C(r, n, p(\cdot))$ such that, for any sequence $\{f_k\}_{k\in\mathbb{N}}$ of measurable functions,*

$$\left\| \left\{ \sum_{k\in\mathbb{N}} [\mathcal{M}_\Theta(f_k)]^r \right\}^{1/r} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \left\| \left( \sum_{k\in\mathbb{N}} |f_k|^r \right)^{1/r} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

*with the usual modification made when $r = \infty$, where $\mathcal{M}_\Theta$ denotes the Hardy–Littlewood maximal operator as in* (4.2).

**Lemma 4.7** [27, Proposition 4.7] *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in (p_+, \infty) \cap [1, \infty)$ with $p_+$ as in* (2.3). *Then $\mathcal{H}^{p(\cdot)}(\Theta) \cap L^q(\mathbb{R}^n)$ is dense in $\mathcal{H}^{p(\cdot)}(\Theta)$.*

**Lemma 4.8** [11, Theorem 3.2.7] *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space.*

*Proof* (Proof of Theorem 4.2) For any $f \in \mathcal{H}^{p(\cdot)}(\Theta) \cap L^q(\mathbb{R}^n)$, from Lemma 4.3, it follows that there exist numbers $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and a sequence of $(p(\cdot), \infty, s)$-atom, $\{a_i\}_{i\in\mathbb{N}}$, supported, respectively, on $\{\theta_{x_i,t_i}\}_{i\in\mathbb{N}} \subset \Theta$ such that

$$f = \sum_{i\in\mathbb{N}} \lambda_i a_i \ \text{ in } L^q(\mathbb{R}^n)$$

and

$$\left\| \left\{ \sum_{i\in\mathbb{N}} \left[ \frac{|\lambda_i| \chi_{\theta_{x_i,t_i}}}{\|\chi_{\theta_{x_i,t_i}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \|f\|_{\mathcal{H}^{p(\cdot)}(\Theta)}. \tag{4.5}$$

By the assumption that the linear operator $T$ is bounded on $L^q(\mathbb{R}^n)$, we further have

$$T(f) = \sum_{i\in\mathbb{N}} \lambda_i (Ta_i) \ \text{ in } L^q(\mathbb{R}^n)$$

and hence in $\mathcal{S}'$. Then, by Lemma 4.4(ii), we obtain that

$$\begin{aligned}
\|T(f)\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} &= \left\| \sum_{i\in\mathbb{N}} \lambda_i T(a_i) \right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\le \left\| \sum_{i\in\mathbb{N}} |\lambda_i| T(a_i) \chi_{\theta_{x_i,t_i-J}} \right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} + \left\| \sum_{i\in\mathbb{N}} |\lambda_i| T(a_i) \chi_{\theta^{\complement}_{x_i,t_i-J}} \right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \left\| \left\{ \sum_{i\in\mathbb{N}} \left[ |\lambda_i| T(a_i) \chi_{\theta_{x_i,t_i-J}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} + \left\| \sum_{i\in\mathbb{N}} |\lambda_i| T(a_i) \chi_{\theta^{\complement}_{x_i,t_i-J}} \right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&=: \mathrm{L}_1 + \mathrm{L}_2.
\end{aligned}$$

For the term $\mathrm{L}_1$, by the boundedness of $T$ on $L^q(\mathbb{R}^n)$ with $q \in [1, \infty] \cap (p_+, \infty]$, size condition of $a_i$, Lemma 4.5 and (4.5), we conclude that

$$\mathrm{L}_1 \lesssim \left\| \left\{ \sum_{i\in\mathbb{N}} \left[ \frac{|\lambda_i| \chi_{\theta_{x_i,t_i}}}{\|\chi_{\theta_{x_i,t_i}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|^{\underline{p}}_{\mathcal{H}^{p(\cdot)}(\Theta)}.$$

For the term $\mathrm{L}_2$, by (4.3), Lemmas 4.4(i), 4.6 and the fact that $\beta > 1/p_-$, we obtain that

$$\mathrm{L}_2 \lesssim \left\| \sum_{i\in\mathbb{N}} \frac{|\lambda_i|}{\|\chi_{\theta_{x_i,t_i}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[ \mathcal{M}_\Theta(\chi_{\theta_{x_i,t_i}}) \right]^\beta \right\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$
\begin{aligned}
&\lesssim \left\| \left\{ \sum_{i\in\mathbb{N}} \frac{|\lambda_i|}{\|\chi_{\theta_{x_i,t_i}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[\mathcal{M}_\Theta(\chi_{\theta_{x_i,t_i}})\right]^\beta \right\}^{1/\beta} \right\|_{L^{\beta p(\cdot)}(\mathbb{R}^n)}^{\beta \underline{p}} \\
&\lesssim \left\| \left\{ \sum_{i\in\mathbb{N}} \frac{|\lambda_i|\chi_{\theta_{x_i,t_i}}}{\|\chi_{\theta_{x_i,t_i}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right\}^{1/\beta} \right\|_{L^{\beta p(\cdot)}(\mathbb{R}^n)}^{\beta \underline{p}} \sim \left\| \sum_{i\in\mathbb{N}} \frac{|\lambda_i|\chi_{\theta_{x_i,t_i}}}{\|\chi_{\theta_{x_i,t_i}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{\underline{p}} \\
&\lesssim \left\| \left\{ \sum_{i\in\mathbb{N}} \left[ \frac{|\lambda_i|\chi_{\theta_{x_i,t_i}}}{\|\chi_{\theta_{x_i,t_i}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{\underline{p}} \sim \|f\|_{\mathcal{H}^{p(\cdot)}(\Theta)}^{\underline{p}}.
\end{aligned}
$$

Combining the estimates of $\mathrm{L}_1$ and $\mathrm{L}_2$, we further conclude that, for any $f \in \mathcal{H}^{p(\cdot)}(\Theta) \cap L^q$,

$$
\|T(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{H}^{p(\cdot)}(\Theta)}.
$$

Next, we prove that (4.4) also holds true for any $f \in \mathcal{H}^{p(\cdot)}(\Theta)$. For any given $f \in \mathcal{H}^{p(\cdot)}(\Theta)$, by Lemma 4.7, we know that there exists a sequence $\{f_j\}_{j\in\mathbb{Z}_+} \subset \mathcal{H}^{p(\cdot)}(\Theta) \cap L^q$ such that $f_j \to f$ as $j \to \infty$ in $\mathcal{H}^{p(\cdot)}(\Theta)$. Therefore, $\{f_j\}_{j\in\mathbb{Z}_+}$ is a Cauchy sequence in $\mathcal{H}^{p(\cdot)}(\Theta)$. By this, we see that, for any $j, k \in \mathbb{Z}_+$,

$$
\|T(f_j) - T(f_k)\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|T(f_j - f_k)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f_j - f_k\|_{\mathcal{H}^{p(\cdot)}(\Theta)}.
$$

Notice that $\{T(f_j)\}_{j\in\mathbb{Z}_+}$ is also a Cauchy sequence in $L^{p(\cdot)}(\mathbb{R}^n)$. Applying Lemma 4.8, we conclude that there exist a $g \in L^{p(\cdot)}(\mathbb{R}^n)$ such that $T(f_j) \to g$ as $j \to \infty$ in $L^{p(\cdot)}(\mathbb{R}^n)$. Let $T(f) := g$. We claim that $T(f)$ is well defined. Indeed, for any other sequence $\{h_j\}_{j\in\mathbb{Z}_+} \subset \mathcal{H}^{p(\cdot)}(\Theta) \cap L^2$ satisfying $h_j \to f$ as $j \to \infty$ in $\mathcal{H}^{p(\cdot)}(\Theta)$, by Lemma 4.4(ii), we have

$$
\begin{aligned}
&\|T(h_j) - T(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{\underline{p}} \\
&\quad \le \|T(h_j) - T(f_j)\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{\underline{p}} + \|T(f_j) - g\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{\underline{p}}. \\
&\quad \lesssim \|h_j - f_j\|_{\mathcal{H}^{p(\cdot)}(\Theta)}^{\underline{p}} + \|T(f_j) - g\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{\underline{p}} \\
&\quad \lesssim \|h_j - f\|_{\mathcal{H}^{p(\cdot)}(\Theta)}^{\underline{p}} + \|f - f_j\|_{\mathcal{H}^{p(\cdot)}(\Theta)}^{\underline{p}} + \|T(f_j) - g\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{\underline{p}} \to 0 \quad \text{as } j \to 0.
\end{aligned}
$$

By this, we obtain that, for any $f \in \mathcal{H}^{p(\cdot)}(\Theta)$,

$$
\|T(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|g\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \lim_{j\to\infty} \|T(f_j)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \lim_{j\to\infty} \|f_j\|_{\mathcal{H}^{p(\cdot)}(\Theta)} \sim \|f\|_{\mathcal{H}^{p(\cdot)}(\Theta)},
$$

which implies that (4.4) also holds true for any $f \in \mathcal{H}^{p(\cdot)}(\Theta)$ and hence finishes the proof of Theorem 4.2. □

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**Conflict of interest** The authors declare that they have no conflict of interest.

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