Indian J Pure Appl Math (2025) 56:1577–1592
https://doi.org/10.1007/s13226-024-00557-9

ORIGINAL RESEARCH

Pell and Pell-Lucas numbers which are concatenations of three repdigits

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Received: 7 July 2022 / Accepted: 30 January 2024 / Published online: 22 February 2024

Abstract In this study, we focus on finding the Pell and Pell-Lucas numbers which are concatenations of three repdigits. We show that these numbers are $169, 408, 985$ and $198, 478, 1154$, respectively. We use a lemma that provides a large upper bound for the subscript n in the equations and Baker's theory of lower bounds for a nonzero linear form in logarithms of algebraic numbers. In addition, continued fraction expansions of some irrational numbers were calculated to show that some inequalities have no solution.

Keywords Linear forms in logarithms · Pell number · Pell-Lucas number · Repdigit · Diophantine equations · Concatenations

AMS Subject Classification 11J86 · 11K31 · 11D61

1 Introduction

Let the sequence of Pell numbers represented by $(P_n)_{n\geq 0}$ be defined as

$$P_0 = 0,\ \ P_1 = 1,\ P_n = 2P_{n-1} + P_{n-2}$$

and the sequence of Pell-Lucas numbers represented by $(Q_n)_{n\geq 0}$ be defined as

$$Q_0 = 2,\ \ Q_1 = 2,\ Q_n = 2Q_{n-1} + Q_{n-2}$$

for $n \geq 2$. If α and β are the roots of the characteristic equation $x^2 - 2x - 1 = 0$, then

$$\alpha = 1 + \sqrt{2} \text{ and } \beta = 1 - \sqrt{2}.$$

Moreover, it is well known that

$$2 < \alpha < 3, -1 < \beta < 0,$$

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}} \text{ and } Q_n = \alpha^n + \beta^n,$$

Communicated by B. Sury.

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$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1} \text{ for } n \geq 1, \tag{1}$$

and

$$\alpha^{n-1} \leq Q_n < 2\alpha^n \text{ for } n \geq 1. \tag{2}$$

By induction method, (1) and (2) can be shown. The base b-repdigit is a positive integer with all digits equal. Simply N is said to a repdigit when $b = 10$, and also it is of the form

$$N = d \cdot \frac{(10^s - 1)}{9} = \overline{\underbrace{d...d}_{s\ times}}$$

for some $d, s \in \mathbb{Z}^+$ with $s \geq 1$ and $1 \leq d \leq 9$. We say that N is a concatenations of k repdigits if N can be written in the form

$$\overline{\underbrace{d_1...d_1}_{s_1 \text{ times}}\underbrace{d_2...d_2}_{s_2 \text{ times}}...\underbrace{d_k...d_k}_{s_k \text{ times}}},$$

where $i, d_i, s_i \in \mathbb{Z}$, $i \geq 1$, $k \geq 2$, $s_1, s_2, ..., s_k \geq 1$, $1 \leq d_1 \leq 9$ and $0 \leq d_2, d_3, ..., d_k \leq 9$. Firstly, some authors searched for numbers with repdigits for special number sequences. In [22] and [19], the authors showed that Fibonacci numbers in this form are only 0, 1, 2, 3, 5, 8, 55, Lucas numbers in this form are only 2, 1, 3, 4, 7, 11 and Pell numbers in this form are only 0, 1, 3, 7 and Pell-Lucas numbers in this form are only 2, 6, respectively. Later, some special number sequences that can be written as a concatenations of two repdigits were investigated by several authors. In [24], Rayaguru and Panda examined that the balancing number in this form is only 35. In [1], the Alahmadi et al. showed that Fibonacci numbers in this form are only 13, 21, 34, 55, 89, 144, 233 and 377. In [15,17,18,28], Keskin et al. worked on the problems of finding Fibonacci or Lucas numbers which are product, sum, difference, or concatenations of two or three repdigits. Moreover, in [13,14,16,27], Keskin et al. determined repdigits base b or 10 as products or sums of two Fibonacci or Lucas numbers. Specially, in [17] and [18], Erduvan and Keskin determined that all Lucas numbers which are concatenations of two or three repdigits are only 18, 29, 47, 76, 199, 322 and concatenations of three repdigits are only 123, 199, 322, 521, 843, 2207, 5778. In [3], the authors determined that the only Perrin numbers which are concatenations of two distinct repdigits are $P_n \in \{10, 12, 17, 29, 39, 51, 68, 90, 119, 277, 644\}$. In [8], Chalebgwa and Ddamulira showed that the only Padovan numbers which are palindromic concatenations of two distinct repdigits are $P_n \in \{151, 616\}$. In [10], Ddamulira found all repdigits as sums of three Padovan numbers. In [4], the authors studied Perrin numbers expressible as sums of two base b repdigits. In [25,26], the authors studied repdigits as products of consecutive Padovan or Perrin numbers and Padovan and Perrin numbers as product of two repdigits, respectively. In [21], Lomelí and Hernández determined repdigits as sums of two Padovan numbers. Moreover, the Diophantine equation

$$F_n = \overline{a\underbrace{b...b}_{s \text{ times}}\underbrace{c...c}_{r \text{ times}}}, \text{ with } 2 \leq r \leq s, \text{ where } 1 \leq a \leq 9 \text{ and } 0 \leq b, c \leq 9$$

in positive integer numbers s, n, and r was solved by Trojovsky [29].

Let $d_1, d_2, d_3, m_1, m_2, m_3 \in \mathbb{Z}$, $m_1, m_2, m_3 \geq 1$, $1 \leq d_1 \leq 9$ and $0 \leq d_2, d_3 \leq 9$. In this paper, we solve the equations

$$P_n = \overline{\underbrace{d_1...d_1}_{m_1 \text{ times}}\underbrace{d_2...d_2}_{m_2 \text{ times}}\underbrace{d_3...d_3}_{m_3 \text{ times}}}, \tag{3}$$

and

$$Q_n = \overline{\underbrace{d_1...d_1}_{m_1 \text{ times}}\underbrace{d_2...d_2}_{m_2 \text{ times}}\underbrace{d_3...d_3}_{m_3 \text{ times}}}, \tag{4}$$

and find the solutions of equations (3) and (4) as 169, 408, 985 and 198, 478, 1154, respectively. In Section 2, we give the definitions and lemmas necessary to find solutions to our equations. In Section 3, we prove our main theorems. While proving these theorems, we made calculations with equations or inequalities with many unknowns, sometimes requiring many operations. MATLAB program was used to make these calculations easily and without errors. Thus, we are time also saved and operations with decimal numbers became easy. Besides, we calculate continued fraction expansions of some irrational numbers.

2 Auxiliary results

In this section, we will give the definitions and lemmas necessary to find solutions to our equations. In [18], in order to solve Diophantine equations of the similar form, the authors have used Baker's theory of lower bounds for a nonzero linear form in logarithms of algebraic numbers.

Let η be an algebraic number of degree d with minimal polynomial

$$a_0x^d + a_1x^{d-1} + ... + a_d = a_0\prod_{i=1}^{d}\left(x - \eta^{(i)}\right) \in \mathbb{Z}[x],$$

where the a_i's are relatively prime integers with $a_0 > 0$ and the $\eta^{(i)}$'s are conjugates of η. $h(\eta)$ is called the logarithmic height of η and

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d}\log\left(\max\left\{|\eta^{(i)}|, 1\right\}\right)\right).$$

Let η and γ be algebraic numbers. Then it is well known that (see [7]):

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2,$$
$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma),$$

and

$$h(\eta^m) = |m|h(\eta).$$

In addition, if $\eta = \frac{a}{b} \in \mathbb{Q}$, $\gcd(a, b) = 1$ and $b \geq 1$, then

$$h(\eta) = \log\left(\max\{|a|, b\}\right).$$

The following lemma is derived from Corollary 2.3 of Matveev [23] and allows to find a large upper bound for the subscript n. Also see in [6] for this lemma.

Lemma 1 *Assume that $\gamma_1, \gamma_2, ..., \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, b_2, ..., b_t$ are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1$$

is not zero. Then

$$|\Lambda| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1A_2 \cdots A_t\right),$$

where

$$B \geq \max\{|b_1|, ..., |b_t|\},$$

and

$$A_i \geq \max\{Dh(\gamma_i), |\log\gamma_i|, 0.16\}$$

for all $i = 1, ..., t$.

Dujella and Pethò [12] revealed a version of a lemma of Baker and Davenport [2]. Later, in [5], the authors put forward a different version of conclusion of Dujella and Pethò's lemma.

Lemma 2 *Suppose $||x||$ denote the distance from x to the nearest integer, i.e., $||x|| = \min\{|x - n| : n \in \mathbb{Z}\}$ for any real number x. Suppose that p/q is a convergent of the continued fraction of the irrational number γ such that $q > 6M$ where M is a positive integer and A, B, μ are some real numbers with $A > 0$ and $B > 1$. If $||\mu q|| - M||\gamma q|| : \epsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

in positive integers u, v, and w with $u \leq M$ and $w \geq \frac{\log(Aq/\epsilon)}{\log B}$.

The following lemma is given in [11].

Lemma 3 *Suppose* $a, x \in \mathbb{R}$, $|x| < a$ *and* $0 < a < 1$. *Then,*

$$|\log(1+x)| < |x| \cdot \frac{-\log(1-a)}{a}$$

and

$$|x| < \left|e^x - 1\right| \cdot \frac{a}{1-e^{-a}}.$$

Lemma 4 *[9] Let* $\tau = [a_0; a_1, a_2, a_3, ...]$ *be an irrational number,* p_i/q_i *for* $i \geq 0$ *be all the convergents of the continued fraction expansion of* τ. *Suppose* $M, N \in \mathbb{Z}^+$ *such that* $q_N > M$. *Then, the inequality*

$$\left|\tau - \frac{r}{s}\right| > \frac{1}{(a(M)+2)s^2}$$

with $0 < s < M$, $r \in \mathbb{Z}^+$ *and* $a(M) := \max\{a_i : i = 0, 1, 2, ..., N\}$ *is valid for all pairs* (r, s) *of positive integer.*

Lemma 5 *[19] The only repdigits in the Pell and Pell-Lucas sequence are* 0, 1, 2, 5 *and* 2, 6, *respectively.*

Lemma 6 *[20] The only Pell and Pell-Lucas numbers which are concatenations of two repdigits are* 12, 29, 70 *and* 14, 34, 84*, respectively.*

3 Main theorems

Theorem 7 *The only Pell numbers which are concatenations of three repdigits are* 169, 408, 985.

Proof Suppose that the equation (3) is valid and

$$\frac{\alpha^n - \beta^n}{2\sqrt{2}} = P_n = \overline{\underbrace{d_1...d_1}_{m_1 \text{ times}}\underbrace{d_2...d_2}_{m_2 \text{ times}}\underbrace{d_3...d_3}_{m_3 \text{ times}}}. \tag{5}$$

We looked the first 100 Pell numbers and we saw $P_n \in \{169, 408, 985\}$, which are all the solutions to the Diophantine equation (3). From now on, suppose that $n > 100$ and we ignore $d_1 = d_2 \neq d_3$ and $d_1 \neq d_2 = d_3$ in equation (3). Because these cases were given in Lemma 6. Besides, the case $d_1 = d_2 = d_3$ is impossible according to Lemma 5. Then, by (5), we can write

$$\begin{aligned}\frac{\alpha^n - \beta^n}{2\sqrt{2}} = P_n &= \underbrace{d_1...d_1}_{m_1 \text{ times}} \cdot 10^{m_2+m_3} + \underbrace{d_2...d_2}_{m_2 \text{ times}} \cdot 10^{m_3} + \underbrace{d_3...d_3}_{m_3 \text{ times}} \\ &= \frac{d_1(10^{m_1}-1)}{9}10^{m_2+m_3} + \frac{d_2(10^{m_2}-1)}{9}10^{m_3} + \frac{d_3(10^{m_3}-1)}{9} \\ &= \frac{1}{9}\left(10^{m_1+m_2+m_3}d_1 - 10^{m_2+m_3}(d_1-d_2) - 10^{m_3}(d_2-d_3) - d_3\right), \end{aligned} \tag{6}$$

i.e.,

$$\frac{9\alpha^n}{2\sqrt{2}} - 10^{m_1+m_2+m_3}d_1 = \frac{9\beta^n}{2\sqrt{2}} - 10^{m_2+m_3}(d_1-d_2) - 10^{m_3}(d_2-d_3) - d_3. \tag{7}$$

By using (1) and (6), it can be seen that

$$10^{m_1+m_2+m_3-1} < P_n \leq \alpha^{n-1} < 10^{n-1}.$$

Then, we get $m_1 + m_2 + m_3 < n$. If we take absolute values of the equation (7), we obtain

$$\left|\frac{9\alpha^n}{2\sqrt{2}} - d_1 10^{m_1+m_2+m_3}\right| \leq \frac{9|\beta|^n}{2\sqrt{2}} + |d_1-d_2| \cdot 10^{m_2+m_3} + |d_2-d_3| \cdot 10^{m_3} + d_3$$

$$
\begin{aligned}
&\leq \frac{9\alpha^{-n}}{2\sqrt{2}} + 9 \cdot 10^{m_2+m_3} + 9 \cdot 10^{m_3} + 9 \\
&\leq \frac{9\alpha^{-n}}{2\sqrt{2}} + (9 \cdot 10^{m_2+m_3} + 9 \cdot 10^{m_3} + 0.9 \cdot 10^{m_3}) \\
&= \frac{9\alpha^{-n}}{2\sqrt{2}} + 10^{m_3}(9 \cdot 10^{m_2} + 9.9) \\
&\leq \frac{9\alpha^{-n}}{2\sqrt{2}} + 10^{m_3}(9 \cdot 10^{m_2} + 0.99 \cdot 10^{m_2}) \\
&< \frac{0.09 \cdot \alpha^{-n} \cdot 10^{m_2+m_3}}{2\sqrt{2}} + 9.99 \cdot 10^{m_2+m_3} \\
&< 9.991 \cdot 10^{m_2+m_3}, \qquad (8)
\end{aligned}
$$

for $n > 100$. If (8) is divided by $d_1 10^{m_1+m_2+m_3}$, then

$$\left| \frac{9}{2\sqrt{2}d_1} \alpha^n 10^{-m_1-m_2-m_3} - 1 \right| < \frac{9.991}{10^{m_1}}. \qquad (9)$$

Now, let's take $\gamma_1 := 9/(2\sqrt{2}d_1)$, $\gamma_2 := \alpha$, $\gamma_3 := 10$ and $b_1 := 1$, $b_2 := n$, $b_3 := -m_1 - m_2 - m_3$ and apply Lemma 1. The numbers γ_1, γ_2, and γ_3 which are elements of the field $K = \mathbb{Q}(\sqrt{2})$ are positive real numbers, then we have $[K : Q] = 2$. So $D = 2$. Let

$$\Lambda_1 := \frac{9}{2\sqrt{2}d_1} \alpha^n 10^{-m_1-m_2-m_3} - 1.$$

Assume that $\Lambda_1 = 0$. Then we have

$$\alpha^n = \frac{2\sqrt{2}d_1}{9} \cdot 10^{m_1+m_2+m_3}.$$

Since α^{2n} is irrational for $n \geq 1$, this is impossible. So, Λ_1 is nonzero. Moreover,

$$
\begin{aligned}
h(\gamma_1) = h\left(\frac{9}{2\sqrt{2}d_1}\right) &\leq h(9/d_1) + h(2\sqrt{2}) \\
&\leq \log 9 + \frac{1}{2}\log 8 < 3.24,
\end{aligned}
$$

$$
\begin{aligned}
h(\gamma_2) = h(\alpha) &= \frac{\log \alpha}{2} < 0.45, \\
h(\gamma_3) = h(10) &= \log 10 < 2.31,
\end{aligned}
$$

$m_1 + m_2 + m_3 < n$ and $B \geq \max\{|1|, |n|, |-m_1 - m_2 - m_3|\}$. We can say $A_1 := 6.48$, $A_2 := 0.9$, $A_3 := 4.62$ and $B := n$. Then we obtain

$$9.991 \cdot 10^{-m_1} > |\Lambda_1| > \exp\left((1 + \log n) \cdot C\right).$$

where $C := -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2) \cdot 0.9 \cdot 4.62 \cdot 6.48$ according to (9) and Lemma 1. From the last inequality, we find

$$m_1 \log 10 < 2.62 \cdot 10^{13} \cdot (1 + \log n) + \log 9.991, \qquad (10)$$

by a simple computation. Now, let's rewrite the equation (6) as

$$\frac{9\alpha^n}{2\sqrt{2}} - \left(d_1 10^{m_1} - (d_1 - d_2)\right) 10^{m_2+m_3} = \frac{9\beta^n}{2\sqrt{2}} - (d_2 - d_3)\, 10^{m_3} - d_3 \qquad (11)$$

and take absolute values of the equation (11). Then,

$$\begin{aligned}\left|\frac{9\alpha^n}{2\sqrt{2}}-\left(d_1 10^{m_1}-(d_1-d_2)\right)10^{m_2+m_3}\right| &\leq \frac{9|\beta|^n}{2\sqrt{2}}+|d_2-d_3|\cdot 10^{m_3}+d_3\\ &\leq \frac{9\alpha^{-n}}{2\sqrt{2}}+9\cdot 10^{m_3}+9\\ &\leq \frac{9\alpha^{-n}}{2\sqrt{2}}+(9\cdot 10^{m_3}+0.9\cdot 10^{m_3})\\ &< \frac{(0.9)\alpha^{-n}10^{m_3}}{2\sqrt{2}}+9.9\cdot 10^{m_3}\\ &< 9.91\cdot 10^{m_3}. \end{aligned} \tag{12}$$

If this inequality is divided by $(d_1 10^{m_1}-(d_1-d_2))\,10^{m_2+m_3}$, then

$$\left|1-\left(\frac{9}{2\sqrt{2}\,(d_1 10^{m_1}-(d_1-d_2))}\right)\alpha^n 10^{-m_2-m_3}\right| < \frac{1.11}{10^{m_2}}. \tag{13}$$

Let's take

$$\gamma_1 := \frac{9}{2\sqrt{2}\,(d_1 10^{m_1}-(d_1-d_2))},\ \gamma_2 := \alpha,\ \gamma_3 := 10$$

and $b_1 := 1, b_2 := n, b_3 := -m_2-m_3$ to apply Lemma 1. The numbers γ_1, γ_2, and γ_3 which are elements of the field $K=\mathbb{Q}(\sqrt{2})$ are positive real numbers and so $D=2$. Let

$$\Lambda_2 := 1-\left(\frac{9}{2\sqrt{2}\,(d_1 10^{m_1}-(d_1-d_2))}\right)\alpha^n 10^{-m_2-m_3}.$$

It can be easily seen that $\Lambda_2 \neq 0$. We have

$$\begin{aligned} h(\gamma_1) &= h\left(\frac{9}{2\sqrt{2}\,(d_1 10^{m_1}-(d_1-d_2))}\right)\\ &\leq h(9)+h(2\sqrt{2})+h(d_1 10^{m_1})+h\,(d_1-d_2)+\log 2\\ &< 3\log 9+\frac{1}{2}\log 8+m_1\log 10+\log 2\\ &< 8.33+m_1\cdot\log 10, \end{aligned}$$

by using the properties of the function h. So, we can take

$$A_1 := 16.66+2m_1\log 10,\ A_2 := 0.9,\ A_3 := 4.62,$$

also $B := n$. Because $m_2+m_3<n$ and $B \geq \max\{|1|, |-n|, |-m_2-m_3|\}$. By (13) and Lemma 1, it can be written that

$$1.11\cdot 10^{-m_2} > |\Lambda_2| > \exp\left(-4.04\cdot 10^{12}\cdot(1+\log n)\cdot(16.66+2m_1\log 10)\right),$$

i.e.,

$$m_2\log 10 < 4.04\cdot 10^{12}\cdot(1+\log n)\cdot(16.66+2m_1\log 10)+\log 1.11. \tag{14}$$

Now, we consider the equation (6) again. Then we can write

$$\frac{9\alpha^n}{2\sqrt{2}}-\left(d_1 10^{m_1+m_2}-(d_1-d_2)\,10^{m_2}-(d_2-d_3)\right)10^{m_3} = \frac{9\beta^n}{2\sqrt{2}}-d_3, \tag{15}$$

later, get its absolute values

$$\left|\frac{9\alpha^n}{2\sqrt{2}}-\left(d_1 10^{m_1+m_2}-(d_1-d_2)\,10^{m_2}-(d_2-d_3)\right)10^{m_3}\right| \leq \frac{9|\beta|^n}{2\sqrt{2}}+d_3$$

$$= \frac{9\alpha^{-n}}{2\sqrt{2}} + 9$$
$$< 9.1. \tag{16}$$

If this inequality is divided by $9\alpha^n/(2\sqrt{2})$, then

$$\left|1 - \left(\frac{2\sqrt{2}\left(d_1 10^{m_1+m_2} - (d_1 - d_2)\, 10^{m_2} - (d_2 - d_3)\right)}{9}\right)\alpha^{-n} 10^{m_3}\right| \le 2.86 \cdot \alpha^{-n}. \tag{17}$$

Let's take

$$\gamma_1 := \left(\frac{2\sqrt{2}\left(d_1 10^{m_1+m_2} - (d_1 - d_2)\, 10^{m_2} - (d_2 - d_3)\right)}{9}\right), \gamma_2 := \alpha, \gamma_3 := 10,$$

and $b_1 := 1, b_2 := -n, b_3 := m_3$ to apply Lemma 1. The numbers γ_1, γ_2, and γ_3 which are positive real numbers are elements of the field $K = \mathbb{Q}(\sqrt{2})$, also $D = 2$. Let

$$\Lambda_3 := 1 - \left(\frac{2\sqrt{2}\left(d_1 10^{m_1+m_2} - (d_1 - d_2)\, 10^{m_2} - (d_2 - d_3)\right)}{9}\right)\alpha^{-n} 10^{m_3}.$$

It can be easily seen that $\Lambda_3 \ne 0$. Also, we can find

$$\begin{aligned} h(\gamma_1) &= h\left(\frac{2\sqrt{2}\left(d_1 10^{m_1+m_2} - (d_1 - d_2)\, 10^{m_2} - (d_2 - d_3)\right)}{9}\right) \\ &\le h(9) + h(2\sqrt{2}) + h(d_1 10^{m_1+m_2}) + h\left((d_1 - d_2)\, 10^{m_2}\right) + h\left(d_2 - d_3\right) + 2\log 2 \\ &\le 4\log 9 + \frac{1}{2}\log 8 + (m_1 + m_2)\log 10 + m_2 \log 10 + 2\log 2 \\ &< 11.22 + m_1 \cdot \log 10 + 2 \cdot m_2 \cdot \log 10. \end{aligned}$$

So, we can choose $A_1 := 22.44 + 2m_1 \log 10 + 4m_2 \log 10,\ A_2 := 0.9,\ A_3 := 4.62$ and $B := n$. Because $m_3 < n$ and $B \ge \max\{|1|, |-n|, |m_3|\}$. By using (17) and Lemma 1, we have

$$2.86 \cdot \alpha^{-n} > |\Lambda_3| > \exp\left(-4.04 \cdot 10^{12} \cdot (1 + \log n)\,(22.44 + 2m_1 \log 10 + 4m_2 \log 10)\right),$$

i.e.,

$$n \log\alpha - \log(2.86) < 4.04 \cdot 10^{12} \cdot (1 + \log n)(22.44 + 2m_1 \log 10 + 4m_2 \log 10). \tag{18}$$

By using (10), (14), and (18), we find $n < 4.65 \cdot 10^{45}$ with MATLAB. Suppose

$$z_1 := (m_1 + m_2 + m_3) \cdot \log 10 - n \cdot \log\alpha - \log(9/(2\sqrt{2}d_1)).$$

We get

$$|\Lambda_1| = \left|e^{-z_1} - 1\right| < \frac{9.991}{10^{m_1}} < 0.9995, \text{ for } m_1 \ge 1$$

from (9). Then, if we choose $a := 0.9995$, then

$$|z_1| = |\log(\Lambda_1 + 1)| < \frac{\log 2000}{0.9995} \cdot \frac{9.991}{10^{m_1}} < \frac{75.98}{10^{m_1}},$$

by Lemma 3. Then we have

$$0 < \left|(m_1 + m_2 + m_3) \cdot \log 10 - n \cdot \log\alpha - \log\left(\frac{9}{2\sqrt{2}d_1}\right)\right| < \frac{75.98}{10^{m_1}},$$

i.e.,

$$0 < \left|(m_1+m_2+m_3)\cdot\frac{\log 10}{\log\alpha} - n - \frac{\log(9/(2\sqrt{2}d_1))}{\log\alpha}\right| < 86.21\cdot 10^{-m_1}, \tag{19}$$

dividing this inequality by $\log\alpha$. We can use Lemma 2. Assume that

$$\gamma := \frac{\log 10}{\log\alpha}\notin\mathbb{Q},\ \mu := -\frac{\log(9/(2\sqrt{2}d_1))}{\log\alpha},\ A := 86.21,\ B := 10,\ w := m_1$$

and $m_1+m_2+m_3 < M = 4.65\cdot 10^{45}$. Then, the denominator of the 99 th convergent of γ exceeds $6M$. Besides,

$$0.10911 < ||\mu q_{99}|| - M||\gamma q_{99}|| < 0.48819.$$

But, the inequality (19) has no solutions for

$$m_1 \geq 50.58 > \frac{\log\left(Aq_{99}/\epsilon\right)}{\log B}.$$

So, $m_1 \leq 50$. Then, by using (14) and (18), we find $n < 1.03\cdot 10^{32}$. Now, let

$$z_2 := (m_2+m_3)\log 10 - n\log\alpha - \log\left(\frac{9}{2\sqrt{2}\left(d_1 10^{m_1} - (d_1-d_2)\right)}\right).$$

We obtain

$$|\Lambda_2| = \left|e^{-z_2}-1\right| < (1.11)\cdot 10^{-m_2} < 0.2 \ \text{ for } m_2 \geq 1$$

from (13). If we choose $a := 0.2$, then

$$|z_2| = |\log(\Lambda_2+1)| < \frac{\log(5/4)}{0.2}\cdot\frac{1.11}{10^{m_2}} < \frac{1.24}{10^{m_2}},$$

by Lemma 3. From here, we can write

$$0 < \left|(m_2+m_3)\log 10 - n\log\alpha - \log\left(\frac{9}{2\sqrt{2}\left(d_1 10^{m_1} - (d_1-d_2)\right)}\right)\right| < 1.24\cdot 10^{-m_2}.$$

If this inequality is divided by $\log\alpha$, we obtain

$$0 < \left|\frac{(m_2+m_3)\log 10}{\log\alpha} - n - \frac{\log\left(9/(2\sqrt{2})(d_1 10^{m_1} - (d_1-d_2))\right)}{\log\alpha}\right| < 1.41\cdot 10^{-m_2}. \tag{20}$$

Let $\gamma := \frac{\log 10}{\log\alpha}$, $\mu := -\frac{\log\left(9/(2\sqrt{2})(d_1 10^{m_1}-(d_1-d_2))\right)}{\log\alpha}$ and $m_2+m_3 < M := 1.03\cdot 10^{32}$. Then it can bee seen that q_{79}, the denominator of the 79 th convergent of γ exceeds $6M$. Since $1\leq m_1\leq 50$, $d_1\neq d_2$, $1\leq d_1\leq 9$ and $0\leq d_2\leq 9$, we obtain

$$0.00018 < ||\mu q_{79}|| - M||\gamma q_{79}|| < 0.499694.$$

Moereover, there is no solution to the inequality (20) for

$$m_2 \geq 40.164 > \frac{\log(Aq_{79}/\epsilon)}{\log B},$$

for $A := 1.41$, $B := 10$, and $w := m_2$, by using Lemma 2. So, $m_2 \leq 40$. Since $m_1 \leq 50$ and $m_2 \leq 40$, by using (18), we have $n < 1.15\cdot 10^{17}$. Now, let

$$z_3 := m_3\log 10 - n\log\alpha + \log\left(\frac{2\sqrt{2}\left(d_1 10^{m_1+m_2} - (d_1-d_2)\,10^{m_2} - (d_2-d_3)\right)}{9}\right).$$

From (17), we can write

$$|\Lambda_3| = \left|e^{z_3} - 1\right| < 1.02 \cdot \alpha^{-n} < 0.01, \text{ for } n > 100.$$

If we choose $a := 0.01$, then we have

$$|z_3| = |\log(\Lambda_3 + 1)| < \frac{\log(100/99)}{0.01} \cdot \frac{2.86}{\alpha^n} < \frac{2.88}{\alpha^n},$$

by Lemma 3. So,

$$0 < \left| m_3 \log 10 - n \log\alpha + \log\left(\frac{2\sqrt{2}\left(d_1 10^{m_1+m_2} - (d_1 - d_2)\, 10^{m_2} - (d_2 - d_3)\right)}{9}\right)\right| < 2.88 \cdot \alpha^{-n}.$$

If this equation is divided by $\log\alpha$, then

$$0 < \left| m_3 \cdot \frac{\log 10}{\log\alpha} - n + \frac{\log\left(2\sqrt{2}\left(d_1 10^{m_1+m_2} - (d_1 - d_2)\, 10^{m_2} - (d_2 - d_3)\right)/9\right)}{\log\alpha}\right|$$
$$< 3.27 \cdot \alpha^{-n}. \qquad (21)$$

Put

$$\mu := \frac{\log\left(2\sqrt{2}\left(d_1 10^{m_1+m_2} - (d_1 - d_2)\, 10^{m_2} - (d_2 - d_3)\right)/9\right)}{\log\alpha},$$

$\gamma := \dfrac{\log 10}{\log\alpha}$ and $m_3 < M := 1.15 \cdot 10^{17}$. It can be seen that q_{51}, the denominator of the 51st convergent of γ exceeds $6M$ and also

$$7.5 \cdot 10^{-7} < ||\mu q_{51}|| - M||\gamma q_{51}|| < 0.499997$$

where $m_1 \le 50$, $m_2 \le 40$, $1 \le d_1 \le 9$ and $0 \le d_2, d_3 \le 9$, except for the case $d_1 = d_2 \ne d_3$, $d_1 \ne d_2 = d_3$ and $d_1 = d_2 = d_3$. Moreover, let $A := 3.27$, $B := \alpha$, and $w := n$ to apply Lemma 2. Then, we can say that the inequality (21) has no solution for

$$n \ge 76.55 > \frac{\log(Aq_{51}/\epsilon)}{\log B},$$

so, $n \le 76$. This is impossible since $n > 100$. □

Theorem 8 *The only Pell-Lucas numbers which are concatenations of three repdigits are* 198, 478, 1154.

Proof Suppose that the equation (4) is valid and

$$\alpha^n + \beta^n = Q_n = \overline{\underbrace{d_1 \ldots d_1}_{m_1 \text{ times}}\,\underbrace{d_2 \ldots d_2}_{m_2 \text{ times}}\,\underbrace{d_3 \ldots d_3}_{m_3 \text{ times}}}. \qquad (22)$$

Firstly, we check the first 139 Pell-Lucas numbers. Then we find $Q_n \in \{169, 408, 985\}$ which are all the solutions to the this equation. Suppose $n \ge 140$ and ignore $d_1 = d_2 \ne d_3$ and $d_1 \ne d_2 = d_3$ for this equation. Because it is given in Lemma 6. If $d_1 = d_2 = d_3$, this equation has no solution according to Lemma 5. Furthermore, the identity

$$Q_n \equiv 2, 4, 6, 8 (\bmod 10)$$

is well known for $n \ge 0$. Thus, we take $d_3 = 2, 4, 6, 8$.

From (22), we obtain

$$Q_n = \overline{\underbrace{d_1 \ldots d_1}_{m_1 \text{ times}}\underbrace{d_2 \ldots d_2}_{m_2 \text{ times}}\underbrace{d_3 \ldots d_3}_{m_3 \text{ times}}}$$

$$= \underbrace{d_1 \ldots d_1}_{m_1 \text{ times}} \times 10^{m_2+m_3} + \underbrace{d_2 \ldots d_2}_{m_2 \text{ times}} \times 10^{m_3} + \underbrace{d_3 \ldots d_3}_{m_3 \text{ times}},$$

we get

$$Q_n = \frac{d_1(10^{m_1}-1)}{9}10^{m_2+m_3} + \frac{d_2(10^{m_2}-1)}{9}10^{m_3} + \frac{d_3(10^{m_3}-1)}{9}, \tag{23}$$

$$Q_n = \frac{1}{9}\left(d_1 10^{m_1+m_2+m_3} - (d_1-d_2)\,10^{m_2+m_3} - (d_2-d_3)\,10^{m_3} - d_3\right). \tag{24}$$

Then we can write

$$9\alpha^n - d_1 10^{m_1+m_2+m_3} = -9\beta^n - (d_1-d_2)\,10^{m_2+m_3} - (d_2-d_3)\,10^{m_3} - d_3. \tag{25}$$

If we divide by $d_1 10^{m_1+m_2+m_3}$ after take absolute values of this equation, then

$$\left|\frac{9}{d_1}\alpha^n 10^{-m_1-m_2-m_3} - 1\right| < \frac{9.992}{10^{m_1}}. \tag{26}$$

Now, let us apply Lemma 1 with

$$\Lambda_1 := \frac{9}{d_1}\alpha^n 10^{-m_1-m_2-m_3} - 1,$$
$$\gamma_1 := \frac{9}{d_1}, \gamma_2 := \alpha, \gamma_3 := 10,$$

and

$$b_1 := 1, b_2 := n, b_3 := -m_1 - m_2 - m_3.$$

It can be shown that $\Lambda_1 \neq 0$, $D = 2$,

$$h(\gamma_1) = h\left(\frac{9}{d_1}\right) \leq h\left(\frac{9}{d_1}\right) \leq \log 9 < 2.2,$$
$$h(\gamma_2) = h(\alpha) = \frac{\log\alpha}{2} < 0.45,$$

and

$$h(\gamma_3) = h(10) = \log 10 < 2.31.$$

So, we can choose $A_1 := 4.4$, $A_2 := 0.9$, and $A_3 := 4.62$. In addition, from (2) and (23), it can be written that

$$10^{m_1+m_2+m_3-1} < Q_n \leq 2\alpha^n < 10^{n+1}.$$

So, $m_1 + m_2 + m_3 < n + 2$. Then, we can take $B := n + 2$. According to (26) and Lemma 1,

$$(9.992) \cdot 10^{-m_1} > |\Lambda_1| > \exp\left(1.78 \cdot 10^{13} \cdot (1 + \log(n+2)) \cdot 4.4\right),$$

i.e.,

$$m_1 \log 10 < 1.78 \cdot 10^{13} \cdot (1 + \log(n+2)) + \log(9.992). \tag{27}$$

If we rearrange the equation (24) as

$$9\alpha^n - \left(d_1 10^{m_1} - (d_1-d_2)\right)10^{m_2+m_3} = -9\beta^n - (d_2-d_3)\,10^{m_3} - d_3 \tag{28}$$

and divide by $(d_1 10^{m_1} - (d_1-d_2))\,10^{m_2+m_3}$ after take absolute values of this equation, then

$$\left|1 - \left(\frac{9}{(d_1 10^{m_1} - (d_1-d_2))}\right)\alpha^n 10^{-m_2-m_3}\right| < \frac{1.11}{10^{m_2}}. \tag{29}$$

Let

$$\gamma_1 := \frac{9}{(d_1 10^{m_1} - (d_1 - d_2))}, \gamma_2 := \alpha, \gamma_3 := 10,$$
$$\Lambda_2 := 1 - \left(\frac{9}{(d_1 10^{m_1} - (d_1 - d_2))}\right) \alpha^n 10^{-m_2 - m_3},$$

and

$$b_1 := 1, b_2 := n, b_3 := -m_2 - m_3.$$

Then, we can choose $B := n + 1$ since $B \geq \max\{|1|, |-n|, |-m_2 - m_3|\}$ and $m_2 + m_3 < n + 1$ to apply Lemma 1. It can be shown that $\Lambda_2 \neq 0$, $D = 2$,

$$h(\gamma_1) = h\left(\frac{9}{(d_1 10^{m_1} - (d_1 - d_2))}\right)$$
$$< 7.3 + m_1 \log 10.$$

So, suppose $A_1 := 14.6 + 2m_1 \log 10$, $A_2 := 0.9$, and $A_3 := 4.62$. From (29) and Lemma 1,

$$1.11 \cdot 10^{-m_2} > |\Lambda_2| > \exp\left(4.04 \cdot 10^{12} \cdot (1 + \log(n + 1))\,(14.6 + 2m_1 \log 10)\right),$$

i.e.,

$$m_2 \log 10 < 4.04 \cdot 10^{12} \cdot (1 + \log(n + 1))\,(14.6 + 2m_1 \log 10) + \log(1.11). \tag{30}$$

We can rewrite the equation (24) as

$$9\alpha^n - \left(d_1 10^{m_1 + m_2} - (d_1 - d_2)\,10^{m_2} - (d_2 - d_3)\right) \cdot 10^{m_3} = -9\beta^n - d_3. \tag{31}$$

If we divide by $9\alpha^n$ after take absolute values of this equation, then

$$\left|1 - \left(\frac{\left(d_1 10^{m_1 + m_2} - (d_1 - d_2)\,10^{m_2} - (d_2 - d_3)\right)}{9}\right) \alpha^{-n} 10^{m_3}\right| \leq 1.02 \cdot \alpha^{-n}. \tag{32}$$

Put

$$\gamma_1 := \left(\frac{\left(d_1 10^{m_1 + m_2} - (d_1 - d_2)\,10^{m_2} - (d_2 - d_3)\right)}{9}\right), \gamma_2 := \alpha, \gamma_3 := 10,$$
$$\Lambda_3 := 1 - \left(\frac{\left(d_1 10^{m_1 + m_2} - (d_1 - d_2)\,10^{m_2} - (d_2 - d_3)\right)}{9}\right) \alpha^{-n} 10^{m_3},$$

and

$$b_1 := 1, b_2 := -n, b_3 := m_3$$

to apply Lemma 1. The numbers γ_1, γ_2, and γ_3 which are elements of the field $K = \mathbb{Q}(\sqrt{2})$ are positive real numbers. Similar to what has been done before, it can be seen that $\Lambda_3 \neq 0$, $D = 2$,

$$h(\gamma_1) = h\left(\frac{\left(d_1 10^{m_1 + m_2} - (d_1 - d_2)\,10^{m_2} - (d_2 - d_3)\right)}{9}\right)$$
$$< 10.18 + m_1 \log 10 + 2m_2 \log 10.$$

So, $A_1 := 20.36 + 2m_1 \log 10 + 4m_2 \log 10$, $A_2 := 0.9$, and $A_3 := 4.62$. As $m_3 < n$ and $B \geq \max\{|1|, |-n|, |m_3|\}$, we can choose $B := n$. Now, by using (17) and Lemma 1,

$$1.02 \cdot \alpha^{-n} > |\Lambda_3| > \exp\left(C \cdot (1 + \log n)\,(20.36 + 2m_1 \log 10 + 4m_2 \log 10)\right),$$

i.e.,

$$n \cdot \log \alpha - \log(1.02) < 4.04 \cdot 10^{12} \cdot (1 + \log n)(20.36 + 2m_1 \log 10 + 4m_2 \log 10). \tag{33}$$

We find $n < 3.12 \cdot 10^{45}$ by using the inequalities (27), (30) and (33).

In this step, we will use Lemma 2 to reduce the upper bound on n. First, let

$$z_1 := (m_1 + m_2 + m_3) \log 10 - n \log \alpha - \log\left(\frac{9}{d_1}\right).$$

It is seen that

$$|\Lambda_1| = \left|e^{-z_1} - 1\right| < \frac{9.992}{10^{m_1}} < 0.9995 \text{ for } m_1 \geq 1$$

from (26). Choosing $a := 0.9995$, by Lemma 3, we have the inequality

$$|z_1| = |\log(\Lambda_1 + 1)| < \frac{\log 2000}{0.9995} \cdot \frac{9.992}{10^{m_1}} < \frac{75.99}{10^{m_1}},$$

i.e.,

$$0 < \left|(m_1 + m_2 + m_3) \log 10 - n \log \alpha - \log\left(\frac{9}{d_1}\right)\right| < \frac{75.99}{10^{m_1}}.$$

From here, we can write

$$0 < \left|(m_1 + m_2 + m_3)\frac{\log 10}{\log \alpha} - n - \frac{\log(9/d_1)}{\log \alpha}\right| < 86.22 \cdot 10^{-m_1}. \tag{34}$$

Now, we can apply Lemma 2. Let

$$\gamma := \frac{\log 10}{\log \alpha} \notin \mathbb{Q},\ M := 3.12 \cdot 10^{45},\ \mu := -\frac{\log(9/d_1)}{\log \alpha},\ A := 86.22,\ B := 10,\ w := m_1.$$

We find q_{90}, denominator of the 90 th convergent of γ, exceeds $6M$ and

$$0.21675 < \epsilon := ||\mu q_{90}|| - M||\gamma q_{90}|| < 0.38018$$

except for the case $d_1 = 9$. So, the inequality (34) has no solutions for

$$\frac{\log(Aq_{90}/\epsilon)}{\log B} < 51.28 \leq m_1.$$

Thus $m_1 \leq 51$. If $d_1 = 9$, the from (34), we have

$$0 < \left|(m_1 + m_2 + m_3)\frac{\log 10}{\log \alpha} - n\right| < 86.22 \cdot 10^{-m_1},$$

i.e.,

$$0 < \left|\frac{\log 10}{\log \alpha} - \frac{n}{m_1 + m_2 + m_3}\right| < \frac{86.22}{(m_1 + m_2 + m_3) \cdot 10^{m_1}}. \tag{35}$$

Assume that $m_1 \geq 52$. Then it can be seen that

$$\frac{10^{m_1}}{172.44} > 5.8 \cdot 10^{49} > n + 2 > m_1 + m_2 + m_3,$$

and so we have

$$\left|\frac{\log 10}{\log \alpha} - \frac{n}{m_1 + m_2 + m_3}\right| < \frac{86.22}{10^{m_1} \cdot (m_1 + m_2 + m_3)} < \frac{1}{2 \cdot (m_1 + m_2 + m_3)^2}.$$

It is seen that the rational number $\frac{n}{m_1+m_2+m_3}$ is a convergent to $\gamma = \frac{\log 10}{\log \alpha}$ by using the known properties of continued fraction. Suppose $\frac{p_r}{q_r}$ is r-th convergent of the continued fraction of γ and

$$\frac{n}{m_1+m_2+m_3} = \frac{p_t}{q_t} \text{ for some } t.$$

Then it can be seen that $q_{98} > 12 \cdot 10^{45} > n+2 > m_1+m_2+m_3$. Thus $t \in \{0, 1, 2, ..., 97\}$ and $a_M = \max\{a_i | i = 0, 1, ..., 97\} = 52$. It is seen that

$$\left|\gamma - \frac{p_t}{q_t}\right| > \frac{1}{(a_M+2)(m_1+m_2+m_3)^2} = \frac{1}{54 \cdot (m_1+m_2+m_3)^2}$$

by Lemma 4 and

$$\frac{86.22}{(m_1+m_2+m_3) \cdot 10^{m_1}} > \frac{1}{54 \cdot (m_1+m_2+m_3)^2}$$

from (35). From here,

$$\frac{8.622}{10^{51}} \geq \frac{86.22}{10^{m_1}} > \frac{1}{54 \cdot (m_1+m_2+m_3)} > \frac{1}{6.48 \cdot 10^{47}}.$$

This is impossible. So, $m_1 \leq 51$. In this case, it can be seen that $n < 1.04 \cdot 10^{32}$ by using (30) and (33). Now, suppose

$$z_2 := (m_2+m_3)\log 10 - n\log\alpha - \log\left(\frac{9}{(d_1 10^{m_1} - (d_1-d_2))}\right).$$

It is seen that

$$|\Lambda_2| = \left|e^{-z_2} - 1\right| < 1.11 \cdot 10^{-m_2} < 0.2 := a \text{ for } m_2 \geq 1$$

from (29) and

$$|z_2| = |\log(\Lambda_2+1)| < \frac{\log(10/8)}{0.2} \cdot \frac{1.11}{10^{m_2}} < \frac{1.24}{10^{m_2}}$$

by Lemma 3. Then,

$$0 < \left|(m_2+m_3)\log 10 - n\log\alpha - \log\left(\frac{9}{(d_1 10^{m_1} - (d_1-d_2))}\right)\right| < 1.24 \cdot 10^{-m_2}.$$

Dividing of the this inequality by $\log\alpha$, we find

$$0 < \left|(m_2+m_3)\frac{\log 10}{\log\alpha} - n + \frac{\log\left(9/(d_1 10^{m_1} - (d_1-d_2))\right)}{\log\alpha}\right| < 1.41 \cdot 10^{-m_2}. \tag{36}$$

Taking $m_2+m_3 < M := 1.04 \cdot 10^{32}$ and $\gamma := \frac{\log 10}{\log\alpha}$, then we find that q_{90}, the denominator of the 90 th convergent of γ exceeds $6M$. Now, taking

$$\mu := \frac{\log\left(9/(d_1 10^{m_1} - (d_1-d_2))\right)}{\log\alpha}$$

and considering the fact that $1 \leq m_1 \leq 51$, $d_1 \neq d_2$, $1 \leq d_1 \leq 9$ and $0 \leq d_2 \leq 9$, it can be shown that

$$3.08 \cdot 10^{-11} < \epsilon := ||\mu q_{90}|| - M||\gamma q_{90}|| < 0.4999$$

except for the case $(d_1, d_2, m_1) = (1, 0, 1)$. In Lemma 2, suppose $A := 1.41$, $B := 10$, and $w := m_2$. It can be shown that the inequality (36) has no solution for

$$\frac{\log(Aq_{90}/\epsilon)}{\log B} < 52.61 \leq m_2.$$

If $(d_1, d_2, m_1) = (1, 0, 1)$, then from (36), we have

$$0 < \left| (m_2 + m_3) \frac{\log 10}{\log \alpha} - n \right| < 1.41 \cdot 10^{-m_2}.$$

If this inequality is divided by $(m_2 + m_3)$, then

$$0 < \left| \frac{\log 10}{\log \alpha} - \frac{n}{m_2 + m_3} \right| < \frac{1.41}{(m_2 + m_3) \cdot 10^{m_2}}. \tag{37}$$

Suppose $m_2 \geq 53$. Then,

$$\frac{10^{m_2}}{2.82} > 3.54 \cdot 10^{52} > n + 2 > m_2 + m_3,$$

and so,

$$\left| \frac{\log 10}{\log \alpha} - \frac{n}{m_2 + m_3} \right| < \frac{1.41}{(m_2 + m_3) \cdot 10^{m_2}} < \frac{1}{2 \cdot (m_2 + m_3)^2}.$$

The rational number $\frac{n}{m_2+m_3}$ is a convergent to $\gamma = \frac{\log 10}{\log \alpha}$. Suppose $\frac{n}{m_2+m_3} = \frac{p_t}{q_t}$ for some t and $\frac{p_t}{q_t}$ is t-th convergent of the continued fraction of γ. Then,

$$q_{70} > 19 \cdot 10^{31} > n + 1 > m_2 + m_3.$$

Thence, $t \in \{0, 1, 2, ..., 69\}$, $a_M = \max\{a_i | i = 0, 1, 2, ..., 69\} = 52$ and

$$\left| \gamma - \frac{p_t}{q_t} \right| > \frac{1}{(a_M + 2)(m_2 + m_3)^2} = \frac{1}{54 \cdot (m_2 + m_3)^2}$$

by Lemma 4, Also,

$$\frac{1.41}{(m_2 + m_3) \cdot 10^{m_2}} > \frac{1}{54 \cdot (m_2 + m_3)^2}$$

from (37). Then we find

$$\frac{1.41}{10^{52}} \geq \frac{1.41}{10^{m_2}} > \frac{1}{54 \cdot (m_2 + m_3)} > \frac{1}{(1.92) \cdot 10^{53}}.$$

This is impossible. So, $m_2 \leq 52$. If m_1 and m_2 are substituted in (33), then $n < 1.37 \cdot 10^{17}$ is found. Now, let

$$z_3 := m_3 \log 10 - n \log \alpha + \log \left(\frac{\left(d_1 10^{m_1+m_2} - (d_1 - d_2) 10^{m_2} - (d_2 - d_3) \right)}{9} \right).$$

It can be written

$$|\Lambda_3| = \left| e^{z_3} - 1 \right| < 1.02 \cdot \alpha^{-n} < 0.01 := a \text{ for } n \geq 140$$

from (32) and

$$|z_3| = |\log(\Lambda_3 + 1)| < \frac{\log(100/99)}{0.01} \cdot \frac{1.02}{\alpha^n} < \frac{1.03}{\alpha^n}$$

by Lemma 3 and

$$0 < \left| m_3 \frac{\log 10}{\log \alpha} - n + \frac{\log \left(\left(d_1 10^{m_1+m_2} - (d_1 - d_2) 10^{m_2} - (d_2 - d_3) \right) / 9 \right)}{\log \alpha} \right| < 1.16 \cdot \alpha^{-n}. \tag{38}$$

Let's take $M := 1.37 \cdot 10^{17}$, $\gamma := \dfrac{\log 10}{\log \alpha}$ and

$$\mu := \frac{\log\left(\left(d_1 10^{m_1+m_2} - (d_1 - d_2)\, 10^{m_2} - (d_2 - d_3)\right)/9\right)}{\log \alpha}.$$

It can be shown that q_{78} exceeds $6M$ and

$$3.3 \cdot 10^{-18} < \epsilon < 0.499997$$

for $1 \le m_1 \le 51$, $1 \le m_2 \le 52$, $1 \le d_1 \le 9$, $0 \le d_2 \le 9$, and $d_3 = 2, 4, 6, 8$ except for the case $d_1 = d_2 \ne d_3$, $d_1 \ne d_2 = d_3$ and $d_1 = d_2 = d_3$. Then suppose $A := 1.16$, $B := \alpha$, and $w := n$ to apply Lemma 2. It can be seen that the inequality (38) has no solution for

$$\frac{\log(Aq_{78}/\epsilon)}{\log B} < 137.64 \le n,$$

i.e., $n \le 137$. This is impossible since $n \ge 140$. This completes the proof. $\square$

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