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ORIGINAL RESEARCH

Fractal generation via generalized Fibonacci–Mann iteration with s-convexity

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Abstract Recently, the generalized Fibonacci–Mann iteration scheme has been defined and used to develop an escape criterion to study mutants of the classical fractals for a function $\sin(z^n) + az + c$, $a, c \in \mathbb{C}$, $n \geq 2$, and z is a complex variable. In the current work, we use generalized Fibonacci–Mann iteration extended further via the notion of s-convex combination in the exploration of new mutants of celebrated Mandelbrot and Julia sets. Further, we provide a few graphical and numerical examples obtained by the use of the derived criteria.

Keywords Escape criterion · Fixed point · Generalized Fibonacci–Mann orbit · Generalized Fibonacci number

1 Introduction

The notion of s-convex combination and various iteration schemes have been extensively used in many recent studies (see, for example, [2, 3, 11, 18, 20, 22, 26, 27] and the references therein). In the current work, we define the notion of generalized Fibonacci–Mann iteration extended further with s-convexity and apply it in the exploration of mutants of celebrated Mandelbrot and Julia sets. We substitute the convex combination in the generalized Fibonacci–Mann iteration with the s-convex one. Recall that the s-convex combination is one of the generalizations of the complex convex combination. There are several papers using the notion of s-convexity in the well-known iteration processes for the exploration of complex fractals. For example, in [3] Jungck-Ishikawa iteration extended with s-convexity, in [26] the Picard–Mann iteration extended with s-convexity, in [27] Jungck-Mann iteration extended with s-convexity, and so on were used in the exploration of the mutants of celebrated Mandelbrot and Julia sets.

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First, we briefly familiarize with the notions utilized in the next section.

Definition 1 (s-convex combination [25]) Let $z_1, z_2, \ldots, z_n \in \mathbb{C}$ and $s \in (0, 1]$. The s-convex combination is described as

$$\lambda_1^s z_1 + \lambda_2^s z_2 + \ldots + \lambda_n^s z_n,$$

where $\lambda_j \geq 0$ for $j \in \{1, 2, \ldots, n\}$ and $\sum_{j=1}^{n} \lambda_j = 1$.

Let k and l be nonzero real numbers. The generalized Fibonacci sequence $\{u(n)\}$ is defined by the recursive relation

$$u(n+1) = ku(n) + lu(n-1), n \geq 1, \tag{1}$$

with the initial conditions $u(0) = u(1) = 1$. As special cases, if we set $k = l = 1$, we obtain the classical Fibonacci sequence $\{u(n)\}$, and if we fix $l = 1$, we have the one-parameter generalization of the Fibonacci sequence known as the k-Fibonacci sequence (for more details about these sequences see, for instance, [4] and [13]). The general expression of the first eleven generalized Fibonacci numbers is presented in Table 1.

Our paper is organized as follows. In Sect. 2, we present an escape criterion for a function of the form $\sin(z^n) + az + c$. In Sect. 3, we furnish a few examples of Julia and Mandelbrot sets explored utilizing the introduced iteration, developed algorithm, established escape criterion, colormap, and MATLAB software some of which are similar to beautiful natural objects. Then, in Sect. 4, we analyze the dependency between the iteration's parameters and two numerical measures (average number of iterations and generation time). Finally, in Sect. 5, we give some concluding remarks.

2 Application of generalized Fibonacci–Mann iteration extended with s-convexity in establishing escape criterion

Suppose that

$$\left|1 - \frac{z^{2n}}{3!} + \frac{z^{4n}}{5!} - \ldots\right| \geq |\omega_1|,$$

where $|\omega_1| \in (0, 1]$, except for those values of z for which $|\omega_1| = 0$. Then, we have

$$\left|\sin\left(z^n\right)\right| = \left|z^n - \frac{z^{3n}}{3!} + \frac{z^{5n}}{5!} - \ldots\right| = \left|z^n\right| \left|1 - \frac{z^{2n}}{3!} + \frac{z^{4n}}{5!} - \ldots\right|$$

and so

$$\left|\sin\left(z^n\right)\right| \geq \left|z^n\right| |\omega_1|, \tag{2}$$

where $z \in \mathbb{C}$ except the values of z for which $|\omega_1| = 0$, $|\omega_1| \in (0, 1]$.

A new iteration scheme was defined as follows in [28].

Table 1 The list of the first eleven generalized Fibonacci numbers

n	$u(n)$
0	1
1	1
2	$k + l$
3	$k^2 + kl + l$
4	$k^3 + k^2l + 2kl + l^2$
5	$k^4 + k^3l + 3k^2l + 2kl^2 + l^2$
6	$k^5 + k^4l + 4k^3l + 3k^2l^2 + 3kl^2 + l^3$
7	$k^6 + k^5l + 5k^4l + 4k^3l^2 + 6k^2l^2 + 3kl^3 + l^3$
8	$k^7 + k^6l + 6k^5l + 5k^4l^2 + 10k^3l^2 + 6k^2l^3 + 4kl^3 + l^4$
9	$k^8 + k^7l + 7k^6l + 6k^5l^2 + 15k^4l^2 + 10k^3l^3 + 10k^2l^3 + 4kl^4 + l^4$
10	$k^9 + k^8l + 8k^7l + 7k^6l^2 + 21k^5l^2 + 15k^4l^3 + 20k^3l^3 + 10k^2l^4 + 5kl^4 + l^5$
11	$k^{10} + k^9l + 9k^8l + 8k^7l^2 + 28k^6l^2 + 21k^5l^3 + 35k^4l^3 + 20k^3l^4 + 15k^2l^4 + 5kl^5 + l^5$

Definition 2 (Generalized Fibonacci–Mann Iteration [28]) Let $T:\mathbb{C}\to\mathbb{C}$ be a complex-valued map, $z_0\in\mathbb{C}$ and $t_j\in[0,1]$, $j\in\mathbb{N}$. Suppose $\mathrm{t}=\inf\left\{t_j\right\}>0$.

We define the generalized Fibonacci–Mann iteration as follows:

$$z_{j+1}=t_j T^{u(j)}\left(z_j\right)+\left(1-t_j\right)z_j, \tag{3}$$

where $u(j)$ is the generalized Fibonacci number.

For the case $k=l=1$, we get the definition of the Fibonacci–Mann iteration [1]. An escape criterion for a function

$$T_{a,c}(z)=\sin\left(z^n\right)+az+c, \tag{4}$$

where $n\geq 2$, $a,c\in\mathbb{C}$ and z is a complex variable, was obtained by applying the Fibonacci–Mann iteration process to obtain fascinating Julia fractals in [23]. Also, an escape criterion was presented for a function of the form (4) via the generalized Fibonacci–Mann iteration in [28].

Now, we define and use the generalized Fibonacci–Mann iteration process extended with s-convexity to prove an escape criterion of $T_{a,c}(z)$ defined in (4).

Definition 3 Let $T:\mathbb{C}\to\mathbb{C}$ be a complex-valued map, $z_0\in\mathbb{C}$ and $t_j\in[0,1]$, $s\in(0,1]$, $j\in\mathbb{N}$. Suppose $\mathrm{t}=\inf\left\{t_j\right\}>0$. We define the generalized Fibonacci–Mann iteration with s-convexity as follows:

$$z_{j+1}=t_j^s T^{u(j)}\left(z_j\right)+\left(1-t_j\right)^s z_j, \tag{5}$$

where $u(j)$ is the generalized Fibonacci number.

If we set $k=l=1$, we have the definition of the Fibonacci–Mann iteration with s-convexity.

Theorem 1 *Let $T_{a,c}(z)=\sin\left(z^n\right)+az+c$, where $a,c\in\mathbb{C}$, $n\geq 2$. Let the sequence $\left\{z_j\right\}_{j\in\mathbb{N}}$ be the generalized Fibonacci–Mann iteration extended with s-convexity. Suppose*

$$|z|\geq|c|>\left(\frac{2\left(\mathfrak{T}|a|+1\right)}{s\mathrm{t}\left|\omega_1\right|}\right)^{\frac{1}{n-1}}. \tag{6}$$

Then, we have $\left|z_j\right|\to\infty$ as $j\to\infty$.

Proof Let $T_{a,c}(z)=\sin\left(z^n\right)+az+c$, $z_0=z$ and

$$z_{j+1}=t_j^s T^{u(j)}\left(z_j\right)+\left(1-t_j\right)^s z_j.$$

For $j=0$, we have $u(0)=1$. Expanding $(1-t_0)^s$ using binomial theorem up to the linear terms of t_0, we get $(1-t_0)^s\leq 1-st_0$. Also, since $\mathrm{t}=\inf\left\{t_j\right\}>0$ we have $t_0\in(0,1]$ and hence, $t_0^s\geq st_0$. Using these facts and considering the assumption $|z|\geq|c|$, by inequality (2) we get

$$\begin{aligned}
|z_1| &= \left|t_0^s T_{a,c}^{u(0)}(z)+(1-t_0)^s z\right| = \left|t_0^s\left[\sin\left(z^n\right)+az+c\right]+(1-t_0)^s z\right|\\
&\geq \left|st_0\left[\sin\left(z^n\right)+az+c\right]+(1-t_0)^s z\right|\\
&\geq st_0\left|\sin\left(z^n\right)\right|-\left|(1-t_0)^s z\right|-st_0|az|-st_0|c|\\
&\geq st_0\left|\sin\left(z^n\right)\right|-\left|(1-t_0)^s\right||z|-st_0|a||z|-st_0|c|\\
&\geq st_0\left|\sin\left(z^n\right)\right|-(1-st_0)|z|-st_0|a||z|-st_0|z|\\
&\geq st_0|\omega_1|\left|z^n\right|-(1-st_0)|z|-st_0|a||z|-st_0|z|\\
&= st_0|\omega_1|\left|z^n\right|-|z|\left((1-st_0)+st_0|a|+st_0\right)\\
&= st_0|\omega_1|\left|z^n\right|-|z|\left(1+st_0|a|\right)\\
&\geq s\mathrm{t}|\omega_1|\left|z^n\right|-|z|\left(1+s\mathfrak{T}|a|\right)\geq s\mathrm{t}|\omega_1|\left|z^n\right|-|z|\left(\mathfrak{T}|a|+1\right)\\
&\geq |z|\left(\mathfrak{T}|a|+1\right)\left(\frac{s\mathrm{t}|\omega_1|\left|z^{n-1}\right|}{\mathfrak{T}|a|+1}-1\right).
\end{aligned}$$

So, we obtain

$$|z_1| \geq \frac{|z_1|}{\mathfrak{T}|a|+1} \geq |z| \left(\frac{s\mathfrak{t}|\omega_1| \left|z^{n-1}\right|}{\mathfrak{T}|a|+1} - 1 \right). \tag{7}$$

For $j = 1$, we have $u(1) = 1$. Using similar arguments and by inequality (7), we obtain

$$\begin{aligned} |z_2| &\geq |z_1| \left(\frac{s\mathfrak{t}|\omega_1| \left|z_1^{n-1}\right|}{\mathfrak{T}|a|+1} - 1 \right) \\ &\geq |z| \left(\frac{s\mathfrak{t}|\omega_1| \left|z^{n-1}\right|}{\mathfrak{T}|a|+1} - 1 \right) \left(\frac{s\mathfrak{t}|\omega_1| \left|z_1^{n-1}\right|}{\mathfrak{T}|a|+1} - 1 \right) \\ &\geq |z| \left(\frac{s\mathfrak{t}|\omega_1| \left|z^{n-1}\right|}{\mathfrak{T}|a|+1} - 1 \right)^2 \end{aligned} \tag{8}$$

and, so

$$|z_2| \geq |z| \left(\frac{s\mathfrak{t}|\omega_1| \left|z^{n-1}\right|}{\mathfrak{T}|a|+1} - 1 \right)^2. \tag{9}$$

Because of inequality (7) and the fact that $|z| \geq |c| > \left(\frac{2(\mathfrak{T}|a|+1)}{s\mathfrak{t}|\omega_1|}\right)^{\frac{1}{n-1}}$, we have $|z_1| \geq |z|$, and this implies

$$|z_1| \left(\frac{s\mathfrak{t}|\omega_1| \left|z_1^{n-1}\right|}{\mathfrak{T}|a|+1} - 1 \right) \geq |z| \left(\frac{s\mathfrak{t}|\omega_1| \left|z^{n-1}\right|}{\mathfrak{T}|a|+1} - 1 \right). \tag{10}$$

Again, using the inequalities $|z_1| \geq |z| \geq |c| > \left(\frac{2(\mathfrak{T}|a|+1)}{s\mathfrak{t}|\omega_1|}\right)^{\frac{1}{n-1}}$ and (8), we find $|z_2| \geq |z_1|$.

Let $j = 2$ and set $\alpha_1 = T_{a,c}(z_2)$, $\alpha_2 = T_{a,c}^2(z_2), \ldots, \alpha_{u(2)-1} = T_{a,c}^{u(2)-1}(z_2)$. By inequality (6), it is easy to see that

$$\left|z_2^{n-1}\right| |s\omega_1| \geq |a| + 2. \tag{11}$$

Using inequalities (11) and (2), we obtain

$$\begin{aligned} \frac{|\alpha_1|}{|z_2|} &= \frac{\left|\sin\left(z_2^n\right) + az_2 + c\right|}{|z_2|} \geq \frac{\left|\sin\left(z_2^n\right)\right| - |a||z_2| - |c|}{|z_2|} \\ &\geq \frac{\left|z_2^n\right||\omega_1| - |a||z_2| - |z_2|}{|z_2|} = \left|z_2^{n-1}\right||\omega_1| - |a| - 1 \\ &\geq 1 \end{aligned}$$

and hence

$$|\alpha_1| \geq |z_2|.$$

Using similar arguments, it is easy to see that $\left|\alpha_{u(2)-1}\right| \geq |z_2|$. Then, we get

$$\begin{aligned} |z_3| &= \left|t_2^s T_{a,c}^{u(2)}(z_2) + (1-t_2)^s z_2\right| = \left|t_2^s \left[\sin\left(\alpha_{u(2)-1}^n\right) + a\alpha_{u(2)-1} + c\right] + (1-t_2)^s z_2\right| \\ &\geq \left|st_2 \left[\sin\left(\alpha_{u(2)-1}^n\right) + a\alpha_{u(2)-1} + c\right] + (1-t_2)^s z_2\right| \\ &\geq st_2 \left|\sin\left(\alpha_{u(2)-1}^n\right)\right| - \left|(1-t_2)^s z_2\right| - st_2\left|a\alpha_{u(2)-1}\right| - st_2|c| \end{aligned}$$

$$
\begin{aligned}
&\geq st_2 \left|\sin\left(\alpha_{u(2)-1}^n\right)\right| - \left|(1-t_2)^s\right| |z_2| - st_2 |a| \left|\alpha_{u(2)-1}\right| - st_2 |c| \\
&\geq st_2 \left|\sin\left(\alpha_{u(2)-1}^n\right)\right| - (1-st_2) |z_2| - st_2 |a| \left|\alpha_{u(2)-1}\right| - st_2 |c| \\
&\geq st_2 |\omega_1| \left|\alpha_{u(2)-1}^n\right| - (1-st_2) \left|\alpha_{u(2)-1}\right| - st_2 |a| \left|\alpha_{u(2)-1}\right| - st_2 \left|\alpha_{u(2)-1}\right| \\
&= st_2 |\omega_1| \left|\alpha_{u(2)-1}^n\right| - \left|\alpha_{u(2)-1}\right| ((1-st_2) + st_2 |a| + st_2) \\
&= st_2 |\omega_1| \left|\alpha_{u(2)-1}^n\right| - \left|\alpha_{u(2)-1}\right| (1 + st_2 |a|) \\
&\geq s\mathfrak{t} |\omega_1| \left|\alpha_{u(2)-1}^n\right| - \left|\alpha_{u(2)-1}\right| (1 + \mathfrak{T} |a|) \\
&\geq \left|\alpha_{u(2)-1}\right| (\mathfrak{T} |a| + 1) \left(\frac{s\mathfrak{t} |\omega_1| \left|\alpha_{u(2)-1}^{n-1}\right|}{(\mathfrak{T} |a| + 1)} - 1\right)
\end{aligned}
$$

and hence,

$$
|z_3| \geq \frac{|z_3|}{\mathfrak{T} |a| + 1} \geq \left|\alpha_{u(2)-1}\right| \left(\frac{s\mathfrak{t} |\omega_1| \left|\alpha_{u(2)-1}^{n-1}\right|}{\mathfrak{T} |a| + 1} - 1\right). \tag{12}
$$

Similarly, by inequalities (9) and (12), we get

$$
|z_3| \geq |z| \left(\frac{s\mathfrak{t} |\omega_1| \left|z^{n-1}\right|}{\mathfrak{T} |a| + 1} - 1\right)^3. \tag{13}
$$

Repeating this till mth term, we find

$$
|z_m| \geq |z| \left(\frac{s\mathfrak{t} |\omega_1| \left|z^{n-1}\right|}{\mathfrak{T} |a| + 1} - 1\right)^m. \tag{14}
$$

Then, because of inequality (6), we have

$$
\frac{s\mathfrak{t} |\omega_1| \left|z^{n-1}\right|}{\mathfrak{T} |a| + 1} - 1 > 1,
$$

where $|\omega_1| \in (0, 1]$. Consequently, the orbit of z tends to infinity, that is, $|z_k| \to \infty$ as $k \to \infty$. □

Corollary 2 *If* $\left|z_j\right| > |c| > \left(\frac{2(\mathfrak{T}|a|+1)}{s\mathfrak{t}|\omega_1|}\right)^{\frac{1}{n-1}}$ *for some* $j \geq 0$*, then the generalized Fibonacci–Mann orbit extended with* s*-convexity escapes to infinity.*

Remark 1 (i) In Theorem 1, we have obtained the escape criterion using generalized Fibonacci–Mann iteration endowed with s-convexity for a complex sine function of the form $T_{a,c}^z = \sin(z^n) + az + c$. For proving the escape criterion we have utilized the inequalities $(1-t_0)^s \leq 1-st_0$ and $t_0^s \geq st_0$ which holds in (0, 1]. On the other hand, Nazeer et al. [22] proved the escape criterion for a complex-valued polynomial $f(z) = z^n - az + c$, via one- step Jungck-Mann, and two-step Jungck-Ishikawa fixed point iterations extended with s-convexity. The authors applied binomial expansion which is not true in (0, 1] for the underlying set of parameters. It is easy to verify that none of $(1-(1-t_0))^s \geq 1-s(1-t_0)$ or $(1-t_0)^s \geq 1-st_0$ hold in (0, 1]. This type of mistake may also be noticed in the proof of theorems of some recent papers exploring nonclassical fractals for complex-valued polynomials endowed with convexity parameter s, such as Kumari et al. [17], Kwun et al. [19], Gdawiec et al. [11], Kang et al. [14], Li et al. [20], Nazeer et al. [22] and Zhang et al. [29], etc. Recently, Antal et al. [3], and Tomar et al. [27] have also utilized similar inequalities to establish escape criteria for the visualization of mutants of celebrated fractals for higher-order complex-valued polynomials via Jungck-Ishikawa, Jungck-Mann and Jungck-Noor iterations equipped with s-convexity respectively.

(ii) The results of the generalized Fibonacci–Mann iteration extended with s-convexity are expected to inspire those interested in generating automatically aesthetic patterns. Because the generalized Fibonacci sequence has two (or one) parameters k and l (or k), and hence, the parameters selected in the above theorem together with these may lead to visually interesting fractal shapes. For a fixed k (respectively l), various values of l (respectively k) can be used to analyze the structure of the obtained fractals.

3 Variants of Julia and Mandelbrot sets

We utilize MATLAB 8.5.0 (R2015a) for exploring mutants of celebrated fractals for transcendental sine function (4) via the generalized version of the Fibonacci–Mann iteration process extended with s-convexity defined in (5). First, we present two algorithms, one for investigating the geometry of the Julia set (Algorithm 1), another for the Mandelbrot set (Algorithm 2) and using the escape criterion of $T_{a,c}(z) = \sin(z^n) + az + c$ established in Sect. 2 in Theorem 1.

Algorithm 1 Julia-Set

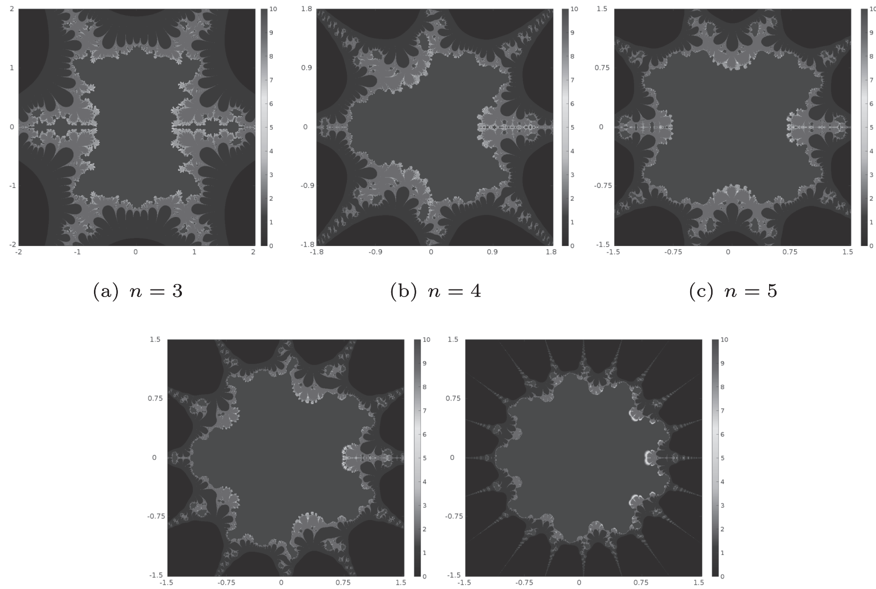
Input: $T_{a,c}(z) = \sin(z^n) + az + c$, $a, c \in \mathbb{C}$ and $n = 2, 3, \ldots$; $A \subset \mathbb{C}$ – area; K – maximum number of iterations; $t_n, \omega_1 \in (0, 1]$ – parameters of the generalized Fibonacci–Mann iteration extended with s-convexity; k and l be nonzero real numbers; $colormap[0..C-1]$ – a color map with C colors.
Output: Julia set for area A.

for $z \in A$ **do**
  $R_1 = \left(\frac{2(\Im|a|+1)}{st|\omega_1|}\right)^{\frac{1}{n-1}}$
  $R = \max(|c|, R_1)$
  $j = 0$
  $u(0) = 1$
  $u(1) = 1$
  while $j \leq K$ **do**
    if $j > 1$ **then**
      $u(j) = ku(j-1) + lu(j-2)$
    end if
    $z_{j+1} = t_j^s T_{a,c}^{u(j)}(z_j) + (1 - t_j)^s z_j$
    if $|z_{j+1}| > R$ **then**
      break
    end if
    $j = j + 1$
  end while
  $i = \lfloor (C-1)\frac{j}{K} \rfloor$
  color z with $colormap[i]$
end for

Algorithm 2 Mandelbrot-Set

Input: $T_{a,c}(z) = \sin(z^n) + az + c$, $a, c \in \mathbb{C}$ and $n = 2, 3, \ldots$; $A \subset \mathbb{C}$ – area; K – maximum number of iterations; $t_n, \omega_1 \in (0, 1]$ – parameters of the generalized Fibonacci–Mann iteration extended with s-convexity; k and l be nonzero real numbers; $colormap[0..C-1]$ – a color map with C colors.
Output: Mandelbrot set for area A.

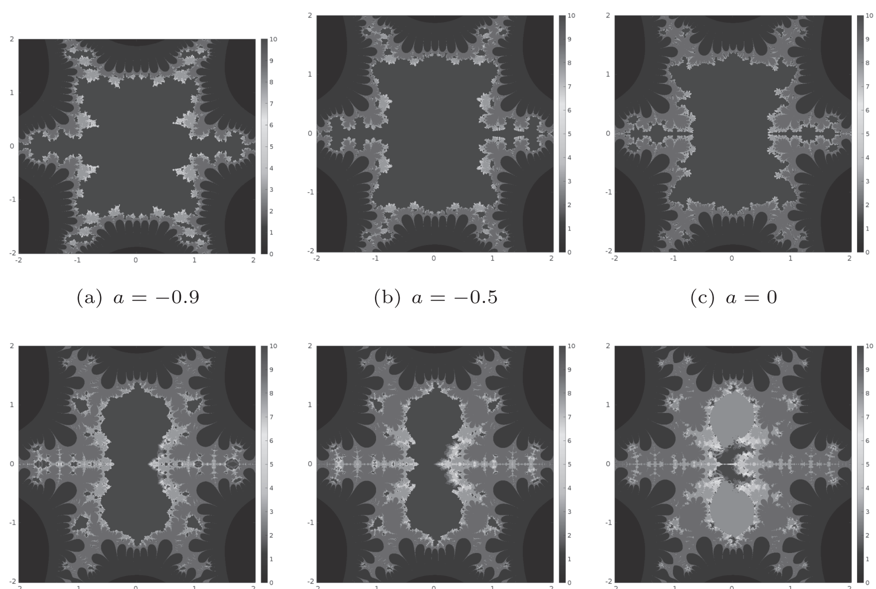
for $c \in A$ **do**
  $R_1 = \left(\frac{2(\Im|a|+1)}{st|\omega_1|}\right)^{\frac{1}{n-1}}$
  $R = \max(|c|, R_1)$
  $j = 0$
  $z_0 = 0$
  $u(0) = 1$
  $u(1) = 1$
  while $j \leq K$ **do**
    if $j > 1$ **then**
      $u(j) = ku(j-1) + lu(j-2)$
    end if
    $z_{j+1} = t_j^s T_{a,c}^{u(j)}(z_j) + (1 - t_j)^s z_j$
    if $|z_{j+1}| > R$ **then**
      break
    end if
    $j = j + 1$
  end while
  $i = \lfloor (C-1)\frac{j}{K} \rfloor$
  color z with $colormap[i]$
end for



(a) $n = 3$ (b) $n = 4$ (c) $n = 5$

(d) $n = 6$ (e) $n = 10$

Fig. 1 Julia sets generated via Fibonacci–Mann iteration scheme with varying values of n and fixed other parameters



(a) $a = -0.9$ (b) $a = -0.5$ (c) $a = 0$

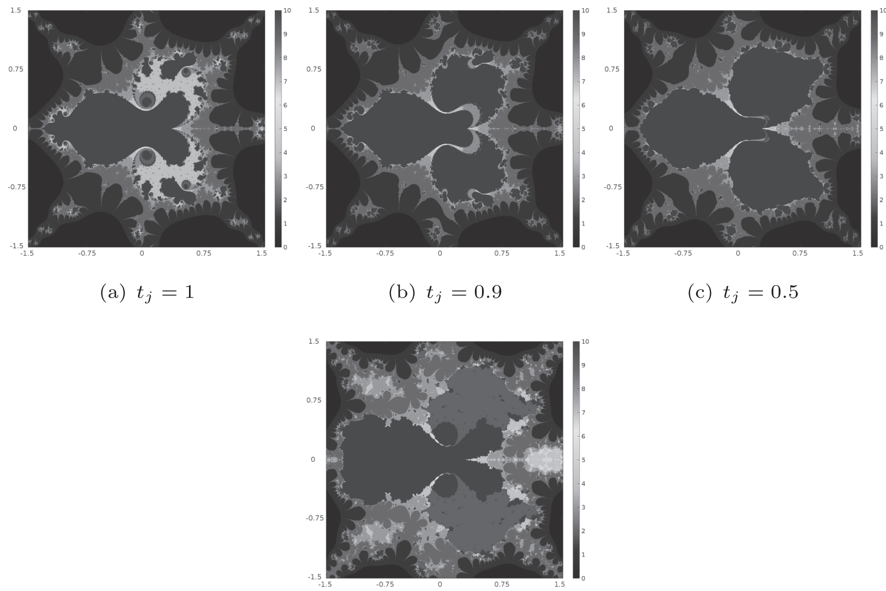
(d) $a = 0.5$ (e) $a = 0.7$ (f) $a = 0.9$

Fig. 2 Julia sets generated via Fibonacci–Mann iteration scheme with varying values of a and fixed other parameters

3.1 Variants of Julia sets

To obtain aesthetic Julia sets presented in Figure 1, we fixed the values of parameters $a = 0$, $c = 0.05$, $t_j = 0.01$, $\omega_1 = 0.05$, $s = 0.2$, $k = l = 1$ and vary the value of parameter n. Because t_j is a constant sequence, so $\mathfrak{T} = \mathfrak{t} = 0.01$.

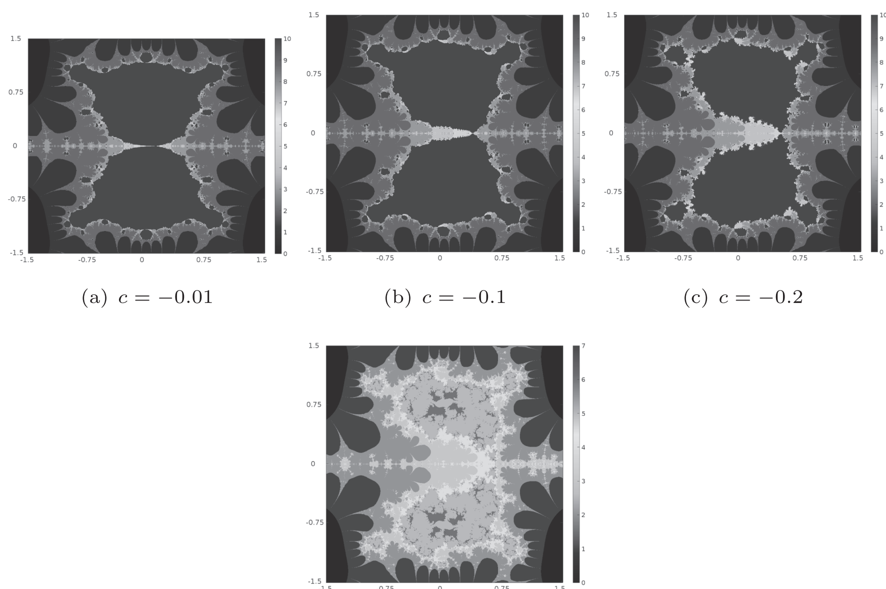
The next example, presented in Figure 2, shows aesthetic cubic Julia fractals for which we fixed the values of parameters $n = 3$, $c = 0.05$, $t_j = 0.01$, $\omega_1 = 0.05$, $s = 0.2$, $k = l = 1$ and vary the value of parameter a. Because t_j is a constant sequence, so $\mathfrak{T} = \mathfrak{t} = 0.01$.



(a) $t_j = 1$ (b) $t_j = 0.9$ (c) $t_j = 0.5$

(d) $t_j = 0.1$

Fig. 3 Julia sets generated via Fibonacci–Mann iteration scheme with varying values of t_j and fixed other parameters



(a) $c = -0.01$ (b) $c = -0.1$ (c) $c = -0.2$

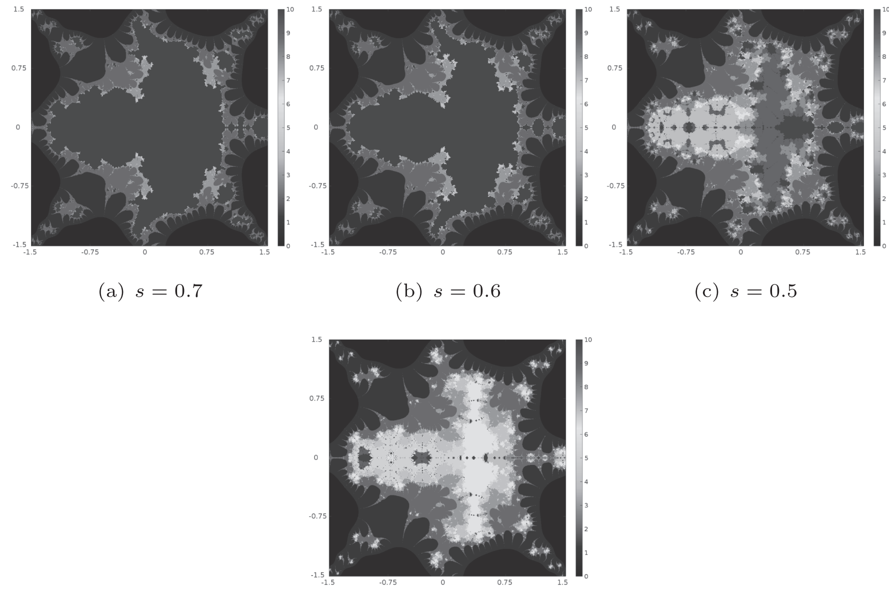
(d) $c = -0.4$

Fig. 4 Julia sets generated via Fibonacci–Mann iteration scheme with varying values of c and fixed other parameters

To show the variation in the shape of Julia sets for varying values of t_j, we fixed the values of parameters $n = 4$, $a = 1$, $c = -0.01$, $\omega_1 = 0.05$, $s = 0.9$, $k = l = 1$ and vary the values of t_j to obtain aesthetic quartic Julia fractals. The obtained Julia set images are presented in Figure 3.

To obtain aesthetic cubic Julia fractals presented in Figure 4, we fixed the values of parameters $a = 1$, $t_j = 0.5$, $\omega_1 = 0.05$, $n = 3$, $k = l = 1$ and vary the value of parameter c. Because t_j is a constant sequence, so $\mathfrak{T} = \mathfrak{t} = 0.5$.

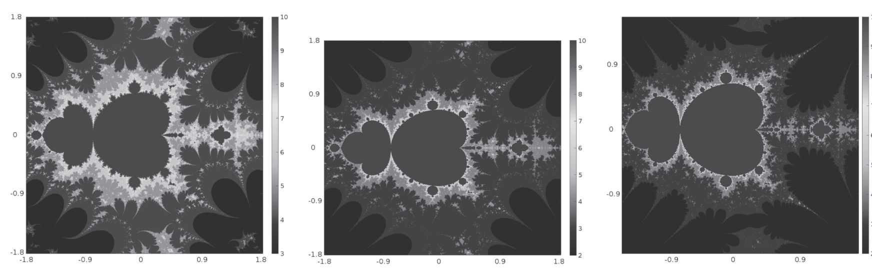
The variation of the Julia set shape for varying values of s is presented in Figure 5. To obtain those images, we fixed the values of parameters $c = -0.5$, $a = 0.4$, $\omega_1 = 0.05$, $n = 4$, $t_j = 0.5$, $k = t = 1$. Because t_j is a constant sequence, so $\mathfrak{T} = \mathfrak{t} = 0.5$.



(a) $s = 0.7$ (b) $s = 0.6$ (c) $s = 0.5$

(d) $s = 0.4$

Fig. 5 Julia sets generated via Fibonacci–Mann iteration scheme with varying values of s and fixed other parameters



(a) $k = 1, l = 1$ (b) $k = 2, l = 1$ (c) $k = 2, l = 2$

Fig. 6 Mandelbrot sets generated via Fibonacci–Mann iteration scheme with varying values of k and l, and fixed other parameters

3.2 Variants of Mandelbrot sets

The quadratic Mandelbrot sets for various values of k and l are presented in Figure 6. The other parameters were fixed to the following values $a = 0$, $t_j = 0.01$, $\omega_1 = 0.05$, $n = 2$, $s = 0.9$. Because t_j is a constant sequence, so $\mathfrak{T} = \mathfrak{t} = 0.01$.

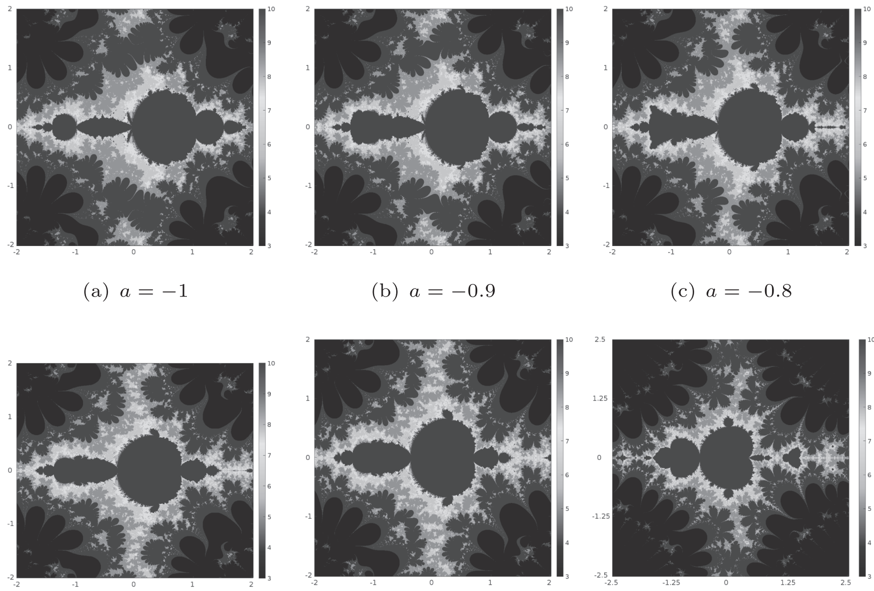
To obtain the quadratic Mandelbrot sets in Figure 7, we fixed the values of parameters $n = 2$, $\omega_1 = 0.05$, $s = 0.9$, $k = l = 1$ and vary the value of parameters a.

The next example, presented in Figure 8, shows aesthetic quadratic Mandelbrot sets for varying t_j. To generate those images, we fixed the values of parameters $a = 0$, $\omega_1 = 0.05$, $n = 2$, $s = 0.9$, $k = l = 1$.

To show the variation in the shape of Mandelbrot sets for varying values of s, we fixed the values of parameters $a = 0$, $t_j = \mathfrak{T} = \mathfrak{t} = \omega_1 = 0.05$, $n = 2$, $k = l = 1$. The obtained images are presented in Figure 9.

In the last graphical example, we present the effect of changing the value of n. The other parameters were set to the following values $a = 0$, $t_j = \mathfrak{T} = \mathfrak{t} = \omega_1 = 0.01$, $\omega_1 = 0.05$, $s = 0.9$, $k = l = 1$. The obtained images of the Mandelbrot sets are presented in Figure 10.

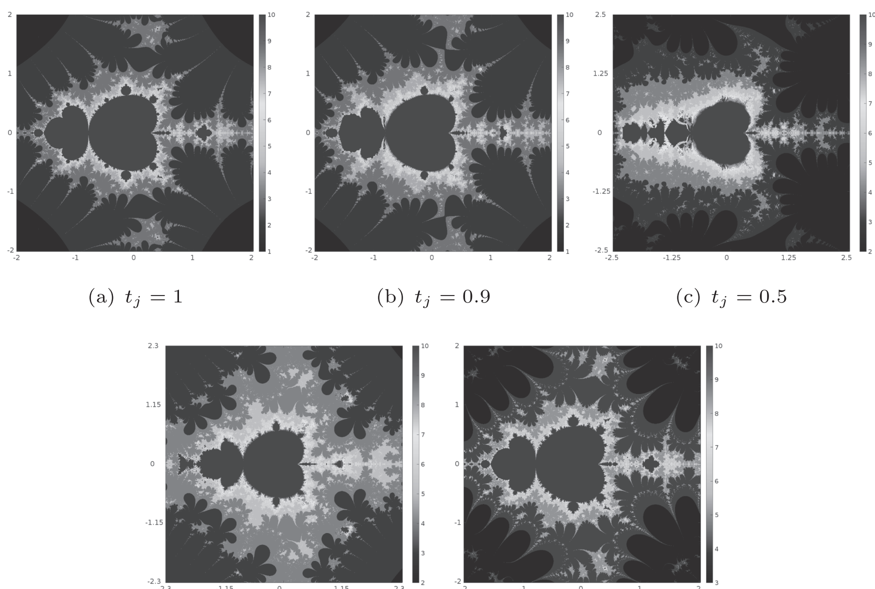
Remark 2 All the explored variants of celebrated Julia and Mandelbrot sets of the sine function by applying generalized Fibonacci-Mann iteration extended further with the notion of s-convexity are aesthetic, novel, pleasing, and symmetrical about the real axis. We have demonstrated only the zoomed fractals because the sine function is unbounded. We have noticed that for the same set of parameters, even a minor change in any of the parameters



(a) $a = -1$ (b) $a = -0.9$ (c) $a = -0.8$

(d) $a = -0.7$ (e) $a = -0.6$ (f) $a = -0.2$

Fig. 7 Mandelbrot sets generated via Fibonacci–Mann iteration scheme with varying values of a and fixed other parameters



(a) $t_j = 1$ (b) $t_j = 0.9$ (c) $t_j = 0.5$

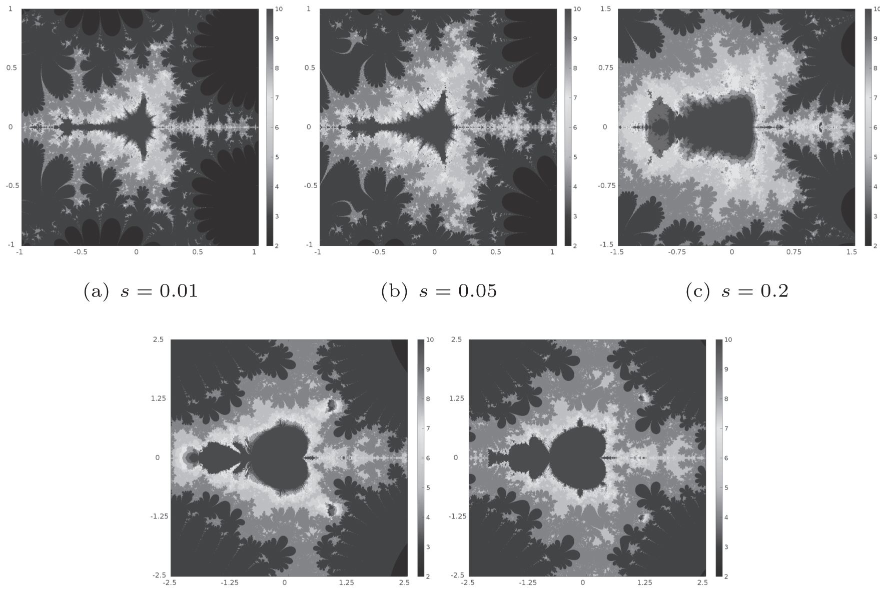
(d) $t_j = 0.05$ (e) $t_j = 0.005$

Fig. 8 Mandelbrot sets generated via Fibonacci–Mann iteration scheme with varying values of t_j and fixed other parameters

has a huge influence on the color, size, and shape of the fractals. As a result, it is essential to choose the values of parameters carefully to explore the fractal with desired characteristics.

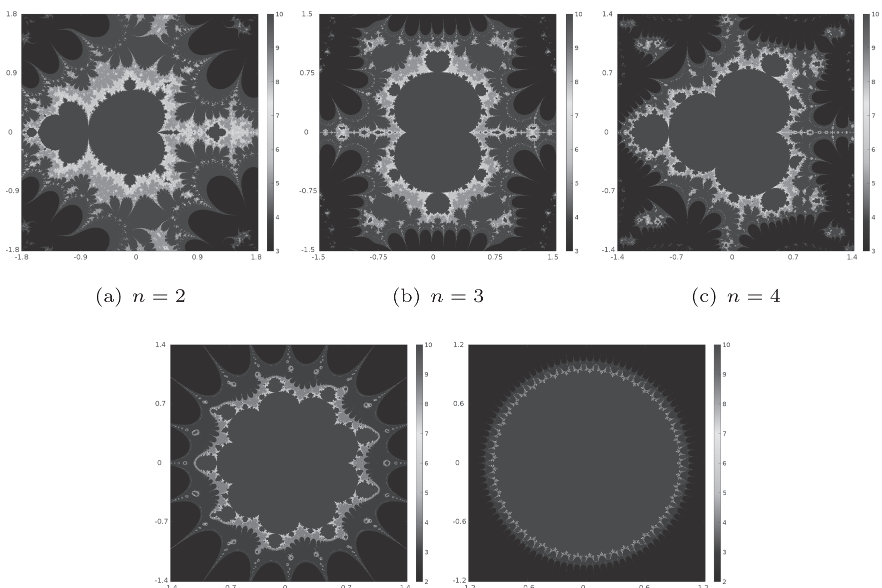
4 Numerical examples

The graphical examples presented in Sect. 3 showed a high variation in the shapes and sizes of the two types of sets (Mandelbrot and Julia sets) obtained with the proposed iteration scheme. Investigating the dependency on the shape and size of those sets and the parameters in the iteration scheme is not a trivial task [15]. Shahid et al.



(a) $s = 0.01$ (b) $s = 0.05$ (c) $s = 0.2$ (d) $s = 0.4$ (e) $s = 0.5$

Fig. 9 Mandelbrot sets generated via Fibonacci–Mann iteration scheme with varying values of s and fixed other parameters



(a) $n = 2$ (b) $n = 3$ (c) $n = 4$ (d) $n = 10$ (e) $n = 50$

Fig. 10 Mandelbrot sets generated via Fibonacci–Mann iteration scheme with varying values of n and fixed other parameters

in [26] proposed two numerical measures to study this dependency. Both numerical measures are obtained using the escape-time algorithm. In the first measure, we measure the generation time of the set, and in the second one, we calculate the average number of iterations (ANI) performed during the generation process.

Measures proposed by Shahid et al. give us relative information about the set size and the speed of computations. The most significant impact on both these measures have the non-escaping points (points for which we perform the maximal number of iterations K) [26]. If the difference between the measures, especially for the ANI, for two sets of parameters is high, then we know that the sizes of the corresponding sets differ in a significant way. The dependencies between the iteration's parameters and the numerical measures can be very complex and non-linear. See, for example, [15,16,26].

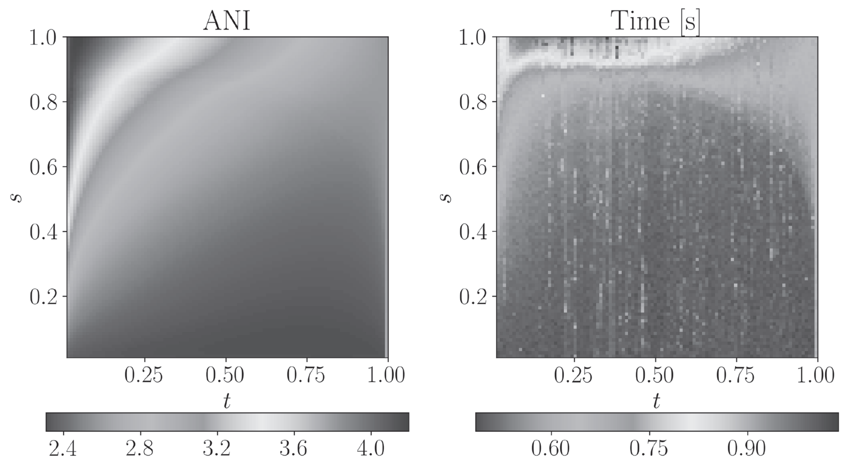


Fig. 11 ANI and time plots for the Julia set with $n = 3$, $a = 2$, $c = 0.1i$ and the Fibonacci–Mann iteration scheme

To perform the numerical experiments for the considered iteration scheme, we have used two sets of parameters defining Mandelbrot and Julia sets. For the Julia sets, we used:

1. $n = 3, a = 2, c = 0.1i, A = [-1.5, 1.5]^2, K = 10, \omega_1 = 0.001, k = l = 1$,
2. $n = 4, a = 0, c = 1, A = [-1.5, 1.5]^2, K = 10, \omega_1 = 0.001, k = l = 1$,

and for the Mandelbrot sets

1. $n = 3, a = -1, A = [-1.5, 1.5]^2, K = 10, \omega_1 = 0.05, k = l = 1$,
2. $n = 4, a = 0.5i, A = [-1.5, 1.5]^2, K = 10, \omega_1 = 0.001, k = l = 1$.

Moreover, we have used a constant sequence $t_j = t = const$ in the considered iteration scheme. For each of the two iteration's parameters (t, s), we generated equally spaced values in $(0, 1]$. The number of generated values for each parameter was set to 100. Therefore, to create a single plot, 10000 images were generated. The resolution of the generated images was set to 800×800 pixels. The experiments were performed on a computer with the following specifications: Intel i5-9600K (@3.70 GHz) processor, 32 GB DDR4 RAM memory, and Windows 10 (64 - bit). The algorithms were implemented in Mathematics 13.2 using the parallelization option of the `Compile` command.

4.1 Julia sets

The numerical results in the case of the cubic Julia sets generated with the Fibonacci–Mann iteration scheme are presented in Figure 11. From the plots, we can observe that the dependency between the iteration's parameters (t, s) and the numerical measures are non-linear. By looking at the ANI plot, we see that the highest values of the measure are obtained in the top-left corner of the parameters' plane, which corresponds to low values of t and high values of s. The maximal value of ANI equal to 4.197 is attained at $t = 0.03$, $s = 1.0$. On the other hand, the low values of ANI are obtained for low values of s and high values of t. The minimal value of ANI equal to 2.309 is attained at $t = 0.72$, $s = 0.04$. In general, the shape of the plot for time is similar to the ANI plot, but it contains noise. This noise is caused by other processes running in the background of the operating system. In the plot, the longest times can be observed for high values of s and low values of t, where the maximal value 1.039 s was attained at $t = 0.38$, $s = 0.97$. The shortest time 0.484 s, on the other hand, was obtained for high values of t and low values of s, namely $t = 0.97$, $s = 0.02$.

In Figure 12, we see numerical results for the quartic Julia sets generated with the Fibonacci–Mann iteration scheme. In this case, we see a very similar dependency between the iteration's parameters and the numerical measures. For both measures, the highest values are obtained for high values of s and low values of t, whereas the low values are obtained for high values of t and low values of s. The extreme values are the following:

- the minimal value of ANI 2.404 is attained at $t = 0.87$, $s = 0.09$,
- the maximal value of ANI 4.783 is attained at $t = 0.02$, $s = 0.96$,
- the shortest time 0.497 s is attained at $t = 0.9$, $s = 0.11$,
- the longest time 1.192 s is attained at $t = 0.08$, $s = 0.99$.

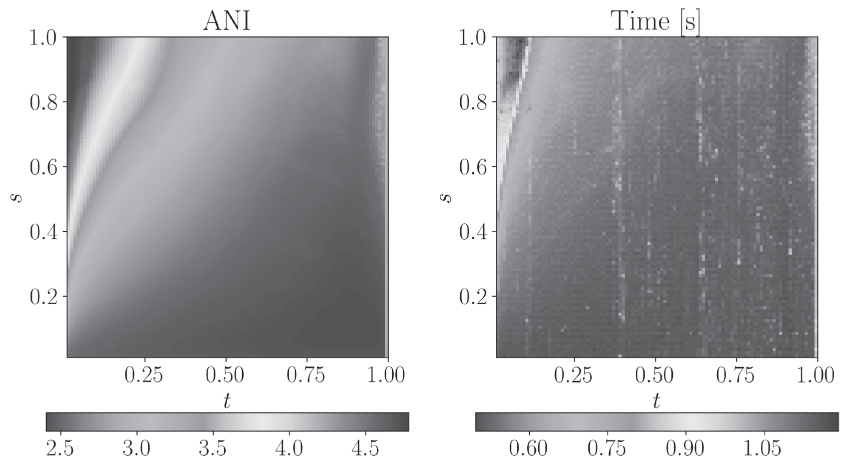


Fig. 12 ANI and time plots for the Julia set with $n = 4$, $a = 0$, $c = 1$ and the Fibonacci–Mann iteration scheme

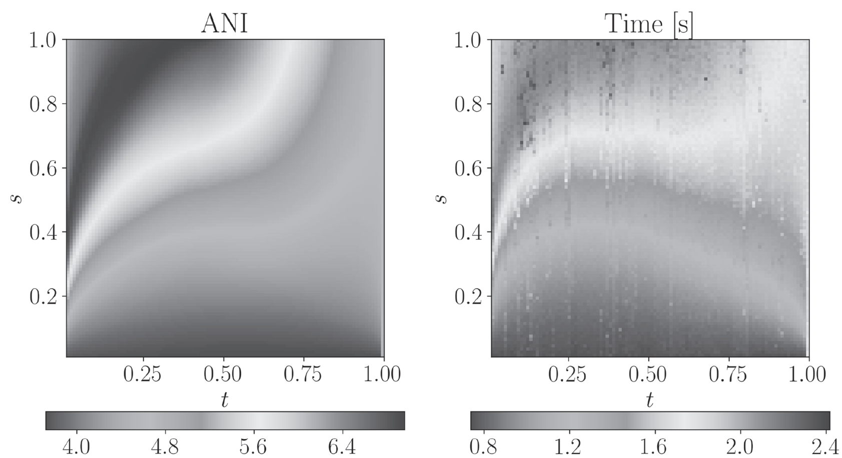


Fig. 13 ANI and time plots for the Mandelbrot set with $n = 3$, $a = -1$ and the Fibonacci–Mann iteration scheme

4.2 Mandelbrot sets

The numerical results for the first example of the Mandelbrot set ($n = 3$) generated using the Fibonacci–Mann iteration scheme are presented in Figure 13. Like in the Julia sets case, the dependency between the iteration's parameters and the numerical measures is non-trivial. Moreover, it is very similar. The only difference is in the size of the areas with high values of the measures. For the Mandelbrot set, the areas are larger in comparison to the Julia sets. The high values of both the measures are obtained for low values of t and high values of s. The maximal values of ANI and time equal to 6.959 and 2.417 s are attained at $t = 0.35$, $s = 0.01$ and $t = 0.34$, $s = 0.99$, respectively. In general, for high values of s, we observe higher values of the measures than for the low values of s. The lowest values are obtained for low values of s attaining the minimal value of 3.721 for ANI and 0.738 s for time at $t = 0.6$, $s = 0.01$ and $t = 0.27$, $s = 0.01$, respectively.

In the last numerical example, we present the results obtained for the quartic Mandelbrot sets generated with Fibonacci–Mann iteration (Figure 14). When we look at the ANI and time plots, then we notice a similar behaviour that we observed in the previous examples. Namely, the areas in the parameters' plane for which we obtain the highest values of the measures are located in the top-left corner (low values of t and high values of s), whereas the lowest values near the bottom-right corner (high values of t and low values of s). The high values of ANI indicate the iterations' parameters for which the biggest Mandelbrot sets were generated. The maximal values of ANI and time equal to 5.733 and 1.912 s were attained at $t = 0.01$, $s = 0.48$ and $t = 0.78$, $s = 0.73$, respectively. On the other hand, the smallest Mandelbrot sets were obtained at $t = 0.73$, $s = 0.01$ and $t = 0.29$, $s = 0.01$ for which the values of ANI equal to 3.245 and time equal to 0.596 s were obtained, respectively.

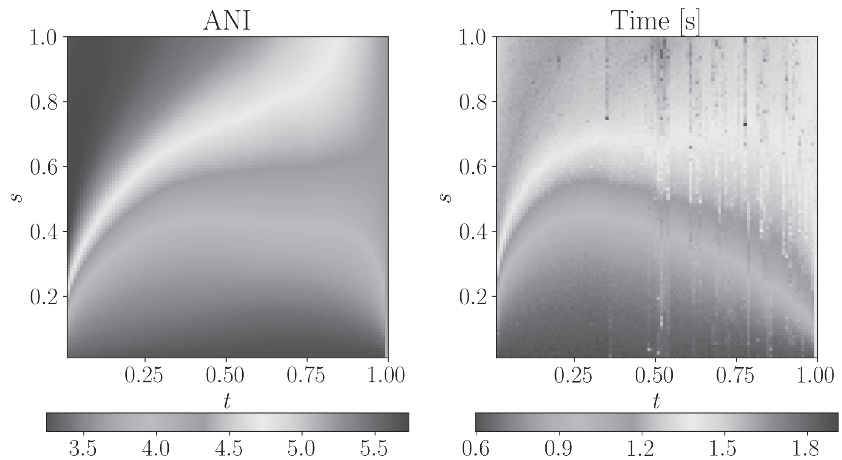


Fig. 14 ANI and time plots for the Mandelbrot set with $n = 4$, $a = 0.5i$ and the Fibonacci–Mann iteration scheme

5 Concluding remarks

Motivated by the reality that iteration processes have a huge impact on the behavior and dynamics of the explored Mandelbrot and Julia sets, which are important from the viewpoints of applications as well as graphical, we have explored distinct variants of celebrated fractals of distinct complex-valued sine functions to display the applicability of the generalized Fibonacci–Mann iteration extended further with the notion of s-convexity. We have studied the mutants of nonclassical fractals for distinct sets of parameters using a color map, algorithm, obtained escape criterion and MATLAB software, and observed that each parameter performs a distinct role in imparting color, size, and shape to different mutants of celebrated fractals. As a result, we have explored fractals for only a limited type of combination of involved parameters used in establishing the escape criterion although we have utilized the majority of combinations that are possible. We have also observed that the shape of each Mandelbrot and Julia sets depends on the complex-valued parameters a and c involved in the underlying sine function. Finally, it is worth to mention that the results of this study may lead to nice applications in the design of textile patterns (for some examples in this direction, see [12,21] and the references therein).

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Data availibility The data that support the findings of this study are available from the corresponding author, upon reasonable request.

Declarations

Conflict of interest The authors declare that they have no conflicts of interest.

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