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ORIGINAL RESEARCH

The r-bi^{q}nomial coefficient and some properties

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Abstract In this paper we give a new generalization of the binomial coefficient: the r-bi^{q}nomial coefficient $\binom{L}{k}_{q,r}$ defined as the coefficient of x^k in the expansion of

$$\left(1+x+\cdots+x^q\right)^L\left(1+2x+\cdots+qx^{q-1}\right)^r.$$

We establish a connection between these coefficients and the partial r-Bell polynomials and we provide many combinatorial properties of these newcoefficients.

Keywords Ordinary multinomial coefficient · Exponential partial Bell partition polynomials

Mathematics Subject Classification 11B68 · 11B39 · 05A10

1 Introduction

Let $q \geq 1$ and $L \geq 0$ be two integers. For $k \geq 0$, the bi^{q}nomial coefficient $\binom{L}{k}_q$ defined by

$$\left(1+x+x^2+\cdots+x^q\right)^L=\sum_{k\geq 0}\binom{L}{k}_q x^k$$

is a natural extension of the binomial coefficient. For an appropriate introduction of these numbers see Smith and Hogatt [1], Bollinger [2] and Andrews and Baxter [3]. This coefficient has been studied by many authors (see [4–6]). In [4], Bondarenko gave a combinatorial interpretation of the bi^{q}nomial coefficient $\binom{L}{k}_q$ as the number of different ways of distributing "k" objects among "L" cells such each cell contains at most "q" objects. In [6],

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Belbachir et al. have established a connection of the bi^{q}nomial coefficient $\binom{L}{k}_q$ with exponential partial Bell partition polynomials

$$B_{n,L}(\varphi_1, \varphi_2, \ldots) := B_{n,L}(\varphi)$$

defined by (see Comtet [7, p. 144])

$$\sum_{n \geq L} B_{n,L}(\varphi) \frac{x^n}{n!} = \frac{1}{L!} (\varphi(x))^L, \quad \varphi(x) = \sum_{j \geq 1} \varphi_j \frac{x^j}{j!}.$$

These authors have shown that we have

$$B_{L+k,L}(1!, 2!, \ldots, q!, 0, 0, \ldots) = \frac{(L+k)!}{L!} \binom{L}{k}_q.$$

Motivated by these studies, we want to introduce in this paper a generalization for numbers $\binom{L}{k}_q$. Indeed, for any non-negative integer r, let the r-bi^{q}nomial coefficient $\binom{L}{k}_{q,r}$ be defined as

$$\left(1 + x + \cdots + x^q\right)^L \left(1 + 2x + \cdots + qx^{q-1}\right)^r = \sum_{k \geq 0} \binom{L}{k}_{q,r} x^k. \tag{1}$$

We have

$$\binom{L}{k}_{q,0} = \binom{L}{k}_q.$$

In [8], Mihoubi and Rahmani defined the partial r-Bell polynomials

$$B_{n+r,L+r}^{(r)}(\varphi_1, \varphi_2, \ldots; \psi_1, \psi_2, \ldots) := B_{n+r,L+r}^{(r)}(\varphi; \psi)$$

by giving a combinatorial definition of the number $B_{n+r,k+r}^{(r)}(\varphi; \psi)$ when $(\varphi_n; n \geq 1)$ and $(\psi_n; n \geq 1)$ are two sequences of nonnegative integers. These authors show that these polynomials can also be defined by their exponential generating function given as

$$\sum_{n \geq k} B_{n+r,L+r}^{(r)}(\varphi; \psi) \frac{x^n}{n!} = \frac{1}{L!} (\varphi(x))^L \left(\frac{d}{dx} \psi(x)\right)^r,$$

$$\varphi(x) = \sum_{j \geq 1} \varphi_j \frac{x^j}{j!}, \quad \psi(x) = \sum_{j \geq 0} \psi_j \frac{x^j}{j!}$$

By choosing $\varphi(x) = x + \cdots + x^{q+1}$ and $\psi(x) = 1 + x + \cdots + x^q$, we get

$$\sum_{k \geq L} B_{k+r,L+r}^{(r)}(\varphi; \psi) \frac{x^k}{k!} = \frac{x^L}{L!} \left(1 + x + \cdots + x^q\right)^L \left(1 + 2x + \cdots + qx^{q-1}\right)^r \tag{2}$$

From (1) and (2), we deduce the following connection between the r-bi^{q}nomial coefficient $\binom{L}{k}_{q,r}$ and the partial r-Bell polynomials $B_{n+r,k+r}^{(r)}(\varphi; \psi)$:

$$B_{L+k+r,L+r}^{(r)}(1!, 2!, \ldots, (q+1)!, 0, 0, \ldots; 1!, 2!, \ldots, q!, 0, 0, \ldots) = \frac{(L+k)!}{L!} \binom{L}{k}_{q,r}.$$

This last identity means that the number $\frac{(L+k)!}{L!} \binom{L}{k}_{q,r}$ counts the number of partitions of the set $\{1, \ldots, L\}$ into k lists such that each list contains at most $q+1$ elements, and, the elements $\{1, \ldots, r\}$ must be in different lists. In particular, the number $\frac{(L+k)!}{L!} \binom{L}{k}_q$ counts the number of partitions of the set $\{1, \ldots, L\}$ into k lists such that each list contains at most $q+1$ elements.

Table 1 Triangle of the 2-bi^{2}nomial coefficient $\binom{L}{k}_{2,2}$

$L \diagdown k$	0	1	2	3	4	5	6	7	8
0	1	4	4						
1	1	5	9	8	4				
2	1	6	15	22	21	12	4		
3	1	7	22	43	58	55	37	16	4

Table 2 Triangle of the 3-bi^{2}nomial coefficient $\binom{L}{k}_{2,3}$

$L \diagdown k$	0	1	2	3	4	5	6	7	8	9
0	1	6	12	8						
1	1	7	19	26	20	8				
2	1	8	27	52	65	54	28	8		
3	1	9	36	87	144	171	147	90	36	8

Table 3 Triangle of the 2-bi^{3}nomial coefficient $\binom{L}{k}_{3,2}$

$L \diagdown k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	4	10	12	9									
1	1	5	15	27	35	31	21	9						
2	1	6	21	48	82	108	114	96	61	30	9			
3	1	7	28	76	157	259	352	400	379	301	196	100	39	9

2 Some immediate properties of the r-bi^{q}nomial coefficients

We start by presenting some values of the coefficient $\binom{L}{k}_{q,r}$ for few values of L, k, r, q (Tables 1, 2, 3). To begin, let's give some immediate results

$$\binom{L}{k}_{q,0} = \binom{L}{k}_q,$$

$$\binom{L}{k}_{q,1} = \frac{k+1}{L+1}\binom{L+1}{k+1}_q,$$

$$\binom{L}{qL+r(q-1)}_{q,r} = q^r,$$

$$\binom{L}{0}_{q,r} = 1,$$

$$\binom{L}{k}_{q,r} = 0 \text{ if } k > qL + r(q-1),$$

$$\binom{L}{0}_{q,r} + \binom{L}{1}_{q,r} = \binom{L+1}{1}_{q,r},$$

$$\binom{1}{k}_{q,0} = 1, \quad 0 \leq k \leq q, \tag{3}$$

$$\binom{0}{k}_{q,1} = k+1, \quad 0 \leq k \leq q-1. \tag{4}$$

In what follows, we will note $[x^n] f(x)$ the coefficient of x^n in $f(x)$, where

$$f(x) = \sum_{n=0}^{\infty} f_n x^n.$$

Theorem 1 *Let L, k be two non-negative integers and r, q be two positive integers such that $k < q$. There holds*

$$\binom{L}{k}_{q,r} = \binom{L+2r+k-1}{k}. \tag{5}$$

Proof We have

$$\begin{aligned}&\left(1+x+\cdots+x^{q}\right)^{L}\left(1+2x+\cdots+qx^{q-1}\right)^{r}(1-x)^{L+2r}\\&=\left(1-x^{q+1}\right)^{L}\left(1-(q+1)\,x^{q}+qx^{q+1}\right)^{r}.\end{aligned}$$

We deduce that for $k < q$, we have in $\mathbb{Q}[[x]]$

$$\binom{L}{k}_{q,r} = \left[x^{k}\right](1-x)^{-L-2r} = (-1)^{k}\binom{-L-2r}{k} = \binom{L+2r+k-1}{k}.$$

□

3 Generalization of Vandermonde's identity

Similarly to the known identity [9]

$$\binom{L_1+L_2}{k}_{q} = \sum_{j=0}^{k}\binom{L_1}{j}_{q}\binom{L_2}{k-j}_{q},$$

the following theorem generalizes this identity.

Theorem 2 *Let L_1, L_2, r_1, r_2 and q be non-negative integers. Then, we have*

$$\binom{L_1+L_2}{k}_{q,r_1+r_2} = \sum_{j=0}^{k}\binom{L_1}{j}_{q,r_1}\binom{L_2}{k-j}_{q,r_2}. \tag{6}$$

Proof From the definition (1), we have

$$\sum_{k\geq 0}\binom{L_1+L_2}{k}_{q,r_1+r_2}x^{k} = \left(\sum_{k\geq 0}\binom{L_1}{k}_{q,r_1}x^{k}\right)\left(\sum_{k\geq 0}\binom{L_2}{k}_{q,r_2}x^{k}\right). \tag{7}$$

By equalizing the coefficient of x^k in each of the members of (7), we obtain the result. □

By setting $L_1 = L_2 = L$ and $r_1 = r_2 = r$ in Proposition 2 in we get

Corollary 1 *There hold*

$$\begin{aligned}&\sum_{j=0}^{k}\binom{L}{j}_{q,r}\binom{L}{k-j}_{q,r} = \binom{2L}{k}_{q,2r},\\&\sum_{j=0}^{\frac{k+1}{2}}\binom{L}{j}_{q,r}\binom{L}{k-j}_{q,r} = \frac{1}{2}\binom{2L}{k}_{q,2r}\ \textit{if } k \textit{ is odd,}\\&\sum_{j=0}^{\frac{k-2}{2}}\binom{L}{j}_{q,r}\binom{L}{k-j}_{q,r} = \frac{1}{2}\binom{2L}{k}_{q,2r} - \frac{1}{2}\binom{2L}{\frac{k}{2}}^{2}_{q,r}\ \textit{if } k \textit{ is even.}\end{aligned}$$

By setting $L_1 = L,\ L_2 = 0$ and $r_1 = r,\ r_2 = 1$ in (6) of Theorem 2 in we get

Corollary 2 *We have*

$$\sum_{j=0}^{k}(k-j+1)\binom{L}{j}_{q,r}=\binom{L}{k}_{q,r+1},\ k<q, \tag{8}$$

$$\sum_{j=0}^{k}\binom{L}{j}_{q,r}=\binom{L+1}{k}_{q,r},\ k\leq q, \tag{9}$$

$$\sum_{j=0}^{k}\binom{L}{j}_{q,r}-\sum_{j=0}^{k-q-1}\binom{L}{j}_{q,r}=\binom{L+1}{k}_{q,r},\ q<k\leq qL+r\left(q-1\right). \tag{10}$$

Using (8) and (9) in we have

Corollary 3 *For all $k<q$, we have*

$$\sum_{j=0}^{k}j\binom{L}{j}_{q,r}=(k+1)\binom{L+1}{k}_{q,r}-\binom{L}{k}_{q,r+1}.$$

4 Study of some summations

Theorem 3 *For all non-negative integers L, r, q, we have*

$$\sum_{j\geq 0}\binom{L}{j}_{q,r}=(q+1)^{L+r}\left(\frac{q}{2}\right)^{r}, \tag{11}$$

$$\sum_{j\geq 0}j\binom{L}{j}_{q,r}=(q+1)^{L+r}\left(\frac{q}{2}\right)^{r}\left(\frac{Lq}{2}+\frac{2r\left(q-1\right)}{3}\right). \tag{12}$$

Proof We have

$$\sum_{j\geq 0}\binom{L}{j}_{q,r}x^{j}=\left(1+x+\cdots+x^{q}\right)^{L}\left(1+2x+\cdots+qx^{q-1}\right)^{r} \tag{13}$$

By deriviting with respect to x, we get

$$\begin{aligned}\sum_{j\geq 1}j\binom{L}{j}_{q,r}x^{j-1}&=L\left(1+x+\cdots+x^{q}\right)^{L-1}\left(1+2x+\cdots+qx^{q-1}\right)^{r+1}\\&\quad+r\left(1+x+\cdots+x^{q}\right)^{L}\left(1+2x+\cdots+qx^{q-1}\right)^{r-1}\\&\quad\times\left(2+6x+\cdots+q\left(q-1\right)x^{q-2}\right).\end{aligned} \tag{14}$$

The desired identities follow by setting $x=1$ in (13) and (14). □

Theorem 4 *For all non-negative integers L, k, r, q there holds*

$$\sum_{j=0}^{k}\binom{L}{j}_{q,r}=\sum_{j=0}^{\lfloor\frac{k}{q+1}\rfloor}\binom{L+1}{k-j\left(q+1\right)}_{q,r}. \tag{15}$$

In particular

$$\sum_{j=0}^{k}\binom{L}{j}_{q,r}=\binom{L+1}{k-q-1}_{q,r}+\binom{L+1}{k}_{q,r}\ \text{if}\ 1\leq\frac{k}{q+1}<2,$$

$$\sum_{j=0}^{k}\binom{L}{j}_{q,r}=\binom{L+1}{k-2q-2}_{q,r}+\binom{L+1}{k-q-1}_{q,r}+\binom{L+1}{k}_{q,r}\ \text{if}\ 2\leq\frac{k}{q+1}<3.$$

Proof From the definition (1) of the r-bi^{q}nomial coefficient $\binom{L}{k}_{q,r}$, we deduce the following identity

$$\frac{1}{1-x}\sum_{s\geq 0}\binom{L}{s}_{q,r}x^s=\frac{1}{1-x^{q+1}}\sum_{k\geq 0}\binom{L+1}{s}_{q,r}x^s.$$

Thus, in $\mathbb{Q}\,[[x]]$, we have

$$\left(\sum_{j\geq 0}x^j\right)\left(\sum_{s\geq 0}\binom{L}{s}_{q,r}x^s\right)=\left(\sum_{j\geq 0}x^{(q+1)j}\right)\sum_{s\geq 0}\binom{L+1}{s}_{q,r}x^s$$

which can be written

$$\sum_{s\geq 0}\binom{L}{s}_{q,r}x^{s+j}=\sum_{k\geq 0}\binom{L+1}{s}_{q,r}x^{s+j(q+1)} \tag{16}$$

By equalizing the coefficients of x^k in each of the members of (16), we obtain the equality

$$\sum_{j=0}^{k}\binom{L}{k-j}_{q,r}=\sum_{j=0}^{\lfloor\frac{k}{q+1}\rfloor}\binom{L+1}{k-j\,(q+1)}_{q,r}$$

from which we deduce (15). □

Theorem 5 *For all non-negative integers L, k, r, q there holds*

$$\sum_{j=0}^{q}\binom{L}{k+j}_{q,r}=\binom{L+1}{k+q}_{q,r}.$$

In particular, for $q=1$ or 2, we get

$$\binom{L}{k}+\binom{L}{k+1}=\binom{L+1}{k+1},$$
$$\binom{L}{k}_{2,r}+\binom{L}{k+1}_{2,r}+\binom{L}{k+2}_{2,r}=\binom{L+1}{k+2}_{2,r}.$$

Proof We have

$$\begin{aligned}\binom{L+1}{k+q}_{q,r}&=\left[x^{k+q}\right]\left(\left(1+x+x^2+\cdots+x^q\right)^{L+1}\left(1+2x+\cdots+qx^{q-1}\right)^r\right)\\&=\left[x^{k+q}\right]\left(\left(1+x+x^2+\cdots+x^q\right)\sum_{i\geq 0}\binom{L}{i}_{q,r}x^i\right)\\&=\left[x^{k+q}\right]\sum_{j=0}^{q}\left(\sum_{i\geq 0}\binom{L}{i}_{q,r}x^{i+j}\right)\\&=\sum_{j=0}^{q}\binom{L}{k+q-j}_{q,r}\\&=\sum_{j=0}^{q}\binom{L}{k+j}_{q,r}.\end{aligned}$$

□

Corollary 4 *There holds*

$$\binom{L+1}{k+q}_{q,r} + \binom{L}{k+1+q}_{q,r} = \binom{L}{k}_{q,r} + \binom{L+1}{k+1+q}_{q,r}.$$

Proof From Theorem 5, we can write

$$\begin{aligned}\binom{L+1}{k+1+q}_{q,r} - \binom{L+1}{k+q}_{q,r} &= \sum_{j=0}^{q}\binom{L}{k+1+j}_{q,r} - \sum_{j=0}^{q}\binom{L}{k+j}_{q,r} \\ &= \sum_{j=1}^{q+1}\binom{L}{k+j}_{q,r} - \sum_{j=0}^{q}\binom{L}{k+j}_{q,r} \\ &= \binom{L}{k+q+1}_{q,r} - \binom{L}{k}_{q,r}.\end{aligned}$$

□

Theorem 6 *We have*

$$\sum_{j=0}^{k} j\binom{L}{j}_{q,r} = k\sum_{j=0}^{\lfloor\frac{k}{q+1}\rfloor}\binom{L+1}{k-j\,(q+1)}_{q,r} - \sum_{\ell=0}^{\lfloor\frac{k-1}{q+1}\rfloor}\left(\sum_{j=0}^{k-1-\ell(q+1)}\binom{L+1}{j}_{q,r}\right).$$

Proof By applying Abel's summation formula

$$\sum_{j=0}^{k} a_j b_j = a_k\sum_{j=0}^{k} b_j - \sum_{j=0}^{k-1}\left(a_{j+1}-a_j\right)\left(\sum_{i=0}^{j} b_i\right)$$

with $a_j = j$ and $b_j = \binom{L}{j}_{q,r}$, we obtain

$$\sum_{j=0}^{k} j\binom{L}{j}_{q,r} = k\sum_{j=0}^{k}\binom{L}{j}_{q,r} - \sum_{j=0}^{k-1}\left(\sum_{i=0}^{j}\binom{L}{i}_{q,r}\right).$$

By relation (15), we have

$$\sum_{j=0}^{k}\binom{L}{j}_{q,r} = \sum_{j=0}^{\lfloor\frac{k}{q+1}\rfloor}\binom{L+1}{k-j\,(q+1)}_{q,r}$$

and

$$\sum_{i=0}^{j}\binom{L}{i}_{q,r} = \sum_{\ell=0}^{\lfloor\frac{j}{q+1}\rfloor}\binom{L+1}{j-\ell\,(q+1)}_{q,r}.$$

We deduce that

$$\sum_{j=0}^{k} j\binom{L}{j}_{q,r} = k\sum_{j=0}^{\lfloor\frac{k}{q+1}\rfloor}\binom{L+1}{k-j\,(q+1)}_{q,r} - \sum_{j=0}^{k-1}\left(\sum_{\ell=0}^{\lfloor\frac{j}{q+1}\rfloor}\binom{L+1}{j-\ell\,(q+1)}_{q,r}\right).$$

But we have

$$\sum_{j=0}^{k-1}\left(\sum_{\ell=0}^{\lfloor\frac{j}{q+1}\rfloor}\binom{L+1}{j-\ell\,(q+1)}_{q,r}\right)=\sum_{\ell=0}^{\left\lfloor\frac{k-1}{q+1}\right\rfloor}\left(\sum_{j=\ell(q+1)}^{k-1}\binom{L+1}{j-\ell\,(q+1)}_{q,r}\right)$$
$$=\sum_{\ell=0}^{\left\lfloor\frac{k-1}{q+1}\right\rfloor}\left(\sum_{j=0}^{k-1-\ell(q+1)}\binom{L+1}{j}_{q,r}\right)$$

It is therefore deduced that

$$\sum_{j=0}^{k}j\binom{L}{j}_{q,r}=k\sum_{j=0}^{\lfloor\frac{k}{q+1}\rfloor}\binom{L+1}{k-j\,(q+1)}_{q,r}-\sum_{\ell=0}^{\left\lfloor\frac{k-1}{q+1}\right\rfloor}\left(\sum_{j=0}^{k-1-\ell(q+1)}\binom{L+1}{j}_{q,r}\right).$$

□

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