Indian J Pure Appl Math (2025) 56:1648–1672
https://doi.org/10.1007/s13226-024-00619-y

ORIGINAL RESEARCH

Phase-field system with two temperatures and a nonlinear coupling term based on the Maxwell-Cattaneo law

Mohamed Ali IPOPA · **Brice Landry DOUMBE BANGOLA** · **Armel ANDAMI OVONO**

Received: 22 September 2022 / Accepted: 31 May 2024 / Published online: 15 July 2024

Abstract We study a generalisation of the Caginalp phase field system derived from the Maxwell-Cattaneo law on the one hand, and from the two-temperature theory of heat conduction on the other. We first show that the problem is well posed and obtain dissipativity properties of the associated solution operators. We then prove estimates $\mathbf{L}^{\infty}$-estimates relying on regularity estimates based on Moser's iterations. We finally analyze a spatial behaviour of the solution in a semi-infinite cylinder, when such solutions exist.

Keywords Caginalp phase-field system · Well-posedness · dissipativity · Moser's iterations · Phragmén-Lindelöf alternative · Amplitude term

1 Introduction

Consider the model proposed by G. Ginalp in [9],

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \theta, \tag{1}$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t}, \tag{2}$$

where, u is the phase-field, θ is the relative temperature. This model describes phase transition phenomena for simple (non-composite) materials, and has been studied from a mathematical point of view (see e.g., [1,2,7,8,12–15,17,19–21,26,32,33]). It is derived from a Ginzburg-Landau type potential using Fourier's law as the heat

Communicated by G.D. Veerappa Gowda.

B. L. DOUMBE BANGOLA and A. ANDAMI OVONO have contributed equally to this work.

M. A. IPOPA (✉)
Département Sciences Générales de l'Ingénieur, Université des Sciences et Techniques de Masuku (USTM), Route Nationale 1, BP 943, Franceville, Haut-0gooué, Gabon
E-mail: ipopa.mohamed@hotmail.fr

B. L. DOUMBE BANGOLA
Département Mathématiques-Informatique, Université des Sciences et Techniques de Masuku (USTM), Route Nationale 1, BP 943, Franceville, Haut-0gooué, Gabon
E-mail: doumbe83@gmail.com

A. ANDAMI OVONO
Département Mathématiques-Informatique, Ecole Normale Supeure (ENS), Avenue des Grandes Ecoles, BP 17009, Libreville, Estuaire, Gabon
E-mail: andami2016@gmail.com

diffusion law. The latter has a major flaw known as the heat conduction paradox (see for example [15]). Several alternatives to Fourier's law have been proposed in order to offer more realistic models: in particular, the Caginalp phase field system using the Maxwell-Cattaneo law or other laws from thermomechanics for example (see [5,22,23]). Today, with the evolution of materials, there are several types of non-simple materials. In order to describe the phase transition phenomena in the case of composite materials, it is essential to consider the heat diffusion law with two types of temperatures (see for example, [4,10,11,16–18,27,28]). More precisely, we consider the conduction temperature θ and the thermodynamic temperature φ. For time-independent problems, the difference between these temperatures is proportional to the heat input, so they coincide when there is no heat input. However, for time-dependent problems, they are usually different, even in the absence of heat input: this is particularly the case for non-simple materials. In this case, the two temperatures are related by (see, for example, [27,28])

$$\theta = \varphi - \Delta\varphi \tag{3}$$

by transferring (3) into (1), we have

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \theta = \varphi - \Delta\varphi. \tag{4}$$

In this work, considering the case of non-simple materials and the Maxwell-Cattaneo heat diffusion law, we obtain the following model

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = g(u)\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right), \tag{5}$$

$$\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial^2 \alpha}{\partial t^2} - \Delta\frac{\partial \alpha}{\partial t} - \Delta\alpha = -g(u)\frac{\partial u}{\partial t} - G(u) + h, \tag{6}$$

with $G(s) = cs - c's^2,\ c > 0,\ c' \geq 0, G' = g$ and

$$\alpha(t) = \int_0^t \varphi(s)\, ds. \tag{7}$$

In this case, the equations (5)-(6) are obtained from the Ginzburg-Landau total free energy :

$$\psi(u,\theta) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + F(u) - G(u)\theta - \frac{1}{2}\theta^2\right) dx, \tag{8}$$

where F is the antiderivative of f, Ω is a regular, bounded domain of $\mathbb{R}^n$, $n = 1, 2, 3$, with boundary $\partial\Omega$, and the enthalpy

$$H = G(u) + \theta = G(u) + \varphi - \Delta\varphi. \tag{9}$$

Then, the evolution equation for the phase-field u is given by

$$\frac{\partial u}{\partial t} = -\partial_u \psi, \tag{10}$$

where ∂_u stands for the variational derivative with respect to u, which yields (5).

And then, we have the energy equation

$$\frac{\partial H}{\partial t} = -\mathrm{div}\, \mathrm{q}, \tag{11}$$

with q the heat flux. Considering the Maxwell-Cattaneo law, the heat flux verifies the relation

$$\left(1 + \frac{\partial}{\partial t}\right) q = -\nabla\varphi, \tag{12}$$

Using (9) and (11), equation (12) becomes

$$\left(1 + \frac{\partial}{\partial t}\right)\frac{\partial \theta}{\partial t} - \Delta\varphi = -\left(1 + \frac{\partial}{\partial t}\right)\frac{\partial G(u)}{\partial t}. \tag{13}$$

Using (7) and remembering that $\theta = \varphi - \Delta\varphi$, we get , integrating (13) between 0 and t ($t \geq 0$),

$$\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -g(u)\frac{\partial u}{\partial t} - G(u) + h, \tag{14}$$

where

$$h = \frac{\partial^2 \alpha}{\partial t^2}(0) + \frac{\partial \alpha}{\partial t}(0) - \Delta \frac{\partial^2 \alpha}{\partial t^2}(0) - \Delta \frac{\partial \alpha}{\partial t}(0) + g(u(0))\frac{\partial u}{\partial t}(0) + G(u(0)). \tag{15}$$

By associating this model with homogeneous Dirichlet boundary conditions,

$$u = \alpha = \frac{\partial \alpha}{\partial t} = 0 \text{ on } \partial\Omega \tag{16}$$

and initial conditions,

$$u|_{t=0} = u_0,\ \alpha|_{t=0} = \alpha_0,\ \left.\frac{\partial \alpha}{\partial t}\right|_{t=0} = \alpha_1, \tag{17}$$

we obtain the following initial and boundary value problem

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = g(u)\left(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\right), \tag{18}$$

$$\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -g(u)\frac{\partial u}{\partial t} - G(u) + h, \tag{19}$$

$$u = \alpha = \frac{\partial \alpha}{\partial t} = 0 \text{ on } \partial\Omega, \tag{20}$$

$$u|_{t=0} = u_0,\ \alpha|_{t=0} = \alpha_0,\ \left.\frac{\partial \alpha}{\partial t}\right|_{t=0} = \alpha_1. \tag{21}$$

For a given space H, we denote the norm in H by $\|.\|_H$. Throughout this paper, the inner product and the norm of the $L^2(\Omega)$ space will be denoted by $(.,.)$ and $\|.\|$, respectively.

Section 2 is devoted to questions of existence and uniqueness of solutions. The next section deals with the dissipativity properties of the system. Section 4 gives estimates of L^∞ on the solutions. The last section analyses the spatial behaviour of the solutions in a semi-infinite cylinder, assuming their existence.

2 Existence and uniqueness of solutions

Main assumptions on f and g

$$c_0|s|^{k+2} - c_1 \leq F(s) \leq f(s)s + c_2 \leq c_3|s|^{k+2} - c_4,$$
$$c_0, c_3 > 0, c_1, c_2, c_4 \geq 0,\ k \text{ an even integer},\ s \in \mathbb{R}, \tag{22}$$

$$|g(s)| \leq c_5(1 + |s|),\ |g'(s)| \leq c_6,\ c_5 > 0, c_6 \geq 0, s \in \mathbb{R}, \tag{23}$$

$$|g(s)s| \leq c_7(1 + |G(s)|),\ c_7 > 0, s \in \mathbb{R}, \tag{24}$$

$$|G(s)|^2 \leq c_8 F(s) + c_9,\ c_8 > 0,\ c_9 \geq 0, s \in \mathbb{R}, \tag{25}$$

$$f(0) = 0,\ f'(s) \geq -c_{10},\ c_{10} \geq 0, s \in \mathbb{R}, \tag{26}$$

$$|f'(s)| \leq c_{11}(1 + |s|^k),\ c_{11} > 0, s \in \mathbb{R}, \tag{27}$$

with $F' = f$ and $k \in \mathbb{N}^*$. Throughout this article, the same letter c and, sometimes, c' and c'', denote positive constants which may change from line to line or even in a same line.

Existence and uniqueness results

Theorem 1 *Let f and g satisfy assumptions (22)-(25) and h belonging to $L^2(\Omega)$. Then, for any initial data satisfying*

$$\|u_0\|^2_{H^1(\Omega)} + \|u_0\|^{k+2}_{k+2} + \|\alpha_0\|^2_{H^2(\Omega)} + \|\alpha_1\|^2_{H^2(\Omega)} < \infty, \tag{28}$$

the problem (18)-(21) possesses a solution $(u, \alpha, \frac{\partial \alpha}{\partial t})$ which satisfies the estimate

$$\begin{aligned} &\|u(t)\|^2_{H^1(\Omega)} + \|u(t)\|^{k+2}_{k+2} + \|\alpha(t)\|^2_{H^2(\Omega)} + \left\|\frac{\partial \alpha}{\partial t}(t)\right\|^2_{H^2(\Omega)} \\ &\le Q(\|u_0\|^2_{H^1(\Omega)} + \|u_0\|^{k+2}_{k+2} + \|\alpha_0\|^2_{H^2(\Omega)} + \|\alpha_1\|^2_{H^2(\Omega)})e^{ct} + c', \end{aligned} \tag{29}$$

where the positive constants c, c', $\|.\|^{k+2}_{k+2} = \|.\|^{k+2}_{L^{k+2}(\Omega)}$ and the monotonic function Q are independent of $(u_0, \alpha_0, \alpha_1)$.

Proof Indeed, the proof is obtained by exploiting the Galerkin scheme technique see, [9] and [10] for details. Now, we just perform a priori estimates. Multiplying (18) by $\frac{\partial u}{\partial t}$, and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|^2 + 2\int_\Omega F(u)\,dx\right) + \left\|\frac{\partial u}{\partial t}\right\|^2 = \int_\Omega g(u)\frac{\partial u}{\partial t}\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)dx. \tag{30}$$

Multiplying (19) by $\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}$, we have, integrating over Ω

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\nabla \alpha\|^2 + \|\Delta\alpha\|^2 + \left\|\frac{\partial \alpha}{\partial t}\right\|^2 + 2\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2\right) \\ &\quad + \left\|\frac{\partial \alpha}{\partial t}\right\|^2 + 2\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2 \\ &\quad = -\int_\Omega g(u)\frac{\partial u}{\partial t}\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)dx + \int_\Omega (h - G(u))\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)dx. \end{aligned} \tag{31}$$

Multiplying (18) by u and integrating over Ω, one has

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 + \int_\Omega f(u)u\,dx = \int_\Omega g(u)u\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)dx. \tag{32}$$

Now, summing (30), (31) and (32), we get

$$\begin{aligned} &\frac{dE_1}{dt} + 2\left(\|\nabla u\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\frac{\partial \alpha}{\partial t}\right\|^2 + 2\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2\right) + 2\int_\Omega f(u)u\,dx \\ &= \int_\Omega g(u)u\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)dx + \int_\Omega (h - G(u))\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)dx, \end{aligned} \tag{33}$$

where

$$\begin{aligned} E_1 =& \|u\|^2 + \|\nabla u\|^2 + 2\int_\Omega F(u)\,dx + \|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 + \left\|\frac{\partial \alpha}{\partial t}\right\|^2 \\ &+ 2\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2, \end{aligned} \tag{34}$$

satisfies

$$E_1 \ge c\left(\|u\|^2_{H^1(\Omega)} + \|u\|^{k+2}_{k+2} + \|\alpha\|^2_{H^2(\Omega)} + \left\|\frac{\partial \alpha}{\partial t}\right\|^2_{H^2(\Omega)}\right) - c' \tag{35}$$

and

$$E_1 \le c''\left(\|u\|^2_{H^1(\Omega)} + \|u\|^{k+2}_{k+2} + \|\alpha\|^2_{H^2(\Omega)} + \left\|\frac{\partial\alpha}{\partial t}\right\|^2_{H^2(\Omega)}\right) + c'''. \tag{36}$$

Multiplying (19) by α and integrating over Ω, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\alpha\|^2 + \|\nabla\alpha\|^2\right) + \|\nabla\alpha\|^2 + \int_\Omega \frac{\partial^2\alpha}{\partial t^2}\alpha\,dx - \int_\Omega \Delta\frac{\partial^2\alpha}{\partial t^2}\alpha\,dx\\ &= -\int_\Omega g(u)\frac{\partial u}{\partial t}\alpha\,dx - \int_\Omega G(u)\alpha\,dx + \int_\Omega h\alpha\,dx.\end{aligned} \tag{37}$$

Observing that

$$\int_\Omega \frac{\partial^2\alpha}{\partial t^2}\alpha\,dx = \frac{d}{dt}\left(\frac{\partial\alpha}{\partial t},\alpha\right) - \left\|\frac{\partial\alpha}{\partial t}\right\|^2, \tag{38}$$

$$-\int_\Omega \Delta\frac{\partial^2\alpha}{\partial t^2}\alpha\,dx = \frac{d}{dt}\left(\nabla\frac{\partial\alpha}{\partial t},\nabla\alpha\right) - \left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2 \tag{39}$$

and

$$\int_\Omega g(u)\frac{\partial u}{\partial t}\alpha\,dx = \frac{d}{dt}\Big(G(u),\alpha\Big) - \int_\Omega G(u)\frac{\partial\alpha}{\partial t}\,dx, \tag{40}$$

we are led to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|\alpha\|^2 + \|\nabla\alpha\|^2 + 2\left(\frac{\partial\alpha}{\partial t},\alpha\right) + 2\left(\nabla\frac{\partial\alpha}{\partial t},\nabla\alpha\right) + 2\Big(G(u),\alpha\Big)\right]\\ &+ \|\nabla\alpha\|^2 = \left\|\frac{\partial\alpha}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2 + \int_\Omega G(u)\frac{\partial\alpha}{\partial t}\,dx - \int_\Omega G(u)\alpha\,dx\\ &+ \int_\Omega h\alpha\,dx.\end{aligned} \tag{41}$$

Thanks to (22), we write

$$2\int_\Omega f(u)u\,dx \ge c\int_\Omega F(u)\,dx - c'. \tag{42}$$

(24) yields

$$\begin{aligned}\left|\int_\Omega g(u)u\frac{\partial\alpha}{\partial t}\,dx\right| &\le \int_\Omega |g(u)u|\left|\frac{\partial\alpha}{\partial t}\right|\,dx\\ &\le c\int_\Omega (|G(u)|+1)\left|\frac{\partial\alpha}{\partial t}\right|\,dx\\ &\le c\int_\Omega |G(u)|^2\,dx + \frac{1}{2}\left\|\frac{\partial\alpha}{\partial t}\right\|^2 + c'\end{aligned} \tag{43}$$

and

$$\begin{aligned}\left|\int_\Omega g(u)u\Delta\frac{\partial\alpha}{\partial t}\,dx\right| &\le \int_\Omega |g(u)u|\left|\Delta\frac{\partial\alpha}{\partial t}\right|\,dx\\ &\le c\int_\Omega (|G(u)|+1)\left|\Delta\frac{\partial\alpha}{\partial t}\right|\,dx\\ &\le c\int_\Omega |G(u)|^2\,dx + \frac{1}{2}\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2 + c'.\end{aligned} \tag{44}$$

Now, applying Cauchy-Schwartz's and Young's inequalities, we get

$$\left|\int_\Omega h\frac{\partial\alpha}{\partial t}\,dx\right| \le c\int_\Omega h^2\,dx + \frac{1}{4}\left\|\frac{\partial\alpha}{\partial t}\right\|^2, \tag{45}$$

$$\left|\int_\Omega h\Delta\frac{\partial\alpha}{\partial t}\,dx\right| \le c\int_\Omega h^2\,dx + \frac{1}{4}\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2, \tag{46}$$

$$\left|\int_\Omega G(u)\frac{\partial\alpha}{\partial t}\,dx\right| \le c\int_\Omega |G(u)|^2\,dx + \frac{1}{4}\left\|\frac{\partial\alpha}{\partial t}\right\|^2, \tag{47}$$

$$\left|\int_\Omega G(u)\Delta\frac{\partial\alpha}{\partial t}\,dx\right| \le c\int_\Omega |G(u)|^2\,dx + \frac{1}{4}\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2. \tag{48}$$

Taking into account (42)-(48), (33) becomes

$$\begin{aligned}&\frac{dE_1}{dt} + 2\left(\|\nabla u\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 + 2\left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2\right) + \left\|\frac{\partial\alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2\\ &\quad + c\int_\Omega F(u)\,dx \le c'\int_\Omega |G(u)|^2\,dx + c''\int_\Omega h^2\,dx + c'''.\end{aligned} \tag{49}$$

Summing (49) and ε_1(41), where $\varepsilon_1 > 0$ is small enough, one has

$$\begin{aligned}&\frac{dE_2}{dt} + 2\left(\|\nabla u\|^2 + \varepsilon_1\|\nabla\alpha\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 + 2\left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2\right) + \left\|\frac{\partial\alpha}{\partial t}\right\|^2\\ &\quad + \left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2 + c\int_\Omega F(u)\,dx \le c'\int_\Omega |G(u)|^2\,dx + c''\|h\|_2^2 + \varepsilon_1\left\|\frac{\partial\alpha}{\partial t}\right\|^2\\ &\quad + \varepsilon_1\left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2 + \varepsilon_1\int_\Omega G(u)\frac{\partial\alpha}{\partial t}\,dx - \varepsilon_1\int_\Omega G(u)\alpha\,dx + \varepsilon_1\int_\Omega h\alpha\,dx + c''',\end{aligned} \tag{50}$$

where

$$E_2 = E_1 + \varepsilon_1\left[\|\alpha\|^2 + \|\nabla\alpha\|^2 + 2\left(\frac{\partial\alpha}{\partial t},\alpha\right) + 2\left(\nabla\frac{\partial\alpha}{\partial t},\nabla\alpha\right) + 2\Big(G(u),\alpha\Big)\right]. \tag{51}$$

Choosing ε_1 such that

$$\begin{aligned}\varepsilon_1\left[\|\alpha\|^2 + 2\left(\frac{\partial\alpha}{\partial t},\alpha\right) + 2\Big(G(u),\alpha\Big)\right] &\ge c\left(\|\alpha\|^2 + \left\|\frac{\partial\alpha}{\partial t}\right\|^2\right)\\ &\quad + c'\int_\Omega |G(u)|^2\,dx - c''\end{aligned} \tag{52}$$

and

$$\varepsilon_1\left[\|\nabla\alpha\|^2 + 2\left(\nabla\frac{\partial\alpha}{\partial t},\nabla\alpha\right)\right] \ge c'\left(\|\nabla\alpha\|^2 + \left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2\right). \tag{53}$$

Collecting (52)-(53) and considering (22), (25), (35) and (36), we then deduce that E_2 enjoys similar estimates as E_1, namely

$$E_2 \le c\left(\|u\|_{H^1(\Omega)}^2 + \|u\|_{k+2}^{k+2} + \|\alpha\|_{H^2(\Omega)}^2 + \left\|\frac{\partial\alpha}{\partial t}\right\|_{H^2(\Omega)}^2\right) + c' \tag{54}$$

and

$$E_2 \ge c''\left(\|u\|_{H^1(\Omega)}^2 + \|u\|_{k+2}^{k+2} + \|\alpha\|_{H^2(\Omega)}^2 + \left\|\frac{\partial\alpha}{\partial t}\right\|_{H^2(\Omega)}^2\right) - c'''. \tag{55}$$

Noting that

$$\varepsilon_1 \int_\Omega G(u)\frac{\partial \alpha}{\partial t}\,dx \le \varepsilon_1 \left\|\frac{\partial \alpha}{\partial t}\right\|^2 + c\int_\Omega |G(u)|^2\,dx, \tag{56}$$

$$\varepsilon_1 \int_\Omega G(u)\alpha\,dx \le \frac{1}{2}\varepsilon_1\|\nabla\alpha\|^2 + c'\int_\Omega |G(u)|^2\,dx \tag{57}$$

and

$$\varepsilon_1 \int_\Omega h\alpha\,dx \le \frac{1}{2}\varepsilon_1\|\nabla\alpha\|^2 + c\|h\|^2. \tag{58}$$

Taking into account (56)-(58) and thanks to (25) and then (22), in view of (50) we deduce that

$$\frac{dE_2}{dt} + c\left\|\frac{\partial u}{\partial t}\right\|^2 \le c'E_2 + c'' \tag{59}$$

and finally by applying Gronwall's lemma, we obtain the result we are looking for. □

Remark 1 Theorem 1 stipulates that solutions $(u, \alpha, \frac{\partial \alpha}{\partial t})$ to problem (18)-(21) have the following regularity $u \in L^\infty(0, T; H_0^1(\Omega) \cap L^{k+2}(\Omega))$, $\alpha \in L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega))$, $\frac{\partial \alpha}{\partial t} \in L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega))$ and $\frac{\partial u}{\partial t} \in L^2(0, T; L^2(\Omega))$, $\forall T > 0$.

Now we state a preliminary regularity result that will allow us to prove the uniqueness of the solution.

Theorem 2 *Under assumptions of theorem 1, assuming that (26) holds true and for $u_0 \in H^2(\Omega)$. Then, for $n = 2$, $\frac{\partial u}{\partial t} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$. Furthermore, for the case $n = 3$ and under the condition $\left\|\nabla \frac{\partial u}{\partial t}\right\| \le 1$, the above regularity remain true.*

Proof Indeed, differentiating (18) with respect to t, we have

$$\frac{\partial^2 u}{\partial t^2} - \Delta\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial t} = g'(u)\frac{\partial u}{\partial t}\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right) + g(u)\left(\frac{\partial^2 \alpha}{\partial t^2} - \Delta\frac{\partial^2 \alpha}{\partial t^2}\right). \tag{60}$$

We deduce from (19) that

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta\frac{\partial^2 \alpha}{\partial t^2} = -g(u)\frac{\partial u}{\partial t} + \Delta\alpha + \Delta\frac{\partial \alpha}{\partial t} - \frac{\partial \alpha}{\partial t} - G(u) + h. \tag{61}$$

Replacing (61) into (60), we arrive at

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} - \Delta\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial t} &= g'(u)\frac{\partial u}{\partial t}\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right) + g(u)\bigg(-g(u)\frac{\partial u}{\partial t}\\ &\quad + \Delta\alpha + \Delta\frac{\partial \alpha}{\partial t} - \frac{\partial \alpha}{\partial t} - G(u) + h\bigg).\end{aligned} \tag{62}$$

Multiplying (62) by $\frac{\partial u}{\partial t}$, we get, integrating over Ω

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial u}{\partial t}\right\|^2 &= -\int_\Omega f'(u)\left|\frac{\partial u}{\partial t}\right|^2 dx\\ &+ \int_\Omega g'(u)\left|\frac{\partial u}{\partial t}\right|^2\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)dx - \int_\Omega \left|g(u)\frac{\partial u}{\partial t}\right|^2 dx\\ &+ \int_\Omega g(u)\frac{\partial u}{\partial t}\Delta\alpha\,dx - \int_\Omega g(u)\frac{\partial u}{\partial t}\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)dx - \int_\Omega g(u)\frac{\partial u}{\partial t}G(u)\,dx\\ &+ \int_\Omega g(u)\frac{\partial u}{\partial t}h\,dx.\end{aligned} \tag{63}$$

From (26), it follows that

$$-\int_{\Omega} f'(u)\left|\frac{\partial u}{\partial t}\right|^2 dx \leq c\left\|\frac{\partial u}{\partial t}\right\|^2. \tag{64}$$

Moreover, we have, leaning on (23), Hölder's, Young's and Ladyzhenskaya's (for n=2) inequalities,

$$\begin{aligned}
\int_{\Omega} g'(u)\left|\frac{\partial u}{\partial t}\right|^2\left(\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\right)dx &\leq c\int_{\Omega}\left|\frac{\partial u}{\partial t}\right|^2\left|\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\right|dx\\
&\leq c\left\|\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\right\|^2+\frac{1}{10}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2, \qquad (65)\\
\int_{\Omega} g(u)\frac{\partial u}{\partial t}\Delta\alpha\, dx &\leq c\int_{\Omega}(|u|+1)\left|\frac{\partial u}{\partial t}\right||\Delta\alpha|\, dx\\
&\leq c(\|u\|^2_{H^1(\Omega)}+1)\|\alpha\|^2_{H^2(\Omega)}+\frac{1}{10}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2, \qquad (66)\\
\int_{\Omega} g(u)\frac{\partial u}{\partial t}\left(\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\right)dx &\leq c\int_{\Omega}(|u|+1)\left|\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\right|\left|\frac{\partial u}{\partial t}\right|dx\\
&\leq c(\|u\|^2_{H^1(\Omega)}+1)\left\|\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\right\|^2\\
&\quad+\frac{1}{10}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2, \qquad (67)\\
\int_{\Omega} g(u)\frac{\partial u}{\partial t}G(u)\, dx &\leq c\int_{\Omega}(|u|+1)|G(u)|\left|\frac{\partial u}{\partial t}\right|dx\\
&\leq c(\|u\|^2_{H^1(\Omega)}+1)\|G(u)\|^2+\frac{1}{10}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 \qquad (68)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\Omega} g(u)\frac{\partial u}{\partial t}h\, dx &\leq c\int_{\Omega}(|u|+1)|h|\left|\frac{\partial u}{\partial t}\right|dx\\
&\leq c(\|u\|^2_{H^1(\Omega)}+1)\|h\|^2+\frac{1}{10}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2.
\end{aligned} \tag{69}$$

Collecting (64)-(69), we arrive, in view of (63) at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|^2+\frac{1}{2}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 \leq c\left\|\frac{\partial u}{\partial t}\right\|^2\\
&\quad+c'(\|u\|^2_{H^1(\Omega)}+1)\left(\|\Delta\alpha\|^2+\left\|\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\right\|^2+\|G(u)\|^2+\|h\|^2+1\right).
\end{aligned} \tag{70}$$

Now, observing that

$$\left\|\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\right\|^2=\left\|\frac{\partial \alpha}{\partial t}\right\|^2+2\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2+\left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2, \tag{71}$$

we deduce from (25) and Gronwall's lemma the claimed result.

Now, for the case $n=3$, Ladyzhenskaya's inequality reads

$$\left\|\nabla\frac{\partial u}{\partial t}\right\|_{L^4(\Omega)} \leq c\left\|\frac{\partial u}{\partial t}\right\|^{1/4}\left\|\nabla\frac{\partial u}{\partial t}\right\|^{3/4} \tag{72}$$

and estimate (65) becomes

$$\begin{aligned}\int_{\Omega} g'(u)\left|\frac{\partial u}{\partial t}\right|^2\left(\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\right)dx &\leq c\int_{\Omega}\left|\frac{\partial u}{\partial t}\right|\left|\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\right|dx\\ &\leq c\left\|\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\right\|^2+\frac{1}{10}\left\|\nabla\frac{\partial u}{\partial t}\right\|^3,\end{aligned} \tag{73}$$

In a similar way to the case of n=2, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|^2+\frac{1}{10}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\left(1-\left\|\nabla\frac{\partial u}{\partial t}\right\|\right)\leq c\left\|\frac{\partial u}{\partial t}\right\|^2\\ &+c'(\|u\|^2_{H^1(\Omega)}+1)\Bigg(\|\Delta\alpha\|^2+\left\|\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\right\|^2+\|G(u)\|^2+\|h\|^2+1\Bigg).\end{aligned} \tag{74}$$

Under the condition $1-\left\|\nabla\frac{\partial u}{\partial t}\right\|\geq 0$, we complete the proof.

□

Consequently, we have the following uniqueness result

Theorem 3 *Let* $(u_0,\alpha_0,\alpha_1)\in H^1(\Omega)\times H^2(\Omega)^2$ *and (22)-(27) be verified. Then, problem (18)-(21) has a unique solution.*

Proof Let

$$y_i=(u^{(i)},\alpha^{(i)},\frac{\partial\alpha^{(i)}}{\partial t}) \tag{75}$$

be any solution to (18)-(20) originating from

$$y_0^{(i)}=(u_0^{(i)},\alpha_0^{(i)},\alpha_1^{(i)}),\ i=1,2. \tag{76}$$

Then, the difference

$$y=(u,\alpha,\frac{\partial\alpha}{\partial t})=y_1-y_2, \tag{77}$$

solves the system

$$\begin{aligned}\frac{\partial u}{\partial t}-\Delta u+f(u^{(1)})-f(u^{(2)})&=g(u^{(1)})\Bigg(\frac{\partial \alpha^{(1)}}{\partial t}-\Delta\frac{\partial \alpha^{(1)}}{\partial t}\Bigg)\\ &\quad-g(u^{(2)})\Bigg(\frac{\partial \alpha^{(2)}}{\partial t}-\Delta\frac{\partial \alpha^{(2)}}{\partial t}\Bigg),\end{aligned} \tag{78}$$

$$\begin{aligned}\frac{\partial^2\alpha}{\partial t^2}+\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial^2\alpha}{\partial t^2}-\Delta\frac{\partial\alpha}{\partial t}-\Delta\alpha&=-g(u^{(1)})\frac{\partial u^{(1)}}{\partial t}+g(u^{(2)})\frac{\partial u^{(2)}}{\partial t}\\ &\quad-G(u^{(1)})+G(u^{(2)}),\end{aligned} \tag{79}$$

with $y(0)=y_0=y_0^{(1)}-y_0^{(2)}$.

After computations and handle right-hand side of above equations, (78)-(79) reads as

$$\begin{aligned}\frac{\partial u}{\partial t}-\Delta u+f(u^{(1)})-f(u^{(2)})&=g(u^{(1)})\Bigg(\frac{\partial \alpha}{\partial t}-\Delta\frac{\partial \alpha}{\partial t}\Bigg)\\ &\quad+(g(u^{(1)})-g(u^{(2)}))\Bigg(\frac{\partial \alpha^{(2)}}{\partial t}-\Delta\frac{\partial \alpha^{(2)}}{\partial t}\Bigg),\end{aligned} \tag{80}$$

$$\frac{\partial^2\alpha}{\partial t^2}+\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial^2\alpha}{\partial t^2}-\Delta\frac{\partial\alpha}{\partial t}-\Delta\alpha=-g(u^{(1)})\frac{\partial u}{\partial t}-(g(u^{(1)})$$

$$-\, g(u^{(2)}))\frac{\partial u^{(2)}}{\partial t} - (G(u^{(1)}) - G(u^{(2)})). \tag{81}$$

Multiplying (80) by $\frac{\partial u}{\partial t}$, we get, integrating over Ω

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 + \int_\Omega (f(u^{(1)}) - f(u^{(2)}))\frac{\partial u}{\partial t}\,dx \\
&= \int_\Omega g(u^{(1)})\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)\frac{\partial u}{\partial t}\,dx \\
&\quad + \int_\Omega (g(u^{(1)}) - g(u^{(2)}))\left(\frac{\partial \alpha^{(2)}}{\partial t} - \Delta\frac{\partial \alpha^{(2)}}{\partial t}\right)\frac{\partial u}{\partial t}\,dx.
\end{aligned} \tag{82}$$

Multiplying (81) by $\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}$, we have, integrating over Ω

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla \alpha\|^2 + \|\Delta \alpha\|^2 + \left\|\frac{\partial \alpha}{\partial t}\right\|^2 + 2\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2\right) + \left\|\frac{\partial \alpha}{\partial t}\right\|^2 \\
&+ 2\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2 = -\int_\Omega g(u^{(1)})\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)\frac{\partial u}{\partial t} \\
&- \int_\Omega (g(u^{(1)}) - g(u^{(2)}))\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)\frac{\partial u^{(2)}}{\partial t}\,dx \\
&- \int_\Omega \left[G(u^{(1)}) - G(u^{(2)})\right]\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)dx.
\end{aligned} \tag{83}$$

Summing (82) and (83), we find

$$\begin{aligned}
&\frac{dE}{dt} + \left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\frac{\partial \alpha}{\partial t}\right\|^2 + 2\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2 \\
&= -\int_\Omega (f(u^{(1)}) - f(u^{(2)}))\frac{\partial u}{\partial t}\,dx \\
&\quad + \int_\Omega (g(u^{(1)}) - g(u^{(2)}))\left(\frac{\partial \alpha^{(2)}}{\partial t} - \Delta\frac{\partial \alpha^{(2)}}{\partial t}\right)\frac{\partial u}{\partial t}\,dx \\
&\quad - \int_\Omega (g(u^{(1)}) - g(u^{(2)}))\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)\frac{\partial u^{(2)}}{\partial t}\,dx \\
&\quad - \int_\Omega \left[G(u^{(1)}) - G(u^{(2)})\right]\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)dx,
\end{aligned} \tag{84}$$

where

$$E = \frac{1}{2}\left(\|\nabla u\|^2 + \|\nabla \alpha\|^2 + \|\Delta \alpha\|^2 + \left\|\frac{\partial \alpha}{\partial t}\right\|^2 + 2\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2\right), \tag{85}$$

satisfies

$$E \geq c\left(\|u\|^2_{H^1(\Omega)} + \|\alpha\|^2_{H^2(\Omega)} + \left\|\frac{\partial \alpha}{\partial t}\right\|^2_{H^2(\Omega)}\right). \tag{86}$$

Using (27) and the Hölder inequality ($k = 2$, when $n = 3$), we show that

$$\begin{aligned}
&\int_\Omega (f(u^{(1)}) - f(u^{(2)}))\frac{\partial u}{\partial t}\,dx \\
&\leq c\int_\Omega (|u^{(1)}|^k + |u^{(2)}|^k + 1)|u|\left|\frac{\partial u}{\partial t}\right|dx \\
&\leq c(\|u^{(1)}\|^{2k}_{H^1(\Omega)} + \|u^{(2)}\|^{2k}_{H^1(\Omega)} + 1)\|u\|^2_{H^1(\Omega)} + \frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|^2.
\end{aligned} \tag{87}$$

A new application of Hölder's inequality, yields, leaning on (23)

$$\begin{aligned}
&\int_\Omega (g(u^{(1)}) - g(u^{(2)}))\Big(\frac{\partial \alpha^{(2)}}{\partial t} - \Delta\frac{\partial \alpha^{(2)}}{\partial t}\Big)\frac{\partial u}{\partial t}\,dx \\
&\leq c\int_\Omega |u|\left|\frac{\partial \alpha^{(2)}}{\partial t} - \Delta\frac{\partial \alpha^{(2)}}{\partial t}\right|\left|\frac{\partial u}{\partial t}\right|\,dx \\
&\leq c\left\|\frac{\partial \alpha^{(2)}}{\partial t} - \Delta\frac{\partial \alpha^{(2)}}{\partial t}\right\|^2 \|u\|^2_{H^1(\Omega)} + \left\|\frac{\partial u}{\partial t}\right\|^2_{H^1(\Omega)}
\end{aligned} \tag{88}$$

and

$$\begin{aligned}
&\int_\Omega \Big[G(u^{(1)}) - G(u^{(2)})\Big]\Big(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\Big)\,dx \\
&\leq c\int_\Omega (|u^{(1)}| + |u^{(2)}| + 1)|u|\left|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right|\,dx \\
&\leq c(\|u^{(1)}\|^2_{H^1(\Omega)} + \|u^{(2)}\|^2_{H^1(\Omega)} + 1)\left\|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right\|^2 + \|u\|^2_{H^1(\Omega)}.
\end{aligned} \tag{89}$$

Taking into account (87)- (89), (84) becomes

$$\frac{dE}{dt} + c\left\|\frac{\partial u}{\partial t}\right\|^2 \leq c'E + \left\|\frac{\partial u}{\partial t}\right\|^2_{H^1(\Omega)}. \tag{90}$$

Finally, the application of Gronwall's lemma to (90), allows us to obtain the desired result. □

3 Dissipativity properties of the system

In this section we make the following assumption :

$$\forall\epsilon > 0,\ |G(s)|^2 \leq \epsilon F(s) + c_\epsilon,\ s \in \mathbb{R}. \tag{91}$$

Taking into account the assumption (91) and the theorem 3, we obtain the following theorem

Theorem 4 *Considering the hypotheses of the theorem 3 and assuming that (91) holds true. Then* $u \in L^\infty(\mathbb{R}^+; H^1_0(\Omega) \cap L^{k+2}(\Omega))$, $\alpha \in L^\infty(\mathbb{R}^+; H^1_0(\Omega) \cap H^2(\Omega))$ *and* $\frac{\partial \alpha}{\partial t} \in L^\infty(\mathbb{R}^+; H^1_0(\Omega) \cap H^2(\Omega))$.

Proof Indeed, starting from (30), (31) and (32), we are led, adding (30), (31) and δ(32), where $\delta > 0$ is small enough, to

$$\begin{aligned}
&\frac{dE_3}{dt} + \delta\|\nabla u\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\frac{\partial \alpha}{\partial t}\right\|^2 + 2\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2 + \delta\int_\Omega f(u)u\,dx \\
&= \delta\int_\Omega g(u)u\Big(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\Big)\,dx + \int_\Omega (h - G(u))\Big(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\Big)\,dx,
\end{aligned} \tag{92}$$

where

$$\begin{aligned}
E_3 = \frac{1}{2}\Big(&\delta\|u\|^2 + \|\nabla u\|^2 + 2\int_\Omega F(u)\,dx + \|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 + \left\|\frac{\partial \alpha}{\partial t}\right\|^2 \\
&+ 2\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2\Big),
\end{aligned} \tag{93}$$

enjoys

$$\begin{aligned} & c\left(\|u\|_{H^1(\Omega)}^2+\|u\|_{k+2}^{k+2}+\|\alpha\|_{H^2(\Omega)}^2+\left\|\frac{\partial \alpha}{\partial t}\right\|_{H^2(\Omega)}^2\right)-c' \leq E_3 \\ & \leq c''\left(\|u\|_{H^1(\Omega)}^2+\|u\|_{k+2}^{k+2}+\|\alpha\|_{H^2(\Omega)}^2+\left\|\frac{\partial \alpha}{\partial t}\right\|_{H^2(\Omega)}^2\right)+c'''. \end{aligned} \tag{94}$$

(22) yields

$$\delta \int_{\Omega} f(u) u \, dx \geq \delta \int_{\Omega} F(u) \, dx-\delta c'. \tag{95}$$

We have, thanks to (24) and Cauchy-Schwartz's inequality

$$\begin{aligned} \delta \int_{\Omega} g(u) u\left(\frac{\partial \alpha}{\partial t}-\Delta \frac{\partial \alpha}{\partial t}\right) dx & \leq \delta \int_{\Omega}|g(u) u|\left|\frac{\partial \alpha}{\partial t}-\Delta \frac{\partial \alpha}{\partial t}\right| dx \\ & \leq \delta c \int_{\Omega}|G(u)|^2 \, dx+\frac{\delta}{2}\left\|\frac{\partial \alpha}{\partial t}-\Delta \frac{\partial \alpha}{\partial t}\right\|^2+\delta c'. \end{aligned} \tag{96}$$

Furthermore,

$$\begin{aligned} \int_{\Omega}(h-G(u))\left(\frac{\partial \alpha}{\partial t}-\Delta \frac{\partial \alpha}{\partial t}\right) dx & \leq \int_{\Omega}(|h|+|G(u)|)\left|\frac{\partial \alpha}{\partial t}-\Delta \frac{\partial \alpha}{\partial t}\right| dx \\ & \leq c \int_{\Omega}|G(u)|^2 \, dx+\frac{1}{2}\left\|\frac{\partial \alpha}{\partial t}-\Delta \frac{\partial \alpha}{\partial t}\right\|^2 \\ & +c'\|h\|^2. \end{aligned} \tag{97}$$

In view of (95)-(97) and going from (92), we find, owing to (91)

$$\begin{aligned} & \frac{d E_3}{d t}+\delta\|\nabla u\|^2+\left(\frac{1}{2}-\delta\right)\left\|\frac{\partial \alpha}{\partial t}-\Delta \frac{\partial \alpha}{\partial t}\right\|^2+\delta \int_{\Omega} F(u) \, dx \\ & +\left\|\frac{\partial u}{\partial t}\right\|^2 \leq C \int_{\Omega}|G(u)|^2 \, dx+C', \end{aligned} \tag{98}$$

for any fixed $\delta \in (0,\ \frac{1}{2})$, $C>0$ and C' positive constant.

$$\text{(Indeed, } \left\|\tfrac{\partial \alpha}{\partial t}-\Delta \tfrac{\partial \alpha}{\partial t}\right\|^2=\left\|\tfrac{\partial \alpha}{\partial t}\right\|^2+2\left\|\nabla \tfrac{\partial \alpha}{\partial t}\right\|^2+\left\|\Delta \tfrac{\partial \alpha}{\partial t}\right\|^2\text{)}$$

Having fixed δ, on account of (91), for $\epsilon>0$, we get

$$\begin{aligned} & \frac{d E_3}{d t}+\delta\|\nabla u\|^2+\left(\frac{1}{2}-\delta\right)\left\|\frac{\partial \alpha}{\partial t}-\Delta \frac{\partial \alpha}{\partial t}\right\|^2+\delta \int_{\Omega} F(u) \, dx \\ & +\left\|\frac{\partial u}{\partial t}\right\|^2 \leq \epsilon c \int_{\Omega} F(u) dx+C_\epsilon \end{aligned} \tag{99}$$

So

$$\begin{aligned} & \frac{d E_3}{d t}+\delta\|\nabla u\|^2+\left(\frac{1}{2}-\delta\right)\left\|\frac{\partial \alpha}{\partial t}-\delta \frac{\partial \alpha}{\partial t}\right\|^2+(\delta-\epsilon c) \int_{\Omega} F(u) \, dx \\ & +\left\|\frac{\partial u}{\partial t}\right\|^2 \leq C_\epsilon \end{aligned} \tag{100}$$

Choosing $\epsilon>0$ small enough, for $\epsilon c<\delta$ to be satisfied and observing that

$$\delta\|\nabla u\|^2+\left(\frac{1}{2}-\delta\right)\left\|\frac{\partial \alpha}{\partial t}-\delta \frac{\partial \alpha}{\partial t}\right\|^2+(\delta-\epsilon c) \int_{\Omega} F(u) \, dx$$

$$\leq CE_3, \tag{101}$$

we arrive at a inequality of the form

$$\frac{dE_3}{dt} + cE_3 + \left\| \frac{\partial u}{\partial t} \right\|^2 \leq c'. \tag{102}$$

An application of Gronwall's lemma to (102) gives the expected result. □

Remark 2 It follows from theorems 1, 3 and 4 that:

1) We define the family of solving operators as :

$$\begin{aligned} S(t) :& \Phi \longrightarrow \Phi, \\ & (u_0, \alpha_0, \alpha_1) \mapsto (u(t), \alpha(t), \frac{\partial \alpha}{\partial t}(t)), \ \forall \, t \geq 0, \end{aligned} \tag{103}$$

where $\Phi = H_0^1(\Omega) \times (H_0^1(\Omega) \cap H^2(\Omega))^2$, and $(u, \alpha, \frac{\partial \alpha}{\partial t})$ is the unique solution to the problem (18)-(21).

2) Moreover, this family of resolution operators is a continuous semigroup i.e., $S(0) = Id$ and $S(t + \tau) = S(t) \circ S(\tau), \ \forall \, t, \ \tau \geq 0$.
3) Finally, it follows from (102) that $S(t)$ is dissipative in Φ, i.e. that it has a bounded absorbing set $\mathbb{B}_0 \subset \Phi$. In other words, $\forall \, B \subset \Phi$(bounded), $\exists \, t_0 = t_0(B)$ such that $t \geq t_0$ implies $S(t)B \subset \mathbb{B}_0$ (see, e.g., [22,24] for details).

4 L^∞-estimates

In this section, we will prove an estimate L^∞ of the solution, considering the one L^p already obtained. The integer p will be chosen as small as possible such that L^p estimate implies the L^∞ estimate. The interest in obtaining such a result comes from the fact that the space $L^\infty(0, T, L^\infty(\Omega))$ is not continuous embeddings into $L^\infty(0, T, L^q(\Omega))$ which requires a non-standard method of Moser's iterations (see, e.g., [2,3,30]).

Theorem 5 *We assume that the assumptions of Theorem4 hold, $n \geq 3, \frac{\partial \alpha}{\partial t} \in L^\infty(0, \infty, L^q(\Omega))$ and $\Delta \frac{\partial \alpha}{\partial t} \in L^\infty(0, \infty, L^q(\Omega))$. Let $(u, \alpha, \frac{\partial \alpha}{\partial t})$ be a classical solution to (18)-(19) defined in $[0, T]$ and k an even integer. Given $r \geq 1$ such that $\tilde{U}_r = \max\{1, \|u_0\|_\infty, U_r = sup_{t \leq T} \|u(t)\|_r\}$.*

Then the following properties hold.

1. We get

$$\tilde{U}_{2r} \leq C^{\rho(r)} \left\| \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} \right\|_{L^\infty(0,\infty,L^q(\Omega))}^{\rho(r)} r^{\rho(r)} \tilde{U}_r. \tag{104}$$

where C is a positive constant and

$$\rho(r) = \frac{q(2r-1)(n+2)}{2r\big(2nr(q-1) - (n-2)(2r-1)q\big)}, \quad q > \frac{n}{2}.$$

2. If for all $q > \frac{n}{2}$, $U_q < \infty$ then $U_\infty < \infty$.

Proof Multiplying (18) by u^{2r-1} and integrating over Ω, we get

$$\begin{aligned} &\frac{1}{2r} \frac{d}{dt} \int_\Omega u^{2r} dx + \frac{2r-1}{r^2} \int_\Omega |\nabla(u^r)|^2 dx + \int_\Omega f(u) u^{2r-1} \, dx \\ &\leq \int_\Omega g(u) \left(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} \right) u^{2r-1} \, dx. \end{aligned} \tag{105}$$

Using (22) and (23), we have

$$\begin{aligned} &\frac{1}{2r} \frac{d}{dt} \int_\Omega u^{2r} dx + \frac{2r-1}{r^2} \int_\Omega |\nabla(u^r)|^2 dx + c_0 \int_\Omega u^{2r+k} dx - c_1 \int_\Omega u^{2r-2} dx \\ &\leq c_5 \int_\Omega |u|^{2r} \left| \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} \right| dx + c_5 \int_\Omega |u|^{2r-1} \left| \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} \right| dx. \end{aligned} \tag{106}$$

It follows from the Hölder's inequality and the fact that $\int_\Omega u^{k+2r}dx \geq 0$,

$$\frac{1}{2r}\frac{d}{dt}\int_\Omega u^{2r}dx + \frac{2r-1}{r^2}\int_\Omega |\nabla(u^r)|^2 dx \leq c_1|\Omega|^{\frac{1}{Q_1}}\|u^{2r-2}\|_{p\frac{2r-1}{2r-2}} + c_5\left\|\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\right\|_{Q_2}\|u^{2r}\|_{p\frac{2r-1}{2r}} + c_5\left\|\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\right\|_q\|u^{2r-1}\|_p, \tag{107}$$

where p, q, Q_1 and Q_2 are positive constants satisfying

$$\begin{aligned} &\frac{1}{p} + \frac{1}{q} = 1 \quad p,\ q > 1,\\ &\frac{1}{Q_1} + \frac{1}{p\frac{2r-1}{2r-2}} = 1 \quad Q_1 > 1,\\ &\frac{1}{Q_2} + \frac{1}{p\frac{2r-1}{2r}} = 1 \quad Q_2 > 1. \end{aligned}$$

Since

$$\begin{aligned} \|u^{2r-1}\|_p &= \|u^r\|_{p\frac{2r-1}{r}}^{\frac{2r-1}{r}}\\ \left\|u^{2r-2}\right\|_{p\frac{2r-1}{2r-2}} &= \left\|u^r\right\|_{p\frac{2r-1}{r}}^{2r-2}\\ \left\|u^{2r}\right\|_{p\frac{2r-1}{2r}} &= \left\|u^r\right\|_{p\frac{2r-1}{r}}^2. \end{aligned} \tag{108}$$

by taking $u^r = w$, it follows from (107) and (108)

$$\frac{1}{2r}\frac{d}{dt}\|w\|_2^2 + \frac{2r-1}{r^2}\|\nabla w\|_2^2 \leq c_1^{\frac{1}{Q_1}}\|w\|_{p\kappa}^{\kappa-\frac{1}{r}} + c_5\left\|\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\right\|_{Q_2}\|w\|_{p\kappa}^2 + c_5\left\|\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\right\|_q\|w\|_{p\kappa}^\kappa, \tag{109}$$

with $\kappa = \frac{2r-1}{r}$.

Let β be such that

$$\frac{1}{\kappa p} = \beta + \frac{1-\beta}{2^*}, \tag{110}$$

with $2^* = \frac{2n}{n-2}$. We claim that $\beta \in (0, 1)$. In fact, a direct calculus gives

$$\beta = \frac{2nr - (n-2)(2r-1)p}{(n+2)(2r-1)p}.$$

Since $2nr < p(2r-1)(2n)$, we get $\beta < 1$. In addition, we get from $p < \frac{2rn}{(2r-1)(n-2)}$ that $\beta > 0$. It implies $\beta \in (0, 1)$.

Using proper interpolation inequalities, we get from (109) and (110)

$$\frac{1}{2r}\frac{d}{dt}\|w\|_2^2 + \frac{2r-1}{r^2}\|\nabla w\|_2^2 \leq c_1^{\frac{1}{Q_1}}\left(\|w\|_1^\beta\|w\|_{2^*}^{1-\beta}\right)^{\kappa-\frac{1}{r}} + c_5\left\|\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\right\|_{Q_2}\left(\|w\|_1^\beta\|w\|_{2^*}^{1-\beta}\right)^2 + c_5\left\|\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\right\|_q\left(\|w\|_1^\beta\|w\|_{2^*}^{1-\beta}\right)^\kappa, \tag{111}$$

and due to proper Sobolev's embeddings, we obtain

$$\begin{aligned}\frac{1}{2r}\frac{d}{dt}\|w\|_2^2+\frac{2r-1}{r^2}\|\nabla w\|_2^2\leq\\ +c_1\left(\|w\|_1^{\beta(\kappa-\frac{1}{r})}(6r)^{\frac{(\kappa-\frac{1}{r})(1-\beta)}{2}}\right)\left((\frac{1}{6r})^{\frac{(\kappa-\frac{1}{r})(1-\beta)}{2}}\|\nabla w\|_2^{(1-\beta)(\kappa-\frac{1}{r})}\right)\\ +c_2\left\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right\|_{Q_2}\left(\|w\|_1^{2\beta}(6r)^{(1-\beta)}\right)\left((\frac{1}{6r})^{(1-\beta)}\|\nabla w\|_2^{2(1-\beta)}\right)\\ +c_3\left\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right\|_{q}\left(\|w\|_1^{\beta\kappa}(6r)^{\frac{\kappa(1-\beta)}{2}}\right)\left((\frac{1}{6r})^{\frac{\kappa(1-\beta)}{2}}\|\nabla w\|_2^{(1-\beta)\kappa}\right).\end{aligned}\tag{112}$$

Using Young's inequality, we find

$$\begin{aligned}\frac{1}{2r}\frac{d}{dt}\|w\|_2^2+\frac{2r-1}{r^2}\|\nabla w\|_2^2\leq\delta_1\left(c_1\|w\|_1^{\frac{\beta(\kappa-\frac{1}{r})}{\delta_1}}(6r)^{\frac{(\kappa-\frac{1}{r})(1-\beta)}{2\delta_1}}\right)\\ +\frac{(\kappa-\frac{1}{r})(1-\beta)}{2}\left(\frac{1}{6r}\|\nabla w\|_2^2\right)+\beta\left(c_2\left\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right\|_{Q_2}^{1/\beta}\|w\|_1^2(6r)^{\frac{(1-\beta)}{\beta}}\right)\\ +(1-\beta)\left(\frac{1}{6r}\|\nabla w\|_2^2\right)+\delta_2\left(c_3\left\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right\|_{q}^{1/\delta_2}\|w\|_1^{\frac{\beta\kappa}{\delta_2}}(6r)^{\frac{(1-\beta)\kappa}{2\delta_2}}\right)\\ +\frac{\kappa(1-\beta)}{2}\left(\frac{1}{6r}\|\nabla w\|_2^2\right),\end{aligned}\tag{113}$$

with $\delta_1=1-\dfrac{(\kappa-\frac{1}{r})(1-\beta)}{2}$, $\delta_2=1-\dfrac{\kappa(1-\beta)}{2}$ and $\delta_1,\ \delta_2\in(0,1)$.

It follows

$$\begin{aligned}\frac{1}{2r}\frac{d}{dt}\|w\|_2^2+\frac{3r-2}{2r^2}\|\nabla w\|_2^2\leq c_1\|w\|_1^{\frac{\beta(\kappa-\frac{1}{r})}{\delta_1}}(6r)^{\frac{(\alpha-\frac{1}{r})(1-\beta)}{2\delta_1}}\\ +c_2\left\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right\|_{Q_2}^{1/\beta}\|w\|_1^2(6r)^{\frac{(1-\beta)}{\beta}}\\ +c_3\left\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right\|_{q}^{1/\delta_2}\|w\|_1^{\frac{\beta\kappa}{\delta_2}}(6r)^{\frac{(1-\beta)\kappa}{2\delta_2}}.\end{aligned}\tag{114}$$

Setting

$$2\sigma_1(r)=\frac{\beta(\kappa-\frac{1}{r})}{\delta_1},\qquad 2\sigma_2(r)=\frac{\beta\kappa}{\delta_2},\tag{115}$$

we get

$$\sigma_1(r)=\frac{\beta(\kappa-\frac{1}{r})}{2-(1-\beta)(\kappa-\frac{1}{r})},\qquad 2\sigma_2(r)=\frac{\beta\kappa}{2-(1-\beta)\kappa}.\tag{116}$$

A direct calculus gives us

$$\sigma_1(r),\ \sigma_2(r)\in(0,1),\ \forall r.\tag{117}$$

From (114)-(117), we get

$$\frac{1}{2r}\frac{d}{dt}\|w\|_2^2+\frac{3r-2}{2r^2}\|\nabla w\|_2^2 \le c_1\|w\|_1^2\,(6r)^{\frac{(\kappa-\frac{1}{r})(1-\beta)}{2\delta_1}} + c_2\left\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right\|_{Q_2}^{1/\beta}\|w\|_1^2\,(6r)^{\frac{(1-\beta)}{\beta}} + c_3\left\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right\|_{q}^{1/\delta_2}\|w\|_1^2\,(6r)^{\frac{(1-\beta)\kappa}{2\delta_2}}. \tag{118}$$

In addition, since (by a direct calculus)

$$\frac{(1-\beta)}{\beta} \ge \frac{(\kappa-\frac{1}{r})(1-\beta)}{2\delta_1} \quad \forall r\ge 1, \tag{119}$$

$$\frac{1}{\beta}\ge\frac{1}{\delta_2} \quad \forall r \ge 1 \tag{120}$$

and $Q_1 < q$, then

$$\frac{1}{2r}\frac{d}{dt}\|w\|_2^2+\frac{3r-2}{2r^2}\|\nabla w\|_2^2 \le c_1\left\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right\|_{q}^{\frac{1}{\beta}}\|w\|_1^2\,(6r)^{\frac{(1-\beta)}{\beta}}. \tag{121}$$

We set $2r\rho(r)-1=\frac{(1-\beta)}{\beta}$. Then (121) becomes

$$\frac{1}{2r}\frac{d}{dt}\|w\|_2^2+\frac{3r-2}{2r^2}\|\nabla w\|_2^2 \le c_1\left\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right\|_{q}^{\frac{1}{\beta}}\|w\|_1^2\,(6r)^{2r\rho(r)-1}. \tag{122}$$

Hence

$$\frac{d}{dt}\|w\|_2^2+\frac{3r-2}{r}\|\nabla w\|_2^2 \le c_1\left\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right\|_{q}^{\frac{1}{\beta}}\|w\|_1^2\,(6r)^{2r\rho(r)}. \tag{123}$$

We have, owing to Poincaré's inequality

$$\frac{d}{dt}\|w\|_2^2+c_0\|w\|_2^2 \le c_1\left\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right\|_{q}^{\frac{1}{\beta}}\|w\|_1^2\,(6r)^{2r\rho(r)}. \tag{124}$$

Integrating (124) between 0 and t

$$\|w(t)\|_2^2 \le \|w(0)\|_2^2 + c_0\left\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right\|_{L^\infty(0,\infty,L^q(\Omega))}^{\frac{1}{\beta}}\|w(t)\|_1^2\,(6r)^{2r\rho(r)}. \tag{125}$$

Since

$$\|w(0)\|_2^2=\int_\Omega w(0)^2dx=\int_\Omega u(0)^{2r}dx \le |\Omega|\|u(0)\|_\infty^{2r}\le|\Omega|\tilde{U}_r^{2r}, \tag{126}$$

we derive from (125) and (126)

$$\tilde{U}_{2r}^{2r}\le|\Omega|\tilde{U}_r^{2r}+c_0\left\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right\|_{L^\infty(0,\infty,L^q(\Omega))}^{\frac{1}{\beta}}(6r)^{2r\rho(r)}\tilde{U}_r^{2r}. \tag{127}$$

Remembering $\frac{1}{\beta}>0$, $2r\rho(r)>0$ and $\rho(r)=\frac{1}{2r\beta}$.

We deduce

$$\tilde{U}_{2r}\le C^{\rho(r)}\left\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right\|_{L^\infty(0,\infty,L^q(\Omega))}^{\rho(r)} r^{\rho(r)}\tilde{U}_r. \tag{128}$$

Fuerthermore, we note that

$$\rho(r)=\frac{q(2r-1)(n+2)}{2r\big(2nr(q-1)-(n-2)(2r-1)q\big)=\gamma(r)}.$$

Hence

$$\rho(r)<\frac{q(n+2)}{r\big(2(2q-n)\big)+q(n-2)}=\frac{1}{k_0r+k_1}, \tag{129}$$

with $k_0=\frac{2(2q-n)}{q(n+2)}$ and $k_1=\frac{n-2}{n+2}$.

Using Moser's iterations to (128) with $r=h$, $r=2h$, $r=2^2h$, etc, we get

$$\tilde{U}_{2^{n+1}h}\le C^{\lambda_1}\left\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\right\|^{\lambda_1}_{L^\infty(0,\infty,L^q(\Omega))}2^{\lambda_2}h^{\lambda_1}\,\tilde{U}_h, \tag{130}$$

with

$$\begin{aligned}\lambda_1&:=\rho(h)+\rho(2h)+\rho(2^2h)+\cdots+\rho(2^{n-1}h)+\rho(2^nr),\\ \lambda_2&:=\rho(2h)+2\rho(2^2h)+3\rho(2^3h)+\cdots+(n-1)\rho(2^{n-1}h)+n\rho(2^nr).\end{aligned}$$

Since $\rho(2^{n+1}h)\le\gamma(2^{n+1}h)$, a direct computation gives us $\lambda_1,\lambda_2<+\infty$ at infinity. This achieves the proof of the theorem. □

Remark 3 It is also possible to consider the case $n=2$ by choosing β such that $\frac{1}{\kappa p}=\beta+\frac{1-\beta}{6}$ into (110).

5 Spatial behavior of solutions

Phragmén-Lindelöf alternative

The objective of this subsection is to study the spatial behaviour of the solutions of (18)-(19). For this purpose, we place ourselves in the semi-infinite cylinder $R=(0,+\infty)\times D$, where D is a two-dimensional bounded domain which is smooth enough to allow the use of the divergence theorem. Moreover, we assume that such solutions exist and that $h\equiv 0$. Finally, we consider the following boundary conditions

$$u=\alpha=0 \text{ on } (0,T)\times(0,+\infty)\times\partial D, \tag{131}$$

$$u(t,0,x_2,x_3)=h_1(t,x_2,x_3),\ \alpha(t,0,x_2,x_3)=m(t,x_2,x_3) \text{ on } (0,T)\times\{0\}\times D, \tag{132}$$

$T>0$ is a given final time, and initial conditions

$$u|_{t=0}=\alpha|_{t=0}=\left.\frac{\partial\alpha}{\partial t}\right|_{t=0}=0 \quad\text{on}\quad R. \tag{133}$$

Now, we assume that the nonlinearity f is such that there exists a positive constant d so that

$$f(s)s+ds^2\ge 0 \text{ and } F(s)+ds^2\ge 0. \tag{134}$$

Remark 4 Here, the function F is as defined in the previous section. We can easily see that the function $f(s)=s^3-s$ satisfies these conditions. Actually, any function of the form $f(s)=a|s|^ks-bs,\ a,k>0$, satisfies these assumptions.

We first consider the function

$$F_\omega(t,z)=\int_0^t\int_{D(z)}e^{-\omega s}\Big(u_s u_{,1}+\alpha_s(\alpha_{,1}+\alpha_{s,1})\Big)da\,ds, \tag{135}$$

where $D(z)=\{x\in R, x_1=z\}$ and ω is an arbitrary positive constant to be fixed later; here, $\upsilon_s=\frac{\partial\upsilon}{\partial s}$ and $\upsilon_{,1}=\frac{\partial\upsilon}{\partial x_1}$. We have, after using the divergence theorem and taking into account (131)-(133),

$$\begin{aligned}F_\omega(t,z+h)-F_\omega(t,z)&=\frac{e^{-\omega t}}{2}\int_{R(z,z+h)}(|\nabla u|^2+2F(u)\\&+(\alpha_t-\Delta\alpha_t)^2+|\nabla\alpha|^2+|\Delta\alpha|^2)\,dx+\int_0^t\int_{R(z,z+h)}e^{-\omega s}\Big(|u_s|^2+|\nabla\alpha_s|^2\\&+(\alpha_s-\Delta\alpha_s)^2+\alpha_s\Delta\alpha_s+G(u)(\alpha_s-\Delta\alpha_s)\Big)\,dx\,ds\\&+\frac{\omega}{2}\int_0^t\int_{R(z,z+h)}e^{-\omega s}\Big(|\nabla u|^2+2F(u)+(\alpha_s-\Delta\alpha_s)^2\\&+|\nabla\alpha|^2+|\Delta\alpha|^2)\Big)\,dx\,ds\end{aligned} \tag{136}$$

and a direct differentiation yields

$$\begin{aligned}\frac{\partial F_\omega}{\partial z}(t,z)&=\frac{e^{-\omega t}}{2}\int_{D(z)}(|\nabla u|^2+2F(u)+(\alpha_t-\Delta\alpha_t)^2+|\nabla\alpha|^2+|\Delta\alpha|^2)\,da\\&+\int_0^t\int_{D(z)}e^{-\omega s}\Big(|u_s|^2+|\nabla\alpha_s|^2+(\alpha_s-\Delta\alpha_s)^2+\alpha_s\Delta\alpha_s\\&+G(u)(\alpha_s-\Delta\alpha_s)\Big)\,da\,ds+\frac{\omega}{2}\int_0^t\int_{D(z)}e^{-\omega s}\Big(|\nabla u|^2+2F(u)\\&+(\alpha_s-\Delta\alpha_s)^2+|\nabla\alpha|^2+|\Delta\alpha|^2)\Big)\,da\,ds.\end{aligned} \tag{137}$$

We now consider a second functional namely,

$$G_\omega(t,z)=\int_0^t\int_{D(z)}e^{-\omega s}uu_{,1}\,da\,ds. \tag{138}$$

We find, proceeding as above,

$$\begin{aligned}G_\omega(t,z+h)-G_\omega(t,z)&=\frac{e^{-\omega t}}{2}\int_{R(z,z+h)}|u|^2\,dx\\&+\int_0^t\int_{R(z,z+h)}e^{-\omega s}\Big(|\nabla u|^2+f(u)u+\frac{\omega}{2}|u|^2-g(u)u(\alpha_s-\Delta\alpha_s)\Big)\,dx\,ds\end{aligned} \tag{139}$$

and it is clear that

$$\begin{aligned}\frac{\partial G_\omega}{\partial z}(t,z)&=\frac{e^{-\omega t}}{2}\int_{D(z)}|u|^2\,da\\&+\int_0^t\int_{D(z)}e^{-\omega s}\Big(|\nabla u|^2+f(u)u+\frac{\omega}{2}|u|^2-g(u)u(\alpha_s-\Delta\alpha_s)\Big)\,da\,ds.\end{aligned} \tag{140}$$

In the sequel, we consider a positive constant τ to be fixed later and introduce the function $H_\omega = F_\omega + \tau G_\omega$. We have

$$\begin{aligned}\frac{\partial H_\omega}{\partial z}(t,z) = \frac{e^{-\omega t}}{2}\int_{D(z)}\Big(&|\nabla u|^2 + \tau|u|^2 + 2F(u) + |\nabla\alpha|^2 + |\Delta\alpha|^2 \\ &+ (\alpha_t - \Delta\alpha_t)^2\Big)\,da + \int_0^t\int_{D(z)} e^{-\omega s}\Big(|u_s|^2 + |\nabla\alpha_s|^2 + (\alpha_s - \Delta\alpha_s)^2 \\ &+ \alpha_s\Delta\alpha_s + G(u)(\alpha_s - \Delta\alpha_s)\Big)\,da\,ds + \tau\int_0^t\int_{D(z)} e^{-\omega s}\Big(|\nabla u|^2 + f(u)u \\ &- G(u)u(\alpha_s - \Delta\alpha_s)\Big)\,da\,ds + \frac{\omega}{2}\int_0^t\int_{D(z)} e^{-\omega s}\Big(|\nabla u|^2 + \tau|u|^2 + 2F(u) \\ &+ |\nabla\alpha|^2 + |\Delta\alpha|^2 + (\alpha_s - \Delta\alpha_s)^2\Big)\,da\,ds.\end{aligned} \tag{141}$$

We can choose a positive constant τ which is large enough to guarantee that

$$\tau|u|^2 + 2F(u) \geq 0 \tag{142}$$

and such that

$$\begin{aligned}&\tau f(u)u + \frac{\omega}{2}(\alpha_s - \Delta\alpha_s)^2 + (G(u) - \tau g(u)u)(\alpha_s - \Delta\alpha_s) \\ &\quad + \omega F(u) + \frac{\tau\omega}{2}|u|^2 \geq K_1(|u|^2 + |\alpha_s|^2 + |\Delta\alpha_s|^2),\ K_1 > 0.\end{aligned} \tag{143}$$

Indeed, to justify the second condition, we can note that the determinant of the matrix

$$\begin{pmatrix}\tau\omega - 2d\omega - d\tau & \frac{\tau}{2} \\ \frac{\tau}{2} & \omega\end{pmatrix} \tag{144}$$

reads $\tau\omega^2 - 2d\omega^2 - d\tau\omega - \frac{\tau^2}{4}$. It appears that we can choose ω and τ such that the above matrix is positive definite and both conditions are enjoyed. We then have the existence of a positive constant K_2 such that

$$\begin{aligned}\frac{\partial H_\omega}{\partial z}(t,z) \geq & K_2\int_0^t\int_{D(z)} e^{-\omega s}\Big(|u|^2 + |\nabla u|^2 + F(u) + |\nabla\alpha|^2 + |\Delta\alpha|^2 \\ &+ |u_s|^2 + |\nabla\alpha_s|^2 + (\alpha_s - \Delta\alpha_s)^2\Big)\,da\,ds.\end{aligned} \tag{145}$$

The next step consists of getting an estimate of $|H_\omega|$ in terms of the spatial derivative of H_ω.

It is clear that we can find, applying Cauchy-Schwarz's inequality, positive constants K_3 and K_4 such that

$$|F_\omega| \leq K_3\int_0^t\int_{D(z)} e^{-\omega s}\Big(|\nabla u|^2 + |\nabla\alpha|^2 + |u_s|^2 + |\alpha_s|^2 + |\nabla\alpha_s|^2\Big)\,da\,ds \tag{146}$$

and

$$|G_\omega| \leq K_4\int_0^t\int_{D(z)} e^{-\omega s}\Big(|u|^2 + |\nabla u|^2\Big)\,da\,ds. \tag{147}$$

It then follows that there exists a positive constant $K_5 = \frac{K_3 + \tau K_4}{K_2}$ such that

$$|H_\omega(t,z)| \leq K_5\frac{\partial H_\omega(t,z)}{\partial z}, \tag{148}$$

for every t and $z \geq 0$.

Inequality (148) is classical in the study of spatial estimates (see e.g., [29]) and yields a Phragmén-Lindelöf alternative. If there exists $z_0 \geq 0$ such that $H_\omega(t,z_0) > 0$, then the solution satisfies the estimate

$$H_\omega(t,z) \geq H_\omega(t,z_0)\exp\big(K_5^{-1}(z - z_0)\big),\ z \geq z_0. \tag{149}$$

This estimate gives information in terms of the measure defined in the cylinder. Indeed, it follows that

$$
\begin{aligned}
&\frac{e^{-\omega t}}{2}\int_{R(0,z)}\Big(|\nabla u|^2+\tau|u|^2+2F(u)+|\nabla\alpha|^2+|\Delta\alpha|^2+(\alpha_t-\Delta\alpha_t)^2\Big)\,dx\\
&+\int_0^t\int_{R(0,z)}e^{-\omega s}\Big(|u_s|^2+|\nabla\alpha_s|^2+(\alpha_s-\Delta\alpha_s)^2+\alpha_s\Delta\alpha_s\\
&+G(u)(\alpha_s-\Delta\alpha_s)\Big)\,dx\,ds+\tau\int_0^t\int_{R(0,z)}e^{-\omega s}\Big(|\nabla u|^2+f(u)u\\
&-g(u)u(\alpha_s-\Delta\alpha_s)\Big)\,dx\,ds+\frac{\omega}{2}\int_0^t\int_{R(0,z)}e^{-\omega s}\Big(|\nabla u|^2+\tau|u|^2\\
&+2F(u)+|\nabla\alpha|^2+|\Delta\alpha|^2+(\alpha_s-\Delta\alpha_s)^2\Big)\,dx\,ds,
\end{aligned}
\tag{150}
$$

tends to infinity exponentially fast, where $R(0,z)=\{x\in R, x_1\leq z\}$.

On contrary, when $H_\omega(t,z)\leq 0$, for every $z\geq 0$, we deduce that the solution decays and we can obtain an estimate of the form

$$
-H_\omega(t,z)\leq -H_\omega(t,0)\exp(-K_5^{-1}z),\ z\geq 0. \tag{151}
$$

This inequality implies that $H_\omega(t,z)$ tends to zero as z goes to infinity. Furthermore, in view of (151), it appears that

$$
E_\omega(t,z)\leq E_\omega(t,0)\exp(-K_5^{-1}z),\ z\geq 0, \tag{152}
$$

where

$$
\begin{aligned}
E_\omega(t,z)=&\frac{e^{-\omega t}}{2}\int_{R(z)}\Big(|\nabla u|^2+\tau|u|^2+2F(u)+|\nabla\alpha|^2+|\Delta\alpha|^2\\
&+(\alpha_t-\Delta\alpha_t)^2\Big)\,dx+\int_0^t\int_{R(z)}e^{-\omega s}\Big(|u_s|^2+|\nabla\alpha_s|^2+(\alpha_s-\Delta\alpha_s)^2\\
&+\alpha_s\Delta\alpha_s+G(u)(\alpha_s-\Delta\alpha_s)\Big)\,dx\,ds+\tau\int_0^t\int_{R(z)}e^{-\omega s}\Big(|\nabla u|^2+f(u)u\\
&-G(u)u(\alpha_s-\Delta\alpha_s)\Big)\,dx\,ds+\frac{\omega}{2}\int_0^t\int_{R(z)}e^{-\omega s}\Big(|\nabla u|^2+\tau|u|^2+2F(u)\\
&+|\nabla\alpha|^2+|\Delta\alpha|^2+(\alpha_s-\Delta\alpha_s)^2\Big)\,dx\,ds
\end{aligned}
\tag{153}
$$

and $R(z)=\{x\in R, x_1>z\}$.

Hence, the prove.

Theorem 6 *Let $(u,\alpha,\frac{\partial\alpha}{\partial t})$ be a solution to the problem defined by (18)-(19), the boundary conditions (131)-(132) and the initial conditions (133). Then, either this solution satisfies the growth estimate (149) or it satisfies the decay estimate*

$$
\mathcal{E}_\omega(t,z)\leq E_\omega(t,0)\exp(\omega t-K_5^{-1}z), \tag{154}
$$

where the energy $\mathcal{E}_\omega$ is defined by

$$
\begin{aligned}
\mathcal{E}_\omega = &\frac{1}{2}\int_{R(z)}\Big(|\nabla u|^2+\tau|u|^2+2F(u)+|\nabla\alpha|^2+|\Delta\alpha|^2\\
&+(\alpha_t-\Delta\alpha_t)^2\Big)\,dx+\int_0^t\int_{R(z)}\Big(|u_s|^2+|\nabla\alpha_s|^2+(\alpha_s-\Delta\alpha_s)^2+\alpha_s\Delta\alpha_s\\
&+G(u)(\alpha_s-\Delta\alpha_s)\Big)\,dx\,ds+\tau\int_0^t\int_{R(z)}\Big(|\nabla u|^2+f(u)u\\
&-g(u)u(\alpha_s-\Delta\alpha_s)\Big)\,dx\,ds+\frac{\omega}{2}\int_0^t\int_{R(z)}\Big(|\nabla u|^2+\tau|u|^2+2F(u)\\
&+|\nabla\alpha|^2+|\Delta\alpha|^2+(\alpha_s-\Delta\alpha_s)^2\Big)\,dx\,ds.
\end{aligned}
\tag{155}
$$

The amplitude term

The spatial decay estimate obtained in the previous subsection is of limited use unless we have an upper bound on the amplitude term $E_\omega(t,0)$ in terms of the boundary conditions. The aim of this subsection is then to obtain such a bound. To this end, we argue as in [22,31]. More precisely, we assume from now on that there exists a positive constant d_1 such that

$$(f(s)+d_1 s)^p \le m(F(s)+ds^2),\ p>1,\ m\ge 0. \tag{156}$$

Remark 5 We can note that the functions of the form $f(s)=a|s|^k s-d_1 s$, $a,\ k>0$, satisfy this condition.

We first note that

$$E_\omega(t,0)=-\int_0^t\int_D e^{-\omega s}\Big(u_{,1}(\rho_s+\tau\rho)+(\alpha_{s,1}+\alpha_{,1})\ell_s\Big)\,da\,ds, \tag{157}$$

whenever ρ and ℓ satisfy the same boundary conditions as u and α, respectively.

Now, functions ρ and ℓ are given by

$$\rho(s,x)=h(s,x_2,x_3)e^{-bx_1},\ \ell(s,x)=m(s,x_2,x_3)e^{-bx_1}, \tag{158}$$

with b an arbitrary positive constant. Then, using the divergence theorem, we obtain that

$$
\begin{aligned}
E_\omega(t,0)=&\int_0^t\int_R e^{-\omega s}\Big(u_{,i}(\rho_{s,i}+\tau\rho_{,i})+(\alpha_{s,i}+\alpha_{,i})\ell_{s,i}\Big)\,dx\,ds\\
&+\int_0^t\int_R e^{-\omega s}\Gamma\,dx\,ds,\ i=2,3,
\end{aligned}
\tag{159}
$$

where

$$
\begin{aligned}
\Gamma =&u_{,ii}(\rho_s+\tau\rho)+(\alpha_{s,ii}+\alpha_{,ii})\ell_s\\
=&\big(u_s+f(u)-(\alpha_s-\Delta\alpha_s)\big)(\rho_s+\tau\rho)+\big(\alpha_{ss}-\Delta\alpha_{ss}+u_s\big)\ell_s\\
=&\big(u_s+(f(u)+d_1u)-d_1u-(\alpha_s-\Delta\alpha_s)\big)(\rho_s+\tau\rho)\\
&+\big(\alpha_{ss}+\alpha_s-\Delta\alpha_{ss}+g(u)u_s+G(u)\big)\ell_s.
\end{aligned}
\tag{160}
$$

Noting that

$$\big(\alpha_{ss}-\Delta\alpha_{ss}\big)\ell_s=\frac{d}{ds}[(\alpha_s-\Delta\alpha_s)\ell_s]-(\alpha_s-\Delta\alpha_s)\ell_{ss}, \tag{161}$$

then,

$$
\begin{aligned}
\Gamma =&\big(u_s+(f(u)+d_1u)-d_1u-(\alpha_s-\Delta\alpha_s)\big)(\rho_s+\tau\rho)\\
&+\frac{d}{ds}[(\alpha_s-\Delta\alpha_s)\ell_s]-(\alpha_s-\Delta\alpha_s)\ell_{ss}+\ell_s(\alpha_s+g(u)u_s+G(u)).
\end{aligned}
\tag{162}
$$

This yields

$$\begin{aligned} E_\omega(t,0) = \int_0^t \int_R e^{-\omega s}\Big(u_{,i}(\rho_{s,i}+\tau\rho_{,i}) + (\alpha_{s,i}+\alpha_{,i})\ell_{s,i}\Big)\,dx\,ds \\ + e^{-\omega t}\int_R (\alpha_t - \Delta\alpha_t)\ell_t\,dx + \int_0^t \int_R e^{-\omega s}\Big(u_s + (f(u)+d_1u) - d_1u \\ - (\alpha_s - \Delta\alpha_s)(\rho_s+\tau\rho)\Big)\,dx\,ds + \int_0^t \int_R e^{-\omega s}\Big(\omega(\alpha_s-\Delta\alpha_s) + \alpha_s \\ + g(u)u_s + G(u)\Big)\ell_s\,dx\,ds - \int_0^t \int_R e^{-\omega s}\Big(\alpha_s - \Delta\alpha_s\Big)\ell_{ss}\,dx\,ds. \end{aligned} \tag{163}$$

Furthermore,

$$\begin{aligned} \int_0^t \int_R e^{-\omega s} u_{,i}(\rho_{s,i}+\tau\rho_{,i})\,dx\,ds \le \varepsilon_1 E_\omega(t,0) \\ +C_1\int_0^t \int_R e^{-\omega s}(\rho_{s,i}+\tau\rho_{,i})^2\,dx\,ds, \end{aligned} \tag{164}$$

$$\int_0^t \int_R e^{-\omega s}(\alpha_{s,i}+\alpha_{,i})\ell_{s,i}\,dx\,ds \le \varepsilon_2 E_\omega(t,0) + C_2\int_0^t \int_R e^{-\omega s}\ell_{s,i}^2\,dx\,ds, \tag{165}$$

$$\begin{aligned} \int_0^t \int_R e^{-\omega s}(u_s - d_1u)(\rho_s+\tau\rho)\,dx\,ds \le \varepsilon_3 E_\omega(t,0) \\ +C_3\int_0^t \int_R e^{-\omega s}(\rho_s+\tau\rho)^2\,dx\,ds, \end{aligned} \tag{166}$$

$$\begin{aligned} \int_0^t \int_R e^{-\omega s}(f(u)+d_1u)(\rho_s+\tau\rho)\,dx\,ds &\le \Big(\int_0^t \int_R e^{-\omega s}(f(u)+d_1u)^p\,dx\,ds\Big)^{1/p} \\ &\quad\times\Big(\int_0^t \int_R e^{-\omega s}(\rho_s+\tau\rho)^q\,dx\,ds\Big)^{1/q} \\ &\le \Big(\int_0^t \int_R e^{-\omega s} m(F(u)+du^2)\,dx\,ds\Big)^{1/p} \\ &\quad\times\Big(\int_0^t \int_R e^{-\omega s}(\rho_s+\tau\rho)^q\,dx\,ds\Big)^{1/q}. \end{aligned} \tag{167}$$

Using the inequality

$$a.b \le \frac{(\epsilon a)^\alpha}{\alpha} + \frac{(\epsilon^{-1}b)^\beta}{\beta},\ \text{when}\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\ \text{for any}\ \epsilon>0, \tag{168}$$

we obtain that

$$\begin{aligned} &\int_0^t \int_R e^{-\omega s}(f(u)+d_1u)(\rho_s+\tau\rho)\,dx\,ds \\ &\le \Big(\int_0^t \int_R e^{-\omega s} m(F(u)+du^2)\,dx\,ds\Big)^{1/p}\Big(\int_0^t \int_R e^{-\omega s}(\rho_s+\tau\rho)^q\,dx\,ds\Big)^{1/q} \\ &\le \varepsilon_4 E_\omega(t,0) + C_4\int_0^t \int_R e^{-\omega s}(\rho_s+\tau\rho)^q\,dx\,ds, \end{aligned} \tag{169}$$

$$\begin{aligned} &-\int_0^t \int_R e^{-\omega s}(\alpha_s-\Delta\alpha_s)(\rho_s+\tau\rho)\,dx\,ds \le \varepsilon_5 E_\omega(t,0) \\ &\quad + C_5\int_0^t \int_R e^{-\omega s}(g_s+\tau g)^2\,dx\,ds, \end{aligned} \tag{170}$$

$$-\int_0^t \int_R e^{-\omega s}(\alpha_s-\Delta\alpha_s)\ell_{ss}\,dx\,ds \le \varepsilon_6 E_\omega(t,0) + C_6\int_0^t \int_R e^{-\omega s}\ell_{ss}^2\,dx\,ds, \tag{171}$$

$$\int_0^t \int_R e^{-\omega s}\Big(\omega(\alpha_s - \Delta\alpha_s) + \alpha_s + g(u)u_s + G(u)\Big)\ell_s \, dx\, ds$$
$$\leq \varepsilon_7 E_\omega(t,0) + C_7 \int_0^t \int_R e^{-\omega s}\ell_s^2 \, dx\, ds, \tag{172}$$
$$e^{-\omega t}\int_R (\alpha_t - \Delta\alpha_t)\ell_t \, dx \leq \varepsilon_8 E_\omega(t,0) + C_8 \int_R e^{-\omega t}\ell_t^2 \, dx. \tag{173}$$

Here ε_i, $i = 1, \cdots, 8$, are positive constants to be fixed later and C_i, $i = 1, \cdots, 8$, are calculable positive constants.

Now, the ε_i's are choosen such that $\varepsilon_1 + \cdots + \varepsilon_8 = \frac{1}{2}$.

We then obtain

$$\begin{aligned} E_\omega(t,0) \leq{}& 2C_1 \int_0^t \int_R e^{-\omega s}(\rho_{s,i} + \tau\rho_{,i})^2 \, dx\, ds + 2C_2 \int_0^t \int_R e^{-\omega s}\ell_{s,i}^2 \, dx\, ds \\ &+ 2(C_3 + C_5)\int_0^t \int_R e^{-\omega s}(\rho_s + \tau\rho)^2 \, dx\, ds \\ &+ 2C_4 \int_0^t \int_R e^{-\omega s}(\rho_s + \tau\rho)^q \, dx\, ds + 2C_6 \int_0^t \int_R e^{-\omega s}\ell_{ss}^2 \, dx\, ds \\ &+ 2C_7 \int_0^t \int_R e^{-\omega s}\ell_s^2 \, dx\, ds + 2C_8 \int_R e^{-\omega t}\ell_t^2 \, dx. \end{aligned} \tag{174}$$

Moreover,

$$\begin{aligned} \int_0^t \int_R e^{-\omega s}(\rho_{s,i} + \tau\rho_{,i})^2 \, dx\, ds ={}& \frac{1}{2b}\int_0^t \int_D e^{-\omega s}(h_{s,\alpha} + \tau h_{,\alpha})^2 \, da\, ds \\ &+ \frac{b}{2}\int_0^t \int_D e^{-\omega s}(h_s + \tau h)^2 \, da\, ds, \end{aligned} \tag{175}$$

where, from now on, the index α only takes the values 2 and 3, and

$$\int_0^t \int_R e^{-\omega s}\ell_{s,i}^2 \, dx\, ds = \frac{1}{2b}\int_0^t \int_D e^{-\omega s} m_{s,\alpha}^2 \, da\, ds + \frac{b}{2}\int_0^t \int_D e^{-\omega s} m_s^2 \, da\, ds, \tag{176}$$
$$\int_0^t \int_R e^{-\omega s}(\rho_s + \tau\rho)^2 \, dx\, ds = \frac{1}{2b}\int_0^t \int_D e^{-\omega s}(h_s + \tau h)^2 \, da\, ds, \tag{177}$$
$$\int_0^t \int_R e^{-\omega s}(\rho_s + \tau\rho)^q \, dx\, ds = \frac{1}{qb}\int_0^t \int_D e^{-\omega s}(h_s + \tau h)^q \, da\, ds, \tag{178}$$
$$\int_0^t \int_R e^{-\omega s}\ell_{ss}^2 \, dx\, ds = \frac{1}{2b}\int_0^t \int_D e^{-\omega s} m_{ss}^2 \, da\, ds, \tag{179}$$
$$\int_0^t \int_R e^{-\omega s}\ell_s^2 \, dx\, ds = \frac{1}{2b}\int_0^t \int_D e^{-\omega s} m_s^2 \, da\, ds, \tag{180}$$
$$\int_R e^{-\omega t}\ell_t^2 \, dx = \frac{1}{2b}\int_D e^{-\omega t} m_t^2 \, da. \tag{181}$$

Finally, we are led to

$$
\begin{aligned}
E_\omega(t,0) \leq\; & C_1\Big(\frac{1}{b}\int_0^t\int_D e^{-\omega s}(h_{s,\alpha}+\tau h_{,\alpha})^2\,da\,ds \\
& + b\int_0^t\int_D e^{-\omega s}(h_s+\tau h)^2\,da\,ds,\Big) + C_2\Big(\frac{1}{b}\int_0^t\int_D e^{-\omega s}m_{s,\alpha}^2\,da\,ds \\
& + b\int_0^t\int_D e^{-\omega s}m_s^2\,da\,ds\Big) + \frac{C_3+C_5}{b}\int_0^t\int_D e^{-\omega s}(h_s+\tau h)^2\,da\,ds \\
& + \frac{2C_4}{qb}\int_0^t\int_D e^{-\omega s}(h_s+\tau h)^q\,da\,ds + \frac{C_6}{b}\int_0^t\int_D e^{-\omega s}m_{ss}^2\,da\,ds \\
& + \frac{C_7}{b}\int_0^t\int_D e^{-\omega s}m_s^2\,da\,ds + \frac{C_8}{b}\int_D e^{-\omega t}m_t^2\,da.
\end{aligned}
\tag{182}
$$

Remark 6 We could optimize the right-hand side of (182) by taking, for example, a suitable value of the parameter b, but it does not seem to be an easy task.

6 Conclusion

In this work we analyzed a phase-field system based on the Maxwell-Cattaneo law with two temperatures and nonlinear coupling which can describe phase transition phenomena in certain class of composite materials.

We discussed L^∞-estimate and the spatial behaviours of solutions, when such solutions exist. It has been proven more precisely:

- An energy estimate which yields a Phragmen Lindelöf alternative giving relevant information on the conditions for obtaining the exponential spatial growth and decay of solutions
- In the case of spatial decay estimate, and with additional condition into the nonlinear term, an upper bound of energy estimates has been obtained.

In our model, it would be interesting to optimize this upper bound by a suitable value of parameter taking into account numerical considerations and, to show the existence of global and exponential attractors from a theorical and numerical point of view. All of these considerations are explored in our ongoing research.

Acknowledgements A special thanks to Alain MIRANVILLE for having inspired the work of this paper.

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