Indian J Pure Appl Math (2025) 56:1673–1686
https://doi.org/10.1007/s13226-024-00551-1

ORIGINAL RESEARCH

Total graph of a lattice

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Dedicated to Late Professor T. T. Raghunathan

Received: 28 September 2022 / Accepted: 19 January 2024 / Published online: 28 March 2024
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Abstract In this paper, we prove that the study of the subgraph $T(Z^*(L))$ of the total graph $T(L)$ of a lattice L is essentially the study of the zero-divisor graph of a poset. Also, we prove that the graph $T^c(Z^*(L))$ is weakly perfect whereas $T(Z^*(L))$ is not weakly perfect. The graph $T(Z^*(L))$ and its complement $T^c(Z^*(L))$ are shown to be a perfect graph if and only if L has at most four atoms. In the concluding section, we establish that, in the context of a commutative reduced ring R, the total graph, the annihilating ideal graph, the complement of the co-annihilating ideal graph, and the complement of the comaximal ideal graph coincide.

Keywords Zero-divisor graph · Total graph · Pseudocomplemented poset · Reduced ring · Perfect graph · Comaximal ideal graph · Co-annihilating ideal graph

Mathematics Subject Classification 05C25, 06A07, 05C17, 13A70

1 Introduction

A lot of research has been carried out on assigning a graph to an algebraic structure as well as an ordered structure and investigating algebraic and combinatorial properties of that structure using the associated graph. The interplay between ring theory and graph theory has been the motivation to define and investigate many other graphs associated with algebraic structures and ordered structures. In the last decade, D. F. Anderson and A. Badawi [5] introduced a new graph associated with a commutative ring R with a non-zero unity known as a total graph of R. Let R be a commutative ring with a non-zero unity and $Z(R)$ be the set of all zero-divisors in R. The *total graph* of R is the simple graph with all elements of R as vertices, and two distinct vertices x and y are adjacent if and only if $x + y \in Z(R)$. The total graph of R is denoted by $T(\Gamma(R))$. Many induced sub-graphs of $T(\Gamma(R))$ are studied in [5], as $Nil(\Gamma(R))$, $Z(\Gamma(R))$ and $\mathrm{Reg}(\Gamma(R))$ of $T(\Gamma(R))$ with the vertex sets $Nil(R)$, $Z(R)$, and $\mathrm{Reg}(R)$, respectively. Note that $Nil(R)$ is the set of all nilpotent elements, and $\mathrm{Reg}(R)$ is the set of all regular elements of R. Analogously, Atani et al. [7] introduced the total graph of a lattice L, denoted by $T(G(L))$, where the vertex set is L. Also, they considered many subgraphs of $T(G(L))$ namely, $Z(G(L))$, $\mathrm{Reg}(G(L))$ and $Z^*(G(L))$, where the vertex sets are $Z(L)$, $L \setminus Z(L) = D(L)$ and $Z(L)^* = Z(L) \setminus \{0\}$, respectively. In this paper, we modify these notations of total graph $T(G(L))$ and its induced subgraphs $Z(G(L))$, $\mathrm{Reg}(G(L))$ and

Communicated by Shariefuddin Pirzada.

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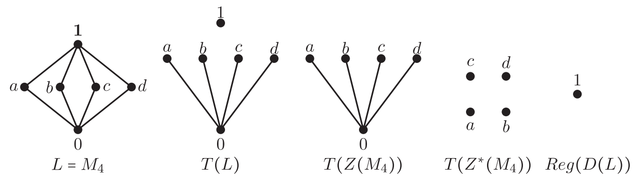

Fig. 1 Total Graph and its subgraph

$Z^*(G(L))$ as $T(L)$ and $T(Z(L))$, $T(D(L))$ and $T(Z^*(L))$, respectively. This is because the vertex set for the underlined graph is clear. With this modified notation, we write the definition given in [7].

Definition 1.1 (Atani et al. [7]) Let L be a lattice with 0. The *total graph* of L is denoted by $T(L)$ with all elements of L as the vertex set, and for any two distinct vertices x and y are adjacent if and only if $x \vee y \in Z(L)$.

We also consider the subgraphs of $T(L)$ such as $T(Z(L))$, $T(Z^*(L))$ and $T(D(L))$ with the vertex set $Z(L)$, $Z^*(L) = Z(L) \setminus \{0\}$ and $L \setminus Z(L) = D(L)$, respectively. The complement of the total graph and their induced subgraphs are denoted by $T^c(L)$, $T^c(Z(L))$, $T^c(Z^*(L))$ and $T^c(D(L))$, respectively. We illustrate the concept of a total graph with an example shown in Fig. 1.

In the second section, we prove that the graph $T^c(Z^*(L))$, that is, the complement of the subgraph $T(Z^*(L))$ of the total graph of L is a compact graph. Hence, $T^c(Z^*(L))$ is isomorphic to a zero-divisor graph of a poset P. In the third section, we prove the perfectness and weakly perfectness of the total graph and its complement. In the last section, we discuss some applications of the total graph to the co-annihilating ideal graph, the annihilating ideal graph and the comaximal ideal graph of a reduced ring R.

Throughout this paper, L denotes a bounded lattice.

2 Preliminary concepts and definitions

In this section, we quote a few definitions and notations from West [27] and Grätzer [15], which are required for the further development of the paper.

For a graph G, by $E(G)$ and $V(G)$, we denote the set of all edges and vertices, respectively. We recall that a graph is *connected* if any two distinct vertices are connected by a path. A graph G is said to be *totally disconnected* (also a *null graph*) if it has no edge, that is, G consists of only isolated vertices. The *distance* between two distinct vertices a and b, is denoted by $d(a, b)$, is the length of a shortest path connecting them (if such a path does not exist, then $d(a, b) = \infty$). The *diameter* of a graph G, denoted by $\operatorname{diam}(G)$, is equal to $\sup\{d(a, b) : a, b \in V(G)\}$. If a and b are two adjacent vertices of G, then we write $a - b$. The *eccentricity* of a vertex a is defined as $e(a) = \max\{d(a, b) : b \in V(G)\}$ and the *radius* of G is given by $\operatorname{rad}(G) = min \{e(x) : x \in V(G)\}$. A graph is *complete* if it is connected with a diameter less than or equal to one. We denote the complete graph on n vertices by K_n. The *girth* of a graph G, denoted by $gr(G)$, is the length of a shortest cycle in G, provided G contains a cycle; otherwise, $gr(G) = \infty$. A *complete bipartite* graph with part sizes m and n is denoted by $K_{m,n}$. A *clique* of a graph is its maximal complete subgraph, and the number of vertices in the largest clique of a graph G, denoted by $\omega(G)$, is called the *clique number* of G. A graph G is called *hyper-triangulated* if each edge of G is an edge of a triangle. In a graph $G = (V, E)$, a set $S \subseteq V$ is an *independent set* if the subgraph induced by S is totally disconnected. The *independence number* $\alpha(G)$ is the maximum size of an independent set in G.

A non-empty subset I of a lattice L is called a *semi-ideal* if for $a \in I, b \in L, b \leq a$ implies $b \in I$. A semi-ideal I of a lattice L is called an *ideal* of L if $a, b \in I$ implies $a \vee b \in I$. Dually, we have the concepts of a *semi-filter* and a *filter*. Clearly, $Z(L)$ and $D(L)$ are a semi-ideal and a semi-filter, respectively. A proper semi-ideal (ideal) I of a lattice L is called *prime* if $a, b \in L$ and $a \wedge b \in I$ imply $a \in I$ or $b \in I$. A proper ideal I of a lattice L is said to be a *semiprime* if $a \wedge b \in I$ and $a \wedge c \in I$ together imply that $a \wedge (b \vee c) \in I$. Dually, we have the concept of a *semiprime filter* and *prime filter*. A prime ideal (respectively, semi-ideal) P of a lattice L is said to be a *minimal prime ideal (respectively, semi-ideal)* belonging to an ideal I if $I \subseteq P$ and there exists no prime ideal (respectively, semi-ideal) Q such that $I \subsetneqq Q \subsetneqq P$. The set of all minimal prime ideals (respectively semi-ideals) of L with 0 is denoted by $\operatorname{Min}(L)$ (respectively, $\operatorname{Min}_s(L)$). An element a of a lattice L with 0 is called an *atom* if there is no $y \in L$ such that $0 < y < a$. A 0-*distributive* lattice L is a lattice with 0 in which $a \wedge b = 0 = a \wedge c$ implies $a \wedge (b \vee c) = 0$. Dually, we have the concept of a 1-*distributive* lattice. Note that every distributive lattice with 0 is a 0-distributive lattice. A lattice L is called *bounded* if L has both the least element 0 and the greatest element 1. An element y of a bounded lattice L is said to be *complement* of $x \in L$, if $x \wedge y = 0$ and $x \vee y = 1$. A bounded lattice L is called *Boolean* if L is distributive and complemented. An element $x^* \in L$ is said to be the *pseudocomplement* of $x \in L$ if $x \wedge x^* = 0$ and for $y \in L$, $x \wedge y = 0$ implies that $y \leq x^*$. A lattice L with 0 is called *pseudocomplemented* if each element of L has the pseudocomplement. Let L be a lattice 0 and let $A \subseteq L$. The *annihilator* of A is denoted by $\operatorname{ann}(A) = \{y \in L \mid x \wedge y = 0 \text{ for all } x \in A\}$. In particular, if $A = \{x\}$ then the annihilator of x is $\operatorname{ann}(x) = \{y \in L \mid x \wedge y = 0\}$.

A prime ideal P containing an ideal I of a lattice L is said to be a *maximal N-prime* of I if P is maximal with respect to the property of being contained in $Z_I(L) = \{x \in L \setminus I \mid x \wedge y \in I \text{ for some } y \in L \setminus I\}$. If $I = \{0\}$, then we say that P is a maximal N-prime ideal. We denote the set of all maximal N-prime ideals of L by $\operatorname{Max}_N(L)$.

From Atani et al. [7], it is clear that many properties of the total graph of 0-distributive lattices are similar to that of the complement of the zero-divisor graph of posets. This motivated us to study the relationship between the total graph and the zero-divisor graph of a lattice L. The zero-divisor graph with respect to an ideal was first defined in the context of a poset P with 0 by Joshi [18]. We write this definition when P is a lattice.

Definition 2.1 (Joshi [18, Definition 2.1]) Let I be an ideal of a lattice L. The *zero-divisor graph* of L with respect to I is denoted by $G_I(L)$. The set of vertices of $G_I(L)$ is $V(G_I L) = \{x \in L \setminus I \mid x \wedge y \in I \text{ for some } y \in L \setminus I\}$ and two distinct vertices x, y are adjacent if and only if $x \wedge y \in I$.

If $I = \{0\}$, then the zero-divisor graph is denoted by $G(L)$. The graph complement of $G(L)$ is denoted by $G^c(L)$.

In the following result, we prove the relationship between the total graph and the zero-divisor graph when L is a Boolean lattice.

Lemma 2.2 *Let L be a Boolean lattice. Then the subgraph $T(Z^*(L))$ of the total graph $T(L)$ is isomorphic to $G^c(L)$ under the map $x \mapsto x^*$, where x^* is the pseudocomplement of x in L.*

Proof Clearly, $V(G^c(L)) = Z^*(L) = V(T(Z^*(L)))$.

Define a map $\phi : V(T(Z^*(L))) \to V(G^c(L))$ defined by $\phi(x) = x^*$. From the fact that $x^{**} = x$, it is clear that ϕ is bijective.

Let x be adjacent to y in $T(Z^*(L))$. Then $x \vee y \in Z^*(L)$. Since L is pseudocomplemented, $\operatorname{ann}(x) \cap \operatorname{ann}(y) \neq \{0\}$. So there exists $(0 \neq)\ a \in \operatorname{ann}(x) \cap \operatorname{ann}(y)$, that is, $x \wedge a = 0$ and $y \wedge a = 0$. This implies that $a \leq x^*$ and $a \leq y^*$ which further gives $0 \neq a \leq x^* \wedge y^*$. Hence x^* and y^* are adjacent in $G^c(L)$.

Conversely, $\phi(x)$ and $\phi(y)$ are adjacent in $G^c(L)$. Hence $\phi(x) \wedge \phi(y) \neq 0$. Suppose, on the contrary, that x and y are not adjacent in $T(Z^*(L))$. Then $x \vee y \notin Z(L)$. Hence $\operatorname{ann}(x \vee y) = \{0\}$, that is, $\operatorname{ann}(x) \cap \operatorname{ann}(y) = \{0\}$. Further, $\phi(x) \wedge \phi(y) \in \operatorname{ann}(x) \cap \operatorname{ann}(y) = \{0\}$, a contradiction to $\phi(x) \wedge \phi(y) \neq 0$. Thus, x, y are adjacent in $T(Z^*(L))$.

This proves that $T(Z^*(L))$ is isomorphic to $G^c(L)$. □

From the above result, it is clear that in the case of Boolean lattices, the subgraph $T(Z^*(L))$ of the total graph $T(L)$ and the complement of zero-divisor graph $G^c(L)$ are isomorphic. This motivates us to study the total graph of a lattice in terms of the complement of the zero-divisor graph. To investigate this relationship in general, we need the following definition of a compact graph.

Definition 2.3 (Lu and Wu [22]) A simple undirected, connected graph G is called a *compact graph* if for each pair of non-adjacent vertices x, y of G, there is a vertex z with $N(x) \cup N(y) \subseteq N(z)$, where $N(x)$ is the set of all vertices adjacent to x.

The following Lemma 2.4 reveals the structure of $T^c(Z^*(L))$, which has at most one connected component, which is compact.

Lemma 2.4 *The graph $T^c(Z^*(L))$ has at most one connected component, say G, with $|G| > 1$, which is compact and $T^c(Z^*(L)) \cong G \bigcup I$, where I is an independent set (may be empty) of $T^c(Z^*(L))$. Moreover, if I is nonempty, then $I \cup \{0\}$ is an ideal of L.*

Proof We consider the following two cases.

(a) $Z(L)$ **is an ideal**: In this case, the result follows from [7, Theorem 2.4 (2)].

(b) $Z(L)$ **is not an ideal**: Suppose $Z(L)$ is not an ideal. Then there exists $a, b \in L$ such that $a \vee b \notin Z(L)$. Hence $a - b$ is an edge in $T^c(Z^*(L))$. We claim that $T^c(Z^*(L))$ has exactly one connected component.

Suppose that $T^c(Z^*(L))$ has two connected components, say G_1 and G_2. Then there is no path between G_1 and G_2. Let $a-b$ be an edge in G_1 and $c-d$ be an edge in G_2. Then $a\vee b \notin Z^*(L)$ and $c\vee d \notin Z^*(L)$. Since there is no path between G_1 and G_2, we have $a\vee c, a\vee d, b\vee c$, and $b\vee d \in Z^*(L)$. Clearly, $a\vee(b\vee c) \notin Z^*(L)$. Otherwise, $a \vee b \in Z^*(L)$, since $Z^*(L)$ is a semi-ideal and $a \vee b \leq a \vee (b \vee c) \in Z^*(L)$. Therefore $a \vee (b \vee c) \notin Z^*(L)$. Hence there is an edge from a to $b \vee c$ in $T^c(Z^*(L))$. Also, $(b \vee c) \vee d \notin Z^*(L)$. But then there is an edge from vertex d to vertex $b \vee c$. Therefore, there is a path from a to d, a contradiction to $T^c(Z^*(L))$ has two connected components. Thus, $T^c(Z^*(L))$ has exactly one connected component, say G.

We now prove that G is a compact graph. Note that, for every $x \in G$, $N_G(x) = N_{T^c(Z^*(L))}(x)$. Let x and y be any two non-adjacent vertices in G, i.e., x and y are adjacent in $T(Z^*(L))$. Hence $x \vee y \in Z^*(L)$. Let $t \in N_{T^c(Z^*(L))}(x) \cup N_{T^c(Z^*(L))}(y)$. Without loss of generality, we may assume that t is adjacent to x in $T^c(Z^*(L))$. Hence $t \vee x \notin Z^*(L)$. We claim that $t \in N_{T^c(Z^*(L))}(x \vee y)$. On the contrary assume that $t \notin N_{T^c(Z^*(L))}(x \vee y)$. So $t \vee (x \vee y) \in Z^*(L)$. This together with $Z^*(L)$ is a semi-ideal of L, we have $t \vee x \in Z^*(L)$, a contradiction. Hence $t \in N_{T^c(Z^*(L))}(x \vee y)$. This proves that $N_{T^c(Z^*(L))}(x) \cup N_{T^c(Z^*(L))}(y) \subseteq N_{T^c(Z^*(L))}(x \vee y)$. Therefore, G is a compact graph.

Thus $T^c(Z^*(L)) \cong G \cup I$, where I is an independent set (may be empty) of $T^c(Z^*(L))$. Now, assume that I is non-empty.

Let $a, b \in I$, a and b need not be distinct. Then $a \vee y, b \vee y \in Z^*(L)$ for all $y \in Z^*(L)$. We claim that, $(a \vee b) \vee y \in Z^*(L)$ for all $y \in Z^*(L)$. Since a, b are isolated vertices of $T^c(Z^*(L))$, we have $b \vee y \in Z^*(L)$ for all $y \in Z^*(L)$ and $a \vee (b \vee y) \in Z^*(L)$ for all $y \in Z^*(L)$. Hence $a \vee b \in I$. Let $x \leq a$ and $a \in I$. We claim that $x \in I$. Let $a \in I \subseteq Z^*(L)$. This implies that, for all $y \in Z^*(L)$, $a \vee y \in Z^*(L)$. Therefore $x \vee y \leq a \vee y \in Z^*(L)$ for all $y \in Z^*(L)$. Hence $x \vee y \in Z^*(L)$ for all y in $Z^*(L)$. This implies that $x \in I$, if $x \neq 0$. Hence, $I \cup \{0\}$ is an ideal of L. □

Remark 2.5 From the above proof, it is clear that $T(Z^*(L))$ is complete if and only if $Z(L)$ is ideal if and only if $T(Z(L))$ is complete. This proves Theorem 2.4(2) of [7]. Now, we provide an example of a lattice in which $Z(L)$ is an ideal; consequently, $T(Z^*(L))$ is a complete graph. Let $L = \{X \subseteq \mathbb{N} \mid |X| < \infty\} \cup \{\mathbb{N}\}$. Then L is a lattice under the set inclusion as a partial order. Clearly, $Z(L)$ is an ideal and hence $T(Z^*(L))$ is a complete graph. Note that if $|Z(L)| < \infty$, then $Z(L)$ cannot be an ideal follows from the proof of Theorem 3.2 (3) of [7].

Remark 2.6 (1) From Lemma 2.4, if $T^c(Z^*(L)) = G \cup I$, then $T(Z^*(L)) = G^c + K_{|I|}$, where $K_{|I|}$ is the complete graph on $|I|$ many vertices.

(2) Let L be a 0-distributive lattice and $x, y \in L$. Then x is adjacent to y in $T(Z^*(L))$ if and only if $\mathrm{ann}(x) \cap \mathrm{ann}(y) \neq \{0\}$. In fact, in a 0-distributive lattice, $\mathrm{ann}(x \vee y) = \mathrm{ann}(x) \cap \mathrm{ann}(y)$.

Theorem 2.7 *(Lu and Wu [22, Theorem 3.1]) A simple undirected connected graph G is the zero-divisor graph of a poset if and only if G is a compact graph.*

From Lemma 2.4, it is clear that the connected component of $T^c(Z^*(L))$ is a compact graph. Hence, by Theorem 2.7, there exists a poset P whose zero-divisor graph is isomorphic to G. Further, it is known that the diameter of the zero-divisor graph of a poset is at most 3 (c.f. [22, Proposition 2.1]) and girth is 3 or 4 if the zero-divisor graph contains a cycle, otherwise ∞ (cf [4, Theorem 3.2]). Hence, we have the following result.

Corollary 2.8 *The* $diam(T^c(Z^*(L))) \leq 3$ *and* $gr(T^c(Z^*(L))) \in \{3, 4, \infty\}$.

Remark 2.9 From [7, Theorem 2.5 (3)], it is clear that $\mathrm{gr}(T(Z(L))) = 3$ if and only if $|Z(L)| \geq 4$, where L is a 0-distributive lattice. We provide an example to show that 0-distributivity is necessary to have $\mathrm{gr}(T(Z(L))) = 3$, if $|Z(L)| \geq 4$. Consider the lattice $L = M_4$ (see Fig. 1). Clearly, L is not a 0-distributive lattice. One can easily verify that $|Z(L)| = 5$ but $gr(T(Z(L))) = \infty$. In fact, $T(Z(L)) \cong K_{1,4}$.

In view of Corollary 2.8, it is natural to ask what is the diameter of $T(Z^*(L))$. In the following result, we provide an example of a poset given in [12, Example 2.3] such that $\mathrm{diam}(T(Z^*(L))) = n$ for any $n \in \mathbb{N}$.

Proposition 2.10 *There exist a lattice L such that* $diam(T(Z^*(L))) = n$ *for any* $n \in \mathbb{N}$.

Proof Let $P = \mathbb{N} \cup \{0\}$ be a meet-semilattice defined such that $x \leq y$ for $x, y \in P$ if and only if either $x = 0$, $x = y$, or $x \neq 0$ is even and $y \in \{x-1, x+1\}$. Consider the lattice $L = \mathbf{1} \oplus P$. Then one can verify that $V(T(Z^*(L))) = L \setminus \{0, 1\}$ and $T(Z^*(L))$ is the graph such that $x \in N(y)$ if and only if either $\{x, y\} = \{k, k+1\}$ or $\{x, y\} = \{2k-2, 2k+2\}$ for some $k \in \mathbb{N} \setminus \{1\}$. Thus, if $x, y \in V(G^c(P))$ then $\mathrm{d}(x, y) = \lceil (|x-y|+1)/2 \rceil$ if x and y are both even, and otherwise $\mathrm{d}(x, y) = \lceil |x-y|/2 \rceil$ (where $\lceil q \rceil$ is the least integer n such that $n \geq q$). In particular, $T(Z^*(L))$ is connected. Moreover, given any $n \in \mathbb{N}$ with $n \geq 3$, if L_n is the subposet of L induced by $\{0, 1, \ldots, 2n-1\}$, then $\mathrm{diam}(T(Z^*(L))) = n$ (where the assumption $n \geq 3$ is necessary, so that $V(T(Z^*(L_n))) = \{1, \ldots, 2n-1\}$). □

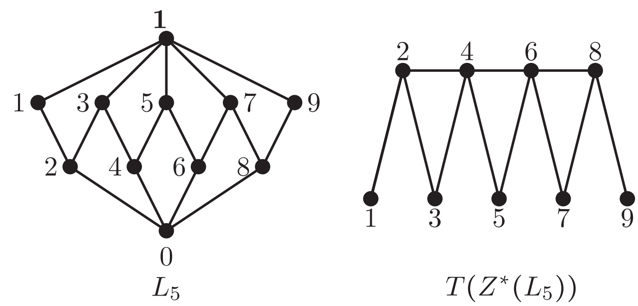

Fig. 2 Total graph with diameter 5

In Fig. 2, a lattice L_5 is depicted and the diameter of $T(Z^*(L_5))$ is 5. In the graph $T(Z(L))$, the vertex 0 is adjacent to every other vertex of the graph $T(Z(L))$. Therefore, the following result is a corollary of Lemma 2.4 and Remark 2.5.

Corollary 2.11 *(Atani et al. [7, Theorem 2.4]) The graph $T(Z(L))$ is connected with diam$(T(Z(L))) \leq 2$. Moreover, diam$(T(Z(L))) = 2$ if and only if $Z(L)$ is not an ideal.*

Lemma 2.12 *Let $D(L)$ be the set of all dense elements of L. Then $D(L)$ forms a filter in L.*

Proof Let $x \leq a$ and $x \in D(L)$. Then clearly $a \in D(L)$. Assume that $x, y \in D(L)$. Then $\mathrm{ann}(x) = \{0\}$ and $\mathrm{ann}(y) = \{0\}$. We claim that $x \wedge y \in D(L)$, that is, $\mathrm{ann}(x \wedge y) = \{0\}$. First, we claim that $\mathrm{ann}(\mathrm{ann}(x \wedge y)) = \mathrm{ann}(\mathrm{ann}(x)) \cap \mathrm{ann}(\mathrm{ann}(y))$. Clearly, $\mathrm{ann}(\mathrm{ann}(x \wedge y)) \subseteq \mathrm{ann}(\mathrm{ann}(x)) \cap \mathrm{ann}(\mathrm{ann}(y))$. Let $t \in \mathrm{ann}(\mathrm{ann}(x)) \cap \mathrm{ann}(\mathrm{ann}(y))$. Let $t_1 \in \mathrm{ann}(x \wedge y)$. This implies that $t_1 \wedge (x \wedge y) = 0$. Hence $t_1 \wedge x \in \mathrm{ann}(y)$. Therefore $t_1 \wedge x \wedge t \in \mathrm{ann}(y) \cap \mathrm{ann}(\mathrm{ann}(y)) = \{0\}$. Hence $t_1 \wedge t \in \mathrm{ann}(x)$. Also, we have $t_1 \wedge t \in \mathrm{ann}(x) \cap \mathrm{ann}(\mathrm{ann}(x)) = \{0\}$ which further implies that $t_1 \wedge t = 0$. Thus $\mathrm{ann}(\mathrm{ann}(x)) \cap \mathrm{ann}(\mathrm{ann}(y)) \subseteq \mathrm{ann}(\mathrm{ann}(x \wedge y))$. Therefore $\mathrm{ann}(\mathrm{ann}(x)) \cap \mathrm{ann}(\mathrm{ann}(y)) = \mathrm{ann}(\mathrm{ann}(x \wedge y))$. Now, $\mathrm{ann}(\mathrm{ann}(x \wedge y)) = \mathrm{ann}(\mathrm{ann}(x)) \cap \mathrm{ann}(\mathrm{ann}(y)) = \mathrm{ann}(0) \cap \mathrm{ann}(0) = L$. Therefore $\mathrm{ann}(x \wedge y) = \{0\}$, i.e., $x \wedge y \in D(L)$. Thus, $D(L)$ is a filter in L. □

In the following result, we prove that if L is a 0-distributive lattice, then $D(L)$ is a semiprime filter.

Lemma 2.13 *Let L be a 0-distributive lattice. Then $D(L)$ forms a semiprime filter in L.*

Proof By Lemma 2.12, $D(L)$ forms a filter in L. Let $x \vee y, x \vee z \in D(L)$ for $x, y, z \in L$. This implies that $\mathrm{ann}(x \vee y) = \{0\}$ and $\mathrm{ann}(x \vee z) = \{0\}$. To prove $D(L)$ is a semiprime filter, we need to show that $\mathrm{ann}(x \vee (y \wedge z)) = \{0\}$, that is, $\mathrm{ann}(x) \cap \mathrm{ann}(y \wedge z) = \{0\}$. Let $t \in \mathrm{ann}(x) \cap \mathrm{ann}(y \wedge z)$. Then $t \wedge y \in \mathrm{ann}(z)$. Since $\mathrm{ann}(x)$ is a semi-ideal and $t \in \mathrm{ann}(x)$, we have $t \wedge y \in \mathrm{ann}(x)$. Thus $t \wedge y \in \mathrm{ann}(x) \cap \mathrm{ann}(z)$. However, $\mathrm{ann}(x) \cap \mathrm{ann}(z) = \{0\}$ implies that $t \wedge y = 0$. Hence $t \in \mathrm{ann}(y)$, which further gives $t \in \mathrm{ann}(x) \cap \mathrm{ann}(y)$. But $\mathrm{ann}(x) \cap \mathrm{ann}(y) = \{0\}$. Therefore $t = 0$. Thus, $\mathrm{ann}(x \vee (y \wedge z)) = \{0\}$, i.e., $x \vee (y \wedge x) \in D(L)$. This proves that $D(L)$ is a semiprime filter in L. □

As observed in Lemma 2.4, the connected component of $T^c(Z^*(L))$ is a zero-divisor graph of some poset. Now, we explicitly construct a required poset as follows.

Construction 2.14 *Let $D(L)$ be the set of all dense elements in L. Consider L^∂, the dual lattice of L. In L, $D(L)$ forms a filter by Lemma 2.12 and hence, in L^∂, the set $D(L)$ forms an ideal. Note that in Fig. 3, $D(L) = \{d, f, g, 1_L\}$ and hence in L^∂, the set $D(L)$ is an ideal of L^∂, where 1_L is the greatest element of L and 0_L is the least element of L.*

Henceforth, we denote the ideal $D(L)$ in L^∂ by $(D(L))^\partial$ and the least element and the greatest element of L^∂ by 1_L^∂ and 0_L^∂ respectively. If there is no ambiguity, then we write 1^∂ and 0^∂.
Note that $(D(L))^\partial = \{x \in L^\partial \mid x \vee y \neq 0_L^\partial$ for any $y \in L^\partial$, where 0_L^∂ is the greatest element of $L^\partial\}$.

Let $(D(L))^\partial$ be an ideal in L^∂. Let $\mathbb{P} = (L \setminus D(L)) \cup \{1_L\}$. Then $\mathbb{P}$ is a poset under the induced partial order of L. Let $\mathbb{P}^\partial = (L^\partial \setminus (D(L))^\partial) \cup \{1_L^\partial\}$ be the dual of the poset $\mathbb{P}$. Clearly, the greatest element 1_L of L is the least element of $\mathbb{P}^\partial$ denoted by 1_L^∂ and the least element of 0_L of L is the greatest element of $\mathbb{P}^\partial$ denoted by 0_L^∂. Further, one can verify that $\mathbb{P}$ is a lattice in which $x \vee_{\mathbb{P}} y = \begin{cases} 1_L & \text{if } x \vee_L y \in D(L); \\ x \vee_L y & \text{if } x \vee_L y \notin D(L); \end{cases}$ and $x \wedge_{\mathbb{P}} y = x \wedge_L y$.

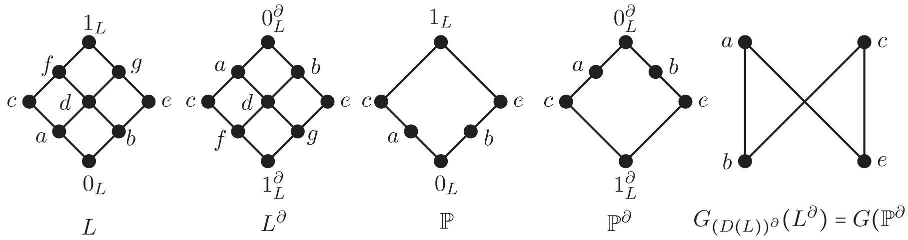

Fig. 3 Illustration of Construction 2.14

Consider the zero-divisor graph on L^∂ with respect to an ideal $(D(L))^\partial$ of L^∂, denoted by $G_{(D(L))^\partial}(L^\partial)$. Note that 1^∂ is the least element of L^∂. The vertex set of $G_{(D(L))^\partial}(L^\partial)$ is $V(G_{(D(L))^\partial}(L^\partial)) = \{x \in L^\partial \backslash (D(L))^\partial \,|\, x \wedge y \in (D(L))^\partial$ for some $y \in L^\partial \backslash (D(L))^\partial\}$, and two distinct vertices x and y are adjacent if and only if $x \wedge y \in (D(L))^\partial$ in L^∂.

Then, the following result is immediate from Joshi [18, Lemma 2.6].

Lemma 2.15 *Let $D(L)$ be the set of all dense elements of L. Then $G_{(D(L))^\partial}(L^\partial) \cong G(\mathbb{P}^\partial)$, where $\mathbb{P}^\partial$ is the dual of the poset $\mathbb{P} = (L \setminus D(L)) \cup \{1_L\}$.*

Theorem 2.16 *The connected component G of $T^c(Z^*(L))$ is isomorphic to the zero-divisor graph of L^∂ with respect to an ideal $(D(L))^\partial$, i.e., $G \cong G(\mathbb{P}^\partial)$.*

Proof Let $x \in V(G)$. Then there exists some $y \in V(G)$ such that x is adjacent to y in G, as G is a connected component of $T^c(Z^*(L))$. Hence $x \vee y \notin Z^*(L)$ for $y \in Z^*(L)$, that is, $x \vee y \in D(L)$ in L, i.e., $x \wedge y \in (D(L))^\partial$ in L^∂ for $y \in L^\partial \setminus (D(L))^\partial$. This gives that $x \in V(G(\mathbb{P}^\partial))$. Conversely, suppose $x \in V(G(\mathbb{P}^\partial))$. Then $x \wedge y \in (D(L))^\partial$ in L^∂ for some $y \in L^\partial \setminus (D(L))^\partial$. This implies that $x \vee y \in D(L)$ in L, i.e., $x \vee y \notin Z^*(L)$. Therefore $x \in V(G)$. Hence $V(G) = V(G(\mathbb{P}^\partial))$. Let x, y are adjacent in G. Then $x \vee y \notin Z^*(L)$. We prove that x and y are adjacent in $G(\mathbb{P}^\partial)$, i.e., $x \wedge y \in (D(L))^\partial$ in L^∂. Suppose $x \wedge y \notin (D(L))^\partial$ in L^∂. This implies that $x \vee y \notin D(L)$ in L. This gives that $x \vee y \in Z^*(L)$, a contradiction to $x \vee y \notin Z^*(L)$. Hence x and y are adjacent in $G(\mathbb{P}^\partial)$. Now, let x, y are adjacent in $G(\mathbb{P}^\partial)$. Hence $x \wedge y \in (D(L))^\partial$ in L^∂. Suppose x is not adjacent to y in G. Then $x \vee y \in Z^*(L)$ in L implies that, $x \wedge y \notin (D(L))^\partial$ in L^∂, a contradiction to $x \wedge y \in (D(L))^\partial$ in L^∂. Therefore x is adjacent to y in G. This proves that G is isomorphic to $G(\mathbb{P}^\partial)$. □

We illustrate Theorem 2.16 with the help of an example. Consider a lattice L depicted in Figure 4. Let $T(Z^*(L))$ be the subgraph of the total graph $T(L)$ of L and $T^c(Z^*(L))$ is $G \bigcup I$. One can easily observe that the connected component G and the zero-divisor graph of $G(\mathbb{P}^\partial)$ are the same, and $I = \{b\}$ is an independent set of $T^c(Z^*(L))$.

Remark 2.17 By Lemma 2.4 and Theorem 2.16, we have the result $T^c(Z^*(L)) = G(\mathbb{P}^\partial) \bigcup I$. In general, I need not be empty (see Figure 4).

We give a class of lattices for which I is empty. For that purpose, we need the following definition of a quasicomplemented lattice. The concept of a quasicomplemented lattice was introduced and studied by Grillet and Varlet [14].

Definition 2.18 (Grillet and Varlet [14]) A lattice L with 0 is called *quasicomplemented*, if for any $x \in L$, there exists y distinct from x such that $x \wedge y = 0$ and $\mathrm{ann}(x \vee y) = \{0\}$.

Lemma 2.19 *(Grillet and Varlet [14]) Every pseudocomplemented lattice is quasicomplemented.*

Lemma 2.20 *Let L be a quasicomplemented lattice. Then the independent set I of $T^c(Z^*(L))$ is empty. Hence $T^c(Z^*(L)) \cong G(\mathbb{P}^\partial)$.*

Proof Suppose, on the contrary, that I is non-empty. Let $x \in I \subseteq L$. Since L is quasicomplemented, there exists $y \in L$ such that $x \wedge y = 0$ and $\mathrm{ann}(x \vee y) = \{0\}$. However, as $x \in I$ and I is an independent set of $T^c(Z^*(L))$, $x \vee y \in Z^*(L)$ for all $y \in Z^*(L)$. Since $x \vee y \in Z^*(L)$, there exists some $z \in Z^*(L)$ such that, $z \wedge (x \vee y) = 0$. But then $z \in \mathrm{ann}(x \vee y) \setminus \{0\}$, a contradiction to $\mathrm{ann}(x \vee y) = \{0\}$. Therefore, I is empty. By Lemma 2.4 and Theorem 2.16, we have $T^c(Z^*(L)) \cong G(\mathbb{P}^\partial)$. □

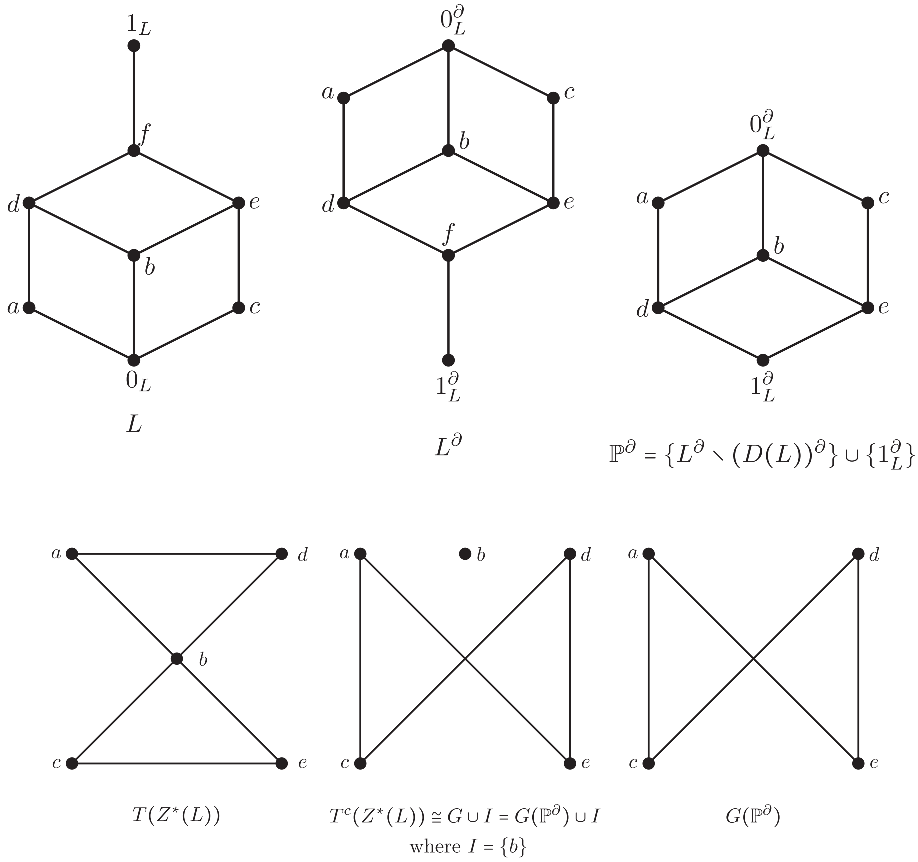

Fig. 4 Illustration of Theorem 2.16

The theory of the zero-divisor graph is well-established and has many connections with the set of prime ideals. To explore more relations with the total graph, we need the following results.

Proposition 2.21 *(Balasubramani and Venkatanarasimhan [8, Theorem 3.1]) Let L be a* 0*-distributive lattice and* $\{P_\alpha\}_{\alpha\in\Lambda}$ *the set of all prime ideals of L. Then* $\bigcap_{\alpha\in\Lambda}\{P_\alpha\} = \{0\}$*. Moreover, if* $P_1, P_2, \ldots, P_n$ *are the only distinct minimal prime ideals of L, then* $\bigcap\limits_{i=0}^{n} P_i = \{0\}$.

Theorem 2.22 *(Atani et al. [7, Theorem 3.2]) Let L be a* 0*-distributive lattice. Then* $Z(L) = \bigcup\limits_{\alpha\in\Lambda} P_\alpha$*, where* $Min(L) = \{P_\Lambda\}_{\alpha\in\Lambda}$.

The following Theorem 2.23 is well known. For the sake of completeness, we state the proof.

Theorem 2.23 ***(Prime Avoidance Theorem)*** *Let P be an ideal and* $P_1, P_2, \ldots, P_n$ *be prime ideals of a lattice L such that*

$$P \subseteq \bigcup_{i=1}^{n} P_i.$$

Then $P \subseteq P_k$ *for some k,* $1 \le k \le n$*. Moreover, if* P_i*'s are minimal prime ideals and let P be a prime ideal, then* $P = P_k$.

Proof On the contrary, assume that $P \nsubseteq P_i$ for all $1 \le i \le n$. Then there exists $x_i \in P$ such that $x_i \notin P_i$ for all $1 \le i \le n$. Now, $\bigvee\limits_{i=1}^{n} x_i \notin P_i$ and $\bigvee\limits_{i=1}^{n} x_i \in P \subseteq \bigcup\limits_{i=1}^{n} P_i$. This implies that $\bigvee\limits_{i=1}^{n} x_i \in P_k$ for some $1 \le k \le n$. Also,

$x_k \leq \bigvee_{i=1}^{n} x_i \in P_k$. Therefore, $x_k \in P_k$ is a contradiction. Hence $P \subseteq P_k$ for some k. Moreover, if P is prime, by the minimality of P_k, we have $P = P_k$. □

Remark 2.24 If a lattice L is complete, then $P \subseteq \bigcup_{i \in \Lambda} P_i$, (where Λ is an index set) implies that $P \subseteq P_k$ for some k. Since L is complete, then $\vee x_i$ exists, and the proof follows on similar lines.

Lemma 2.25 *Let L be a 0-distributive lattice. If $|Min(L)| < \infty$, then an independent set I of $T^c(Z^*(L))$ is empty. Hence $T^c(Z^*(L)) \cong G(\mathbb{P}^\partial)$.*

Proof Let $\mathrm{Min}(L) = \{P_1, P_2, \ldots, P_n\}$. On the contrary, assume that I is non-empty. Let $x \in I$. Since $|V(T^c(Z^*(L)))| \geq 2$, we have $x \vee y \in Z^*(L) = \bigcup_{i=1}^{n} P_n$, for all $y \in Z^*(L) \setminus \{x\}$. As $x \neq 0$, $x \notin P_i$ for some $P_i \in \mathrm{Min}(L)$, by Proposition 2.21. First, we claim that $P_i \setminus \left(\bigcup_{j \neq i, P_j \in \mathrm{Min}(L)} P_j \right) \neq \emptyset$. If $P_i \setminus \left(\bigcup_{j \neq i, P_j \in \mathrm{Min}(L)} P_j \right) = \emptyset$, then $P_i \subseteq \bigcup_{j \neq i, P_j \in \mathrm{Min}(L)} P_j$, and so $P_i = P_j$ for some $i \neq j$, by Theorem 2.23, a contradiction. Thus $P_i \setminus \left(\bigcup_{j \neq i, P_j \in \mathrm{Min}(L)} P_j \right) \neq \emptyset$. Let $y \in P_i \setminus \left(\bigcup_{j \neq i,\ P_j \in \mathrm{Min}(L)} P_j \right)$. As $x \vee y \in Z(L)$, we have either $x \vee y \in P_i$ or $x \vee y \in \bigcup_{j \neq i, P_j \in \mathrm{Min}(L)} P_j$. This is a contradiction, as $x \notin P_i$ and $y \notin \bigcup_{i \neq j} P_j$. Hence, I is empty. By Lemma 2.4 and Theorem 2.16, we get, $T^c(Z^*(L)) \cong G(\mathbb{P}^\partial)$. □

Since every finite 0-distributive lattice is pseudocomplemented and a pseudocomplemented lattice is quasicomplemented, we have by Lemma 2.20, the following Corollary 2.26. Also, if a lattice L is finite, then clearly $|\mathrm{Min}(L)| < \infty$. Hence, the following result also follows from Lemma 2.25.

Corollary 2.26 *Let L be a finite 0-distributive lattice. Then the independent set I of $T^c(Z^*(L))$ is empty. Hence $T^c(Z^*(L)) \cong G(\mathbb{P}^\partial)$.*

The following example shows that if $|\mathrm{Min}(L)|$ is infinite, then I need not be empty.

Remark 2.27 Consider the set $\mathbb{N}$ of natural numbers. Define a relation on $\mathbb{N}$ as $a \leq b$ if and only if $a \mid b$. Then $\mathbb{N}$ is a lattice with $x \wedge y = gcd(x, y)$, $x \vee y = lcm(x, y)$. Put $L = \mathbb{N} \oplus \mathbf{1}$. Then L is a lattice under induced partial order of $\mathbb{N}$. Clearly, $\mathbf{1}$ is the largest element of L. Further, we can see that L is a 0-distributive lattice. If $1 < |\mathrm{Min}(L)| < \infty$, then $Z(L)$ can not be an ideal. For this, let $I = Z(L) = \bigcup_{i=1}^{n} P_i$. By Prime Avoidance Theorem, $I \subseteq P_k$ for some k. Hence $\bigcup_{i \neq k} P_i \subseteq P_k$. But then $P_i = P_k$, for all i, $i \neq k$. This contradicts the fact that $|\mathrm{Min}(L)| > 1$. Thus, $Z(L)$ can not be an ideal. By Remark 2.5, $T^c(Z^*(L))$ is not a complete graph. By Lemma 2.25, $G(\mathbb{P}^\partial)$ cannot be complete. One can show that $Z^*(L) = \mathbb{N}$ and the number of minimal prime ideals are infinite, and $T(Z^*(L))$ is a complete graph. Hence, I is non-empty.

Now, we see the relation between the 0-distributive lattice L and the constructed poset $\mathbb{P}^\partial$. For this purpose, we state Rav's Theorem [25], when $|\mathrm{Min}(L)| < \infty$.

Theorem 2.28 *(Rav's Theorem [25]) Let I be a semiprime ideal of a lattice L with $|Min(L)| < \infty$. Then $I = \bigcap_{i=1}^{n} P_i$, where $P_i \in Min(L)$.*

Theorem 2.29 *Let L be a lattice such that $|Min(L)| < \infty$. Then $Min(L) = Max_N(L)$.*

Proof Let $P \in \mathrm{Min}(L)$. Since $|\mathrm{Min}(L)| < \infty$, by Theorem 2.22, we have $Z(L) = \bigcup_{i=1}^{n} P_i$, where $P_i \in \mathrm{Min}(L)$. Hence $P \subseteq Z(L)$. Suppose $P \notin \mathrm{Max}_N(L)$. Then there exists a maximal N-prime ideal Q of L such that $P \subsetneqq Q \subseteq Z(L) = \bigcup_{i=1}^{n} P_i$, where $P_i \in \mathrm{Min}(L)$. Then, by the Prime Avoidance Theorem, $Q = P_k$ for some k, $1 \leq k \leq n$. Therefore, we have $P \subsetneqq P_k \subseteq Z(L)$, a contradiction to the minimality of P_k. Hence $P \in \mathrm{Max}_N(L)$.

Conversely, suppose $R \in \mathrm{Max}_N(L)$. Let $R \notin \mathrm{Min}(L)$. Then there exists some $P \in \mathrm{Min}(L)$ such that $P \subsetneqq R \subseteq Z(L) = \bigcup_{i=1}^{n} P_i$. Then, by the Prime Avoidance Theorem, it is clear that $R = P_k$ for some $1 \leq k \leq n$. Therefore we have $P \subsetneqq R = P_k$, a contradiction to the minimality of P_k. Hence $R \in \mathrm{Min}(L)$. □

Theorem 2.30 *Let L be a* 0*-distributive lattice such that* $|Min(L)| < \infty$. *Then* $|Min(L)| = \#$ *Minimal prime filters of-* $L = |Max_N(L)|$.

Proof By Lemma 2.13, $D(L)$ forms a semiprime ideal in L^∂. Hence, by the Duality Principle and Theorem 2.28, we have $D(L) = \bigcap_{i\in\Lambda} F_i$, where F_i's are minimal prime filters of L and Λ is an index set. First, we prove that $|\{F_i \mid D(L) = \bigcap_{i\in\Lambda} F_i\}| < \infty$. Since $D(L) = \bigcap_{i\in\Lambda} F_i$, where F_i are minimal prime filters. By Theorem 2.22, we have $\bigcup_{i=1}^{n} P_i = Z(L) = L \setminus D(L) = L \setminus \bigcap_{i\in\Lambda} F_i = \bigcup_{i\in\Lambda} (L \setminus F_i)$, where $P_i \in \mathrm{Min}(L)$. Clearly, $L \setminus F_i$ is a prime ideal for every i. This gives that $L \setminus F_j \subseteq \bigcup_{i=1}^{n} P_i$ for all j, $j \in \Lambda$ and $P_i \in \mathrm{Min}(L)$. Now, by the Prime Avoidance Theorem, we have $L \setminus F_j = P_k$ for some k. This gives that $|\{F_i | D(L) = \bigcap_{i\in\Lambda} F_i\}| < \infty$.

Hence we assume that $D(L) = \bigcap_{i=1}^{m} F_i$. This implies that $Z(L) = L \setminus D(L) = L \setminus \bigcap_{i=1}^{m} F_i$, this gives that $Z(L) = \bigcup_{i=1}^{m} (L \setminus F_i)$. Now, we prove that $L \setminus F_i$'s are maximal N-prime ideals of L. Clearly, $L \setminus F_i \subseteq Z(L)$. Now, assume that $L \setminus F_i \notin \mathrm{Max}_N(L)$. Then there exists a maximal N-prime ideal, say R, such that $L \setminus F_i \subsetneqq R \subseteq Z(L) = \bigcup_{i=1}^{n} P_i$. Then, by the Prime Avoidance Theorem, we get $R = P_k$ for some $P_k \in \mathrm{Min}(L)$. This gives that $L \setminus F_i \subsetneqq R = P_k$, a contradiction to the minimality of P_k. Thus $L \setminus F_i$ is a maximal N-prime ideal of L for every i, $1 \leq i \leq m$. By Theorem 2.29, $L \setminus F_i$ is a minimal prime ideal. Further, $Z(L) = \bigcup_{i=1}^{n} P_i$, where $P_i \in \mathrm{Min}(L)$. Without loss of generality, we claim that $P_i = L \setminus F_i$ and $m = n$. Without loss of generality, assume that $m < n$. Since $P_j \subseteq Z(L) = \bigcup_{i=1}^{m} (L \setminus F_i)$ for all j, $1 \leq j \leq n$. By the Prime Avoidance Theorem, we have $P_j = L \setminus F_k$ for some k, and this is true for every P_j. Since $m < n$, then there exists k_1, k_2 such that $k_1 \neq k_2$ and $P_{k_1} = P_{k_2} = L \setminus F_i$ for some i. This contradicts the fact that $P_i \neq P_j$ for all $i \neq j$. Hence, we have $m = n$, and without loss of generality, we may assume that $P_i = L \setminus F_i$ for all i. This proves that $|\mathrm{Min}(L)| = \#$ Minimal prime filters of $L = |\mathrm{Max}_N(L)|$. □

Remark 2.31 From the above proof, we see that, F_i's are minimal prime filters of L and F_i's are minimal prime ideals of $\mathbb{P}^\partial$, and hence $|\mathrm{Min}(\mathbb{P}^\partial)| = |\mathrm{Min}(L)| = |\mathrm{Max}_N(L)| = \#$ Minimal prime filters of $L = n$.

Lemma 2.32 *A lattice L is a* 0*-distributive lattice if and only if* $\mathbb{P}^\partial$ *is* 0*-distributive.*

Proof Assume that L is 0-distributive. Let $a, b, c \in \mathbb{P}^\partial$ with $a \wedge b = 1^\partial$ and $a \wedge c = 1^\partial$, where 1^∂ is the least element of $\mathbb{P}^\partial$. This means that $a \vee b \in D(L)$ and $a \vee c \in D(L)$ in L. By Lemma 2.13, we have $a \vee (b \wedge c) \in D(L)$ in L. Then by the duality, we have $a \wedge (b \vee c) = 1^\partial$ in $\mathbb{P}^\partial$. Hence $\mathbb{P}^\partial$ is a 0-distributive lattice. Conversely, assume that $\mathbb{P}^\partial$ is 0-distributive and let $a, b, c \in L$ with $a \wedge b = 0$ and $a \wedge c = 0$. This means that $a \vee b = 0^\partial$ and $a \vee c = 0^\partial$ in $\mathbb{P}^\partial$ and $D(\mathbb{P}^\partial) = \{0^\partial\}$, where 0^∂ is the greatest element of $\mathbb{P}^\partial$. By Lemma 2.13, we have $a \vee (b \wedge c) = 0^\partial$. Then, by the duality, we get $a \wedge (b \vee c) = 0$ in L. □

By Lemma 2.13, $(D(L))^\partial$ is a semiprime ideal in L^∂, whenever L is 0-distributive. Hence, the following result easily follows from Joshi [18, Theorem 2.14] and Lemma 2.13.

Corollary 2.33 *The zero-divisor graph* $G_{(D(L))^\partial}(L^\partial)$ *is a complete bipartite graph if and only if there exist prime ideals* P_1 *and* P_2 *of* L^∂ *such that* $P_1 \bigcap P_2 = (D(L))^\partial$. *Equivalently, the zero-divisor graph* $G(\mathbb{P}^\partial)$ *is a complete bipartite graph if and only if there exist prime ideals* P_1 *and* P_2 *of* $\mathbb{P}^\partial$ *such that* $P_1 \bigcap P_2 = \{1^\partial\}$, *where* 1^∂ *is the least element of* $\mathbb{P}^\partial$.

Theorem 2.34 *Let L be a* 0*-distributive lattice with* $|Min(L)| < \infty$. *Then the following statements are equivalent,*

(1) $G(L)$ *is complete bipartite.*
(2) $G(\mathbb{P}^{\partial})$ *is complete bipartite.*
(3) $|Min(L)| = 2$.
(4) $|Min(\mathbb{P}^{\partial})| = 2$.

Proof The proof follows from Lemma 2.13 and Corollary 2.33. □

By Lemma 2.13, $D(L)$ forms a semiprime filter in L and hence in L^{∂}, $(D(L))^{\partial}$ is a semiprime ideal. In [19], Joshi and Khiste studied the connectivity of the complement of the zero-divisor graph with respect to a semiprime ideal of a lattice L. In fact, they proved the following result.

Theorem 2.35 *(Joshi, Khiste [19, Theorem 2.10 and Main Theorem]) Let L be a lattice and I be a semiprime ideal. If L has exactly two maximal N-primes of I, say* P_1 *and* P_2*. Then the following statements hold:*

(1) $(\Gamma_I(L))^c$ *is connected if and only if* $P_1 \bigcap P_2 \neq I$.
(2) If $(\Gamma_I(L))^c$ *is connected, then* $diam(\Gamma_I(L))^c = 2$.
(3) $e(x) = 2$ *for all* $x \in V((\Gamma_I(L))^c)$*. Hence* $rad((\Gamma_I(L))^c) = 2$ *and the center of* $(\Gamma_I(L))^c$ *is* $V((\Gamma_I(L))^c)$.
(4) $gr((\Gamma_I(L))^c) = 3$*. In fact,* $(\Gamma_I(L))^c$ *is hyper-triangulated.*

Hence, in view of the above result, if L is a 0-distributive lattice with $|\text{Min}(L)| < \infty$, we have $(D(L))^{\partial}$ is a semiprime ideal. Hence, we have the following result.

Theorem 2.36 *Let L be a* 0*-distributive lattice with* $|Min(L)| < \infty$*. Then the graph* $(G^c_{(D(L))^{\partial}}(L^{\partial})) = G^c(\mathbb{P}^{\partial}) = T(Z^*(L))$ *is connected if and only if* L^{∂} *does not have exactly two maximal N-prime ideals* P_1, P_2 *of* $(D(L))^{\partial}$ *such that* $P_1 \bigcap P_2 = (D(L))^{\partial}$ *if and only if* $\mathbb{P}^{\partial}$ *does not have exactly two maximal N-prime ideals* P_1 *and* P_2 *(equivalently, minimal prime ideals of* $\mathbb{P}^{\partial}$*) such that* $P_1 \bigcap P_2 = \{1^{\partial}\}$*. Furthermore, if* $T(Z^*(L))$ *is connected, then the following statements hold.*

(1) $diam\ (T(Z^*(L))) = 2$.
(2) $e(x) = 2$ *for all* $x \in V(T(Z^*(L)))$*. Hence* $rad(T(Z^*(L))) = 2$ *and the center of* $T(Z^*(L))$ *is* $V(T(Z^*(L)))$.
(3) $gr(T(Z^*(L))) = 3$*. In fact,* $T(Z^*(L))$ *is hyper-triangulated.*

Proof The proof follows from Lemma 2.15, Theorem 2.16, Remark 2.17, Lemma 2.25 and Theorem 2.30. □

From Theorem 2.36, Proposition 2.21 and Remark 2.31, we have the following result.

Corollary 2.37 *(Atani et al. [7, Theorem 3.2 (2)]) Let L be a* 0*-distributive lattice. Then* $T(Z^*(L))$ *is connected if and only if* $|Min(L)| \neq 2$*. Moreover, if* $T(Z^*(L))$ *is connected, then we have* $diam(T(Z^*(L))) \in \{1, 2\}$.

The following result immediately follows from Remark 2.27 and Theorem 2.36.

Corollary 2.38 *(Atani et al. [7, Proposition 3.6]) Let L be a* 0*-distributive lattice with* $|Min(L)| < \infty$*. Then the following statements hold.*

(1) There is no vertex of $T(Z^*(L))$ *which is adjacent to every other vertex of* $T(Z^*(L))$.
(2) If $T(Z^*(L))$ *is connected, then* $rad(Z^*(L)) = 2$.

The following result can be found in Joshi et al. [20]. Note that if $|\text{Min}(L)| = 1$, then the zero-divisor graph $G(L)$ of a 0-distributive lattice L is empty. Using the arguments given in Remark 2.27, it is clear that if $|\text{Min}(L)| < \infty$, then $G(L)$ can not be complete. Hence, we have the following result.

Theorem 2.39 *(Joshi, Waphare and Pourali [20, Lemma 4.2 and Theorem 4.2]) Let L be a* 0*-distributive lattice such that* $|Min(L)| < \infty$*. Then*

(1) $diam(G(L)) = 2$ *if and only if* $|Min(L)| = 2$.
(2) $diam(G(L)) = 3$ *if and only if* $|Min(L)| \geq 3$.

The following result immediately follows from Theorem 2.39.

Corollary 2.40 *(Atani et al. [7, Proposition 3.3]) Let L be a* 0*-distributive lattice with* $|Min(L)| < \infty$*. Then* $diam(G(L)) = 2$ *if and only if* $|Min(L)| = 2$.

Corollary 2.41 *[Atani et al. [7, Theorem 3.4 (2)]] Let L be a* 0*-distributive lattice with* $|Min(L)| < \infty$*. Then, the following statements hold.*

(1) $diam(G(L)) = 2$ if and only if $T(Z^*(L))$ is not connected and $|Z^*(L)| \geq 3$.
(2) $diam(G(L)) = 3$ if and only if $T(Z^*(L))$ is connected.

Proof (1): Suppose $\text{diam}(G(L)) = 2$. By Theorem 2.39, we have $|\text{Min}(L)| = 2$. But then by Corollary 2.37, we get $T(Z^*(L))$ is disconnected. Also, note that $\text{diam}(G(L)) = 2$, hence $|Z^*(L)| \geq 3$. Conversely, suppose $T(Z^*(L))$ is not connected and $|Z^*(L)| \geq 3$. Then by Corollary 2.37, we have $|\text{Min}(L)| = 2$ and by Theorem 2.39, we have $\text{diam}(G(L)) = 2$.
(2): Suppose $\text{diam}(G(L)) = 3$. By Theorem 2.39, $|\text{Min}(L)| \geq 3$. By Corollary 2.37, $T(Z^*(L))$ is connected. Conversely, assume that $T(Z^*(L))$ is connected. Then by Corollary 2.37, we have $|\text{Min}(L)| \neq 2$. If $|\text{Min}(L)| = 1$, then $G(L)$ is empty. Hence $|\text{Min}(L)| \geq 3$. Thus by Theorem 2.39, we have $\text{diam}(G(L)) = 3$. □

3 Coloring of a total graph

In this section, we provide a counter-example of a lattice L for which $T(Z^*(L))$ is not weakly perfect. Moreover, we prove that if $|\text{Min}(L)| < \infty$, then $T^c(Z^*(L))$ is weakly perfect. For this purpose, we need the following results.

Theorem 3.1 *(Joshi and Mundlik [17, Theorem 4.2]) Let L be a 0-distributive lattice. Then P is a minimal prime ideal of L if and only if P is a minimal prime semi-ideal of L.*

Theorem 3.2 *(Halaš and Jukl [16, Theorem 2.9], Joshi [18, Corollary 2.11]) Let P be a poset and $G(P)$ be its zero-divisor graph. If $\omega(G(P)) < \infty$, then P has finite number of minimal prime semi-ideals and if n is this number, then $\chi(G(P)) = \omega(G(P)) = n$. Moreover, if P is an atomic poset, then $\chi(G(P)) = \omega(G(P)) = |At(P)| = n$, where $At(P)$ is the set of all atoms of P.*

In Theorem 2.16, it is proved that the connected component G of $T^c(Z^*(L))$ is isomorphic to the zero-divisor graph $G(\mathbb{P}^\partial)$ of the poset $\mathbb{P}^\partial$, i.e., $G \cong G(\mathbb{P}^\partial)$. In fact, we have $T^c(Z^*(L)) = G \cup I = G(\mathbb{P}^\partial) \cup I$, where I is an independent set of $T^c(Z^*(L))$. Therefore if $\omega(G(\mathbb{P}^\partial)) < \infty$, then $G(\mathbb{P}^\partial)$ is weakly perfect. Hence, $T^c(Z^*(L))$ is weakly perfect.
Now, we compute $\chi(T^c(Z^*(L)))$ in terms of I-atoms.

Definition 3.3 (Beran [10]) Let I be an ideal of a lattice L. An element $i \in L$ is called an I-atom, if the following conditions hold,

(1) $i \notin I$, and,
(2) for $x \in L$, if $x < i$, then $x \in I$.

Dually, we have the concept of F-coatom for a given filter F of P. If $I = \{0\}$, then I-atom is nothing but an atom in the usual sense. As noted earlier, $(D(L))^\partial$ is a semiprime ideal of L^∂ whenever L is 0-distributive, we have the following remark.

Remark 3.4 An element x is a $(D(L))^\partial$-atom of L^∂ if and only if x is an atom of $\mathbb{P}^\partial$. Hence, we have the number of $(D(L))^\partial$-atoms is equal to the number of atoms of $\mathbb{P}^\partial$.

Note that, if $\omega(T^c(Z^*(L)))$ is finite, then $\omega(G(\mathbb{P}^\partial))$ is finite. Hence, by Theorem 3.2, $\mathbb{P}^\partial$ has finitely many minimal prime semi-ideals (equivalently, finitely many minimal prime ideals, if $\mathbb{P}^\partial$ is 0-distributive), say n. Hence, by Lemma 2.25 and Lemma 2.32, we have the following result.

Theorem 3.5 *Assume that $\omega(T^c(Z^*(L))) < \infty$. Then $T^c(Z^*(L))$ is weakly perfect. Moreover, if L is a 0-distributive lattice with $|Min(L)| = n$, then $\omega(T^c(Z^*(L))) = \chi(T^c(Z^*(L))) = n = \omega(G(\mathbb{P}^\partial)) = \chi(G(\mathbb{P}^\partial))$.*

The following example was suggested by Professor Peter Cameron in different context. However, the same example shows that the total graph is not weakly perfect and hence by Theorem 3.5, the complement of the zero-divisor graphs of a ordered set is not weakly perfect.

Example 3.6 Consider the lattice depicted in Fig. 5. Then $\omega(T(Z^*(L))) = 7$ and $\chi(T(Z^*(L))) = 8$. Therefore, $T(Z^*(L))$ is not weakly perfect and hence by Theorem 3.5, $G^c(L^\partial)$ is not weakly perfect.

We close this section with a characterization of $T(Z^*(L))$ to be perfect.

Definition 3.7 (Gallai [13]) A graph G is *perfect* if the clique number $\omega(H)$ is the same as the chromatic number $\chi(H)$ for every induced subgraph H of G.

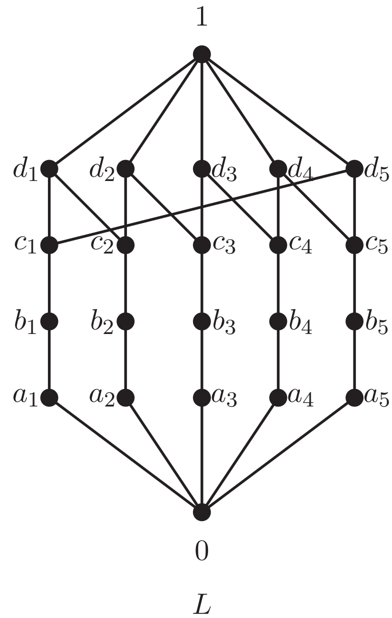

Fig. 5 A lattice whose total graph is not weakly perfect

The following result essentially follows from Khandekar and Joshi [21] (see also [24]).

Theorem 3.8 *(Khandekar and Joshi [21]) Let L be a finite 0-distributive lattice. Then the zero-divisor graph $G(L)$ is perfect if and only if $\omega(G(L)) \leq 4$.*

Theorem 3.9 *Let L be a finite 0-distributive lattice. Then*

(1) $T(Z^*(L))$ is perfect.
(2) $T^c(Z^*(L))$ is perfect.
(3) $|Min(L)| \leq 4$.
(4) $|Min(\mathbb{P}^{\partial})| \leq 4$.
(5) $|At(L)| \leq 4$.
(6) $|(D(L))^{\partial} - atoms| \leq 4$.
(7) $\omega(T(Z^*(L)) \leq 4$.
(8) $\omega(T^c(Z^*(L)) \leq 4$.

4 Applications to the co-annihilating ideal graphs

Let R be a commutative ring with identity, and $Id(R)$ denotes the set of all ideals of R. As we know, $Id(R)$ forms a complete lattice with respect to inclusion, with (0) and R as its least and greatest elements, respectively. If R is a reduced ring, then $(Id(R), \leq)$ is a modular as well as 0-1 distributive lattice. Now, we denote the lattice $Id(R)$ by $\mathbb{L}$. With this preparation, we prove that in the case of a reduced Artinian ring, the complement of the co-annihilating ideal graph $\mathfrak{A}'_R$ of R is the same as the total graph of $\mathbb{L}$.

Definition 4.1 (Akbari et al. [3]) Let R be a commutative ring with identity. The *co-annihilating ideal graph* of R, denoted by $\mathfrak{A}_R$ is a graph whose vertex set is the set of all non-zero proper ideals of R and two distinct vertices I and J are adjacent whenever $\mathrm{Ann}(I) \cap \mathrm{Ann}(J) = \{0\}$, where $\mathrm{Ann}(I) = \{x \in R \mid xi = 0 \text{ for all } i \in I\}$.

Observation 4.2 *(Akbari et al. [3, Observation 1.1]) Let R be an Artinian ring. Then for any non-zero proper ideal I of R, $\mathrm{Ann}(I) \neq \{0\}$. Hence, if $\mathrm{Ann}(I) \cap \mathrm{Ann}(J) = \{0\} = \mathrm{Ann}(I + J)$, then $I + J = R$.*

Theorem 4.3 *Let R be an Artinian reduced commutative ring with identity and $\mathbb{L}$ be the lattice $Id(R)$ of all ideals of R. Then $\mathfrak{A}^c_R = T(Z^*(\mathbb{L}))$.*

Proof Let R be an Artinian reduced commutative ring with identity. Let $\mathbb{L}$ be the lattice $Id(R)$ of all ideals of R. Clearly, $V(\mathfrak{A}_R^c) = V(T(Z^*(\mathbb{L})))=$ {non-zero proper ideals of R}. Assume that $I, J \in V(T(Z^*(\mathbb{L})))$. Suppose I is adjacent to J in $T(Z^*(\mathbb{L}))$, that is, $I \vee J \in Z^*(L)$. We claim that I is adjacent to J in $\mathfrak{A}_R^c$. As $I \vee J \in Z^*(\mathbb{L})$, so there exists some non-zero proper ideal K of R, such that $K \wedge (I \vee J) = 0$. Note that $K \wedge I \leq K \wedge (I \vee J) = 0$. This gives that $K \wedge I = 0$. Similarly, $K \wedge J = 0$. As R is a reduced ring, therefore we have $K \wedge I = KI = 0$ and $K \wedge J = KJ = 0$. This implies that $K \in \mathrm{Ann}(I)$ and $K \in \mathrm{Ann}(J)$. Thus $K \in \mathrm{Ann}(I) \cap \mathrm{Ann}(J)$. Hence $\mathrm{Ann}(I) \cap \mathrm{Ann}(J) \neq \{0\}$, that is, I and J are adjacent in $\mathfrak{A}_R^c$.

Suppose I and J are adjacent in $\mathfrak{A}_R^c$ for any non-zero proper ideals I and J of R. We claim that I is adjacent to J in $T(Z^*(\mathbb{L}))$. Let $\mathrm{Ann}(I) \cap \mathrm{Ann}(J) \neq \{0\}$. Then $K = \mathrm{Ann}(I) \cap \mathrm{Ann}(J)$ gives that $KI = \{0\} = KJ$. As R is reduced, therefore, $K \cap I = \{0\} = K \cap J$, i.e., $K \wedge I = 0 = K \wedge J$. Since $Id(R)$ is 0-distributive, $K \wedge (I \vee J) = 0$. This implies that $I \vee J \in Z^*(\mathbb{L})$, that is, I and J are adjacent in $T(Z^*(\mathbb{L}))$. Therefore, $T(Z^*(\mathbb{L})) = \mathfrak{A}_R^c$. □

The *comaximal ideal graph* $CG(R)$ of a commutative ring R with identity was introduced in [28]. The graph $CG(R)$ is a simple graph with its vertices the non-zero proper ideals of R and two distinct vertices I and J are adjacent if and only if $I + J = R$. If R is an Artinian ring, then the following result is due to Akbari et al. [3].

Corollary 4.4 *(Akbari et al. [3, Corollary 1.2]) Let R be an Artinian ring. Then $\mathfrak{A}_R = CG(R)$.*

In [26], Visweswaran and Patel introduced the annihilating ideal graph (also known as the sum annihilating ideal graph) of a commutative ring R with identity, denoted by $\Omega(R)$. The vertex set of $\Omega(R)$ equals the set of all nonzero annihilating ideals of R and two distinct vertices I and J are adjacent if and only if $I + J$ is an annihilating ideal of R.
The following Lemma 4.5 is due to Aghapouramin and Nikmehr [2].

Lemma 4.5 *(Aghapouramin and Nikmehr [2, Lemma 2.1]) Let I, J be ideals of R. Then $I - J$ is an edge of $\Omega(R)$ if and only if* $\mathrm{Ann}(I) \cap \mathrm{Ann}(J) \neq \{0\}$.

The following Observation 4.6 is an easy consequence of Lemma 4.5.

Observation 4.6 *(1) Let R be a commutative ring with identity. Then $\Omega(R) = \mathfrak{A}_R^c$.*
(2) If R is an Artinian ring, then $\Omega(R) = \mathfrak{A}_R^c = (CG(R))^c$.

The following observation is a clear consequence of Theorem 4.3 and Corollary 4.4.

Corollary 4.7 *Let R be a finite reduced commutative ring with identity. Then $T^c(Z^*(\mathbb{L})) \cong \mathfrak{A}_R \cong CG(R) \cong (\Omega(R))^c$. Hence $CG(R)$, $(CG(R))^c$, $\Omega(R)$ and $(\Omega(R))^c$ are perfect graphs if and only if $\omega(T^c(Z^*(\mathbb{L}))) \leq 4$.*

Acknowledgements The authors are thankful to the referee for his/her suggestions that improved the presentation of the paper. The first author is grateful to the Department of Mathematics, Savitribai Phule Pune University, Pune, for awarding Professor T. T. Raghunathan fellowship during his Master's Degree (2010–2012). Also, the authors are thankful to Mr. Nilesh Khandekar for his valuable suggestions and fruitful discussion during the completion of this paper.

Author contributions Both authors contributed equally to the study of the total graph of a lattice. Both authors read and approved the final version of the manuscript.

Data availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that there is no conflict of interests regarding the publishing of this paper.

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