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ORIGINAL RESEARCH

Inequalities for operators and operator pairs in Hilbert spaces

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Abstract In this paper, our goal is to establish novel inequalities for the Euclidean operator radius of a pair of bounded linear operators defined on a complex Hilbert space. Additionally, we use the Heilbronn inequality to derive further inequalities relevant to both single and pairs of Hilbert space operators.

Keywords Numerical radius · Operator norm · Hilbert sapce · Euclidean operator radius · Heilbronn inequality

Mathematics Subject Classification 47A63 · 47A12 · 47A05 · 47A30

1 Introduction

Consider a complex Hilbert space $\mathbb{H}$ with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. Let $\mathcal{B}(\mathbb{H})$ represent the C^*-algebra, which consists of all bounded linear operators on $\mathbb{H}$, including the identity operator I. An operator $A \in \mathcal{B}(\mathbb{H})$ is called positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathbb{H}$, denoted as $A \geq 0$. For any positive bounded linear operator A, there exists a unique positive bounded linear operator $A^{\frac{1}{2}}$ such that $A = (A^{\frac{1}{2}})^2$. Furthermore, we define the absolute value of A as $|A| = (A^*A)^{\frac{1}{2}}$. It is important to note that $|A| \geq 0$. Let $\mathbb{S}^1$ denote the unit sphere of $\mathbb{H}$, defined as

$$\mathbb{S}^1 := \left\{ x \in \mathbb{H};\ \|x\| = \sqrt{\langle x, x \rangle} = 1 \right\}.$$

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Let $A \in \mathcal{B}(\mathbb{H})$. Consider the operator norm of A by $\|A\|$, defined as $\|A\| = \sup\left\{\|Ax\|;\ x \in \mathbb{S}^1\right\}$, and the numerical radius of A, denoted by $\omega(A)$, defined as $\omega(A) = \sup\left\{\left|\langle Ax, x\rangle\right|;\ x \in \mathbb{S}^1\right\}$. It is easy to check that $\omega(A) \leq \|A\|$, where the equality holds if A is a normal operator, i.e., $A^*A = AA^*$, A^* denote the adjoint operator of A. Additionally, the operator norm and the numerical radius are equivalent, as $\frac{\|A\|}{2} \leq \omega(A) \leq \|A\|$ for any $A \in \mathcal{B}(\mathbb{H})$. Importantly, we also have $\omega(A) \leq \omega(|A|)$. We define the operator A as a normaloid when its numerical radius equals its operator norm, expressed as $\omega(A) = \|A\|$. Further details and properties of the numerical radius can be explored in references [3–7,10,18,21,22,26].

In the work by Popescu [24], the concept of the *Euclidean operator radius* for a pair (B, C) of bounded linear operators on a Hilbert space $\mathbb{H}$ was introduced.

Consider a pair of bounded linear operators (B, C) on $\mathbb{H}$. The Euclidean operator radius is defined as

$$\omega_e(B, C) := \sup\left\{\sqrt{\left|\langle Bx, x\rangle\right|^2 + \left|\langle Cx, x\rangle\right|^2}\ ;\ \ x \in \mathbb{S}^1\right\}.$$

The mapping $\omega_e : \mathcal{B}(\mathbb{H})^2 := \mathcal{B}(\mathbb{H}) \times \mathcal{B}(\mathbb{H}) \to \mathbb{R}$ is a norm defined on the space of bounded operators $\mathcal{B}(\mathbb{H})^2$. Notably, in the work of Popescu [24], it has been established that for any $B, C \in \mathcal{B}(\mathbb{H})$, the following inequality holds:

$$\frac{\sqrt{2}}{4}\sqrt{\|B^*B + C^*C\|} \leq \omega_e(B, C) \leq \sqrt{\|B^*B + C^*C\|}, \tag{1}$$

It is crucial to highlight that the constants $\frac{\sqrt{2}}{4}$ and 1 have been rigorously demonstrated in [24] as the optimal choices ensuring the validity of (1).

For $A \in \mathcal{B}(\mathbb{H})$, it is observed that

$$\omega_e(A, A^*) = \sqrt{2}\omega(A).$$

In the case where $A \in \mathcal{B}(\mathbb{H})$ and $A = B + iC$ represents the Cartesian decomposition of A, the following equality holds:

$$\begin{aligned}\omega_e^2(B, C) &= \sup\left\{\left|\langle Bx, x\rangle\right|^2 + \left|\langle Cx, x\rangle\right|^2\ ;\ \ x \in \mathbb{S}^1\right\}\\ &= \sup\left\{\left|\langle Ax, x\rangle\right|^2\ ;\ \ x \in \mathbb{S}^1\right\} = \omega^2(A).\end{aligned}$$

By defining $B = \frac{1}{2}(A + A^*)$ and $C = \frac{1}{2i}(A - A^*)$, both of which are self-adjoint operators, it follows that

$$2(B^2 + C^2) = A^*A + AA^* = |A|^2 + |A^*|^2.$$

For further exploration of the Euclidean operator radius and related inequalities, readers are encouraged to consult [15,16,20,25].

The Heilbronn inequality, as established in [14] or [19, p. 395], is a potent expression within Hilbert spaces. It takes the form:

$$\sum_{i=1}^{n}\left|\langle x, y_i\rangle\right| \leq \|x\|\left(\sum_{i,j=1}^{n}\left|\langle y_i, y_j\rangle\right|\right)^{\frac{1}{2}}$$

where x and y_i are elements of $\mathbb{H}$ for $i = 1, ..., n$. A noteworthy consequence arises when we specialize to the case $n = 2$, where $y_1 = y$ and $y_2 = z$. This yields the inequality:

$$\left|\langle x, y\rangle\right| + \left|\langle x, z\rangle\right| \leq \|x\|\left(\|y\|^2 + \|z\|^2 + 2\left|\langle y, z\rangle\right|\right)^{\frac{1}{2}} \tag{2}$$

valid for all $x,\ y,\ z \in \mathbb{H}$. The present paper aims to explore various applications stemming from the Heilbronn inequality, particularly in the context of single and pairs of Hilbert space operators.

This paper is structured as follows. In Section 2, we prove new bounds for $\omega_e(B, C)$, where B and C are two bounded linear operators on $\mathbb{H}$. Section 3 explores how the Heilbronn inequality can be used to find more bounds for both single Hilbert space operators and pairs of them.

2 New inequalities involving $\omega_e(B, C)$

In this section, we establish new bounds for the Euclidean operator radius of a pair (B, C) of bounded linear operators defined on a Hilbert space $\mathbb{H}$. To support our first investigation, we recall the following inequality established by Dragomir in [10], which states that for any $x, y, z \in \mathbb{H}$, the following holds:

$$\left|\langle x, y\rangle\right|^2 + \left|\langle x, z\rangle\right|^2 \leq \|x\|^2\Big(\max\left\{\|y\|^2, \|z\|^2\right\} + \left|\langle y, z\rangle\right|\Big) \tag{3}$$

Now, we are ready to prove our first result.

Theorem 1 *Let $B, C \in \mathcal{B}(\mathbb{H})$. Then the inequality*

$$\omega_e^2(B, C) \leq \frac{1}{2}\left(\|B\|^2 + \|C\|^2 + \omega(B^2) + \omega(C^2)\right)$$

holds.

Proof Let x be any element in $\mathbb{H}$. If we choose $y = Bx$ and $z = B^*x$ in inequality (3), then we deduce

$$\begin{aligned} 2\left|\langle Bx, x\rangle\right|^2 &= \left|\langle Bx, x\rangle\right|^2 + \left|\langle B^*x, x\rangle\right|^2 \\ &\leq \|x\|^2\Big(\max\left\{\|Bx\|^2, \|B^*x\|^2\right\} + \left|\langle Bx, B^*x\rangle\right|\Big) \\ &\leq \|x\|^2\Big(\|x\|^2 \max\left\{\|B\|^2, \|B^*\|^2\right\} + \left|\langle B^2x, x\rangle\right|\Big) \\ &\leq \|x\|^4\Big(\|B\|^2 + \omega(B^2)\Big). \end{aligned}$$

This yields that

$$\left|\langle Bx, x\rangle\right|^2 \leq \frac{1}{2}\|x\|^4\Big(\|B\|^2 + \omega(B^2)\Big).$$

In the same way, we find

$$\left|\langle Cx, x\rangle\right|^2 \leq \frac{1}{2}\|x\|^4\Big(\|C\|^2 + \omega(C^2)\Big).$$

Taking the supremum over $x \in \mathbb{S}^1$ in the sum of the above inequalities, we obtain the inequality of the statement. □

To establish our second result in this section, let us revisit a recently improved version of the Cauchy–Schwarz inequality, as introduced in [2]. It is stated as follows:

$$\left|\langle x, y\rangle\right|^2 \leq \nu\|x\|^2\|y\|^2 + (1-\nu)\left|\langle x, y\rangle\right|\|x\|\|y\| \leq \|x\|^2\|y\|^2, \tag{4}$$

where $x, y \in \mathbb{H}$ and $\nu \in [0, 1]$.

Theorem 2 *Let $B, C \in \mathcal{B}(\mathbb{H})$ and $\nu \in [0, 1]$. Then the inequalities*

$$\omega_e^2(B, C) \leq \nu\||B|^2 + |C|^2\| + (1-\nu)\omega_e(B, C)\||B|^2 + |C|^2\|^{\frac{1}{2}} \tag{5}$$

and

$$\omega_e^2(B, C) \leq \nu(\|B\|^2 + \|C\|^2) + (1-\nu)\omega_e(B, C)(\|B\|^2 + \|C\|^2)^{\frac{1}{2}} \tag{6}$$

hold.

Proof Let $x, y, z \in \mathbb{H}$ and $\nu \in [0, 1]$. From inequality (4), applied twice, we have

$$\begin{aligned}\left|\langle x, y\rangle\right|^2 + \left|\langle x, z\rangle\right|^2 &\le \nu\|x\|^2(\|y\|^2 + \|z\|^2)\\ &\quad + (1-\nu)\|x\|(\left|\langle x, y\rangle\right|\|y\| + \left|\langle x, z\rangle\right|\|z\|)\end{aligned}$$

By using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\left|\langle x, y\rangle\right|^2 + \left|\langle x, z\rangle\right|^2 \le \nu\|x\|^2(\|y\|^2 + \|z\|^2)\\ &+ (1-\nu)\|x\|\left(\left|\langle x, y\rangle\right|^2 + \left|\langle x, z\rangle\right|^2\right)^{\frac{1}{2}}\left(\|y\|^2 + \|z\|^2\right)^{\frac{1}{2}}\end{aligned} \tag{7}$$

for any $x, y, z \in \mathbb{H}$ and $\nu \in [0, 1]$.

If we choose $y = Bx$ and $z = Cx$ in inequality (7), then we deduce

$$\|Bx\|^2 + \|Cx\|^2 = \langle B^*Bx, x\rangle + \langle C^*Cx, x\rangle = \langle |B|^2 + |C|^2x, x\rangle,$$

we find

$$\begin{aligned}&\left|\langle Bx, x\rangle\right|^2 + \left|\langle Cx, x\rangle\right|^2\\ &\le \nu\|x\|^2(\|Bx\|^2 + \|Cx\|^2) + (1-\nu)\|x\|(\left|\langle Bx, x\rangle\right|^2\\ &+ \left|\langle Cx, x\rangle\right|^2)^{\frac{1}{2}}(\|Bx\|^2 + \|Cx\|^2)^{\frac{1}{2}}\\ &\le \nu\|x\|^4\||B|^2 + |C|^2\| + (1-\nu)\|x\|^4\omega_e(B, C)\||B|^2 + |C|^2\|^{\frac{1}{2}}.\end{aligned}$$

Taking the supremum over $x \in \mathbb{S}^1$ in the above inequality, we prove the first inequality of the statement. Inequality (6) follows easily from inequality (5). □

Remark 3 For $C = B^*$ in inequality (6) and taking into account that $\omega_e(A, A^*) = \sqrt{2}\omega(A)$, we deduce:

$$\omega^2(B) \le \nu\|B\|^2 + (1-\nu)\omega(B)\|B\|$$

for any operator $B \in \mathcal{B}(\mathbb{H})$ and $\nu \in [0, 1]$.

For $C = B^*$ in inequality (5), we deduce:

$$\omega^2(B) \le \nu\||B|^2 + |B^*|^2\| + (1-\nu)\omega(B)\||B|^2 + |B^*|^2\|^{\frac{1}{2}}$$

for any operator $B \in \mathcal{B}(\mathbb{H})$ and $\nu \in [0, 1]$.

To prove our next result in this section, we need to recall a well-known inequality attributed to Buzano, established in [8]. The inequality is stated as follows: for $x, y, e \in \mathbb{H}$ with e having unit norm,

$$|\langle x, e\rangle\ \langle e, y\rangle| \le \frac{1}{2}\left(\|x\|\ \|y\| + |\langle x, y\rangle|\right). \tag{8}$$

Now, we are ready to present our result.

Theorem 4 *Let $B, C \in \mathcal{B}(\mathbb{H})$, $t, \theta \in \mathbb{R}$. Then the inequality*

$$\omega(e^{it}B + e^{i\theta}C) \le \frac{1}{2}\left(\|B\| + \|C\|\right) + \frac{1}{\sqrt{2}}\omega_e(B, C)$$

holds.

Proof If $x, y, z, e \in \mathbb{H}$ and e has unit norm, applying Buzano's inequality (8) we prove:

$$|\langle x, e\rangle \langle e, y\rangle| \le \frac{1}{2}\left(\|x\|\, \|y\| + |\langle x, y\rangle|\right),$$
$$|\langle x, e\rangle \langle e, z\rangle| \le \frac{1}{2}\left(\|x\|\, \|z\| + |\langle x, z\rangle|\right).$$

Adding the two inequalities we get

$$\begin{aligned}|\langle x, e\rangle| \left(|\langle y, e\rangle| + |\langle z, e\rangle|\right) &\le \frac{\|x\|}{2}\left(\|y\| + \|z\|\right) + \frac{|\langle x, y\rangle| + |\langle x, z\rangle|}{2}\\ &\le \frac{\|x\|}{2}\left(\|y\| + \|z\|\right) + \frac{1}{\sqrt{2}}\sqrt{|\langle x, y\rangle|^2 + |\langle x, z\rangle|^2}.\end{aligned}$$

If we choose $y = Bx$ and $z = Cx$ in the above inequality, then we obtain

$$\begin{aligned}&|\langle x, e\rangle| \left(|\langle Bx, e\rangle| + |\langle Cx, e\rangle|\right)\\ &\le \frac{\|x\|}{2}\left(\|Bx\| + \|Cx\|\right) + \frac{1}{\sqrt{2}}\sqrt{|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2},\end{aligned}$$

it follows that

$$\begin{aligned}|\langle x, e\rangle||\langle (e^{it}B + e^{i\theta}C)x, e\rangle| &\le |\langle x, e\rangle| \left(|\langle e^{it}Bx, e\rangle| + |\langle e^{i\theta}Cx, e\rangle|\right)\\ &= |\langle x, e\rangle| \left(|\langle Bx, e\rangle| + |\langle Cx, e\rangle|\right)\\ &\le \frac{\|x\|}{2}\left(\|Bx\| + \|Cx\|\right)\\ &+ \frac{1}{\sqrt{2}}\sqrt{|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2}.\end{aligned}$$

If we take $e = \frac{x}{\|x\|}$ in the above relation and take the supremum over $x \in \mathbb{S}^1$ in the above inequality, we obtain the inequality of the statement. □

Remark 5 For $t = 0$ and $\theta = \frac{\pi}{2}$ in Theorem 4, we deduce the following:

$$\omega(B + iC) \le \frac{1}{2}\left(\|B\| + \|C\|\right) + \frac{1}{\sqrt{2}}\omega_e(B, C). \tag{9}$$

For $B = \frac{A+A^*}{2}$ and $C = \frac{A-A^*}{2i}$ in inequality (9), we deduce

$$\omega(A) \le \frac{1}{2(2-\sqrt{2})}\left(\|A + A^*\| + \|A - A^*\|\right).$$

3 Heilbronn inequality and its applications

This section aims to explore various applications stemming from the Heilbronn inequality (2), particularly in the context of single and pairs of Hilbert space operators.

3.1 One operator applications of Heilbronn inequality

In the next result, the second author established the inequality between the first term and the last one in [11], employing Buzano's inequality. The enhancement in the first inequality was achieved by Abu-Omar and Kittaneh in [1].

Theorem 6 *For all $A \in \mathcal{B}(\mathbb{H})$, we have the inequality*

$$\omega^2(A) \le \frac{1}{2}\left(\left\|\frac{|A|^2 + |A^*|^2}{2}\right\| + \omega\left(A^2\right)\right) \le \frac{1}{2}\left(\|A\|^2 + \omega\left(A^2\right)\right). \tag{10}$$

Proof In the following we give a different proof by using the Heilbronn inequality (2). Let x be any element in $\mathbb{H}$. If we take $y = Ax$ and $z = A^*x$, then we get

$$\left|\langle x, Ax\rangle\right| + \left|\langle x, A^*x\rangle\right| \leq \|x\| \left(\|Ax\|^2 + \left\|A^*x\right\|^2 + 2\left|\langle Ax, A^*x\rangle\right|\right)^{\frac{1}{2}},$$

namely

$$\left|\langle Ax, x\rangle\right| \leq \frac{1}{2}\|x\| \left(\|Ax\|^2 + \left\|A^*x\right\|^2 + 2\left|\langle A^2x, x\rangle\right|\right)^{\frac{1}{2}},$$

for all $x \in \mathbb{H}$.

Since, we have the following equality:

$$\|Ax\|^2 + \left\|A^*x\right\|^2 = \langle \left(|A|^2 + \left|A^*\right|^2\right) x, x\rangle,$$

then we deduce

$$\left|\langle Ax, x\rangle\right| \leq \frac{1}{2}\|x\| \left(\langle \left(|A|^2 + \left|A^*\right|^2\right) x, x\rangle + 2\left|\langle A^2x, x\rangle\right|\right)^{\frac{1}{2}}. \tag{11}$$

By taking the supremum, we obtain

$$\begin{aligned}\omega(A) = \sup_{x\in\mathbb{S}^1}\left|\langle Ax, x\rangle\right| &\leq \frac{1}{2}\sup_{x\in\mathbb{S}^1}\left(\langle \left(|A|^2 + \left|A^*\right|^2\right) x, x\rangle + 2\left|\langle A^2x, x\rangle\right|\right)^{\frac{1}{2}}\\ &\leq \frac{1}{2}\left(\sup_{x\in\mathbb{S}^1}\langle \left(|A|^2 + \left|A^*\right|^2\right) x, x\rangle + 2\sup_{x\in\mathbb{S}^1}\left|\langle A^2x, x\rangle\right|\right)^{\frac{1}{2}}\\ &= \frac{1}{2}\left(\left\||A|^2 + \left|A^*\right|^2\right\| + 2\omega\left(A^2\right)\right)^{\frac{1}{2}} = \frac{\sqrt{2}}{2}\left(\left\|\frac{|A|^2 + |A^*|^2}{2}\right\| + \omega\left(A^2\right)\right)^{\frac{1}{2}},\end{aligned}$$

which proves (10). □

The following extension involving powers is noteworthy. To establish it, we rely on McCarthy's well-known inequality (refer to [12, Theorem 1.4]). This inequality states that if A is a positive operator in $\mathcal{B}(\mathbb{H})$ and $x \in \mathbb{S}^1$, then for every $r \geq 1$, the following relation holds:

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle. \tag{12}$$

Theorem 7 *For all $A \in \mathcal{B}(\mathbb{H})$ and $r \geq 1$ we have the inequality*

$$\omega^{2r}(A) \leq \frac{1}{2}\left(\left\|\frac{|A|^2 + |A^*|^2}{2}\right\|^r + \omega^{2r}(A)\right) \leq \frac{1}{2}\left(\|A\|^{2r} + \omega^{2r}(A)\right). \tag{13}$$

Proof The bounds obtained in this theorem follows easily from Theorem 6 by taking r^{th} power of the bounds obtained in Theorem 6 and using convexity of the power function. □

If we take the power $r \geq 1$, then we get using the convexity of the power function that

$$\begin{aligned}\left|\langle Ax, x\rangle\right|^{2r} &\leq \|x\|^{2r}\left[\frac{1}{2}\left(\langle\left(\frac{|A|^2 + |A^*|^2}{2}\right) x, x\rangle + \left|\langle A^2x, x\rangle\right|\right)\right]^r\\ &\leq \frac{1}{2}\|x\|^{2r}\left[\langle\left(\frac{|A|^2 + |A^*|^2}{2}\right) x, x\rangle^r + \left|\langle A^2x, x\rangle\right|^r\right]\end{aligned} \tag{14}$$

for all for $x \in \mathbb{H}$.

From McCarthy inequality (12), we get that

$$\left\langle \left(\frac{|A|^2+|A^*|^2}{2}\right) x, x\right\rangle^r \leq \left\langle \left(\frac{|A|^2+|A^*|^2}{2}\right)^r x, x\right\rangle$$

for $x \in \mathbb{S}^1$.

Therefore, from (14) we get that

$$\left|\langle Ax, x\rangle\right|^{2r} \leq \frac{1}{2}\left[\left\langle \left(\frac{|A|^2+|A^*|^2}{2}\right)^r x, x\right\rangle + \left|\langle A^2x, x\rangle\right|^r\right] \tag{15}$$

for $x \in \mathbb{S}^1$.

To highlight a crucial consequence of Theorem 7, let's revisit a key lemma presented in [13, Lemma 2.5].

Lemma 8 *Let h be a non-negative nondecreasing convex function on $[0, \infty)$, and let $A, B \in \mathcal{B}(\mathbb{H})$ be positive operators. Then it holds that*

$$h\left(\left\|\frac{A+B}{2}\right\|\right) \leq \left\|\frac{h(A)+h(B)}{2}\right\|.$$

In particular, for every $r \geq 1$, the following inequality is valid:

$$\left\|\left(\frac{A+B}{2}\right)^r\right\| = \left\|\frac{A+B}{2}\right\|^r \leq \left\|\frac{A^r+B^r}{2}\right\|. \tag{16}$$

Corollary 9 *For all $A \in \mathcal{B}(\mathbb{H})$ and $1 \leq r \leq 2$ we have the inequality*

$$\omega^{2r}(A) \leq \frac{1}{2}\left(\left\|\frac{|A|^{2r}+|A^*|^{2r}}{2}\right\| + \omega^{2r}(A)\right) \leq \frac{1}{2}\left(\|A\|^{2r} + \omega^{2r}(A)\right). \tag{17}$$

Proof Since the function $f(t) = t^r$ is operator convex for $r \in [1, 2]$, then (see [12,23])

$$0 \leq \left(\frac{|A|^2+|A^*|^2}{2}\right)^r \leq \frac{|A|^{2r}+|A^*|^{2r}}{2},$$

which gives, through (16) that

$$\left\|\frac{|A|^2+|A^*|^2}{2}\right\|^r = \left\|\left(\frac{|A|^2+|A^*|^2}{2}\right)^r\right\| \leq \left\|\frac{|A|^{2r}+|A^*|^{2r}}{2}\right\| \leq \|A\|^{2r}$$

and the inequality (17) is proved. □

Remark 10 In [7] the authors obtained the folowing vector inequality for $x \in \mathbb{S}^1$ and $r \geq 1$,

$$\left|\langle Ax, x\rangle\right|^{2r} \leq \frac{1}{2}\left[\left\langle \frac{|A|^{2r}+|A^*|^{2r}}{2} x, x\right\rangle + \left|\langle A^2x, x\rangle\right|^r\right]. \tag{18}$$

This implies that

$$\omega^{2r}(A) \leq \frac{1}{2}\left(\left\|\frac{|A|^{2r}+|A^*|^{2r}}{2}\right\| + \omega^{2r}(A)\right). \tag{19}$$

We observe that for $1 \leq r \leq 2$,

$$\left\langle \left(\frac{|A|^2+|A^*|^2}{2}\right)^r x, x\right\rangle \leq \left\langle \frac{|A|^{2r}+|A^*|^{2r}}{2} x, x\right\rangle$$

for all $x \in \mathbb{S}^1$, which shows that our inequality (15) is better in this case than (18). Also, the first inequality in (17) is better than (19) for $1 \leq r \leq 2$.

3.2 Applications of the Heilbronn inequality to pairs of operators

In this subsection, our primary objective is to explore applications of the Heilbronn Inequality in the context of pairs of Hilbert space operators.

To achieve our goals, we introduce two new concepts for a pair (B, C) of bounded linear operators defined on a Hilbert space $\mathbb{H}$. The first concept is of the operator norm type and is defined as

$$n_1(A, B) := \sup_{x,u\in\mathbb{S}^1}\left(\left|\langle x, Au\rangle\right| + \left|\langle x, Bu\rangle\right|\right)$$

for $A, B \in \mathcal{B}(\mathbb{H})$.

The second concept is of the numerical radius type and is defined as

$$\omega_1(A, B) := \sup_{x\in\mathbb{S}^1}\left(\left|\langle x, Ax\rangle\right| + \left|\langle x, Bx\rangle\right|\right)$$

for $A, B \in \mathcal{B}(\mathbb{H})$.

Our first result in this subsection is as follows.

Theorem 11 *If* $A, B \in \mathcal{B}(\mathbb{H})$, *then*

$$\left\|\frac{A+B}{2}\right\|^2 \le \frac{1}{4}n_1^2(A, B) \le \frac{1}{2}\left(\left\|\frac{|A|^2+|B|^2}{2}\right\| + \omega\left(B^*A\right)\right) \tag{20}$$

and

$$\omega^2\left(\frac{A+B}{2}\right) \le \frac{1}{4}\omega_1^2(A, B) \le \frac{1}{2}\left(\left\|\frac{|A|^2+|B|^2}{2}\right\| + \omega\left(B^*A\right)\right) \tag{21}$$

Proof Let u be any element in $\mathbb{H}$. From (20) we have for $y = Au$, $z = Bu$ that

$$\left|\langle x, Au\rangle\right| + \left|\langle x, Bu\rangle\right| \le \|x\|\left(\|Au\|^2 + \|Bu\|^2 + 2\left|\langle Au, Bu\rangle\right|\right)^{\frac{1}{2}}$$

and

$$\left|\langle x, (A+B)u\rangle\right| \le \left|\langle x, Au\rangle\right| + \left|\langle x, Bu\rangle\right|,$$

which gives that

$$\begin{aligned}\left|\langle x, (A+B)u\rangle\right| &\le \left|\langle x, Au\rangle\right| + \left|\langle x, Bu\rangle\right| \\ &\le \|x\|\left(\|Au\|^2 + \|Bu\|^2 + 2\left|\langle Au, Bu\rangle\right|\right)^{\frac{1}{2}} \\ &= \|x\|\left(\left\langle\left(|A|^2+|B|^2\right)u, u\right\rangle + 2\left|\langle B^*Au, u\rangle\right|\right)^{\frac{1}{2}}\end{aligned} \tag{22}$$

for all $x, u \in \mathbb{H}$.

Therefore, we obtain the following:

$$\begin{aligned}\|A+B\| &= \sup_{x,u\in\mathbb{S}^1}\left|\langle x, (A+B)u\rangle\right| \le n_1(A, B) \\ &\le \sup_{u\in\mathbb{S}^1}\left(\left\langle\left(|A|^2+|B|^2\right)u, u\right\rangle + 2\left|\langle B^*Au, u\rangle\right|\right)^{\frac{1}{2}} \\ &\le \left(\sup_{u\in\mathbb{S}^1}\left\langle\left(|A|^2+|B|^2\right)u, u\right\rangle + 2\sup_{u\in\mathbb{S}^1}\left|\langle B^*Au, u\rangle\right|\right)^{\frac{1}{2}} \\ &= \left(\left\||A|^2+|B|^2\right\| + 2\omega\left(B^*A\right)\right)^{\frac{1}{2}},\end{aligned}$$

which proves (20).

From (22) we get

$$\begin{aligned}\left|\langle x,(A+B)\,x\rangle\right|^2 &\le \left(\left|\langle x,Ax\rangle\right|+\left|\langle x,Bx\rangle\right|\right)^2\\ &\le \|x\|^2\left(\left\langle\left(|A|^2+|B|^2\right)x,x\right\rangle+2\left|\langle B^*Ax,x\rangle\right|\right)\end{aligned}$$

for $x\in\mathbb{H}$.

If we take the supremum over $x\in\mathbb{S}^1$, then we get (21). □

Theorem 12 *For all $A, B\in\mathcal{B}(\mathbb{H})$ and $r\ge 1$ we have the inequality*

$$\begin{aligned}\left\|\frac{A+B}{2}\right\|^{2r} &\le \frac{1}{4^r}n_1^{2r}(A,B)\le\frac{1}{2}\left(\left\|\frac{|A|^2+|B|^2}{2}\right\|^r+\omega^r\left(B^*A\right)\right)\\ &\le\frac{1}{2}\left(\frac{\|A\|^{2r}+\|B\|^{2r}}{2}+\omega^r\left(B^*A\right)\right)\end{aligned}\tag{23}$$

and

$$\begin{aligned}\omega^{2r}\left(\frac{A+B}{2}\right) &\le \frac{1}{4^r}\omega_1^{2r}(A,B)\le\frac{1}{2}\left(\left\|\frac{|A|^2+|B|^2}{2}\right\|^r+\omega^r\left(B^*A\right)\right)\\ &\le\frac{1}{2}\left(\frac{\|A\|^{2r}+\|B\|^{2r}}{2}+\omega^r\left(B^*A\right)\right).\end{aligned}\tag{24}$$

Proof The bounds obtained in this theorem follows easily from Theorem 11 by taking r^{th} power of the bounds obtained in Theorem 11 and using convexity of the power function. □

Remark 13 With the assumptions of Theorem 12 and if $1\le r\le 2$ we have that

$$\begin{aligned}\left\|\frac{A+B}{2}\right\|^{2r} &\le \frac{1}{4^r}n_1^{2r}(A,B)\le\frac{1}{2}\left(\left\|\frac{|A|^2+|B|^2}{2}\right\|^r+\omega^r\left(B^*A\right)\right)\\ &\le\frac{1}{2}\left(\left\|\frac{|A|^{2r}+|B|^{2r}}{2}\right\|+\omega^r\left(B^*A\right)\right)\\ &\le\frac{1}{2}\left(\frac{\|A\|^{2r}+\|B\|^{2r}}{2}+\omega^r\left(B^*A\right)\right)\end{aligned}$$

and

$$\begin{aligned}\omega^{2r}\left(\frac{A+B}{2}\right) &\le \frac{1}{4^r}\omega_1^{2r}(A,B)\le\frac{1}{2}\left(\left\|\frac{|A|^2+|B|^2}{2}\right\|^r+\omega^r\left(B^*A\right)\right)\\ &\le\frac{1}{2}\left(\left\|\frac{|A|^{2r}+|B|^{2r}}{2}\right\|+\omega^r\left(B^*A\right)\right)\\ &\le\frac{1}{2}\left(\frac{\|A\|^{2r}+\|B\|^{2r}}{2}+\omega^r\left(B^*A\right)\right).\end{aligned}$$

Before introducing our next result, let's begin by revisiting the Crawford number associated with the operator $A\in\mathcal{B}(\mathbb{H})$:

$$c(A)=\inf_{x\in\mathbb{S}^1}\left|\langle Ax,x\rangle\right|.$$

We state now the following theorem.

Theorem 14 *For all $A, B \in \mathcal{B}(\mathbb{H})$, we have the inequality*

$$\omega_e^2(A, B) \le 2\left[\left\|\frac{|A|^2 + |B|^2}{2}\right\| + \omega\left(B^*A\right) - c(A)\,c(B)\right]. \tag{25}$$

Proof If we take the square in (2) we get

$$\left|\langle x, y\rangle\right|^2 + \left|\langle x, z\rangle\right|^2 \le \|x\|^2\left(\|y\|^2 + \|z\|^2\right) + 2\|x\|^2\left|\langle y, z\rangle\right| - 2\left|\langle x, y\rangle\right|\left|\langle x, z\rangle\right|$$

for all $x, y, z \in \mathbb{H}$.

If we take $y = Ax$, $z = Bx$ and $x \in \mathbb{S}^1$, then we get

$$\begin{aligned}
&\left|\langle x, Ax\rangle\right|^2 + \left|\langle x, Bx\rangle\right|^2 \\
&\le \|Ax\|^2 + \|Bx\|^2 + 2\left|\langle Ax, Bx\rangle\right| - 2\left|\langle x, Ax\rangle\right|\left|\langle x, Bx\rangle\right| \\
&= \left\langle \left(|A|^2 + |B|^2\right)x, x\right\rangle + 2\left|\langle B^*Ax, x\rangle\right| - 2\left|\langle x, Ax\rangle\right|\left|\langle x, Bx\rangle\right| \\
&\le \left\langle \left(|A|^2 + |B|^2\right)x, x\right\rangle + 2\omega\left(B^*A\right) - 2c(A)\,c(B)
\end{aligned}$$

and by taking the supremum over $x \in \mathbb{S}^1$, we get (25). □

Remark 15 If we take $B = A^*$ in relation (25), then we have the inequality

$$\omega^2(A) \le \left\|\frac{|A|^2 + |A^*|^2}{2}\right\| + \omega\left(A^2\right) - c^2(A)$$

for all $A \in \mathcal{B}(\mathbb{H})$.

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Declarations

Competing interests The authors declare that they have no competing interests.

References

1. A. Abu-Omar and F. Kittaneh, *Upper and lower bounds for the numerical radius with applications to involution operators*, Rocky Mountain J. Math., **45** (4) (2015), 1055-1065.
2. M.W. Alomari, *On Cauchy– Schwarz type inequalities and applications to numerical radius inequalities*, Ricerche mat (2022).
3. S. Bag, P. Bhunia and K. Paul, *Bounds of numerical radius of bounded linear operators using t-Aluthge transform*, Math. Inequal. Appl., **23** (3) (2020), 991–1004.
4. P. Bhunia and K. Paul, *Furtherance of numerical radius inequalities of Hilbert space operators*, Arch. Math. (Basel), **117** (5) (2021), 537–546.
5. P. Bhunia and K. Paul, Development of inequalities and characterization of equality conditions for the numerical radius, Linear Algebra Appl., 630 (2021), 306–315.
6. P. Bhunia and K. Paul, New upper bounds for the numerical radius of Hilbert space operators, Bull. Sci. Math., 167 (2021), Paper No. 102959, 11 pp.
7. P. Bhunia and K. Paul, *Proper improvement of well-known numerical radius inequalities and their applications*, Results Math. **76** (4) (2021), Paper no. 177.
8. M. L. Buzano, *Generalizzazione della diseguaglianza di Cauchy-Schwarz* (Italian), Rend. Sem. Mat. Univ. e Politech. Torino **31** (1974), 405-409.

9. C. Conde, K. Feki and F. Kittaneh, *On some Berezin number and norm inequalities for operators in Hilbert and semi-Hilbert spaces*, submitted.
10. S.S. Dragomir, *Some inequalities for the Euclidean operator radius of two operators in Hilbert spaces*, Linear Algebra and its Applications **419** (2006), 256–264.
11. S. S. Dragomir, *Inequalities for the norm and the numerical radius of linear operators in Hilbert spaces*, Demonstr. Math. **30** (2) (2007), 411-417.
12. T. Furuta, J. Mićić, J. Pečarić, and Y. Seo, *Mound–Pečarić method in operator inequalities*, Element, Zagreb, 2005.
13. M. Ghaderi Aghideh, M.S. Moslehian and J. Rooin, *Sharp inequalities for the numerical radii of block operator matrices*, Analysis Math., **45** (4) (2019), 687-703.
14. H. Heilbronn, *On the averages of some arithmetical functions of two variables*, Mathematica, **5** (1958), 1-7.
15. S. Jana, P. Bhunia and K. Paul, *Euclidean operator radius inequalities of a pair of bounded linear operators and their applications*, Bull. Braz. Math. Soc. (N.S.) **54** (2023), no. 1, 14 pp.
16. S. Jana, P. Bhunia and K. Paul, *Euclidean operator radius inequalities of d–tuple operators and operator matrices*, Math. Slovaca, (2024), in press.
17. T. Kosem, *Inequalities between* $\|f(A+B)\|$ and $\|f(A)+f(B)\|$, Linear Algebra Appl. **418** (2006), no. 1, 153–160.
18. F. Kittaneh, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math., **158** (1) (2003), 11–17.
19. D.S. Mitrinović, J.E. Pečarić, and A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993.
20. M.S. Moslehian, M. Sattari and K. Shebrawi, *Extensions of Euclidean operator radius inequalities*, Math. Scand., **120** (1) (2017), 129–144.
21. M.E. Omidvar and H.R. Moradi, *Better bounds on the numerical radii of Hilbert space operators*, Linear Algebra Appl. **604** (2020), 265–277.
22. M.E. Omidvar and H.R. Moradi, *New estimates for the numerical radius of Hilbert space operators*, Linear Multilinear Algebra, **69**:5,(2021), 946–956 https://doi.org/10.1080/03081087.2020.1810200
23. J. Pečarić, F. Proschan and Y.L. Tong, *Convex functions, partial orderings, and statistical applications*, Academic Press, Inc., 1992.
24. G. Popescu, *Unitary invariants in multivariable operator theory*, Memoirs of the American Mathematical Society, Volume: **200**(941); 2009; 91 pp.
25. S. Sahoo, N.C. Rout and M. Sababheh, *Some extended numerical radius inequalities*, Linear Multilinear Algebra, **69** (5) (2021), 907–920.
26. T. Yamazaki, *On upper and lower bounds for the numerical radius and an equality condition*, Studia Math., **178** (1) (2007), 83–89.