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A survey on the boxicity and cubicity of graphs

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Abstract
A d-dimensional box (or d-box) is a set $[a_1, b_1] \times \cdots \times [a_d, b_d]$ where $[a_i, b_i]$, for $i \in [d]$, are closed intervals on the real line. A d-dimensional cube (or d-cube) is a d-box with the constraint $b_i - a_i = 1$ for each $i \in [d]$. The boxicity (cubicity) of a graph G is the minimum integer d such that there exists a function that maps every vertex of G to a d-box (or d-cube respectively) so that two distinct vertices of G have an edge in G if and only if their corresponding d-boxes intersect. We survey some key results on the boxicity and cubicity of graphs, including bounds, general techniques, computational hardness results, and relationship with other graph invariants.

Keywords Boxicity · Cubicity · Poset dimension · Separation dimension · Intersection dimension of graphs

1 Introduction

Given a family of sets $\mathcal{F}$, we say that a graph $G = (V, E)$ is an *intersection graph* of sets from $\mathcal{F}$, if there exists a function $f : V(G) \to \mathcal{F}$ such that for distinct $u, v \in V(G)$, we have $uv \in E(G)$ if and only if $f(u) \cap f(v) \neq \emptyset$. When such a function f exists for a graph G, we say that f is an *intersection representation* of G using sets from $\mathcal{F}$. An intersection representation f of G using sets from $\mathcal{F}$ shall also equivalently be considered as the indexed collection of sets $\{f(v)\}_{v \in V(G)}$.

A d-dimensional box (or d-box) is a set $[a_1, b_1] \times [a_2, b_2] \cdots \times [a_d, b_d]$ where $[a_i, b_i]$, for $i \in [d]$, are closed intervals on the real line. A d-dimensional cube (or d-cube) is a d-box with the constraint $b_i - a_i = 1$, for each $i \in [d]$. A *d-dimensional box (or cube) representation* for a graph G is an intersection representation of G using d-boxes (d-cubes respectively).

Definition 1.1 The *boxicity* of a graph G (denoted as $box(G)$) is the minimum integer $d \geq 0$ such that G has a d-dimensional box representation.

Definition 1.2 The *cubicity* of a graph G (denoted as $cub(G)$) is the minimum integer $d \geq 0$ such that G has a d-dimensional cube representation.

By convention, it is assumed that the boxicity and cubicity of complete graphs is 0. Since d-cubes are also d-boxes, it follows that for any graph G, $box(G) \leq cub(G)$.

The study of boxicity and cubicity of graphs was initiated by F.S. Roberts [1]. These concepts have since found application in the study of overlap of ecological niches [2] in ecology and fleet maintenance problems in operations research [3]. From the point of view of computer science, the importance of these notions stems from the fact that several hard problems become easier on graphs with bounded boxicity. For instance, it is possible to solve in polynomial-time the Max Clique problem in graphs with bounded boxicity (this is the problem of deciding

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if there is a clique of size at least k in an input graph, where k is an input integer; this problem is NP-complete for general graphs). This is because the graphs with boxicity at most k can contain at most $(2n)^k$ many maximal cliques (here n denotes the number vertices of the input graph), and there are algorithms that can list all the maximal cliques in any input graph, spending only a polynomial (in n) amount of time per clique (for example the Chiba-Nishizeki algorithm [4]). Another example is the MAX INDEPENDENT SET problem, which asks to decide if there is an independent set of size at least k in an input graph, where k is an input integer. This problem is hard to approximate for general graphs to within a factor of $n^{1-\epsilon}$, for any $\epsilon > 0$, unless $\mathsf{P} = \mathsf{NP}$ [5]. But for graphs having boxicity k, where k is a constant, it is approximable to within a factor of $\left(1 + \frac{1}{c}\log n\right)^{k-1}$ for any constant $c \geq 1$, given a k-dimensional box representation of the input graph [6, 7], in polynomial time.

An intersection representation of a graph G using closed intervals of the real line (when such a representation exists) is called an *interval representation* of G.

Definition 1.3 The graphs that have interval representations are called *interval graphs*.

Note that interval graphs are precisely the graphs having boxicity at most 1. Similarly, graphs having cubicity at most 1 are precisely the *unit-interval graphs* (called "indifference graphs" in [1]), which are defined as the interval graphs that have *unit-interval representations*, i.e. interval representations in which every interval is of unit length. We also consider the interval graphs that have an interval representation in which all intervals have length equal to some $r > 0$ as unit-interval graphs, since such a representation can be converted into a unit-interval representation and vice versa by scaling each interval by a factor $1/r$ or r respectively. We call such a representation an *equal-interval representation* of the given unit-interval graph.

We will denote an interval graph and its interval representation by the same symbol when there is no chance of confusion. Suppose that I is an interval representation of an interval graph. We will often denote by $I(u)$ the interval corresponding to a vertex u of the graph in the interval representation I; i.e. $I = \{I(u)\}_{u \in V(G)}$. When the left and right endpoints of an interval $I(u)$ need to be specified, we shall assume that $I(u) = [l(I(u)), r(I(u))]$; i.e. $l(I(u))$ denotes the left endpoint and $r(I(u))$ denotes the right endpoint of the interval $I(u)$.

Interval (unit-interval) graphs perform a crucial role in the study of boxicity (resp. cubicity). A d-box (resp. d-cube) representation of G can be projected on to every axis of $\mathbb{R}^d$ so that every d-box (resp. d-cube) gets projected to an interval (resp. unit length interval) on each axis. On each axis, the intersection graph of these intervals is a supergraph of G since the intersection of two boxes in $\mathbb{R}^d$ forces the intersection of the corresponding projections on each axis. Furthermore, if there is a pair of vertices that are not adjacent, then there must be at least one axis where the intervals corresponding to them are not intersecting and therefore, there must be one interval (resp. unit-interval) supergraph which does not contain that pair as an edge. This is simply because for any two d-boxes $A = A_1 \times A_2 \times \cdots \times A_d$ and $B = B_1 \times B_2 \times \cdots \times B_d$, we have $A \cap B \neq \emptyset$ if and only if $A_i \cap B_i \neq \emptyset$ for each $i \in [d]$.

For graphs $G, G_1, G_2, \ldots, G_k$, we say that "G is the intersection of the graphs $G_1, G_2, \ldots, G_k$", or $G = G_1 \cap G_2 \cap \cdots \cap G_k$, if $V(G) = V(G_i)$ for each $i \in [k]$ and $E(G) = E(G_1) \cap E(G_2) \cap \cdots \cap E(G_k)$. The following theorem, which can also be taken as an alternate definition of boxicity (resp. cubicity), is a natural corollary of the discussion in the previous paragraph.

Theorem 1.4 ([1]) Let G be a graph that is not a complete graph. Then G has boxicity (cubicity) at most l if and only if it can be represented as the intersection of l many interval (resp. unit-interval) graphs, i.e. there exist interval (resp. unit-interval) supergraphs $I_1, I_2, \ldots, I_l$ of G such that $G = I_1 \cap I_2 \cap \cdots \cap I_l$.

Let K_n denote the complete graph on n vertices and K_n^- denote the graph obtained from K_n by removing any one edge. Kratochvíl and Tuza [8] defined a general notion of "intersection dimensions of graphs" as follows.

Definition 1.5 (Intersection dimension of a graph [8]) Let G be a graph. Let $\mathcal{A}$ be a class of graphs that contains K_n and K_n^-, $\forall n \in \mathbb{N}$. The intersection dimension of G with respect to $\mathcal{A}$ (denoted as $dim_{\mathcal{A}}(G)$) is defined as follows:

$$dim_{\mathcal{A}}(G) = \min\{k\colon \exists G_1, G_2, \ldots, G_k \in \mathcal{A} \text{ such that } G = G_1 \cap G_2 \cap \cdots \cap G_k\}$$

Using the above definition and using Theorem 1.4, it is clear that for any graph G that is not a complete graph, $box(G) = dim_{\mathcal{I}}(G)$ and $cub(G) = dim_{\mathcal{U}}(G)$, where $\mathcal{I}$ and $\mathcal{U}$ denote the classes of interval graphs and unit-interval graphs respectively. Boxicity and cubicity defined this way gives a more combinatorial interpretation to the geometric problem of representing graphs using intersecting boxes and cubes. This allows us to use the properties of interval and unit-interval graphs to analyze the boxicity and cubicity of graphs respectively. For example, it follows from this definition that the boxicity and cubicity of a graph are well defined; i.e. for any graph G, the values $box(G)$ and $cub(G)$ are always bounded. To see this, observe that for any graph G, we have $G = \bigcap_{e \in E(\overline{G})} K_e^-$, where K_e^- is the graph obtained from the complete graph on vertex set $V(G)$ by removing the edge e. Note that K_e^- is a unit-interval graph since it has a unit-interval representation in which one endpoint of e is assigned the interval $[0, 1]$, the other endpoint of e is assigned the interval $[2, 3]$, and all other vertices are assigned the interval $[1, 2]$. Thus we have that G is the intersection of $|E(\overline{G})|$ unit-interval graphs, and therefore, $box(G) \leq cub(G) \leq |E(\overline{G})|$.

For the sake of convenience, whenever we have $G = I_1 \cap I_2 \cap \cdots \cap I_l$, where G is a graph and $I_1, I_2, \ldots, I_l$ are interval (unit-interval) graphs, we shall refer to the collection $\{I_1, I_2, \ldots, I_l\}$ as a “box (resp. cube) representation” of G of dimension l. It is easy to see that if a graph G has a box (resp. cube) representation of dimension l, then it has a box (resp. cube) representation of dimension l' for all $l' \geq l$, since adding any number of complete graphs, each on vertex set $V(G)$, to a box (resp. cube) representation of G still gives a box (resp. cube) representation of G.

There is yet another way to view boxicity and cubicity, which also comes from Theorem 1.4. By the expression $G = G_1 \cup G_2 \cup \cdots \cup G_k$, we mean that $V(G_i) = V(G)$ for each $i \in [k]$ and $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k)$, or in other words, $G_1, G_2, \ldots, G_k$ “cover” the edges of G. Let $\overline{G}$ denote the complement of the graph G. Then we have that $G = G_1 \cap G_2 \cap \cdots \cap G_k$ if and only if $\overline{G} = \overline{G_1} \cup \overline{G_2} \cup \cdots \cup \overline{G_k}$. Thus, the expression $G = I_1 \cap I_2 \cap \cdots \cap I_l$ in Theorem 1.4 can also be written in terms of the complement graph $\overline{G}$ as $\overline{G} = \overline{I_1} \cup \overline{I_2} \cup \cdots \cup \overline{I_l}$. In this view, the boxicity and cubicity of a graph are “edge covering problems” where we are trying to cover the edges of $\overline{G}$ with co-interval and co-unit-interval[1] graphs respectively. The following theorem by Cozzens and Roberts [9] captures this concept.

Theorem 1.6 ([9]) Let G be a graph that is not a complete graph. Then $box(G) \leq k$ (resp. $cub(G) \leq k$) if and only if there exists co-interval (resp. co-unit-interval) spanning subgraphs $G_1, G_2, \ldots, G_k$ of G such that $G = G_1 \cup G_2 \cup \cdots \cup G_k$.

Both the intersection dimension and the edge covering views lend different strategies to deal with the problem at hand and one is preferred over the other depending on the context.

Some notational comments and basic definitions: We assume that G is a simple undirected graph unless mentioned otherwise. We denote the maximum degree of a vertex in G by $\Delta(G)$. The *independence number* of a graph G, denoted as $\alpha(G)$, is the cardinality of the largest independent set in G. The *clique number* of a graph G, denoted as $\omega(G)$, is the order of the largest clique in G. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colours in any proper vertex colouring of G. The *vertex cover number* of a graph G is the cardinality of the minimum vertex cover of G. The *distance* between two distinct vertices of a connected graph G is the number of edges in the shortest path connecting them. The *diameter* of a connected graph is the maximum distance over all pairs of vertices in G. Given a set $S \subseteq V(G)$, we denote the subgraph induced in G by S as $G[S]$. For $S \subseteq V(G)$, we write $G - S$ to denote the graph $G[V(G) \setminus S]$. We denote the complement of a graph G by $\overline{G}$, which is the graph having vertex set $V(G)$ in which two distinct vertices are adjacent if and only if they are not adjacent in G. Given a graph G and a vertex $u \in V(G)$, we denote by $N(u)$ the set of neighbours of u (or the “neighbourhood of u”) in G. The closed neighbourhood of u, defined to be $N(u) \cup \{u\}$, is denoted by $N[u]$. A *universal vertex* in a graph is a vertex that is adjacent to all other vertices. We denote by K_n the complete graph on n vertices. For sets V, A, and B, we write $V = A \sqcup B$ to denote that $V = A \cup B$ and $A \cap B = \emptyset$; i.e. $\{A, B\}$ is a partition of V into

[1] Co-interval and co-unit-interval graphs are the complements of interval and unit-interval graphs respectively.

$$I_u(v) = \begin{cases} [0,1] & \text{if } v = u \\ [1,2] & \text{if } v \in N(u) \\ [2,3] & \text{if } v \notin N[u] \end{cases}$$

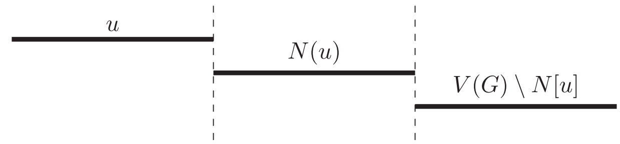

Fig. 1 Interval representation of the graphs I_u

two sets. A graph $G = (A \sqcup B, E)$ is a *bipartite graph* if the sets A and B are independent sets in G. A graph $G = (A \sqcup B, E)$ is a *co-bipartite graph* the sets A and B are cliques in G. Similarly, a graph $G = (K \sqcup S, E)$ is a *split graph* if the set K is a clique in G and the set S is an independent set in G. By log, we mean the logarithm with the base 2. By ln, we mean the natural logarithm. By $\log^*$, we mean the iterated logarithm function that is defined recursively as $\log^*(n) = 1 + \log^*(\log n)$ for $n > 1$ and $\log^*(n) = 0$ for $n \le 1$.

2 Basic results and techniques

2.1 Upper bounds

The general approach to answer questions related to upper bounds on the intersection dimension $dim_{\mathcal{A}}(G)$ of a graph G is to find a collection of graphs $G_1, G_2, \ldots, G_k \in \mathcal{A}$ such that $G = G_1 \cap G_2 \cap \cdots \cap G_k$. The following remark lays down explicitly the necessary and sufficient conditions for a collection of graphs $\{G_1, G_2, \ldots, G_k\}$ to have the property that $G = G_1 \cap G_2 \cap \cdots \cap G_k$.

Remark 2.1 For graphs $G, G_1, G_2, \ldots, G_k$, we have $G = G_1 \cap G_2 \cap \cdots \cap G_k$ if and only if both the conditions below are satisfied.

1. $V(G) = V(G_i)$ and $E(G) \subseteq E(G_i)$ for all $i \in [k]$, i.e. G_i is a supergraph of G for all $i \in [k]$. In other words, if $uv \in E(G)$ then $uv \in E(G_i)$, $\forall i \in [k]$.
2. For distinct vertices $u, v \in V(G)$, if $uv \notin E(G)$ then there exists a $j \in [k]$ such that $uv \notin E(G_j)$, or equivalently, for each edge $uv \in E(\overline{G})$, there exists a $j \in [k]$ such that $uv \notin E(G_j)$.

Because of the intersection dimension view of boxicity and cubicity, the above remark tells us that for finding $box(G)$ and $cub(G)$ of a graph G, we can try to find the least number of interval graphs and unit-interval graphs respectively that satisfy the conditions in Remark 2.1. As observed before, this approach already gives the bound $box(G) \le cub(G) \le |E(\overline{G})|$. It also allows us to derive the bound $box(G) \le cub(G) \le n$ for any graph G on n vertices. To see this, observe that corresponding to every vertex u of G, we can construct a unit-interval graph I_u having the unit-interval representation $\{I_u(v)\}_{v \in V(G)}$ defined as in Figure 1.

It is easy to check that G and the collection of unit-interval graphs $\{I_u : u \in V(G)\}$ defined above satisfy conditions 1 and 2 of Remark 2.1 and therefore $G = \bigcap_{u \in V(G)} I_u$. It follows that $box(G) \le cub(G) \le n$.

This upper bound can be reduced to $\lfloor n/2 \rfloor$ in case of boxicity and to $\lfloor 2n/3 \rfloor$ in case of cubicity as shown by Roberts [1].

Theorem 2.2 ([1]) For any graph G on n vertices, $box(G) \le \lfloor n/2 \rfloor$.

Proof Identify a set $S \subseteq V(G)$ such that S can be partitioned into pairs of non-adjacent vertices and $V(G) \setminus S$ forms a clique in G. Clearly, such a set S exists, since S is nothing but the set of vertices that are matched by a maximal matching in $\overline{G}$. Suppose that the partition of S into pairs of non-adjacent vertices is given by $S = \bigsqcup_{i=1}^{l} S_i$, where $|S| = 2l$. Consider some $i \in [l]$. Let $S_i = \{u_1, u_2\}$. We construct an interval graph I_i having the interval representation $\{I_i(v)\}_{v \in V(G)}$ defined as in Figure 2.

$$I_i(v) = \begin{cases} [0,1] & \text{if } v = u_1 \\ [5,6] & \text{if } v = u_2 \\ [0,3] & \text{if } v \in N(u_1) \setminus N(u_2) \\ [3,6] & \text{if } v \in N(u_2) \setminus N(u_1) \\ [0,6] & \text{if } v \in N(u_1) \cap N(u_2) \\ [2,4] & \text{if } v \notin N[u_1] \cup N[u_2] \end{cases}$$

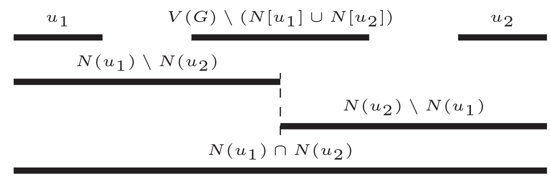

Fig. 2 Interval representation of the graphs I_i in the proof of Theorem 2.2

Clearly, I_i is a supergraph of G for each i. Also, observe that if a pair of vertices $v_1, v_2 \in V(G)$ is not adjacent in G, then either $v_1 \in S$ or $v_2 \in S$ because $V(G) \setminus S$ is a clique in G. Without loss of generality, assume that $v_1 \in S$. Then there is an i such that $v_1 \in S_i$ which ensures that $v_1v_2 \notin E(I_i)$ by the above construction. Hence, both conditions of Remark 2.1 are satisfied by G and the collection of interval graphs $\{I_1, I_2, \ldots, I_l\}$, and so $G = I_1 \cap I_2 \cap \cdots \cap I_l$. Since $2l = |S| \leq n$, we have $l \leq \lfloor n/2 \rfloor$, and therefore $box(G) \leq \lfloor n/2 \rfloor$. □

We now show that the bound given above is tight by exhibiting a graph on $2t$ vertices whose boxicity is exactly t.

Definition 2.3 Let G be the graph on $2t$ vertices obtained by removing a perfect matching from a complete graph on $2t$ vertices. This graph is called the "Roberts' graph" or the "cocktail party graph" on $2t$ vertices, and we denote it as $\overline{tK_2}$, since it is the complement of the graph obtained by the disjoint union of t copies of K_2.

Theorem 2.4 $box(\overline{tK_2}) \geq t$.

Proof Let G be a graph isomorphic to $\overline{tK_2}$. Let $k = box(G)$ and let $\{I_1, I_2, \ldots, I_k\}$ be a box representation of G of dimension k. Then $G = I_1 \cap I_2 \cap \cdots \cap I_k$. Consider I_i, for any $i \in [k]$. By Remark 2.1, we have that I_i is a supergraph of G. If there exist two edges $xy, x'y' \in E(\overline{G})$ (clearly, x, y, x', y' are all pairwise distinct since $\overline{G}$ is isomorphic to tK_2) such that $xy, x'y' \notin E(I_i)$, then since the edges xx', xy', yx', yy' are all in G, we have that there is an induced cycle of length 4 in I_i. This contradicts the fact that I_i is an interval graph (see Figure 4 and the discussion above it). We can thus conclude that at most one edge of $E(\overline{G})$ is absent in I_i. Since we have by Remark 2.1 that for each $xy \in E(\overline{G})$, there exists $i \in [k]$ such that $xy \notin E(I_i)$, we have that $k \geq |E(\overline{G})| = t$. □

Note that it now follows from Theorem 2.2 that $box(\overline{tK_2}) = t$. We now state a theorem (whose proof is straightforward and hence omitted) that relates the boxicity (cubicity) of the intersection of k graphs with the boxicity (resp. cubicity) of the individual graphs.

Theorem 2.5 Suppose $G, G_1, G_2, \ldots G_k$ be graphs such that $G = G_1 \cap G_2 \cap \cdots \cap G_k$. Then $box(G) \leq \sum_{i \in [k]} box(G_i)$ and $cub(G) \leq \sum_{i \in [k]} cub(G_i)$.

For showing an upper bound on the cubicity in terms of the number of vertices, we cannot use the same approach as in Theorem 2.2 directly (why?). Still, it is possible to show a $\left\lfloor \frac{2n}{3} \right\rfloor$ upper bound on the cubicity of any graph on n vertices. Note that the graphs with boxicity at most 2 are precisely the "rectangle intersection graphs", which are the intersection graphs of axis-parallel rectangles in the plane. Similarly, the graphs with cubicity at most 2 are the unit-square graphs, which are the intersection graphs of axis-parallel squares in the plane. Now we are ready to prove the claimed upper bound for cubicity.

Theorem 2.6 ([1]) For any graph G on n vertices, $cub(G) \leq \lfloor 2n/3 \rfloor$.

Proof For vertices x, y and z, we define $A(x) = N(x) \setminus (N(y) \cup N(z))$, $A(x, y) = (N(x) \cap N(y)) \setminus N(z)$ and $A(x, y, z) = N(x) \cap N(y) \cap N(z)$. We identify a set $S \subseteq V(G)$ such that S can be partitioned as $S = \bigsqcup_{i=1}^{l} S_i$ so that: (a) $\forall i \in [l]$, S_i is an independent set in G containing at most 3 vertices, and (b) $V(G) \setminus S$ is a clique in G. Such a

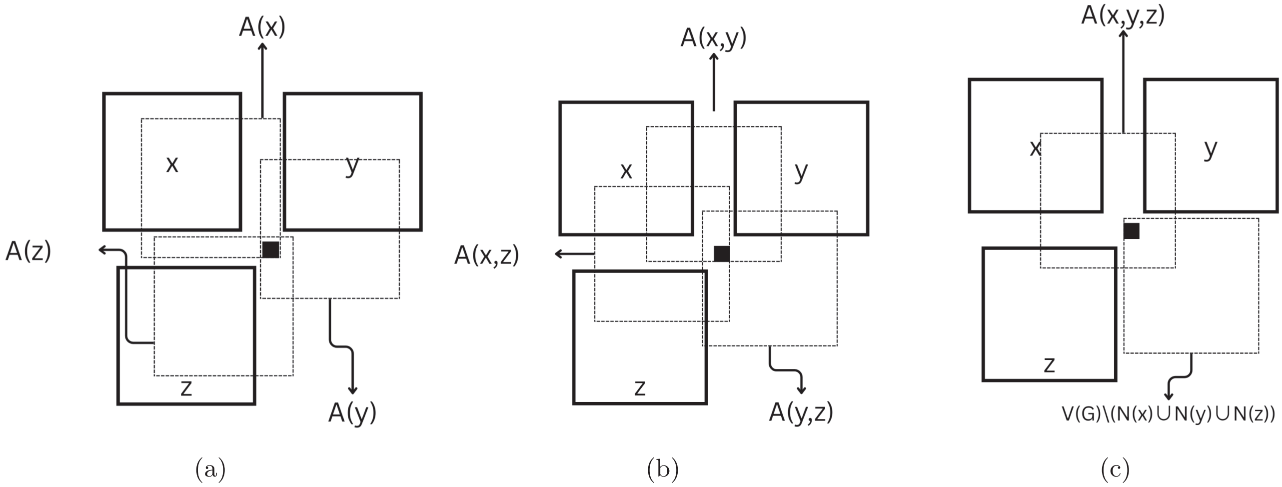

Fig. 3 The relative position of the unit squares corresponding to the vertices of G is illustrated

set always exists because one can iteratively identify (and remove) independent sets of size at most three, choosing sets of size two only when the sets of size three have been exhausted, until a clique is left. Observe that in this iterative process, if some set S_i contains only 2 vertices, i.e. $S_i = \{x, y\}$, then $V(G) \setminus (N(x) \cup N(y)) \subseteq \bigcup_{1 \le j < i} S_j$ because otherwise we could have added a third vertex z from $V(G) \setminus (N(x) \cup N(y))$ into S_i. Also note that this iterative process gives a partition such that if for some i we have $|S_i| = 2$, then for every $j > i$ we have $|S_j| = 2$. Let k be the least number such that for $i > k$, we have $|S_i| = 2$. Consider an i such that $k + 1 \le i \le l$. Let $S_i = \{u_1, u_2\}$. We construct a unit-interval graph I_i having unit-interval representation $\{I_i(v)\}_{v \in V(G)}$ defined as:

$$I_i(v) = \begin{cases} [1, 2] & \text{if } v = u_1 \\ [3, 4] & \text{if } v = u_2 \\ [1.5, 2.5] & \text{if } v \in N(u_1) \setminus N(u_2) \\ [2.5, 3.5] & \text{if } v \in N(u_2) \setminus N(u_1) \\ [2, 3] & \text{otherwise} \end{cases}$$

Clearly, I_i is a supergraph of G. Next, consider an i such that $1 \le i \le k$. Let $S_i = \{x, y, z\}$. We construct a unit square graph R_i by assigning a unit square to each vertex depending on whether it belongs to $A(x)$, $A(y)$, $A(z)$, $A(x, y)$, $A(x, z)$, $A(y, z)$, $A(x, y, z)$ or otherwise. The construction is best presented pictorially as is illustrated in Figures 3a–3c. While three figures are presented for ease of understanding, they should be understood as a single superimposed figure. All the unit squares corresponding to vertices in $V(G) \setminus \{x, y, z\}$ contain the small black square and therefore form a clique. This ensures that the unit square graph R_i is a supergraph of G. Furthermore, observe that if a pair of vertices v_1, v_2 is such that $v_1v_2 \notin E(G)$ then either $v_1 \in S$ or $v_2 \in S$ because $V(G) \setminus S$ is a clique. Consequently, there exists an i such that $S_i \cap \{v_1, v_2\} \neq \emptyset$ which means that $v_1v_2 \notin E(I_i)$ or $v_1v_2 \notin E(R_i)$ depending on whether $i > k$ or not. Therefore, it is clear that conditions 1 and 2 of Remark 2.1 are satisfied by G and the collection of graphs $\{R_1, R_2, \ldots, R_k, I_{k+1}, I_{k+2}, \ldots, I_l\}$. So we have $G = R_1 \cap R_2 \cap \cdots \cap R_k \cap I_{k+1} \cap I_{k+2} \cap \cdots \cap I_l$. By Theorem 2.5, the cubicity of G is at most $2k + l - k = \frac{2}{3}(3k) + \frac{1}{2}(2l - 2k) \le \frac{2}{3}(3k) + \frac{2}{3}(2l - 2k) = \frac{2|S|}{3} \le \left\lfloor \frac{2n}{3} \right\rfloor$. □

Using techniques similar to those employed in the proofs of Theorems 2.2 and 2.6, the following upper bounds on boxicity of split graphs and the cubicity of co-bipartite graphs can be obtained.

Theorem 2.7 ([9]) Let $G = (K \sqcup S, E)$ be a split graph. Then provided $K \neq \emptyset$, $box(G) \le \min(\lceil |K|/2 \rceil, \lceil |S|/2 \rceil)$.

Theorem 2.8 For any co-bipartite graph $G = (A \sqcup B, E)$, $box(G) \le cub(G) \le \min(|A|, |B|)$.

Observation 2.12 shows that the first inequality in the above statement can actually be changed to an equality. Note that this bound is tight since the Roberts' graph on $2t$ vertices is a co-bipartite graph whose vertex set can be partitioned into two cliques each having size t, and has boxicity t.

Chandran, Das and Shah [10] showed the following upper bound on the boxicity of a graph in terms of its vertex cover number. Note that the vertex cover number of any graph on n vertices that is not a complete graph is at most $n-2$. Hence this bound is better than the one given by Theorem 2.2.

Theorem 2.9 ([10]) For any graph G having vertex cover number t, $box(G) \leq \lfloor t/2 \rfloor + 1$.

Proof Let X be a minimum vertex cover of G. Define G' to be the graph $\overline{G}[X]$.

Suppose that M is a maximal matching in G'. Let $|V(M)| = 2k$. Now we can use the construction as in the proof of Theorem 2.2 to construct k many interval supergraphs $I_1, I_2, \ldots, I_k$ so that for any $xy \notin E(G)$ such that at least one of x or y is in $V(M)$, we have that for some $i \in [k]$, $xy \notin E(I_i)$. Define $H = G - V(M)$. It is clear that H is either an independent set or a split graph based on whether $V(M) = X$ or not, respectively. Consider the graph G'' obtained from G by turning the vertices in $V(M)$ into universal vertices. Notice that G'' is a split graph. Clearly, G'' is also a supergraph of G. Observe that if $xy \notin E(G)$ and both x and y are absent in $V(M)$ then we have $xy \notin E(G'')$ by construction. It follows from Remark 2.1 that $E(G) = \left(\bigcap_{i=1}^{k} E(I_i)\right) \cap E(G'')$. By Theorem 2.5, this means that $box(G) \leq k + box(G'')$. Since adding universal vertices to a graph does not change its boxicity, we can conclude that $box(G'') = box(H)$.

Suppose $V(M) = X$. Then H is an independent set. Therefore, $box(G'') = box(H) = 1$ and we have $box(G) \leq k + 1 = \lfloor t/2 \rfloor + 1$.

Next, suppose $V(M) \neq X$. Then $box(G'') = box(H) \leq \lceil \frac{t-2k}{2} \rceil$ by Theorem 2.7. Therefore, $box(G) \leq k + \lceil \frac{t-2k}{2} \rceil \leq \lfloor t/2 \rfloor + 1$. We are done. □

The following result from [10] shows an upper bound on the boxicity of bipartite graphs that is slightly better than the one that can be obtained from Theorem 2.9.

Theorem 2.10 ([10]) For a bipartite graph $G = (V_1 \sqcup V_2, E)$, $box(G) \leq \min(\lceil |V_1|/2 \rceil, \lceil |V_2|/2 \rceil)$.

Chandran, Das and Shah [10] also observe that the upper bound for boxicity in terms of the vertex cover given above can be used to show the following theorem.

Theorem 2.11 ([10]) If $box(G) = \frac{n}{2} - s$, then $\chi(G) \geq \frac{n}{2s+2}$.

Thus, when the boxicity of a graph is close to the upper bound given by Theorem 2.2, its chromatic number must also necessarily be high. Note that in general, the chromatic number does not exhibit a strong relation with the boxicity of a graph. For instance, complete graphs have large chromatic numbers but boxicity equal to 0. Similarly, interval graphs can have large chromatic number even though they have boxicity at most 1. On the other hand, there also exist bipartite graphs whose boxicity is large (see the discussion after Theorem 2.24). Thus there cannot be an upper bound on boxicity based on chromatic number alone. However, as we shall see in Section 3.2, $box(G) \leq 2\chi(G^2)$, where G^2 is the square of the graph G (see the proof Theorem 3.2).

2.1.1 Bounds using co-bipartite graphs

Co-bipartite graphs are routinely used to find bounds on both boxicity and cubicity of graphs. It was shown by Adiga, Bhowmick and Chandran [11] that every co-bipartite interval graph $G = (A \sqcup B, E)$ has a *canonical interval representation*, i.e. an interval representation $\{I(v)\}_{v \in V(G)}$ for which there exists $k_l, k_r \in \mathbb{R}$ such that $\forall u \in V(G)$, $k_l \leq l(I(u)) \leq r(I(u)) \leq k_r$; $\forall u \in A$, $l(I(u)) = k_l$; and $\forall u \in B$, $r(I(u)) = k_r$. From this observation, we get the following folklore result.

Observation 2.12 *For any co-bipartite graph G, $cub(G) = box(G)$.*

Proof Let $box(G) = k$ and suppose that $\{I_1, I_2, \ldots, I_k\}$ is a box representation of G of dimension k (i.e. $I_1, I_2, \ldots, I_k$ are interval graphs such that $G = I_1 \cap I_2 \cap \cdots \cap I_k$). For every $i \in [k]$, I_i is co-bipartite since $E(G) \subseteq E(I_i)$, and therefore we let $\{I_i(v)\}_{v \in V(G)}$ be a canonical interval representation of I_i. For $i \in [k]$, let λ_i be the maximum length of an interval in I_i, i.e. $\lambda_i = \max_{u \in V(G)}(r(I_i(u)) - l(I_i(u)))$. Now for each $i \in [k]$, we construct a unit-interval graph I'_i having the equal-interval representation $\{I'_i(v)\}_{v \in V(G)}$ as follows: $\forall u \in B$ assign $I'_i(u) = [l(I_i(u)), l(I_i(u)) + \lambda_i]$ and $\forall u \in A$ assign $I'_i(u) = [r(I_i(u)) - \lambda_i, r(I_i(u))]$. It is clear that the graph I'_i is isomorphic to I_i for each $i \in [k]$. So $\{I'_1, I'_2, \ldots, I'_k\}$ is a cube representation of dimension k for G. □

The following theorem suggests that the graph obtained by adding edges to a graph G till it becomes a co-bipartite graph does not drastically increase its boxicity.

Theorem 2.13 ([11]) Suppose G is a graph and $A, B \subseteq V(G)$ are disjoint subsets. Define G' to be the co-bipartite graph having $V(G') = V(G)$ and $E(G') = \{uv : u, v \in A\} \cup \{uv : u, v \in B\} \cup E(G)$. Then, $box(G') \leq 2box(G)$.

Proof Suppose $\{I_1, I_2, \ldots, I_k\}$ is an optimal box representation of G where the interval graph I_i, for $i \in [k]$, has the interval representation $\{I_i(v)\}_{v \in V(G)}$, where we assume $I_i(v) = [l_i(v), r_i(v)]$. Let $M_i = \max_{v \in V(G)} r_i(v)$ and $m_i = \min_{v \in V(G)} l_i(v)$.

Define

$$I'_i(v) = \begin{cases} [l_i(v), M_i] & \text{if } v \in A \\ [m_i, r_i(v)] & \text{if } v \in B \\ I_i(v) & \text{otherwise} \end{cases} \qquad I''_i(v) = \begin{cases} [m_i, r_i(v)] & \text{if } v \in A \\ [l_i(v), M_i] & \text{if } v \in B \\ I_i(v) & \text{otherwise} \end{cases}$$

Then the interval graphs $\{I'_1, I'_2, \ldots, I'_k\} \cup \{I''_1, I''_2, \ldots, I''_k\}$ determined by the above representations form a box representation of G' of dimension $2k$. □

If G is a bipartite graph, then we also get a lower bound.

Theorem 2.14 ([11]) Suppose $G = (A \sqcup B, E)$ is a bipartite graph and suppose G' is the co-bipartite graph defined as in Theorem 2.13. If G' is not an interval graph, then $box(G) \leq box(G') \leq 2box(G)$. If G' is an interval graph, then $box(G) \leq 2$.

Proof Let $\{I_1, I_2, \ldots, I_k\}$ be an optimal box representation of G'. We can assume as before that for each $i \in [k]$, $\{I_i(v)\}_{v \in V(G)}$ is a canonical interval representation of the co-bipartite graph I_i (since G' is co-bipartite and I_i is a supergraph of G'). Suppose that $k \geq 2$. Then let I'_1 be a graph having interval representation $\{I'_1(v)\}_{v \in V(G)}$ defined as follows:

$$I'_1(v) = \begin{cases} I_1(v) & \text{if } v \in A \\ [l(I_1(v)), l(I_1(v))] & \text{if } v \in B \end{cases}$$

and let I'_2 be a graph having interval representation $\{I'_2(v)\}_{v \in V(G)}$ defined as follows:

$$I'_2(v) = \begin{cases} [r(I_2(v)), r(I_2(v))] & \text{if } v \in A \\ I_2(v) & \text{if } v \in B \end{cases}$$

It can be seen that I'_1 is the graph obtained from I_1 by removing all edges between vertices in B, and I'_2 is the graph obtained from I_2 by removing all edges between vertices in A. It follows that $I'_1 \cap I'_2 \cap \bigcap_{3 \leq i \leq k} I_i = G$. Thus $box(G) \leq box(G')$. If $k = 1$, then we can create a box representation $\{I_1, I_2\}$ for G' where I_2 is the complete graph with vertex set $V(G)$ and repeat the above argument to obtain a box representation of G of dimension 2. □

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2.1.2 Cubicity in terms of other graph parameters

Chandran and Mathew [12] showed that for an interval graph G with n vertices, $cub(G) \leq \lceil \log n \rceil$. This was improved by Adiga and Chandran [13] by associating cubicity with the number of edges in the largest claw[2] of G, also known as the claw number of G, denoted by $\psi(G)$.

Theorem 2.15 ([13]) For an interval graph G, $cub(G) \leq \lceil \log \psi(G) \rceil + 2$.

In the same paper, the authors observe that if $\psi(G) = \alpha(G)$, then cubicity is at most $\lceil \log \alpha(G) \rceil$. If we add a universal vertex to an interval graph G we get another interval graph G' that has cubicity at least that of G and $\psi(G') = \alpha(G') = \alpha(G)$. Thus, $cub(G) \leq cub(G') \leq \lceil \log \alpha(G') \rceil = \lceil \log \alpha(G) \rceil$. This leads to the following theorem.

Theorem 2.16 ([13]) For an interval graph G, $cub(G) \leq \lceil \log \alpha(G) \rceil$

Since the independence number can only decrease when we add edges to a graph, any interval supergraph I of a graph G has $\alpha(I) \leq \alpha(G)$. This gives the following corollary.

Corollary 2.17 ([13]) $cub(G) \leq \lceil \log \alpha(G) \rceil box(G)$

In general when we remove a vertex from G, its boxicity can decrease by at most 1. This can be generalized as follows.

Theorem 2.18 Let $A \subseteq V(G)$. Then $box(G) \leq |A| + box(G - A)$.

Proof Construct $|A|$ many interval graphs I_u having the interval representation $\{I_u(v)\}_{v \in V(G)}$ defined as follows. For $u \in A$, define

$$I_u(v) = \begin{cases} [0, 1] & \text{if } v = u \\ [1, 2] & \text{if } v \in N(u) \\ [2, 3] & \text{otherwise} \end{cases}$$

Suppose $b = box(G - A)$ and $I'_1, I'_2, \ldots, I'_b$ be an optimal box representation of $G - A$. For $i \in [b]$, let I''_i be the interval graph obtained from I'_i by adding each vertex of A as a universal vertex (so A forms a clique in I''_i).

It is easy to see that $G = \bigcap_{u \in A} I_u \cap \bigcap_{i \in [b]} I''_i$, and therefore $box(G) \leq |A| + b$. □

Corollary 2.19 Let $A \subseteq V(G)$. Then, $cub(G) \leq |A| + box(G - A) \lceil \log(\alpha(G - A)) \rceil$

Proof The interval graphs I_u defined in Theorem 2.18 are unit-interval graphs. The interval graphs I''_i defined in Theorem 2.18 have cubicity at most $\lceil \log \alpha(I''_i) \rceil \leq \lceil \log \alpha(G - A) \rceil$ because of Theorem 2.16. This gives us the result. □

By taking A to be a minimum vertex cover of G and using Corollary 2.19, it can be seen immediately that $cub(G) \leq |A| + box(G - A) \lceil \log(n - |A|) \rceil = |A| + \lceil \log(n - |A|) \rceil$ where the last equality follows as $G - A$ is a graph with no edges and hence has boxicity 1 (unless the vertex cover number is $n - 1$, in which case G is a complete graph and has cubicity 0). The following result due to Chandran, Das and Shah [10] says that a slightly stronger statement is true.

Theorem 2.20 ([10]) For any graph G on n vertices having vertex cover number t, $cub(G) \leq t + \lceil log(n - t) \rceil - 1$.

The edge clique cover number of a graph G, denoted by $\theta(G)$, is the minimum cardinality of a collection of cliques of G such that every edge of G is contained in at least one of the cliques. The edge biclique cover number of a graph G, denoted by $\eta(G)$, is the minimum number of complete bipartite subgraphs of G such that each edge of G is in at least one of them. We then have the following results due to Michael and Quint [14].

---

[2] A claw is an induced subgraph that is a star.

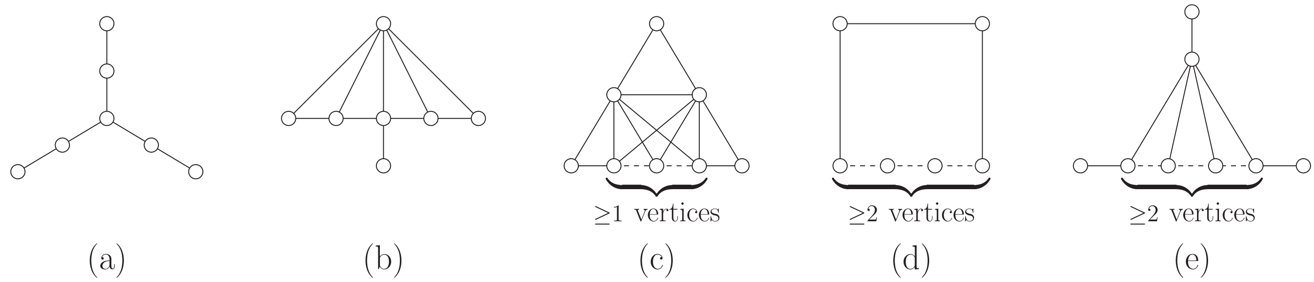

Fig. 4 The forbidden induced subgraphs that characterize interval graphs

Theorem 2.21 ([14]) For any graph G having $\theta(G) > 0$, $cub(G) \leq \theta(G)$.

Theorem 2.22 ([14]) For any graph G, $cub(G) \leq \eta(\overline{G})$.

2.2 Lower bounds for boxicity

The following observation will be crucially used to prove lower bounds.

Observation 2.23 *Let G be a graph and H be an induced subgraph of G. Then $box(G) \geq box(H)$.*

The above observation along with some graphs with boxicity at least 2 will be very useful to find lower bounds. For example, if we know that a graph has an induced cycle of length at least 4, then it cannot be an interval graph. In fact, the classic result of Lekkerkerker and Boland [15] showed that the graphs given in Figure 4 are the minimal forbidden induced subgraphs for interval graphs; in other words, a graph is an interval graph if and only if it does not contain any of the graphs in Figure 4 as induced subgraphs.

We use these forbidden induced subgraphs to observe the following.

Observation 2.24 *Let G be a graph having $V(G) = \{v_1, v_2, \ldots, v_6\}$ such that:*

- *$v_1, v_2, v_3, v_4, v_5, v_6$ forms a cycle in G, and*
- *$v_1v_4, v_2v_5, v_3v_6 \notin E(G)$.*

Then $box(G) > 1$.

The observation follows by noting that graphs defined as above always contain C_4, C_5, C_6 or the Hajós graph (the smallest graph of type (c) in Figure 4) as an induced subgraph, which are all forbidden induced subgraphs for interval graphs. We will use Observation 2.24 to prove the following theorem.

Theorem 2.25 ([9]) Let G be a graph and $p \geq 1$. Suppose that $V(G)$ contains two disjoint sets of vertices $S_1 = \{a_1, a_2, \ldots, a_{2p-1}\}$ and $S_2 = \{b_1, b_2, \ldots, b_{2p-1}\}$, and suppose that the only edges between S_1 and S_2 in $\overline{G}$ are $\{a_ib_i : i \in [2p-1]\}$. Then $box(G) \geq p$.

Proof Suppose $k = box(G) \leq p-1$. Then there exist k interval graphs $I_1, I_2, \ldots, I_k$ such that $G = I_1 \cap I_2 \cap \cdots \cap I_k$. Let $T \subseteq E(\overline{G})$ be defined as $T = \{a_ib_i : i \in [2p-1]\}$. As $E(\overline{G}) \subseteq \bigcup_{i\in[k]} E(\overline{I_i})$, we have that $T \subseteq \bigcup_{i\in[k]} E(\overline{I_i})$. Since $(2p-1)/k \geq (2p-1)/(p-1) > 2$, we have by the pigeonhole principle that $\exists i \in [k]$ such that at least three edges of T must be absent I_i. Without loss of generality, let these edges of T be a_1b_1, a_2b_2, a_3b_3. It can be seen that the graph induced by the vertices $\{a_1, b_1, a_2, b_2, a_3, b_3\}$ is of the form defined in Observation 2.24. This contradicts the fact that I_i is an interval graph. □

It follows from the above theorem that any graph on n vertices whose vertex set can be partitioned into two sets of size $n/2$ such that in the complement of the graph, the collection of all edges between the two sets forms a matching of cardinality $n/2$, has boxicity at least $n/4$. We thus have the following corollary.

Corollary 2.26 ([10], [9]) There exist bipartite graphs and split graphs with boxicity at least $n/4$ where n is the number of vertices.

In fact, for the same reason, one can also conclude that there are co-bipartite graphs on n vertices having boxicity at least $n/4$, but the n-vertex co-bipartite graph that one obtains using the above construction turns out to be the same as the Roberts' graph on n vertices, which has boxicity $n/2$.

The join and disjoint union operations of graphs will be used repeatedly in our analysis. For the sake of completeness, we define these operations below.

Definition 2.27 Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be graphs (here we assume $V_1 \cap V_2 = \emptyset$). The join of G_1 and G_2, denoted as $G_1 \vee G_2$ is the graph $G_3 = (V_3, E_3)$ such that $V_3 = V_1 \cup V_2$ and $E_3 = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$. The disjoint union of G_1 and G_2, denoted as $G_1 \sqcup G_2$ is the graph $G_3 = (V_3, E_3)$ such that $V_3 = V_1 \cup V_2$ and $E_3 = E_1 \cup E_2$.

Observe that if G is a graph and K_t is the complete graph of order t, then $box(G \vee K_t) = box(G)$. This is because if $\{I_1, I_2, \ldots, I_k\}$ is a box representation of G, then $\{I_i \vee K_t\}_{i\in[k]}$ is a box representation for $G \vee K_t$. Similarly, if G is not a complete graph, then $box(G \sqcup K_t) = box(G)$ as $G \sqcup K_t = \bigcap_{i\in k}(I_i \sqcup K_t)$ is a box representation of $G \sqcup K_t$. On the other hand, for any two numbers n and n' we have $box(K_n \sqcup K_{n'}) = 1$ as $K_n \sqcup K_{n'}$ is not a complete graph, but it is an interval graph. Using these observations, and using arguments similar to that in the proof of Theorem 2.4, the following more general theorem can be shown.

Theorem 2.28 Suppose G_1, G_2 are two graphs with disjoint vertex sets. Then

(a) $box(G_1 \vee G_2) = box(G_1) + box(G_2)$
(b) $box(G_1 \sqcup G_2) = \max(box(G_1), box(G_2), 1)$

Note that the Roberts' graph on $2t$ vertices, which is $\overline{tK_2}$, is the same as the graph $\overline{K_2} \vee \overline{K_2} \vee \cdots \vee \overline{K_2}$, and Theorem 2.28 tells us that $box(\overline{tK_2}) = t$, which is consistent with our earlier observation about the boxicity of Roberts' graphs. Recall that Roberts' graphs have boxicity equal to the upper bound given in Theorem 2.2. A complete characterization of all such graphs was given by Trotter [16]. More precisely, a minimal family of graphs $\mathcal{C}_n$ was provided such that a graph with $2n+1$ vertices has boxicity n if and only if it contains a graph from $\mathcal{C}_n$ as an induced subgraph.

Theorem 2.29 ([16]) Let $n \geq 1$ and let G be a graph with $|V(G)| = 2n+1$.
Let $H_n = \overline{(n-2)K_2} \vee C_5$, $W_n = \overline{(n-3)K_2} \vee \overline{C_7}$.

1. If $n = 1$, then $box(G) = n$ if and only if G contains $\overline{K_2}$.
2. If $n = 2$, then $box(G) = n$ if and only if G contains $\overline{2K_2}$ or H_2.
3. If $n > 3$, then $box(G) = n$ if and only if G contains $\overline{nK_2}$, H_n, or W_n.

Note that $\overline{2K_2}$ is isomorphic to C_4 and H_2 is isomorphic to C_5.

2.2.1 Vertex isoperimetric problem and minimum interval supergraphs

Let $G = (V, E)$ be a graph. Define the "vertex boundary" $N(X, G)$ of $X \subseteq V$ as $N(X, G) = \{u \in V \setminus X : \exists v \in X \text{ with } uv \in E\}$. Similarly, define the "strong vertex boundary" $N_S(X, G)$ of X as $N_S(X, G) = \{u \in V \setminus X : uv \in E, \forall v \in X\}$.

For $k \in \mathbb{N}$, the parameters $b_v(k, G)$ and $c_v(k, G)$ are defined as follows.

$$b_v(k, G) = \min_{\substack{X\subseteq V\\|X|=k}} |N(X, G)|$$

$$spsc_v(k, G) = \max_{\substack{X \subseteq V \\ |X|=k}} |N_S(X, G)|$$

It is easy to see that $N_S(X, \overline{G}) = V \setminus X \setminus N(X, G)$ and $c_v(k, \overline{G}) = n - k - b_v(k, G)$.

Adiga, Chandran and Sivadasan [17] showed that $|E(\overline{I_{min}})| \leq \sum_{i=1}^{n-1} c_v(i, \overline{G})$ where I_{min} is an interval supergraph of G with minimum number of edges. It is easy to prove that $box(G) \geq \frac{|E(\overline{G})|}{|E(\overline{I_{min}})|}$ and therefore, the following holds:

Theorem 2.30 ([17]) Let G be a non-complete graph with n vertices. Then,

$$box(G) \geq \frac{|E(\overline{G})|}{\sum_{i=1}^{n-1} c_v(i, \overline{G})}$$

As a consequence, we can find lower bounds on boxicity for some special classes of graphs by estimating the values of c_v as given below.

Theorem 2.31 ([17]) Let $k \geq 1$ be an integer and G be a graph with n vertices.

(1) If G is an $(n - k - 1)$-regular graph for some positive integer k. Then, $box(G) \geq \frac{n}{2k}$.
(2) If G is an $(n - k - 1)$-regular co-planar graph. Then, $box(G) \geq n/8$.
(3) If G is the complement of a k-regular C_4-free graph. Then, $box(G) \geq n/4$.
(4) $box(\overline{C_n}) \geq n/3$.

2.2.2 Lower bounds for random graphs: $\mathcal{G}(n, p)$

Using ideas from subsection 2.2.1 on random graphs, the following lower bounds on the boxicity of random graphs are obtained in [17].

Theorem 2.32 ([17]) Let $p \leq 1 - 40\frac{\log n}{n^2}$. Let G be a graph drawn from the $\mathcal{G}(n, p)$ model of random graphs. Then with high probability, $box(G) = \Omega(np(1 - p))$.

Applying the above theorem with $p = 1/2$, we get that almost all graphs on n vertices have boxicity $\Omega(n)$.

Theorem 2.33 ([17]) Let m be an integer such that $m \leq cn^2/3$, where $c < 1$ is any positive constant.

1. Then for almost all graphs G on n vertices and m edges, $box(G) = \Omega(m/n)$.
2. Then for almost all balanced bipartite graphs[3] G on $2n$ vertices and m edges, $box(G) = \Omega(m/n)$.

2.2.3 Spectral lower bounds

For a graph G and $X \subseteq V(G)$, define $N'(X, G) = \{u \in V(G) \colon \exists v \in X \text{ with } uv \in E\}$. The difference between $N(X, G)$ and $N'(X, G)$ is that $N'(X, G)$ can contain vertices from X as well. Let G be a k-regular graph. Let λ be the second largest eigenvalue in absolute value of the adjacency matrix of G. For any subset $X \subseteq V$, $|N'(X, G)| \geq \frac{k^2|X|}{\lambda^2 + (k^2 - \lambda^2)\frac{|X|}{n}}$. This relationship allows us to derive the following lower bounds from Theorem 2.30.

Theorem 2.34 ([17]) Let $k \in \mathbb{N}$.

1. Let G be a connected k-regular graph on n vertices. Let λ be the second largest eigenvalue in absolute value of the adjacency matrix of G. Then the boxicity of G is at least $\left(\frac{k^2/\lambda^2}{\log(1+k^2/\lambda^2)}\right)\left(\frac{n-k-1}{2n}\right)$.

[3] A balanced bipartite graph G is a bipartite graph with bipartition $V(G) = A \sqcup B$ such that $|A| = |B|$.

2. The boxicity of random k-regular graphs on n vertices is $\Omega(k/\log k)$.

2.2.4 Closed neighourhood

The closed vertex neighbourhood of a set S, is the set $N[S, G] = S \cup N(S, G)$.

Definition 2.35 Let $G = (V, E)$ be a graph. Let $A, B \subseteq V$. Let t be a positive integer such that $t \leq |A|$. Then $n_t(A, B, G) = \min_{S \subseteq A, |S|=t} |N[S, G] \cap B|$. Similarly, let $m_t(A, B, G) = \min_{S \subseteq A, |S|=t} |N'(S, \overline{G}) \cap B|$.

The following result was shown in [17] by analysing the closed neighbourhoods of sets of vertices.

Theorem 2.36 ([17]) Let $G = (V, E)$ be any graph. Let $S_1, S_2 \subseteq V$ such that $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$ and $S_1 \cup S_2 = V$. Suppose that there is no vertex $u \in S_2$ such that $N[u] \cap S_1 = S_1$. Let $b^* = box(G)$. Let t be a fixed positive integer such that $1 \leq t \leq |S_1|$. Let $n_t = n_t(S_1, S_2, G)$. Let $t^* = |S_2| - 2b^*(|S_2| - n_t)$. Let $m^* = m_{t^*}(S_2, S_1, G)$, for $t^* > 0$. For $t^* \leq 0$, we define $m^* = \infty$. Then

$$box(G) \geq \frac{|S_2|}{2((|S_2| - n_t) + (t - 1)\frac{t^*}{m^*})}$$

Theorem 2.36 has the following consequence:

Theorem 2.37 ([17]) The boxicity of a graph on n vertices with n_u many universal vertices and minimum degree δ is at least $\frac{n-n_u}{2(n-\delta-1)}$

Proof Let G be a graph on n vertices containing n_u universal vertices and having minimum degree δ. Let U be the set of universal vertices in G. Consider the graph G' induced in G by the vertices in $V \setminus U$. Let $S_1 = S_2 = V \setminus U$ and fix $t = 1$. Then the result follows by applying Theorem 2.36 on G' after computing $n_t(S_1, S_2, G')$ and $|S_2|$ in terms of minimum degree δ and n_u. □

2.3 Lower bounds for cubicity

Observation 2.38 *Let G be a graph and H be an induced subgraph of G. Then $cub(G) \geq cub(H)$.*

Roberts [1] gave a formula for the cubicity of a complete k-partite graph.

Theorem 2.39 ([1]) Let G be the complete k-partite graph with parts of order $n_1, n_2, \ldots, n_k$. Then $cub(G) = \sum_{i=1}^{k} \lceil \log n_i \rceil$.

Consider a graph that has the complete k-partite graph with parts of order $n_1, n_2, \ldots, n_k$ as an induced subgraph. Then by the theorem above, the cubicity of the graph is at least $\sum_{i=1}^{k} \lceil \log(n_i) \rceil$. By taking the join of this graph with the empty graph containing n_{k+1} vertices and no edges, where $n_{k+1} \geq 3$, we get a graph that contains the complete k-partite graph with parts of order $n_1, n_2, \ldots, n_{k+1}$ as an induced subgraph. Thus, the cubicity of the graph increases by at least $\lceil \log(n_{k+1}) \rceil > 1$. Since the cubicity of any graph without edges is at most 1, this tells us that it is possible that for two graphs G_1 and G_2, $cub(G_1 \vee G_2) > cub(G_1) + cub(G_2)$.

The following theorem is an analogue of Theorem 2.28 for cubicity, albeit weaker.

Theorem 2.40 Suppose G_1, G_2 are two graphs with $V(G_i) \neq \emptyset$ for $i \in 1, 2$. Then

(a) $cub(G_1 \vee G_2) \geq cub(G_1) + cub(G_2)$
(b) $cub(G_1 \sqcup G_2) = \max(cub(G_1), cub(G_2), 1)$

For cubicity, the geometry of the cube representation becomes very important. This is because the constraint that every side must be of equal length, forces non-intersecting cubes to compete for space. Therefore, we can exploit the volume taken up by an independent set to obtain lower bounds. The easiest application of this idea is the lower bound on cubicity in terms of the claw number. Recall, that the claw number (denoted as $\psi(G)$) is the number of edges in the largest claw of G. The unique vertex with degree strictly greater than 1 in the claw is called the central vertex of the claw. Adiga and Chandran [13] showed that for any graph G, $cub(G) \geq \lceil \log \psi(G) \rceil$. This is clear since in any cube representation of G in $cub(G)$ dimensions, the cubes associated with every non-central vertex of a claw in G are all non-intersecting and each of them must contain at least one corner of the cube corresponding to the central vertex. Since there are at most $2^{cub(G)}$ many corners for any cube in the representation, we have $\psi(G) \leq 2^{cub(G)}$.

We can generalize this idea to the largest independent set as well, as shown in the following theorem by Chandran, Mannino and Oriolo [18].

Theorem 2.41 ([18]) Let G be a graph on n vertices with diameter D. Then $cub(G) \geq \frac{\log \alpha(G)}{\log(D+1)}$.

Proof It is not hard to prove that any unit cube representation of G is contained in a cubical volume of side length $D + 1$. Therefore, the total volume that any optimal unit cube representation of G can take is at most $(D+1)^{cub(G)}$. In particular, the volume occupied by the largest independent set is at most $(D+1)^{cub(G)}$. In other words, $(D+1)^{cub(G)} \geq \alpha(G)$. □

We also have the following lower bound in terms of clique number:

Theorem 2.42 ([18]) For any graph G having n vertices and diameter D, $cub(G) \geq \frac{\log n - \log \omega(G)}{\log D}$.

Proof Subdivide the cubical volume of side length $D + 1$ described above into an integral grid. Then every cube contains at least one interior point in this grid. Since there are at most $D^{cub(G)}$ many such points, there is a point which is contained in at least $n/D^{cub(G)}$ cubes. The vertices corresponding to these cubes are all pairwise adjacent, and therefore they form a clique of order at least $n/D^{cub(G)}$. Therefore, $n/D^{cub(G)} \leq \omega(G)$. □

3 Techniques based on colourings

In this section, we will highlight some techniques that involve colourings (not necessarily proper) of graphs. The approach will be to create colourings in such a way that the colour classes satisfy certain desirable structural properties. These structural properties will then be exploited to bound the boxicity and cubicity of the graph. We list some examples.

3.1 Boxicity and acyclic chromatic number

An acyclic colouring $c : V(G) \rightarrow [k]$ of a graph G is a proper colouring of G using k colours that has the additional property that the subgraph induced by the union of any two colour classes is a forest. The minimum number of colours required in any acyclic colouring of a graph G is called the *acyclic chromatic number* of G, and is denoted by $\chi_a(G)$. Since the boxicity of any forest is at most 2 (why?), we get the following result of Esperet and Joret [19].

Theorem 3.1 ([19]) Suppose G is a graph and $\chi_a(G)$ is the acyclic chromatic number of G. Suppose $\chi_a(G) \geq 2$. Then $box(G) \leq \chi_a(G)(\chi_a(G) - 1)$.

Proof Suppose $\chi_a(G) = k$ and suppose $c : V(G) \rightarrow [k]$ is an optimal acyclic colouring of G. Suppose $C_1, C_2, \ldots, C_k$ are the colour classes of G under c. For $1 \leq i < j \leq k$, define $H_{ij} = C_i \cup C_j$. By the definition of an acyclic colouring, the graph induced on the vertices in H_{ij} is a forest in G. For $1 \leq i < j \leq k$, define the graph $G_{ij} = (V, E_{ij})$ to be the graph $V = V(G)$ and $E_{ij} = E(G) \cup \{uv \colon u \in V(G) \setminus H_{ij}, v \in V(G)\}$. Then it is easy to see that $G = \bigcap_{1 \leq i < j \leq k} G_{ij}$. Note that H_{ij} is a forest in G_{ij} and the vertices of $V(G) \setminus H_{ij}$ are universal vertices in G_{ij}. Thus, it is clear that $box(G_{ij}) = box(H_{ij}) \leq 2$. It follows from Theorem 2.5 that $box(G) \leq \sum_{1 \leq i < j \leq k} box(G_{ij}) \leq 2 \times \binom{k}{2} = k(k-1)$. □

3.2 Boxicity and maximum degree

Suppose G is a graph. Consider the graph G^2, the graph obtained from G by adding an edge between every pair of vertices at distance 2 in G. Chandran, Francis and Sivadasan [20] used an optimal proper colouring c of G^2 to obtain degree based upper bounds on the boxicity of G. Observe that c is also a proper colouring of G. Also, if there is a path uvw in G, then the edge uw is present in G^2 and therefore with $c(u) \neq c(w)$. Thus c satisfies the property that there is no path uvw in G with $c(u) = c(w)$.

Theorem 3.2 ([20]) Suppose G is a graph with maximum degree Δ. Then $box(G) \leq 2\Delta^2$.

Proof Suppose c is an optimal proper colouring of G^2 using k colours. Observe that c is also a proper colouring of G. Suppose $C_1, C_2, \ldots, C_k$ are the colour classes of G under c.

For each $i \in [k]$, define the graph $G_i = (V, E_i)$ as $V = V(G)$ and $E_i = E(G) \setminus \{uv \colon u, v \in V \setminus C_i\}$. By the above discussion, for $i \in [k]$ and distinct $u, w \in C_i$, there does not exist a vertex v that is a neighbour of both u and w in G, or in other words, for any vertex $v \in V$, we have $|N_G(v) \cap C_i| \leq 1$. It follows that G_i is a collection of stars and therefore, has boxicity at most 1. Define the graph $G'_i = (V, E'_i)$ as $V = V(G)$ and $E'_i = E_i \cup \{uv \colon u, v \in V \setminus C_i\}$. By Lemma 2.13, $box(G'_i) \leq 2box(G_i) \leq 2$. It is easy to show that $G = G'_1 \cap G'_2 \cap \cdots \cap G'_k$. Thus by Theorem 2.5, we have $box(G) \leq \sum_{i \in [k]} box(G'_i) \leq 2k$.

Let n denote the number of vertices in G. Denote by $\Delta(G^2)$, the maximum degree of any vertex in G^2. Suppose G^2 is not an odd cycle or a clique. Then by Brooks' theorem, we must have $k = \chi(G^2) \leq \Delta(G^2)$. Since $\Delta(G^2) \leq \Delta^2$, it follows that $box(G) \leq 2k \leq 2\Delta^2$. If G^2 is a clique, then $\Delta^2 \geq \Delta(G^2) \geq n - 1 \geq \lfloor n/2 \rfloor \geq box(G)$. If G^2 is an odd cycle (in fact, any cycle), then $n = 3$ and the statement follows trivially. This completes all the cases. □

Esperet [21] improved the upper bound in Theorem 3.2 by a factor of 2.

Theorem 3.3 ([21]) Suppose G is a graph with maximum degree Δ. Then $box(G) \leq \Delta^2 + 2$

Adiga, Bhowmick and Chandran [11] reduced the upper bound considerably by showing $box(G) \leq O(\Delta \log^2 \Delta)$ by relating boxicity with poset dimension (see Section 6 for a more detailed discussion). In the same paper, it was also shown that there exist graphs with $box(G) = \Omega(\Delta \log \Delta)$. Scott and Wood [22] later reduced the upper bound to $O(\Delta \log^{1+o(1)} \Delta)$, which is currently the best known upper bound for boxicity purely as a function of maximum degree.

Theorem 3.4 ([22]) For every graph with maximum degree Δ, as $\Delta \to \infty$

$$box(G) \leq (180 + o(1))\Delta \log(\Delta)(2e)^{\sqrt{\log \log \Delta}} \log \log \Delta$$

3.3 Cubicity and chromatic number

When the chromatic number is low, we can get a good upper bound for cubicity in terms of boxicity, as given by the following theorem of Chandran, Mathew and Rajendraprasad [23].

Theorem 3.5 ([23]) For any graph G, $cub(G) \leq 2\lceil \log(\chi(G)) \rceil box(G) + \chi(G) \lceil \log(\alpha(G)) \rceil$.

Proof Suppose $k = \chi(G)$ and c is an optimal proper colouring of the vertex set $V(G)$ using colours from $\{1, 2, \ldots, k\}$. Define $b_i : V(G) \to \{0, 1\}$ to be the function such that $b_i(v)$ is the ith bit in the binary encoding of $c(v)$. We then create $\lceil \log k \rceil$ many co-bipartite graphs G_i defined as follows: $V(G_i) = V(G)$ and $E(G_i) = E(G) \cup \{uv \colon b_i(u) = b_i(v)\}$.

For co-bipartite graphs, cubicity is equal to boxicity (see Observation 2.12). Using Theorem 2.13, we then have $cub(G_i) = box(G_i) \leq 2box(G)$.

For each $i \in [k]$, define G'_i as the graph $V(G'_i) = V(G)$ and $E(G'_i) = \{uv : c(u) \neq i, v \in V(G)\}$. Then each G'_i is an interval graph, and by Theorem 2.16, $cub(G'_i) \leq \lceil \log(\alpha(G'_i)) \rceil \leq \lceil \log(\alpha(G)) \rceil$.

It is easy to see that $E(G) = \bigcap_{i=1}^{\lceil \log k \rceil} E(G_i) \cap \bigcap_{i=1}^{k} E(G'_i)$. This gives us the result. □

Corollary 3.6 ([23]) If G is a bipartite graph, then $cub(G) \leq 2(box(G) + \lceil \log(\alpha(G)) \rceil)$.

3.4 Boxicity and treewidth

A tree decomposition of a graph G is a pair $(\{X_i \subseteq V(G) : i \in I\}, T\})$ where I is a set of indices, and T is a connected tree with nodes labelled by I such that the following is satisfied:

1. $\bigcup_{i \in I} X_i = V(G)$.
2. $\forall uv \in E(G), \exists i \in I$ such that $u, v \in X_i$.
3. $\forall v \in V(G)$, the collection $\{i \in I : v \in X_i\}$ forms a subtree of T.

The width of a tree decomposition $(\{X_i \subseteq V(G) : i \in I\}, T\})$ is defined as $\max_{i \in I} |X_i| - 1$ and the treewidth of a graph (denoted as $tw(G)$) is the minimum width over all possible tree decompositions of G. The sets X_i in the definition of a tree decomposition $(\{X_i \subseteq V(G) : i \in I\}, T\})$ are called the "bags" of T.

Suppose $(\{X_i : i \in I\}, T)$ is an optimal tree decomposition of a graph G, i.e. a tree decomposition having width $tw(G)$. Define the colouring θ to be an optimal proper colouring of the graph G' obtained from G by making every bag X_i into a clique. Then G' is a chordal graph, and hence θ uses at most $tw(G) + 1$ colours (the maximum possible size of a bag in the decomposition, and hence the maximum possible size of a clique in G'). Note that θ is also a proper colouring of G. Chandran and Sivadasan [24] used such a colouring to show the following.

Theorem 3.7 ([24]) For any graph G, $box(G) \leq tw(G) + 2$.

Proof The proof uses a special tree decomposition called a normalized tree decomposition, i.e. a triple $(\{X_i : i \in I\}, r \in I, T)$ that satisfies the following properties:

- T is a rooted tree where the subset X_r that corresponds to the root node r contains exactly one vertex.
- For any node i, if i' is a child of i, then $|X_{i'} \setminus X_i| = 1$.

Consider a normalized tree decomposition $(\{X_i : i \in I\}, r \in I, T)$ with width equal to $tw(G)$ (for any graph G, such a tree decomposition always exists). Suppose $height(i)$ is defined as the distance of the node i from the root r in T. Define the function $b : V(G) \to I$ as $b(v) = i$, where i is the (unique) node in I such that $height(i)$ is minimum subject to the condition that $v \in X_i$. It can be seen that b is a bijection from $V(G)$ to the set I. Suppose $\theta : V(G) \to \{1, 2, \ldots, tw(G) + 1\}$ is the colouring of G as defined above. Then clearly, if $u, v \in X_i$ for some $i \in I$ then $\theta(u) \neq \theta(v)$, since $uv \in E(G')$ and θ is a proper colouring of G' too.

Categorize every pair of non-adjacent vertices $u, v \in V(G)$ into three types: (a) pairs of vertices such that neither $b(u)$ is an ancestor of $b(v)$ in T nor $b(v)$ is an ancestor of $b(u)$ in T; (b) pairs of vertices with $\theta(u) = \theta(v)$ and either $b(u)$ is an ancestor of $b(v)$ in T or $b(v)$ is an ancestor of $b(u)$ in T; (c) pairs of vertices such that $\theta(u) \neq \theta(v)$ and either $b(u)$ is an ancestor of $b(v)$ in T or $b(v)$ is an ancestor of $b(u)$ in T.

For $i \in [tw(G)+1]$, construct interval graphs I_i having interval representation $\{I_i(v)\}_{v \in V(G)}$ defined as follows:

- If $\theta(v) = i$, then $I_i(v) = [2 \times height(b(v)), 2 \times height(b(v)) + 1]$.
- If $\theta(v) \neq i$ then let $S = \theta^{-1}(i) \cap N(v)$,
  - If $S = \emptyset$, then $I_i(v) = [3n, 3n]$.
  - If $S \neq \emptyset$, then $I_i(v) = [\min_{w \in S}(2 \times height(b(w)) + 1), 3n]$ (note that the number of bags in a normalized tree decomposition is at most n and therefore $height(b(w)) \leq n - 1$ for any $w \in V(G)$).

Suppose $u, v \in V(G)$ is a non-adjacent pair of type (b). Without loss of generality, assume that $b(u)$ is an ancestor of $b(v)$. Let $i = \theta(u) = \theta(v)$. It is clear that u and v are non-adjacent in I_i since $height(b(u)) < height(b(v))$ and therefore, $2 \times height(b(u)) + 1 < 2 \times height(b(v))$. Now, suppose $u, v \in V(G)$ is a non-adjacent pair of type (c). Without loss of generality, assume that $b(u)$ is an ancestor of $b(v)$. Take $\theta(u) = i \neq \theta(v)$ and $S = \theta^{-1}(i) \cap N(v)$. For every $w \in S$, it is true that either $b(w)$ is an ancestor of $b(v)$ or $b(v)$ is an ancestor of $b(w)$. We claim that for all $w \in S$, $b(u)$ is an ancestor of $b(w)$. Consider some $w \in S$. If $b(v)$ is an ancestor of $b(w)$, then since $b(u)$ is an ancestor of $b(v)$, we can directly conclude that $b(u)$ is an ancestor of $b(w)$. On the other hand, suppose that $b(w)$ is an ancestor of $b(v)$. Then since $b(u)$ is also an ancestor of $b(v)$, we know that either $b(u)$ is an ancestor of $b(w)$ or $b(w)$ is an ancestor of $b(u)$. If $b(w)$ is an ancestor of $b(u)$, then $b(u)$ lies on the path between $b(w)$ and $b(v)$ in T. Since $vw \in E(G)$, we have that $w \in X_{b(v)}$. As we also have $w \in X_{b(w)}$, we now have that $w \in X_{b(u)}$, which means that both u and w occur in the bag $X_{b(u)}$, which contradicts the fact that $\theta(u) = i = \theta(w)$. So we can conclude that $b(u)$ is an ancestor of $b(w)$. It follows that $\forall w \in S$, $b(u)$ is an ancestor of $b(w)$, which implies that $height(b(u)) < height(b(w))$, and consequently, $r(I_i(u)) = 2 \times height(b(u)) + 1 < \min_{w\in S}(2 \times height(b(w)) + 1) = l(I_i(v))$. Therefore, u and v are not adjacent in the interval graph I_i.

For case (a), we construct the interval graph $I_{tw(G)+2}$ having the interval representation $\{I_{tw(G)+2}(v)\}_{v\in V(G)}$ as follows. Consider a depth-first ordering of the nodes of T. We can associate with each node i, two numbers $first(i)$ and $last(i)$ that denote its sequence number in the ordered list corresponding to its first occurrence and last occurrence, respectively. Now, for each vertex $v \in V$, let $I_{tw(G)+2}(v) = [first(b(v)), last(b(v))]$. It is easy to verify that for every non-adjacent pair $u, v \in V(G)$ of type (a), $I_{tw(G)+2}(u) \cap I_{tw(G)+2}(v) = \emptyset$. Also, it is not difficult to check that each of the interval graphs $I_1, I_2, \ldots, I_{tw(G)+2}$ is a supergraph of G, and hence $G = I_1 \cap I_2 \cap \cdots \cap I_{tw(G)+2}$. □

The above proof also suggests an algorithm that is FPT in treewidth for constructing a $tw(G) + 2$ sized box representation of G (see [25]).

The following theorem says that the bound in the above theorem is asymptotically tight.

Theorem 3.8 ([24]) For any integer $k \geq 1$, there exists a graph G with $tw(G) \leq k$ and $box(G) \geq k\left(1 - \frac{2}{\sqrt{k}}\right) = k(1 - o(1))$.

The upper bound has some direct consequences. For example, it implies degree-based upper bounds for many graph classes. These results are summarized in Section 9. Other consequences are listed below.

Theorem 3.9 ([24]) For every planar graph H, there is a constant $c(H)$ such that every graph with boxicity $\geq c(H)$ has a minor isomorphic to H.

Theorem 3.10 ([24]) In any graph G of boxicity b, there exists a simple cycle of length at least $b - 3$. Moreover, there exists a graph G whose boxicity is b, but the length of any simple cycle in it is at most $2b$.

4 Using probabilistic methods

Probabilistic techniques have been used successfully to find several bounds on boxicity and cubicity. Usually these proofs involve generating random permutations or random partitions or both. For a graph G on n vertices, we consider a permutation of $V(G)$ to be a bijection from $V(G)$ to $[n]$.

4.1 Boxicity and maximum degree

For graphs on n vertices with maximum degree $\Delta > \ln n + 2$, the following theorem gives a better bound than the one obtained in Theorem 3.3.

Theorem 4.1 ([26]) Let G be a graph on n vertices and with maximum degree Δ. Then $box(G) \leq \lceil(\Delta + 2)\ln n\rceil$.

Proof The argument is probabilistic. Let $t = \lceil(\Delta+2)\ln n\rceil$. Generate t many random permutations $\pi_1, \pi_2, \ldots, \pi_t$ of $V(G)$. For $i \in [t]$, construct the interval graph I_i having interval representation $\{I_i(v)\}_{v\in V(G)}$ as follows: for $v \in V(G)$, $I_i(v) = [\min_{w\in N[v]} \pi_i(w), \pi_i(v)]$.

Clearly, for each $i \in [t]$, we have that I_i is a supergraph of G. It is not difficult to observe that if $e = uv \in E(\overline{G})$ then, $\Pr[e \in E(I_i)] = \frac{1}{2}\left(\frac{d(u)}{d(u)+2} + \frac{d(v)}{d(v)+2}\right) \leq \frac{\Delta}{\Delta+2}$. For an $e \in E(\overline{G})$, let Z_e be the event that $e \in \bigcap_{j=1}^{t} E(I_j)$. Then we can bound $\Pr[Z_e] = \prod_{j=1}^{t} \Pr[e \in E(I_j)] \leq \left(\frac{\Delta}{\Delta+2}\right)^t \leq \left(1 - \frac{2}{\Delta+2}\right)^t \leq e^{-\frac{2t}{\Delta+2}} \leq \frac{1}{n^2}$. Union bound gives $\Pr\left[\bigvee_{e\in E(\overline{G})} Z_e\right] \leq \frac{n^2}{2}\cdot\frac{1}{n^2} = 1/2$. Thus, $\Pr\left[\bigwedge_{e\in E(\overline{G})} \overline{Z_e}\right] > 1/2$. So there is a non-zero probability that $G = I_1 \cap I_2 \cap \cdots \cap I_t$, which implies that G has a box representation of dimension $t = \lceil(\Delta+2)\ln n\rceil$. □

4.2 Cubicity and maximum degree

Chandran, Francis and Sivadasan [27] showed that the cubicity of any graph on n vertices and having maximum degree Δ is $O(\Delta \log n)$. The proof for this is also probabilistic and differs from Theorem 4.1 only in that here, in addition to using a random permutation, we also use a random partition (A, B) of the vertex set.

Theorem 4.2 ([27]) Let G be a graph on n vertices and with maximum degree Δ. Then $cub(G) \leq \lceil 4(\Delta+1)\ln n\rceil$.

Proof Let $t = \lceil 4(\Delta+1)\ln n\rceil$. Generate uniformly at random t many permutations $\pi_1, \pi_2, \ldots, \pi_t$ of $V(G)$ and t many pairs (A_i, B_i) such that $A_i \cap B_i = \emptyset$ and $A_i \cup B_i = V(G)$, i.e. we generate t many partitions of $V(G)$ into two subsets. For each $i \in [t]$, define I_i as the indifference graph whose equal interval representation $\{I_i(v)\}_{v\in V(G)}$ is defined as follows:

$$I_i(v) = \begin{cases} [\pi_i(v), n + \pi_i(v)] & \text{if } v \in B_i \\ [-n, 0] & \text{if } v \in A_i \text{ and } N(v) \cap B_i = \emptyset \\ \left[-n + \max_{x\in N(v)\cap B} \pi_i(x), \max_{x\in N(v)\cap B} \pi_i(x)\right] & \text{if } v \in A_i \text{ and } N(v) \cap B_i \neq \emptyset \end{cases}$$

Clearly, for each $i \in [t]$, we have that I_i is a supergraph of G. Let $e = uv \in E(\overline{G})$. Then it is easy to see that $\Pr[e \in E(I_i)] \leq \frac{1}{2} + \frac{1}{4}\left(\frac{d(u)}{d(u)+1} + \frac{d(v)}{d(v)+1}\right) \leq \frac{2\Delta+1}{2(\Delta+1)}$. Suppose that for $e \in E(\overline{G})$, Z_e is the event that $e \in \bigcap_{j=1}^{t} E(I_j)$. Since $t = \lceil 4(\Delta+1)\ln n\rceil$, we can bound $\Pr[Z_e] = \prod_{j=1}^{t} \Pr[e \in E(I_j)] \leq \left(\frac{2\Delta+1}{2(\Delta+1)}\right)^t \leq \left(1 - \frac{1}{2(\Delta+1)}\right)^t \leq e^{-\frac{t}{2(\Delta+1)}} \leq \frac{1}{n^2}$. By the union bound, we get $\Pr\left[\bigvee_{e\in E(\overline{G})} Z_e\right] \leq \frac{n^2}{2}\cdot\frac{1}{n^2} = 1/2$. Thus, $\Pr\left[\bigwedge_{e\in E(\overline{G})} \overline{Z_e}\right] > 1/2$. Thus there exists a cube representation of G of dimension at most $\lceil 4(\Delta+1)\ln n\rceil$. □

Remark 4.3 The proofs Theorems 4.1 and 4.2 yield randomized polynomial time algorithms that construct box and cube representations respectively, of an input graph on n vertices of dimension at most $\lceil(\Delta+2)\ln n\rceil$ and $\lceil 4(\Delta+1)\ln n\rceil$ respectively. These algorithms can also be derandomized to obtain deterministic polynomial time algorithms that output a box or cube representation of the input graph in at most the same number of dimensions.

4.3 Cubicity, degeneracy and crossing number

An ordering of the vertices of a graph G is said to be a *k-degenerate ordering* for G if every vertex has at most k neighbours appearing later than itself in the ordering. A *k-degenerate graph* is a graph that has a k-degenerate ordering. The *degeneracy* of a graph G is the smallest integer k for which G has a k-degenerate ordering. A k-degenerate ordering of a graph G having degeneracy k is a called a *degeneracy ordering* of G.

We continue our theme of randomly generating permutations and partitions. Here, the degeneracy ordering is already a desirable permutation for us which we would like to exploit. Therefore, we will focus on randomly generating partitions. We will generate these partitions by random colourings. The following theorem is due to Adiga, Chandran and Mathew [28].

Theorem 4.4 ([28]) Let G be a k-degenerate graph with n vertices. Then, $cub(G) \leq (k+2) \cdot \lfloor 2e \ln n \rfloor$.

Proof Suppose that $v_1, v_2, \ldots, v_n$ is a degeneracy ordering of G. For $1 \leq x < y \leq n$ such that $v_x v_y \notin E(G)$, we define the set $T_{xy} = \{v_z \in N(v_x) : z > y\} \cup \{v_x, v_y\}$. Observe that since $v_1, v_2, \ldots, v_n$ is a k-degenerate ordering of G, we have $|T_{xy}| \leq k+2$. For an arbitrary vertex colouring c of G, not necessarily proper, we say that T_{xy} is "favorably coloured" in c if v_y receives a colour different from the rest of the vertices in T_{xy}.

Suppose that for some $a, b \in \mathbb{N}$, there is a family of colourings $F_{a,b} = \{c_1, c_2, \ldots, c_a\}$ of G such that the following is satisfied: (1) $\forall j \in [a]$, c_j uses at most b colours, and (2) if a pair of vertices v_x, v_y are non-adjacent in G with $x < y$ then there is a $j \in [a]$ such that T_{xy} is favorably coloured in c_j. For every $i \in [b]$, $j \in [a]$, and $x \in [n]$, define $g_{ij}(v_x) = \max\{z\colon v_z \in N(v_x) \text{ and } c_j(v_z) = i\} \cup \{0\}$. Now for $i \in [b]$ and $j \in [a]$, define the indifference supergraph I_{ij} having equal interval representation $\{I_{ij}(v_z)\}_{z \in [n]}$ as follows:

$$I_{ij}(v_z) = \begin{cases} \left[g_{ij}(v_z) - n, g_{ij}(v_z)\right] & \text{if } c_j(v_z) \neq i \\ [z, z+n] & \text{if } c_j(v_z) = i \end{cases}$$

Clearly, I_{ij} are supergraphs of G. Now consider distinct vertices $v_x, v_y \in V(G)$ such that $v_x v_y \notin E(G)$. We can assume without loss of generality that $x < y$. By property (2) of the family of colourings $F_{a,b}$ defined above, T_{xy} is favourably coloured in c_j for some $j \in [a]$. If $c_j(v_y) = i$, then $g_{ij}(v_x) < y$ and consequently, $v_x v_y \notin E(I_{ij})$. Therefore, we can conclude that for every pair of vertices v_x, v_y that are non-adjacent in G, $\exists j \in [a]$ and $i \in [b]$ such that $v_x v_y \notin E(I_{ij})$. It follows that $G = \bigcap_{i \in [b], j \in [a]} I_{ij}$ and we have a cube representation of G of dimension ab.

We claim that for $a = \lfloor 2e \ln n \rfloor$ and $b = k+2$, there exists a family of colourings $F_{a,b}$ as defined above. The proof is probabilistic. For each $j \in [a]$, we randomly generate the colouring c_j as follows: for each vertex $v \in V(G)$, randomly pick a colour q from $\{1, 2, \ldots, k+2\}$ and set $c_j(v) = q$. By construction, the family of colourings $\{c_1, c_2, \ldots, c_a\}$ satisfies the property (1) above. We claim that it also satisfies property (2) with non-zero probability.

Suppose $1 \leq x < y \leq n$. Let $t = |T_{xy}| \leq k+2$. Then it is not difficult to show that $\Pr[T_{xy} \text{ is favorably coloured in } c_j] = \frac{(k+2)(k+1)^{t-1}}{(k+2)^t} = \left(\frac{k+1}{k+2}\right)^{t-1} \geq \left(\frac{k+1}{k+2}\right)^{k+1}$. Let $Z_{xy,j}$ be the event that T_{xy} is not favorably coloured in c_j. Then using the above observation we have $\Pr[Z_{xy,j}] \leq 1 - \left(\frac{k+1}{k+2}\right)^{k+1} < e^{-\left(\frac{k+1}{k+2}\right)^{k+1}} \leq e^{-\frac{1}{e}}$ (since $\left(\frac{k+2}{k+1}\right)^{k+1} = \left(1 + \frac{1}{k+1}\right)^{k+1} \leq e$, we have $\left(\frac{k+1}{k+2}\right)^{k+1} \geq \frac{1}{e}$). Therefore, we have

$$\Pr\left[\bigvee_{\substack{v_x v_y \notin E(G) \\ x<y}} \bigwedge_{j \in [a]} Z_{xy,j}\right] \leq \frac{n^2}{2} e^{-\frac{a}{e}} \leq \frac{1}{2}.$$

Therefore, for $a = \lfloor 2e \ln n \rfloor$ and $b = k+2$, there exists a family of colourings F_{ab} of G having the properties that we need. It follows that $cub(G) \leq (k+2) \cdot \lfloor 2e \ln n \rfloor$. □

The result is tight up to a constant factor. The above proof can also be derandomized and a polynomial time algorithm can be designed that constructs a cube representation of G of dimension $8k(\lfloor 2.42 \ln n \rfloor + 1)$. See [29] for the algorithm.

The *crossing number* of a graph G is the minimum number of edge crossings over all drawings of G in the plane. It can be shown that a graph with crossing number t has degeneracy at most $6.5t^{1/4} + 15$ (see [28]). Using this fact and Theorem 2.13, Adiga, Chandran and Mathew [28] obtain the following result.

Theorem 4.5 ([28]) For any graph G that has crossing number t, $box(G) \leq 66t^{\frac{1}{4}} \lceil \log 4t \rceil^{\frac{3}{4}} + 6$ and $cub(G) \leq 6 \log n + (6.5t^{1/4} + 17) \lceil 2e \ln(4t) \rceil$.

The bound on boxicity in the above theorem is tight up to a factor of $\ln^{1/4}(n)$.

5 Techniques based on layering

A layering of a graph G is a partition $(V_1, V_2, \ldots, V_t)$ of $V(G)$ such that for every edge $uv \in E(G)$, $\exists j \in [t-1]$ such that both u and v are in $V_j \cup V_{j+1}$. A layering of a graph G often provides the necessary structure for us to exploit in our construction of box and cube representations of G.

5.1 Boxicity and layered treewidth

The layered tree-width of a graph G (denoted as $ltw(G)$) is the minimum integer k such that there is a tree decomposition $(\{X_i \subseteq V(G) : i \in I\}, T)$ (see Section 3.4 for the definition of tree decompositions) and a layering $(V_1, V_2, \ldots, V_t)$ of G, such that $\forall i \in I, \forall j \in [t]$, we have $|X_i \cap V_j| \leq k$. The layered treewidth of a graph is clearly at most its one more than its treewidth, and in fact can be much smaller than its treewidth.

Scott and Wood [22] showed the following upper bound on $box(G)$ in terms of $ltw(G)$.

Theorem 5.1 ([22]) For every graph G, $box(G) \leq 6ltw(G) + 4$.

Proof For $i \in \{0, 1, 2\}$ define $G_i = \bigcup_{j \equiv i \pmod 3} G[V_j \cup V_{j+1}]$. Let H_i be the complete graph having vertex set $V(G) \setminus V(G_i)$. Define $G'_i = G_i \vee H_i$. Construct an interval graph I' having interval representation $\{I'(v)\}_{v \in V(G)}$ defined as $I'(v) = [i, i+1]$ for $v \in V_i$. Then it is easy to show that $G = G'_0 \cap G'_1 \cap G'_2 \cap I'$.

Let $k = ltw(G)$. Consider a tree decomposition $(\{X_i \subseteq V(G) : i \in I\}, T)$ of G and layering $(V_1, V_2, \ldots, V_t)$ of G satisfying $\forall i \in I, \forall j \in [t]$, $|X_i \cap V_j| \leq k$. Observe that $(\{X_i \cap (V_j \cup V_{j+1}) \subseteq V(G) : i \in I\}, T)$ is a tree decomposition of G_i with width at most $2k - 1$. Since, $box(G'_i) = box(G_i) \leq tw(G_i) + 2$, the result follows. □

5.2 Cubicity and bandwidth

Suppose G is a graph on n vertices. A linear ordering f of a graph G on n vertices is a bijective function $f : V(G) \to [n]$. The "width" of the linear ordering f is defined as $\max_{uv \in E(G)} |f(u) - f(v)|$. The *bandwidth* $bw(G)$ of G is the minimum possible width achieved by any linear ordering of G. In the following, we think of a linear ordering f of G to be the sequence $f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(n)$ of the vertices of G.

A linear ordering of small width gives us good layerings of G. To see this, we take a linear ordering $v_1, v_2, \ldots, v_n$ of G of width b and partition $V(G)$ into at most $l = \lceil \frac{n}{b} \rceil$ sets $B_1, B_2, \ldots, B_l$ as follows: for $i < l$, define $B_i = \{v_{(i-1)b+1}, v_{(i-1)b+2}, \ldots, v_{ib}\}$ and define B_l to be the set of all remaining vertices; i.e. $B_l = V(G) \setminus \bigcup_{i=1}^{l-1} B_i$. This forms a layering of G. Using this layering, it was shown in [27], that if the bandwidth of a graph is b, then the cubicity of G is $O(\Delta \ln b)$.

Theorem 5.2 ([27]) For any graph G having maximum degree Δ and bandwidth b, $cub(G) \leq \lceil 12(\Delta+1)\ln(2b) \rceil + 1$.

Proof Let $n = |V(G)|$, $l = \lceil \frac{n}{b} \rceil$, and $B_1, B_2, \ldots, B_l$ be the layering of G obtained from a linear ordering of G of width b as described above. Using the construction in Section 4.2, the induced subgraphs $H_i = G[B_i \cup B_{i+1}]$, for $i \in [l-1]$, have a cube representation $\{I_i^1, I_i^2, \ldots, I_i^t\}$ of dimension $t = \lceil 4(\Delta+1)\ln(2b) \rceil$, where I_i^j is an indifference graph having an equal interval representation $\{I_i^j(v)\}_{v \in V(H_i)}$ in which each interval has length n and every interval intersects the interval $[0, n]$, as described in Section 4.2. Note that for all $i \in [l-3]$, no vertex in H_i is adjacent to a vertex in H_{i+3}. For $r \in \{0, 1, 2\}$ and $j \in \{1, 2, \ldots, t\}$, construct indifference graphs $I'_{r,j}$ having the equal interval representation $\{I'_{r,j}(v)\}_{v\in V(G)}$, defined as:

$$I'_{r,j}(v) = \begin{cases} I_i^j(v) & \text{if } v \in H_i \text{ for some } i \equiv r \pmod 3 \\ [0, n] & \text{otherwise} \end{cases}$$

Clearly, these graphs are indifference supergraphs of G. For $r \in \{0, 1, 2\}$, define $G_r = \bigcap_{j=1}^{t} I'_{r,j}$. Finally, define the graph I_0 having the unit-interval representation $\{I_0(v)\}_{v\in V(G)}$ that is defined as $I_0(v) = [i, i+1]$, where $v \in B_i$. Then it is easy to see that $E(G) = E(G_0) \cap E(G_1) \cap E(G_2) \cap E(I_0)$. Therefore, the result follows. This construction can also be made into a polynomial time algorithm. □

In the same paper, another upper bound in terms of the bandwidth alone was also shown.

Theorem 5.3 ([27]) $cub(G) \le bw(G) + 1$

Proof Consider the linear ordering $v_1, v_2, \ldots, v_n$ with width equal to $b = bw(G)$. For each $1 \le i \le b$, consider the indifference graph I_i having the equal interval representation $\{I_i(v_j)\}_{j\in[n]}$, in which each interval has length 2, defined as:

$$I_i(v_j) = \begin{cases} [-1, 1] & \text{if } j < i \\ [t-2, t] & \text{if } j = i + tb \text{ for some } 0 \le t \le \lceil \frac{n-i}{b} \rceil \\ [t, t+2] & \text{if } i + tb < j < i + (t+1)b \text{ and } v_j u_{i+tb} \in E(G) \text{ for some } 0 \le t \le \lceil \frac{n-i}{b} \rceil \\ [t+1, t+3] & \text{if } i + tb < j < i + (t+1)b \text{ and } v_j u_{i+tb} \notin E(G) \text{ for some } 0 \le t \le \lceil \frac{n-i}{b} \rceil \end{cases}$$

We will denote an interval of the form $[l + x, r + x]$ by $[l, r] + x$. Now, construct the equal-interval graph H having equal interval representation $\{H(v_j)\}_{j\in[n]}$ defined (recursively) as:

$$H(v_j) = \begin{cases} [j-b, j] & \text{if } j < b \\ H(v_{j-b}) + b & \text{if } j \ge b \text{ and } v_j v_{j-b} \in E(G) \\ H(v_{j-b}) + b + \frac{1}{n^2} & \text{if } j \ge b \text{ and } v_j v_{j-b} \notin E(G) \end{cases}.$$

It can be shown that each of these are supergraphs of G. Also when $v_p v_q \notin E(G)$, if $|p - q| \ge b$ we have $v_p v_q \notin E(H)$ and if $|p - q| < b$ then $v_p v_q \notin E(I_l)$ where $l = p \pmod b$. Therefore, we have that $G = I_1 \cap I_2 \cap \cdots \cap I_b \cap H$. □

Some bounds derived from bandwidth for the cubicity of some special classes of graphs have been summarized in Section 9.

6 Boxicity and poset dimension

A *poset* $\mathcal{P} = (V, \prec_{\mathcal{P}})$ is a set V with an irreflexive, antisymmetric and transitive binary relation $\prec_{\mathcal{P}}$ on V. For distinct vertices $x, y \in V$, we say that x "is comparable with" y if $x \prec_{\mathcal{P}} y$ or $y \prec_{\mathcal{P}} x$. A graph $G = (V, E)$ is a *comparability graph* if and only if there is a poset $\mathcal{P} = (V, \prec_{\mathcal{P}})$ such that there is an edge between two vertices $x, y \in V$ if and only if x is comparable with y in $\mathcal{P}$. We call such a poset $\mathcal{P}$, an "associated poset" of G and we call G the "underlying comparability graph" of poset $\mathcal{P}$. A poset in which every element is either a minimal element or a maximal element is called a "height-2 poset". A *total order* is a poset in which each pair of distinct elements are comparable with each other. A *linear extension* $\mathcal{L} = (V, \prec_{\mathcal{L}})$ of a poset $\mathcal{P}$ is a total order such that $x \prec_{\mathcal{P}} y \implies x \prec_{\mathcal{L}} y$. A *realizer* $\mathcal{R} = \{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_k\}$ of a poset $\mathcal{P}$ is a set of linear extensions of $\mathcal{P}$ such that $x \prec_{\mathcal{P}} y$ if and only if $\forall i \in [k]$, $x \prec_{\mathcal{L}_i} y$. The poset dimension of a poset $\mathcal{P}$, denoted by $dim(\mathcal{P})$, is the least integer k for which there is a realizer $\mathcal{R}$ of $\mathcal{P}$ of cardinality k.

Poset dimension and boxicity are, at first glance, seemingly unrelated notions. For example, the graph $\overline{nK_2}$ is a comparability graph and has boxicity n but the dimension of any of its associated posets is 2. Yannakakis [30] was the first to discover a connection between poset dimension and boxicity which he used to show that the problem of determining whether the boxicity of a graph is at most 3 is NP hard. After that, for 25 years no progress was made in this direction until a concrete relationship between the two notions was discovered by Adiga, Bhowmick and Chandran [11].

Theorem 6.1 ([11]) Let $\mathcal{P} = (V, \prec_{\mathcal{P}})$ be a poset such that $dim(\mathcal{P}) > 1$, and let $G_{\mathcal{P}}$ be its underlying comparability graph with $\chi(G_{\mathcal{P}}) > 1$. Then,

$$\frac{box(G_{\mathcal{P}})}{\chi(G_{\mathcal{P}}) - 1} \le dim(\mathcal{P}) \le 2box(G_{\mathcal{P}})$$

Proof First we will show that $dim(\mathcal{P}) \le 2box(G_{\mathcal{P}})$. Let $box(G_{\mathcal{P}}) = k$. Suppose $\{I_1, I_2, \ldots, I_k\}$ is a box representation of $G_{\mathcal{P}}$ of dimension k, where for each $i \in [k]$, the interval graph I_i has an interval representation $\{I_i(v)\}_{v \in V}$. For every $i \in [k]$, we define two total orders $\mathcal{L}_i = (V, \prec_{\mathcal{L}_i})$ and $\mathcal{L}'_i = (V, \prec_{\mathcal{L}'_i})$, where $\prec_{\mathcal{L}_i}$ and $\prec_{\mathcal{L}'_i}$ are defined as follows:

- Define posets $\mathcal{P}_i = (V, \prec_{\mathcal{P}_i})$ and $\mathcal{P}'_i = (V, \prec_{\mathcal{P}'_i})$, where $\prec_{\mathcal{P}_i} = \{(a, b) \in V^2 \colon r(I_i(a)) < l(I_i(b))\}$ and $\prec_{\mathcal{P}'_i} = \{(a, b) \in V^2 \colon r(I_i(b)) < l(I_i(a))\}$.
- Define the directed graph G_i as the graph with vertex set $V(G_i) = V$ and $E(G_i) = \prec_{\mathcal{P}} \cup \prec_{\mathcal{P}_i}$. Similarly, define G'_i as the graph with vertex set $V(G'_i) = V$ and $E(G'_i) = \prec_{\mathcal{P}} \cup \prec_{\mathcal{P}'_i}$. It can be shown that both the directed graphs G_i and G'_i are acyclic as follows. We shall give the proof only for G_i as the same proof can easily be modified to work for G'_i too. First notice that for any $a, b \in V$, $a \prec_{\mathcal{P}_i} b \Rightarrow ab \notin E(I_i) \Rightarrow ab \notin E(G_{\mathcal{P}}) \Rightarrow a \not\prec_{\mathcal{P}} b$ and $b \not\prec_{\mathcal{P}} a$. Thus every edge $(a, b) \in E(G_i)$ belongs to exactly one of $\prec_{\mathcal{P}}$ or $\prec_{\mathcal{P}_i}$. Suppose that G_i contains a directed cycle. Then consider a shortest directed cycle C in G_i. Clearly, if any two consecutive edges (a, b) and (b, c) of C both belong to $\prec_{\mathcal{P}}$, then (a, c) belongs to $\prec_{\mathcal{P}}$ and hence $(a, c) \in E(G_i)$. Similarly if both belong to $\prec_{\mathcal{P}_i}$, again $(a, c) \in E(G_i)$ since in that case (a, c) belongs to $\prec_{\mathcal{P}_i}$. But if $(a, c) \in E(G_i)$, then we have a contradiction to the assumption that C is a shortest directed cycle in G_i. Thus the edges of C alternate between edges in $\prec_{\mathcal{P}}$ and edges in $\prec_{\mathcal{P}_i}$. Then C contains some three consecutive edges, say $(a, b), (b, c), (c, d)$ such that $a \prec_{\mathcal{P}_i} b$, $b \prec_{\mathcal{P}} c$, and $c \prec_{\mathcal{P}_i} d$. It follows that $r(I_i(a)) < l(I_i(b))$, $r(I_i(c)) \ge l(I_i(b))$, and $r(I_i(c)) < l(I_i(d))$. Combining, we get $r(I_i(a)) < l(I_i(d))$, and hence $a \prec_{\mathcal{P}_i} d$. Thus $(a, d) \in E(G_i)$, again contradicting the assumption that C is a shortest directed cycle in G. This shows that G_i is acyclic, and in a similar way, it can be shown that G'_i is also acyclic.
- Define $\prec_{\mathcal{L}_i}$ and $\prec_{\mathcal{L}'_i}$ to be some topological ordering of G_i and G'_i respectively; i.e. they satisfy the property that for every edge $(a, b) \in E(G_i)$, we have $a \prec_{\mathcal{L}_i} b$ and for every edge $(a, b) \in E(G'_i)$, we have $a \prec_{\mathcal{L}'_i} b$.

It is clear that for each $i \in [k]$, both $\mathcal{L}_i$ and $\mathcal{L}'_i$ are linear extensions of $\mathcal{P}$. We claim that $\bigcup_{i=1}^{k} \{\mathcal{L}_i, \mathcal{L}'_i\}$ is a realizer for $\mathcal{P}$. This is because if x and y are not comparable in $\mathcal{P}$ then $xy \notin E(G_{\mathcal{P}})$ and therefore, there exists an $i \in [k]$ such that $xy \notin E(I_i)$. Without loss of generality, assume $r(I_i(x)) < l(I_i(y))$. Consequently, $x \prec_{\mathcal{P}_i} y$ and $y \prec_{\mathcal{P}'_i} x$. It follows that $x \prec_{\mathcal{L}_i} y$ and $y \prec_{\mathcal{L}'_i} x$. Hence, we are done.

Now, we will show that $box(G_{\mathcal{P}}) \le (\chi(G_{\mathcal{P}}) - 1)dim(\mathcal{P})$. Let $k = dim(\mathcal{P})$ and $p = \chi(G_{\mathcal{P}}) - 1$. Define a vertex colouring c of $G_{\mathcal{P}}$ as follows: for a vertex $v \in V(G_{\mathcal{P}})$, if γ is the number of elements in a longest chain in $\mathcal{P}$ having v as its maximum element, then assign $c(v) = \gamma$. It is easy to check that this is a proper colouring of $G_{\mathcal{P}}$. This colouring is also an optimal colouring because the number of elements in a longest chain in $\mathcal{P}$ is the clique number of $G_{\mathcal{P}}$.

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Suppose $\mathcal{R} = \{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_k\}$ is a realizer of $\mathcal{P}$. Suppose that the bijection $\Pi_j : V \to [n]$ gives the ordering of the vertices in $\mathcal{L}_j$; i.e. $u \prec_{\mathcal{L}_j} v \Leftrightarrow \Pi_j(u) < \Pi_j(v)$. For every $i \in [p]$ and $j \in [k]$, construct the interval graph I_{ij} having interval representation $\{I_{ij}(v)\}_{v \in V}$ defined as follows:

- If $i \in [p]$ and $j \in [k]$ and $(i, j) \neq (p, k)$, then define

$$I_{i,j}(v) = \begin{cases} [\Pi_j(v), \Pi_j(v)] & \text{if } c(v) = i \\ [0, \Pi_j(v)] & \text{if } c(v) > i \\ [\Pi_j(v), n+1] & \text{if } c(v) < i \end{cases}$$

- If $i = p$ and $j = k$, then define

$$I_{p,k}(v) = \begin{cases} [\Pi_k(v), \Pi_k(v)] & \text{if } c(v) = p+1 \\ [\Pi_k(v), n+1] & \text{otherwise} \end{cases}$$

Using the fact that for each $xy \in E(G_{\mathcal{P}})$ such that $x \prec_{\mathcal{P}} y$, we have $c(y) > c(x)$, it is easy to see that the graphs $I_{i,j}$ are supergraphs of $G_{\mathcal{P}}$. Now suppose $xy \notin E(G_{\mathcal{P}})$. We will show that for some $i \in [p]$ and $j \in [k]$, we have $xy \notin I_{ij}$. If $c(x) = c(y)$ then $xy \notin E(I_{c(x),1})$. Therefore, assume without loss of generality that $c(x) < c(y)$. Since $\mathcal{R}$ is a realizer of $\mathcal{P}$, there exists $j \in [k]$ such that in $y \prec_{\mathcal{L}_j} x$ and therefore, $\Pi_j(x) > \Pi_j(y)$. Since $c(x) < c(y) \leq \chi(G_{\mathcal{P}}) = p + 1$, we have $c(x) \leq p$. If $c(x) < p$ then $xy \notin E(I_{c(x),j})$ because $I_{c(x),j}(x) = [\Pi_j(x), \Pi_j(x)]$ and $I_{c(x),j}(y) = [0, \Pi_j(y)]$. If $c(x) = p$ and if $j < k$ then again $xy \notin E(I_{p,j})$ by the same reasoning. Finally, if $c(x) = p$ and $j = k$ then $xy \notin E(I_{p,k})$ as x is assigned $[\Pi_k(x), n+1]$ and y is assigned $[\Pi_k(y), \Pi_k(y)]$ (note that since $c(x) = p$, we have $c(y) = p + 1$). Therefore, $G = \bigcap_{i \in [p], j \in [k]} I_{ij}$ and the result follows. □

Definition 6.2 ([11]) The extended double cover of G, denoted as G_c, is a bipartite graph with partite sets A and B, which are copies of $V(G)$ such that, corresponding to every $u \in V(G)$, there are two vertices $u_A \in A$ and $u_B \in B$ and $\{u_A, v_B\}$ is an edge in G_c if and only if either $u = v$ or u is adjacent to v in G.

For a bipartite graph G having partite sets A and B, the "natural height-2 poset" associated with G is the poset $\mathcal{P} = (V(G), \prec_{\mathcal{P}})$, where $a \prec_{\mathcal{P}} b$ if and only if $a \in A$, $b \in B$ and $ab \in E(G)$. Note that if $\mathcal{P}$ is the natural height-2 poset associated with a bipartite graph G, then $G = G_{\mathcal{P}}$.

Theorem 6.3 ([11]) Let G be a graph and $\mathcal{P}_c$ be the natural height-2 poset associated with its extended double cover. Then, $\frac{dim(\mathcal{P}_c)}{2} - 2 \leq box(G) \leq 2dim(\mathcal{P}_c)$, and therefore $box(G) = \Theta(dim(\mathcal{P}_c))$.

Proof Suppose G_c^* is the graph with $V(G_c^*) = V(G_c)$ and $E(G_c^*) = E(G_c) \cup \{xy : x, y \in A\} \cup \{xy : x, y \in B\}$. We show that $box(G_c) - 2 \leq box(G) \leq box(G_c^*)$. Then by Theorem 2.14 and Theorem 6.1, we are done.

We will first show $box(G_c) \leq box(G)+2$. Suppose $k = box(G)$ and $\{I_1, I_2, \ldots, I_k\}$ is a box representation of G of dimension k, where for each $i \in [k]$, the interval graph I_i has an interval representation $\{I_i(v)\}_{v \in V(G)}$. Construct, for each $i \in [k]$, an interval graph I'_i on vertex set $V(G_c)$ having interval representation $\{I'_i(v)\}_{v \in V(G_c)}$ defined as $I'_i(u_A) = I'_i(u_B) = I_i(u)$, for each $u \in V(G)$. Define the interval graph I'_{k+1} by taking $V(I'_{k+1}) = V(G_c)$ and $E(I'_{k+1}) = \{xy : x \in B, y \in V(I'_{k+1}) \setminus \{x\}\}$ and $I'_{k+2} = \{xy : x \in A, y \in V(I'_{k+1}) \setminus \{x\}\}$. It is easy to see that $\{I'_1, I'_2, \ldots, I'_{k+2}\}$ forms a box representation for G_c of dimension $k + 2$.

Now we will show $box(G) \leq box(G_c^*)$. Suppose $box(G_c^*) = k_c$ and suppose $\{J_1, J_2, \ldots, J_{k_c}\}$ is a box representation for G_c^* of dimension k_c. Since G_c is co-bipartite, each of $J_1, J_2, \ldots, J_{k_c}$ is co-bipartite, and therefore we can assume that for $i \in [k_c]$, there is a canonical interval representation $\{J_i(v)\}_{v \in V(G_c^*)}$ for J_i (see Section 2 for the definition of a canonical interval representation). Then construct interval graphs J'_i for $i \in [k_c]$, each having vertex set $V(G)$, with interval representation $\{J'_i(v)\}_{v \in V(G)}$ defined as $J'_i(v) = J_i(v_A) \cap J_i(v_B)$. It is easy to see that $G = \bigcap_{i=1}^{k_c} J'_i$. □

The connection between poset dimension and boxicity has several interesting consequences. Algorithmic hardness results for one imply hardness results for the other (see Section 8 for a more precise statement). Similarly, upper bounds for one imply upper bounds for the other. Füredi and Kahn [31] showed that if $\Delta_{\mathcal{P}}$ is the maximum degree of the comparability graph of the poset $\mathcal{P}$, then $dim(\mathcal{P}) \leq O(\Delta_{\mathcal{P}} \log^2 \Delta_{\mathcal{P}})$. Combining this with Theorem 6.3, Adiga, Bhowmick and Chandran [11] concluded that for a graph with maximum degree Δ, we have $box(G) \leq O(\Delta \log^2 \Delta)$. On the other hand, Erdős, Kierstead and Trotter [32] showed that there exist posets $\mathcal{P}$ such that $dim(\mathcal{P}) \geq \Omega(\Delta_{\mathcal{P}} \log \Delta_{\mathcal{P}})$. The problem of finding matching upper and lower bounds for the dimension of a poset $\mathcal{P}$ in terms of $\Delta_{\mathcal{P}}$ was considered an important and difficult problem (see discussion in [22]). In a breakthrough paper, Scott and Wood [22] improved the bound on boxicity obtained by Adiga, Bhowmick and Chandran [11] to $box(G) \leq O(\Delta \log^{1+o(1)} \Delta)$ and they used this result to show that for any poset $\mathcal{P}$, $dim(\mathcal{P}) \leq O(\Delta_{\mathcal{P}} \log^{1+o(1)} \Delta_{\mathcal{P}})$), thus bringing down the gap between the known upper and lower bounds on $dim(\mathcal{P})$ to a factor of $\log^{o(1)} \Delta_{\mathcal{P}}$.

7 Separation dimension and boxicity of line graphs

Suppose V is any set of cardinality n. Then for a permutation $\sigma : V \to [n]$ and disjoint subsets $A, B \subseteq V$, we say that σ separates A and B if $\forall (a, b) \in A \times B, \sigma(a) < \sigma(b)$ or $\forall (a, b) \in A \times B, \sigma(b) < \sigma(a)$.

Basavaraju et al. [33] introduced the notion of a "pairwise-suitable" family of permutations and the "separation dimension" of a hypergraph.

Definition 7.1 ([33]) A family $\mathcal{F}$ of permutations of $V(H)$ is *pairwise-suitable* for a hypergraph H if, for every two disjoint edges $e, f \in E(H)$, there exists a permutation $\sigma \in \mathcal{F}$ which separates e and f. The cardinality of a smallest family of permutations that is pairwise suitable for H is called the *separation dimension* of H and is denoted by $\pi(H)$.

For $v \in H$ and a family of permutations $\mathcal{F} = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$, we can define an embedding f of v in $\mathbb{R}^k$ by taking $f(v) = (\sigma_1(v), \sigma_2(v), \ldots, \sigma_k(v))$. Similarly, given any embedding f of v in $\mathbb{R}^k$, we can obtain a family of permutations $\mathcal{F} = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ by projecting the points of $f(V)$ on each of the k axes and reading them along an axis, breaking ties arbitrarily. This gives us an alternative definition for the separation dimension of a hypergraph.

Definition 7.2 ([33]) Suppose H is a hypergraph. Then $\pi(H)$ is the smallest natural number k such that the vertices of H can be embedded into $\mathbb{R}^k$ with the property that any two disjoint edges of H can be separated by a hyperplane normal to one of the axes.

Basavaraju et al. [33] showed a connection between separation dimension and boxicity.

Theorem 7.3 ([33]) For a hypergraph H, denote by $L(H)$ the line graph of H. Then $\pi(H) = box(L(H))$.

It is not difficult to see that the separation dimension $\pi(G)$ of a graph G is a monotone property. In fact, if H is a subgraph of a graph G, then $L(H)$ is an induced subgraph of $L(G)$, and therefore the boxicity of the line graph of a graph is at least the boxicity of the line graph of any subgraph. Thus, for any $n \in \mathbb{N}$, the graph on n vertices whose line graph has the highest boxicity is K_n. Basavaraju et al. [33] show using a probabilistic argument $\pi(K_n) \leq 6.84 \log n$, and therefore $box(L(G)) = \pi(G) \leq 6.84 \log n$ for any graph G on n vertices. They also show that this bound is tight up to a constant factor by establishing that $box(L(K_n)) = \pi(K_n) \geq \log\lceil \frac{n}{2} \rceil$.

The separation dimension of bounded degree graphs was studied by Alon et al. [34] where it was shown that if a graph G has maximum degree Δ then its separation dimension is at most $2^{9 \log^* \Delta} \Delta$, where $\log^*$ is the iterated logarithm. In the same paper, it was also shown that almost all Δ-regular graphs have separation dimension at least $\lceil \frac{\Delta}{2} \rceil$. The upper bound was later brought down to 20Δ by Scott and Wood [35]. They also showed that graphs with separation dimension 3 have bounded average degree and bounded chromatic number. Alon et al. [36] showed that the separation dimension of a k-degenerate graph on n vertices is at most $O(k \log \log n)$ and there is a family of

2-degenerate graphs with separation dimension at least $\Omega(\log\log n)$. It was also shown that the number of edges in a graph on n vertices having separation dimension s is at most $3(4\log n)^{s-2}\cdot n$. Other questions on the separation dimension of graphs and its variants have been studied in [33–38]. Theorem 7.3 can be used to translate these results about separation dimension of graphs into results about the boxicity of the corresponding line graphs.

8 Algorithms

Since interval graphs and indifference graphs can be recognized in polynomial time, deciding whether $box(G)\le 1$ or $cub(G)\le 1$ is tractable. But results by Cozzens [39], Yannakakis [30] and Kratochvíl [40] established that determining whether $box(G)\le k$ or $cub(G)\le k$ is NP-hard even when $k=2$ or $k=3$. Using the connection between poset dimension and boxicity, Adiga, Bhowmick and Chandran [11] showed that there is no polynomial-time algorithm that can approximate the boxicity or cubicity of a graph on n vertices to within a factor of $O(n^{0.5-\epsilon})$ for any $\epsilon>0$, unless NP = ZPP [11, 41]. This inapproximability bound was strengthened by Chalersmook, Laekhanukit and Nanongkai [42], who showed that boxicity or cubicity cannot be approximated to within a factor of $O(n^{1-\epsilon})$ in polynomial time for any $\epsilon>0$, unless NP = ZPP. Their hardness results apply even to restricted graph families such as bipartite, co-bipartite, and split graphs.

Adiga, Babu and Chandran [43] gave a polynomial time $2\lceil n\sqrt{\log\log n}/\sqrt{\log n}\rceil$ factor approximation algorithm to compute boxicity and a $2\lceil n(\log\log n)^{3/2}/\sqrt{\log n}\rceil$ factor approximation algorithm to compute cubicity. Babu et al. [44] provided a constant factor approximation for the cubicity of trees.

Adiga, Chitnis and Saurabh [45] initiated the study of parameterized algorithms for boxicity. We list some known results about parameterized and approximation algorithms for boxicity and cubicity. In the list below, the k in the running time of an FPT algorithm stands for the parameter.

(1) An FPT algorithm for boxicity that runs in time $2^{O(2^k k^2)}|V(G)|$ when parameterized by the **vertex cover number** [45].
(2) An additive 1-approximation for boxicity when parameterized by the **vertex cover number** running in time $2^{O(k^2\log k)}|V(G)|$ [45] .
(3) An additive 2-approximation for boxicity when parameterized by the **max leaf number** running in time $2^{O(k^3\log k)}|V(G)|^{O(1)}$ [45] .
(4) An FPT algorithm to compute the boxicity of co-bipartite graphs that runs in time $2^{O(k^2\log k)}|V(G)|^2$ when parameterized by the **vertex cover number of the associated bipartite graph**[4] [45].
(5) A kernel of at most $k^{2^{O(k)}}$ vertices for the problem of computing boxicity when parameterized by the **cluster vertex deletion number** [46]. This implies that there is an FPT algorithm for computing boxicity, when parameterized by the cluster vertex deletion number. Note that the cluster vertex deletion number is at most the vertex cover number, and therefore this result can be thought of as a generalization of the result given in (1) above.
(6) An additive 1-approximation algorithm to compute $box(G)$ that, given a graph G and a path decomposition of G of width w as input, runs in time $2^{O(w^2\log w)}|V(G)|$. This implies the existence of an additive 1-approximation FPT algorithm for boxicity when parameterized by the **pathwidth** of the input graph G that runs in time $f(k)|V(G)|$ for a computable function f [46]. Note that the pathwidth of a graph is at least its treewidth and at most its bandwidth. The pathwidth of a graph is also upper bounded by the max leaf number of the graph.

[4] Suppose G is a co-bipartite graph such that $V(G)=S_1\sqcup S_2$ is its partition into cliques. Then the associated bipartite graph of G is the bipartite graph obtained by making S_1 and S_2 into independent sets but retaining the edges of G between the sets S_1 and S_2.

(7) A 2-approximation FPT algorithm for computing the cubicity of a graph, parameterized by the **vertex cover number**, that runs in time $2^{O(k^2 2^k)}|V(G)|^{O(1)}$ [43]. Also, there is a $(1+\epsilon)$-factor FPT approximation scheme for the same problem that runs in time $2^{g(k,\epsilon)}|V(G)|^{O(1)}$, where $g(k,\epsilon) = \frac{1}{\epsilon}k^3 2^{\frac{4k}{\epsilon}}$ [43].
(8) An FPT algorithm for computing the boxicity of a graph G on n vertices that contains a clique of size at least $n-k$, parameterized by k, and runs in time $n^2 2^{O(k^2 \log k)}$ [43].
(9) A polynomial time $\left(2+\frac{1}{k}\right)$-approximation algorithm for computing the boxicity of any circular arc graph that runs in time $O(mn+n^2)$, where m is the number of edges, n the number of vertices, and $k \geq 1$ is the boxicity of the input graph [47]. The algorithm can also compute, in $O(mn+kn^2)$ time, a box representation of the input graph in $2k+1$ dimensions.
(10) A polynomial time $\left(2+\frac{\lceil \log n \rceil}{k}\right)$-factor approximation algorithm for computing the cubicity of any circular arc graph that runs in time $O(mn+n^2)$, where m is the number of edges, n the number of vertices, and $k \geq 1$ is the cubicity of the input graph [47]. The algorithm can be adapted to compute a cube representation of the input graph in $2k+\lceil \log n \rceil$ dimensions in $O(mn+kn^2)$ time.
(11) An additive 2-approximation algorithm for computing the boxicity of a normal circular arc graph on n vertices and having m edges, when given a normal circular arc model of the graph, that runs in time $O(mn+n^2)$ [47]. The algorithm can also compute, in time $O(mn+kn^2)$, a box representation of the input graph in at most $k+2$ dimensions, where k is the boxicity of the graph.

Adiga, Babu and Chandran [43] provided a general method for constructing parameterized algorithms for boxicity. This method is captured by Theorems 8.1 and 8.2.

Suppose $\mathcal{F}$ is a family of graphs. Let the family $\mathcal{F}+ke$ be the set of all graphs that can be obtained by adding at most k edges to a graph in $\mathcal{F}$. Similarly, let the family $\mathcal{F}-ke$ be the set of all graphs that can be obtained by adding at most k edges to a graph in $\mathcal{F}$, and $\mathcal{F}+kv$ be the set of all graphs that can be obtained by adding at most k vertices to a graph in $\mathcal{F}$. The graph classes $\mathcal{F}+ke$ and $\mathcal{F}+k_1e-k_2e$ are defined similarly. This notation was introduced by Cai [48].

A subset $S \subseteq V$ such that $|S| \leq k$ is called a modulator for an $\mathcal{F}+kv$ graph $G=(V,E)$ if $G-S \in \mathcal{F}$. Similarly, a set E_k of pairs of vertices such that $|E_k| \leq k$ is called a modulator for an $\mathcal{F}-ke$ graph $G=(V,E)$ if $G'=(V,E\cup E_k) \in \mathcal{F}$. Modulators for graphs in $\mathcal{F}+ke$ and $\mathcal{F}+k_1e-k_2e$ are defined in a similar manner.

Theorem 8.1 ([43]) Let $\mathcal{F}$ be a family of graphs such that for each graph $G' \in \mathcal{F}$ on n vertices, we have $box(G') \leq b \leq n$. Let $T(n)$ denote the time required to compute a box representation in b dimensions of a graph on n vertices in $\mathcal{F}$. Let G be an $\mathcal{F}+kv$ graph on n vertices. Given a modulator of G, a box representation of G in at most $2box(G)+b$ dimensions can be computed in time $T(n-k)+n^2 2^{O(k^2\log k)}$.

Theorem 8.2 ([43]) Let $\mathcal{F}$ be a family of graphs such that for each graph $G' \in \mathcal{F}$ on n vertices, we have $box(G') \leq b \leq n$. Let $T(n)$ denote the time required to compute a box representation in b dimensions of a graph belonging to $\mathcal{F}$ on n vertices. Let G be an $\mathcal{F}+k_1e-k_2e$ graph on n vertices and let $k=k_1+k_2$. Given a modulator of G, a box representation of G in at most $box(G)+2b$ dimensions can be computed in time $T(n)+O(n^2)+2^{O(k^2\log k)}$.

The consequences of Theorems 8.1 and 8.2 are summarised in Table 1, which is taken from [43].

9 Boxicity and cubicity of special graphs

In this section, we list some results about the boxicity and cubicity of some special classes of graphs. The material in this section is intended as a reference, and hence we do not provide the definitions of many of the graph classes discussed herein.

Table 1 Results obtained from Theorems 8.1 and 8.2 [43]

Parameter k	Approximation guarantee	Running time
Interval completion number	Additive 2	$2^{O(k^2 \log k)} n^{O(1)}$
Feedback vertex set number	$\left(2 + \frac{2}{\text{box}(G)}\right)$ factor	$2^{O(k^2 \log k)} n^{O(1)}$
Proper interval vertex deletion number	$\left(2 + \frac{1}{\text{box}(G)}\right)$ factor	$2^{O(k^2 \log k)} n^{O(1)}$
Proper interval edge deletion number	Additive 2	$2^{O(k^2 \log k)} n^{O(1)}$
Planar vertex deletion number	$\left(2 + \frac{3}{\text{box}(G)}\right)$ factor	$f(k) n^{O(1)}$
Crossing number	Additive 6	$f(k) n^{O(1)}$
Planar edge deletion number	Additive 6	$f(k) n^{O(1)}$

9.1 Planar graphs and related classes

It was shown by Thomassen [49] that the boxicity of planar graphs is at most 3. In fact he proved something stronger: there is a box representation for every planar graph in 3 dimensions such that no two boxes have an interior point in common and such that two boxes which intersect have precisely a 2 dimensional rectangular boundary in common. In other words, planar graphs are strict 3-box graphs[5]. This result was later improved by Felsner and Francis [50] who showed that every planar graph has such a representation in which the strict 3-boxes are strict 3-cubes, where the cubes are not necessarily of the same size. Note that the Roberts' graph on 6 vertices is planar, and hence there exist planar graphs having boxicity 3.

Hartman, Newman and Ziv [51] proved that all planar bipartite graphs have boxicity at most 2. Note that a result of Bellantoni et al. [52] implies that every bipartite graph that has boxicity at most 2 is a "grid intersection graph"; i.e. it has an intersection representation using horizontal and vertical line segments in the plane. Scheinerman [53] showed that the boxicity of outerplanar graphs is at most 2. Bohra, Chandran and Raju [54] showed the existence of a series parallel graph of boxicity 3 (series-parallel graphs form a subclass of planar graphs). Chandran, Francis and Suresh [55] studied the boxicity of Halin graphs and showed it to be equal to 2 unless it is isomorphic to K_4. In fact, they prove the stronger result that if G is a planar graph formed by connecting the leaves of any tree in a simple cycle, then $box(G) = 2$ unless G is isomorphic to K_4.

The Euler genus of a graph G is the minimum Euler genus of a surface in which G embeds with no crossings. Esperet and Joret [19] established that any graph of Euler genus g has boxicity bounded above by $5g + 3$. This result was later improved by Esperet [56], who showed that the upper bound can be reduced to $O(\sqrt{g} \log g)$, while also observing the existence of graphs with Euler genus g whose boxicity is $\Omega(\sqrt{g \log g})$. The gap between the upper and lower bounds was subsequently resolved by Scott and Wood [22], who proved that the boxicity of any graph of Euler genus g is $O(\sqrt{g \log g})$.

9.2 Some graph classes related to interval graphs

The upper bounds on the boxicity and cubicity of the special classes of graphs shown in Tables 2 and 3 follow from Theorems 5.2 and 3.7 respectively.

For some subclasses of the families given in Tables 2 and 3, better upper bounds are known. We list them below.

For split graphs, which form a subclass of chordal graphs, the bound given by Theorem 2.7 is better than the bound for chordal graphs given in Table 3, since for a split graph $G = (K \sqcup S, E)$, we have $\omega(G) \geq |K|$. Since the split graph of high boxicity given by Corollary 2.26 is not a strongly chordal graph, it is natural to ask if the boxicity of strongly chordal graphs can be bounded above by a constant. Spinrad [57] shows that this is not possible in the following way. Since a box representation of an n-vertex graph of dimension k can be encoded

[5] Strict d-box graphs are the intersection graphs of closed d-boxes in $\mathbb{R}^d$ such that no two boxes have an interior point in common and any two boxes which intersect have precisely a $(d-1)$-box in common.

Table 2 Upper bounds on $cub(G)$ for different graph classes derived from bandwidth.

Graph class	Upper bounds on $cub(G)$
Circular Arc Graphs	$2\Delta(G)+1$
AT-free Graphs	$3\Delta(G)-1$
Co-comparability Graphs	$2\Delta(G)$

Table 3 Upper bounds on $box(G)$ for different graph classes derived from treewidth.

Graph class	**Upper bounds on $box(G)$**
Chordal Graphs	$\omega(G)+1$, $\Delta(G)+2$
Circular Arc Graphs	$2\omega(G)+1$
Permutation Graphs	$2\Delta(G)+1$
Any minor closed family which excludes at least one planar graph	constant

using $O(k \cdot n \log n)$ bits, there are at most $2^{O(kn \log n)}$ graphs on n vertices that have boxicity at most k. It then follows from the results of Spinrad [58] that the boxicity of strongly chordal graphs and chordal bipartite graphs cannot be bounded by any constant. Chandran, Francis and Mathew [59, 60] showed how to explicitly construct strongly chordal graphs and chordal bipartite graphs having arbitrarily high boxicity.

Better bounds on the boxicity and cubicity of AT-free graphs and some of their subclasses is given by the following theorem by Bhowmick and Chandran [61].

Theorem 9.1 ([61]) Let G be an AT-free graph.

1. $box(G) \leq \chi(G)$ and this bound is sharp.
2. If G is claw-free, then $box(G) = cub(G) \leq \chi(G)$ and this bound is sharp.
3. If G has girth at least 5, then $box(G) \leq 2$ and $cub(G) \leq 2\lceil \log \psi(G) \rceil + 4$. Moreover, the bound on boxicity is sharp.
4. $cub(G) \leq box(G)(\lceil \log \psi(G) \rceil + 2) \leq \chi(G)(\lceil \log \psi(G) \rceil + 2)$.

The theorem below by Bhowmick and Chandran [62] shows that certain subclasses of circular-arc graphs admit better bounds on boxicity.

Theorem 9.2 ([62]) Let G be a circular arc graph on n vertices.

1. If G admits a circular arc representation on the unit circle in which the length of every arc is less than $\pi \cdot (\frac{\alpha-1}{\alpha})$, for some integer $\alpha \geq 2$, then $box(G) \leq \alpha$.
2. If G has maximum degree less than $\left\lfloor n \cdot (\frac{\alpha-1}{2\alpha}) \right\rfloor$, for some integer $\alpha \geq 2$, then $box(G) \leq \alpha$.
3. If G admits a circular arc representation in which some point on the circle is crossed by at most r arcs, then $box(G) \leq r+1$ and this bound is tight.
4. If G admits a circular arc representation in which no family of $k \leq 3$ arcs covers the circle, then $box(G) \leq 3$ and if G admits a circular arc representation in which no family of $k \leq 4$ arcs covers the circle, then $box(G) \leq 2$. Both these bounds are tight.

9.3 Kneser graphs, line graphs and complements of line graphs

Chandran, Mathew and Sivadasan [63] showed that for a line graph G, $box(G) \leq 2\Delta(G)(\lceil \log \log \Delta(G) \rceil + 3) + 1$. From this result, it follows that $box(G) = O(\chi(G) \log \log \chi(G))$ since $\Delta(G) \leq 2(\chi(G) - 1)$ when G is a line graph. Using bounds on the separation dimension, we have that for any graph G, $box(L(G)) = O(\Delta(G)) = O(\Delta(L(G)))$ and $box(L(G)) = O(\log |V(G)|)$ (see Theorem 7.3 and the discussion following it).

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The *Kneser graph* $K(n,k)$, where k and n are positive integers, is the graph whose vertices are the k-sized subsets of $\{1, 2, \ldots, n\}$ in which two vertices are adjacent if and only if their corresponding sets are disjoint. Caoduro and Lichev [64] studied the boxicity of Kneser graphs and complements of line graphs.

Theorem 9.3 ([64]) $box(K(n,k)) \leq n-2$ if $n \geq 2k+1$, and $box(K(n,k)) \geq n - \frac{13k^2-11k+16}{2}$ if $n \geq 2k^3-2k^2+1$.

Theorem 9.4 ([64]) Let G be any graph having $\Delta(G) \geq 3$. Then $box(\overline{L(G)}) \leq |V(G)| - 2$. Moreover,

- $box(\overline{L(G)}) \geq \frac{|E(L(G))|}{12}$, if $\Delta(G) = 3$
- $box(\overline{L(G)}) \geq \frac{|E(L(G))|}{16}$, if $\Delta(G) = 4$
- $box(\overline{L(G)}) \geq \frac{2|E(L(G))|}{\Delta(G)^2+3\Delta(G)}$, if $\Delta(G) \geq 5$.

9.4 Hypercubes and products of graphs

Let G_1 and G_2 be two graphs. The *strong product*, the *Cartesian product* and the *direct product* of two graphs are denoted by $G_1 \boxtimes G_2$, $G_1 \square G_2$ and $G_1 \times G_2$ respectively, and are defined as follows. Each of these graphs have vertex set $V(G_1) \times V(G_2)$, and have the following edge sets:

$$E(G_1 \boxtimes G_2) = \{(u_1,u_2)(v_1,v_2) \colon u_i = v_i \text{ or } u_iv_i \in E(G_i) \text{ for } i = 1 \text{ and } i = 2\} \quad (1)$$

$$E(G_1 \square G_2) = \{(u_1,u_2)(v_1,v_2) \colon u_1 = v_1, u_2v_2 \in E(G_2)\} \cup \{(u_1,u_2)(v_1,v_2) \colon u_1v_1 \in E(G_1), u_2 = v_2\} \quad (2)$$

$$E(G_1 \times G_2) = \{(u_1,u_2)(v_1,v_2) \colon u_1v_1 \in E(G_1) \text{ and } u_2v_2 \in E(G_2)\}. \quad (3)$$

Chandran et al. [65] proved the following results about the boxicity and cubicity of product graphs.

Theorem 9.5 ([65]) Let $G_1, G_2, \ldots, G_d$ be graphs. Then:

$$\max_{i\in[d]} box(G_i) \leq box(\boxtimes_{i=1}^d G_i) \leq \sum_{i=1}^d box(G_i)$$

$$\max_{i\in[d]} cub(G_i) \leq cub(\boxtimes_{i=1}^d G_i) \leq \sum_{i=1}^d cub(G_i)$$

Furthermore, if each G_i, $i \in [d]$, has a universal vertex, then the second inequality in both the above chains is tight.

Theorem 9.6 ([65]) Let $G_1, G_2, \ldots, G_d$ be graphs. Then:

$$box(\square_{i=1}^d G_i) \leq box(\boxtimes_{i=1}^d G_i) + box(\square_{i=1}^d K_{\chi(G_i)})$$

$$cub(\square_{i=1}^d G_i) \leq cub(\boxtimes_{i=1}^d G_i) + cub(\square_{i=1}^d K_{\chi(G_i)})$$

Theorem 9.7 ([65]) Let $G_1, G_2, \ldots, G_d$ be graphs and let $|V(G_i)| = n_i$ for each $i \in [d]$.

$$box(\square_{i=1}^d G_i) \leq \max_{i\in[d]} cub(G_i) + box(\square_{i=1}^d K_{n_i})$$

$$cub(\square_{i=1}^d G_i) \leq \max_{i\in[d]} cub(G_i) + cub(\square_{i=1}^d K_{n_i})$$

Theorem 9.8 ([65]) Let $G_1, G_2, \ldots, G_d$ be graphs, and let $n = \prod_{i=1}^{d} \chi(G_i)$ be the number of vertices in $\times_{i=1}^{d} K_{\chi(G_i)}$. Then:

$$box(\times_{i=1}^{d} G_i) \leq box(\boxtimes_{i=1}^{d} G_i) + box(\times_{i=1}^{d} K_{\chi(G_i)}) \leq \sum_{i=1}^{d} (box(G_i) + \chi(G_i))$$

$$cub(\times_{i=1}^{d} G_i) \leq cub(\boxtimes_{i=1}^{d} G_i) + cub(\times_{i=1}^{d} K_{\chi(G_i)}) \leq \sum_{i=1}^{d} \left(cub(G_i) + \chi(G_i) \log \frac{n}{\chi(G_i)} \right)$$

Suppose H_d is the d-dimensional hypercube. It was shown by Chandran, Mannino and Oriolo [18] that $\frac{d-1}{\log d} \leq cub(H_d) \leq 2d$. Chandran and Sivadasan [66] later improved the bound to $cub(H_d) = \Theta(\frac{d}{\log d})$ by using a probabilistic argument. Chandran, Mathew and Sivadasan [63] showed that $box(H_d) \geq \frac{1}{2}(\lceil \log \log d \rceil + 1)$ and in [65], the upper bound $box(H_d) \leq \frac{12 \log d}{\log \log d}$ was proven by exploiting the connection of boxicity with the partial order dimension of the d-dimensional Boolean lattice[6].

The Hamming graph $H_{d,q}$ is obtained by taking the Cartesian product of d copies of K_q. Observe that $H_{d,2} = H_d$, the d-dimensional hypercube. We have the following relation between boxicity and cubicity of hypercubes and Hamming graphs:

Theorem 9.9 ([65]) For $d \geq 2$,

$$\log q \leq box(H_{d,q}) \leq \lceil 10 \log q \rceil box(H_d)$$
$$\log q \leq cub(H_{d,q}) \leq \lceil 10 \log q \rceil cub(H_d)$$

9.5 Some algebraically defined graphs

For a ring R, it is possible to define a graph on the set $Z(R)$ of its zero divisors. The zero divisor graph $\Gamma(R)$ for a ring R is defined as the graph with vertex set $V(\Gamma(R)) = Z(R)$ and $E(\Gamma(R)) = \{\{a_i, a_j\} | a_i, a_j \in Z(R) \text{ and } a_i a_j = 0\}$.

Suppose $N = \prod_{i=1}^{a} p_i^{n_i}$ where $p_1, p_2, \ldots, p_a$ are distinct primes. By analysing the reduced graph[7] of $\Gamma(\mathbb{Z}_N)$ it was shown by Kavaskar [67] that $box(\Gamma(\mathbb{Z}_N)) \leq \prod_{i=1}^{a}(n_i + 1) - \prod_{i=1}^{a}(\lfloor n_i/2 \rfloor + 1) - 1$. It was also shown that for a reduced ring R, $box(\Gamma(R)) \leq 2^k - 2$ where $k = \chi(\Gamma(R)$.

The exact boxicity of $\Gamma(\mathbb{Z}_N)$ was computed by Chandran and Sahoo [68] and it was shown to be either a or $a - 1$ depending on the prime factorization of N. In the same paper, it was shown that $\lfloor k/2 \rfloor \leq box(\Gamma(R)) \leq k$ where $k = \chi(\Gamma(R)$.

The divisibility poset $\mathcal{P} = (V, \prec_{\mathcal{P}})$ is a poset defined on a set of positive integers where the partial order is the divisibility relation, i.e. V is a finite subset of $\mathbb{Z}_{>0}$ and for $a, b \in V$ we have $a \prec_{\mathcal{P}} b$ if a divides b. The comparability graph of a divisibility poset is called a divisor graph. For a positive integer n, we denote by $D(n)$, the divisor graph defined on the set of all divisors of n. For $d \geq 2$, and distinct primes $p_1, p_2, \ldots, p_d$, let $n = \prod_{i \in [d]} p_i^{m_i}$ where $m_1 \leq m_2 \leq \cdots \leq m_d$. Chandran and Ghosh [69] showed that $box(D(n)) \leq cub(D(n)) \leq m_1 + m_2 + \cdots + m_{d-1}$ and for the lower bound, they showed that when d is odd, $cub(D(n)) \geq box(D(n)) \geq \sum_{1 \leq j \leq \frac{d-1}{2}} m_{2j-1} + m_{d-1}$ and when d is even, $cub(D(n)) \geq box(D(n)) \geq \sum_{1 \leq j \leq \frac{d-2}{2}} m_{2j} + m_{d-1}$.

Chandran and Ghosh [69] also considered another subclass of algebraically defined comparability graphs, namely *power graphs*: the power graph of a group G, denoted by $Pow(G)$, is a simple, undirected graph with

[6] The d-dimensional Boolean lattice is the poset on $V(H_d)$ such that for $u = (u_1, u_2, \ldots, u_d)$ and $v = (v_1, v_2, \ldots, v_d)$, we have $u \preceq v$ if and only if $u_i \leq v_i, \forall i \in [d]$.

[7] The reduced graph of a graph G is obtained by identifying vertices having the same neighbourhood in G. In other words, we define an equivalence relation $\sim$ on $V(G)$ as $u \sim v$ if and only if $N(u) = N(v)$. Then the equivalence classes of $V(G)$ are the vertices of the reduced graph and the equivalence classes $[u]$ and $[v]$ are adjacent if and only if $uv \in E(G)$.

vertex set G, and with two elements being adjacent if one is a power of another. They showed that $box(Pow(\mathbb{Z}_n)) = box(D(n))$ and $cub(Pow(\mathbb{Z}_n)) = cub(D(n))$. It follows that both the upper and lower bounds for the boxicity (and cubicity) of the divisor graph $D(n)$ also hold for the power graph of the cyclic group of order n.

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