



# Studies on Affine Fibrations and Affine Forms

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## Abstract

Problems on Affine Fibrations and Affine Forms are among the fundamental questions in Affine Algebraic Geometry. They have turned out to be closely, and often symbiotically, related to other challenging problems in Affine Algebraic Geometry, like the Epimorphism Problem and the Zariski Cancellation Problem. In this survey article, we highlight some of the known partial results and open questions on affine fibrations, Laurent polynomial fibrations and affine forms and their connections with other problems, with special emphasis on contributions by Indian mathematicians.

**Keywords** Affine Fibrations · Polynomial Rings · Epimorphism Problem · Zariski Cancellation Problem · Affine Forms · Laurent Polynomial Fibrations

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## 1 Introduction

For integral domains  $R \subset A$ , we shall use the notation “ $A = R^{[n]}$ ” to mean that  $A = R[t_1, \dots, t_n]$  for elements  $t_1, \dots, t_n \in A$  algebraically independent over  $R$ , i.e.,  $A$  is isomorphic as an  $R$ -algebra to a polynomial algebra in  $n$  indeterminates over  $R$ . For a prime ideal  $P$  of  $R$ , the notation  $A_P$  denotes the localisation  $S^{-1}A$ , with  $S = R \setminus P$ , i.e.,  $A_P = A \otimes_R R_P$ , and  $k(P)$  denotes the residue field of the local ring  $R_P$ , i.e.,  $k(P)$  is the field of fractions of the integral domain  $R/P$ . Thus,  $A \otimes_R k(P) = A_P/P A_P$ .

Investigations in affine algebraic geometry often involve determining whether, for an integral domain  $A$  and a certain subring  $R$ , we have  $A = R^{[n]}$ . The following conditions are obviously necessary for the relation  $A = R^{[n]}$ :

- (i)  $A$  is finitely generated over  $R$ .
- (ii)  $A$  is flat over  $R$ .
- (iii) For every prime ideal  $P$  of  $R$ , the fibre ring  $A \otimes_R k(P) = k(P)^{[n]}$ .

Geometrically, denoting  $\text{Spec } R$  to be the set of prime ideals of  $R$  with affine scheme structure, condition (iii) states that for the affine morphism from  $\text{Spec } A$  to  $\text{Spec } R$ , the fibre at each point  $P$  of  $\text{Spec } R$  (which is isomorphic to  $\text{Spec}(A \otimes_R k(P))$ ) is isomorphic to the affine scheme  $\text{Spec}(k(P)^{[n]})$ .

**Definition** Let  $R \subset A$  be commutative rings. We shall call  $A$  to be an *affine  $n$ -fibration* ( $\mathbb{A}^n$ -fibration) over  $R$  if the above conditions (i), (ii) and (iii) hold. An *affine  $n$ -fibration*  $A$  over  $R$  will be called non-trivial if  $A \neq R^{[n]}$ .

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For instance, if  $R$  is an integral domain,  $M$  a finitely generated projective  $R$ -module of rank  $n$  and  $A = \text{Sym}_R(M)$ , then clearly the  $R$ -algebra  $A$  is finitely generated, flat and locally trivial, i.e.,  $A_P = R_P^{[n]}$  for every prime ideal  $P$  of  $R$ ; in particular,  $A$  is an  $\mathbb{A}^n$ -fibration over  $R$ .

In the area of Affine Fibrations, one investigates the structure and properties of an  $R$ -algebra  $A$  under suitable hypotheses on the fibre rings  $A \otimes_R k(P)$  of  $A$ . In Sections 2, 3 and 4, we shall recall some of the pioneering results of A. Sathaye and T. Asanuma on the above concept of  $\mathbb{A}^n$ -fibrations and the subsequent investigations on it by S.M. Bhatwadekar and his school in India. Surveys of results on  $\mathbb{A}^n$ -fibrations had been made earlier in various contexts in [19], [39], [40], [29] and [42].

We mention here that in the literature on Affine Algebraic Geometry, there are two slightly different definitions of  $\mathbb{A}^n$ -fibrations. The above definition was introduced by Sathaye in [84]. In the other definition of the term  $\mathbb{A}^n$ -fibration, in place of the above condition (iii), the fibre rings  $A/mA$  are required to be polynomial rings for almost all maximal ideals  $m$  of  $R$ . Much work has been done on this concept by M. Miyanishi, R.V. Gurjar and their school. However, in this article, we shall discuss the concept of  $\mathbb{A}^n$ -fibration as defined by Sathaye.

In Section 6, we shall also discuss results on Laurent polynomial fibrations, where condition (iii) is replaced by the condition that the fibre rings are Laurent polynomial algebras, and other variants.

Now, let  $k$  be a field and  $A$  be a  $k$ -algebra.

**Definition** We say that  $A$  is an *affine  $n$ -form* ( $\mathbb{A}^n$ -form) over  $k$  if  $A \otimes_k L = L^{[n]}$  for some field extension  $L$  of  $k$ .

More generally, let  $R$  be a ring containing a field  $k$ .

**Definition** We say that the  $R$ -algebra  $A$  is an  $\mathbb{A}^n$ -form over  $R$  with respect to  $k$  if  $A \otimes_k L = (R \otimes_k L)^{[n]}$  for some field extension  $L$  of  $k$ .

In Section 5, we shall highlight a few results on the triviality of such  $\mathbb{A}^n$ -forms.

We recall the definition of a fixed point free locally nilpotent derivation.

**Definition** A  $k$ -linear derivation  $D$  on a  $k$ -algebra  $A$  is said to be a *locally nilpotent derivation* (LND), if for each  $a \in A$ , there exists an integer  $n \geq 0$  (depending on  $a$ ), such that  $D^n(a) = 0$ .

**Definition** A locally nilpotent derivation  $D$  on a ring  $A$  is said to be *fixed point free* if  $(DA) = A$ , where  $(DA)$  is the ideal of  $A$  generated by  $D(A)$ .

**Definition** For a locally nilpotent derivation  $D$  on  $A$ , if there exists  $a \in A$  such that  $D(a) = 1$ , then  $a$  is said to be a *slice* of  $D$ .

The Slice Theorem of Gabriel-Nouazé ([53]) states:

**Theorem 1.1** *Let  $k$  be a field of characteristic zero,  $A$  a  $k$ -algebra and  $D$  an LND on  $A$  with kernel  $C$ . If  $D$  has a slice  $a$ , then  $A = C[a] = C^{[1]}$ .*

Finally, we recall the definition of a retract.

**Definition** Let  $R \subseteq A$  be integral domains. If there exists an  $R$ -linear ring homomorphism from  $A$  to  $R$ , then  $R$  is said to be a *retract* of  $A$ .

## 2 On the Triviality of Affine Fibrations

In this section, we discuss results on affine fibrations over local rings being polynomial algebras or at least stably polynomial algebras.

First, we recall the local-global theorem which was proved by H. Bass, E.H. Connell and D.L. Wright ([15]), and independently by A.A. Suslin ([88]).

**Theorem 2.1** *Let  $R$  be a ring and  $A$  a finitely presented  $R$ -algebra. Suppose that there exists an integer  $n \geq 0$  such that, for each maximal ideal  $m$  of  $R$ ,  $A_m = R_m^{[n]}$ . Then  $A \cong \text{Sym}_R(M)$  as  $R$ -algebras for some finitely generated projective  $R$ -module  $M$  of rank  $n$ .*

Thanks to Theorem 2.1, questions on affine fibrations can often be reduced to the local situation. We first recall the famous problem of B. Veisfeiler and I.V. Dolgačev ([90]), often called the “Affine Fibration Problem”.

**Question 2.2** Is every  $\mathbb{A}^n$ -fibration over a regular local ring  $R$  necessarily a polynomial ring over  $R$ ?

A deep work of Asanuma provides the following stable structure theorem for  $\mathbb{A}^n$ -fibrations over arbitrary Noetherian rings ([5, Theorem 3.4]).

**Theorem 2.3** *Let  $R$  be a Noetherian ring and  $A$  an  $\mathbb{A}^n$ -fibration over  $R$ . Then there exists an integer  $m$  such that  $A$  is an  $R$ -subalgebra of  $R^{[m]}$  for some  $m$  and  $A^{[m]} \cong \text{Sym}_{R^{[m]}}(L)$  as  $R^{[m]}$ -algebras, where  $L$  is a finitely generated projective  $R^{[m]}$ -module of rank  $n$ . In fact,  $\Omega_{A/R}$  is a finitely generated projective  $A$ -module of rank  $n$  and  $L$  is the extended module  $\Omega_{A/R} \otimes_A R^{[m]}$ .*

A response to Question 2.2 was deduced by Asanuma from the above theorem — that over a regular local ring  $R$ , any  $\mathbb{A}^n$ -fibration is at least stably polynomial over  $R$  ([5, Corollary 3.5]):

**Corollary 2.4** *An  $\mathbb{A}^n$ -fibration  $A$  over a regular local ring  $R$  is stably  $\mathbb{A}^n$ , i.e., there exists an integer  $t \geq 0$  such that  $A^{[t]} = R^{[n+t]}$ .*

However, even an  $\mathbb{A}^1$ -fibration over an arbitrary  $R$  need not be stably polynomial as illustrated by the following standard example:

**Example 2.5** Let  $k$  be a field of characteristic zero and  $t$  an indeterminate over  $k$ . Let  $R = k[[t^2, t^3]]$  and  $A = R[X + tX^2] + (t^2, t^3)R[X]$ . Then  $A$  is an  $\mathbb{A}^1$ -fibration over  $R$  but  $A$  is not stably polynomial over  $R$ . For details, see [92, 4.1].

For an arbitrary  $n$ , Corollary 2.4 is so far the best partial answer to Question 2.2 in the affirmative direction. We now focus on the case  $n = 1$  on which much studies have been undertaken.

Recall that an integral domain  $R$  with field of fractions  $K$  is said to be *seminormal* if it satisfies the condition: an element  $t \in K$  will belong to  $R$  if  $t^2, t^3 \in R$ .

For  $n = 1$ , Question 2.2 has a complete affirmative solution even over Noetherian seminormal domains. The triviality of  $\mathbb{A}^1$ -fibrations was first proved over Noetherian UFDs by Kambayashi-Miyaniishi ([72]), then over Noetherian normal local domains by Kambayashi-Wright ([74]), and then over Noetherian seminormal local domains by Greither ([58]). Example 2.5 shows the necessity of the seminormality hypothesis. Subsequently, there have been a series of generalisations of the above results showing that conditions on only the generic and codimension-one fibres determine  $\mathbb{A}^1$ -fibrations. These will be discussed separately in Section 4.

One can also readily deduce (cf. [19, Theorem 3.3]) the triviality of  $\mathbb{A}^1$ -fibrations over Noetherian seminormal local domains from Asanuma’s structure theorem (Theorem 2.3 above) using Swan’s theorem ([89, Theorem 6.1]) that over a seminormal domain  $R$ ,  $\text{Pic}(R^{[m]}) = \text{Pic}(R)$  for every  $m \geq 1$  and the following cancellation theorem of E. Hamann ([65, Theorems 2.6 and 2.8]):

**Theorem 2.6** *Let  $R$  be an integral domain such that either  $R$  is Noetherian seminormal or  $R$  contains  $\mathbb{Q}$  and  $A$  be an integral domain containing  $R$  such that  $A^{[m]} = R^{[m+1]}$ . Then  $A = R^{[1]}$ .*

The above result was proved earlier for a local UFD by S.S. Abhyankar, P. Eakin and W. Heinzer in [3, Corollary 4.7]. One of the necessary conditions for  $A$  to be  $R^{[n]}$  is that the module of differentials  $\Omega_{A/R}$  is a free  $A$ -module of rank  $n$ . Now suppose that  $\Omega_{A/R}$  is free for an  $\mathbb{A}^1$ -fibration  $A$  over a Noetherian domain  $R$ . Then by Asanuma's result (Theorem 2.3), we have  $A^{[m]} = R^{[m+1]}$ . Hence, as noted in [19, Theorem 2.4], by the above cancellation theorem of Hamann, we have:

**Theorem 2.7** *Let  $R$  be a Noetherian domain such that either  $R$  is seminormal or  $R$  contains  $\mathbb{Q}$ . Let  $A$  be an  $\mathbb{A}^1$ -fibration over  $R$  for which  $\Omega_{A/R}$  is free. Then  $A = R^{[1]}$ .*

The necessity of the hypotheses on  $R$  is shown by the following construction of Bhatwadekar and the first author ([17, Example 4.1]):

**Example 2.8** Let  $D = \mathbb{Z}_{(2)}[2\sqrt{2}]$ ,  $A = D[X, Y](= D^{[2]})$ ,  $R = D[F]$ , where

$$F = X - 2Y(\sqrt{2}X - Y^2) + \sqrt{2}(\sqrt{2}X - Y^2)^2 - \sqrt{2}(Y - \sqrt{2}(\sqrt{2}X - Y^2))^4.$$

Then  $A$  is an  $\mathbb{A}^1$ -fibration over  $R$  and  $\Omega_{A/R}$  is free, but  $A/(F) \neq D^{[1]}$ ; in particular,  $A \neq R^{[1]}$ .

Many of the challenges in Affine Algebraic Geometry involve the question of determining whether a polynomial  $F$  in  $R[X_1, \dots, X_n]$  ( $= R^{[n]}$ ) is a coordinate, i.e., whether  $R[X_1, \dots, X_n] = R[F]^{[n-1]}$ . For simplicity, we consider the case  $n = 2$  and the polynomial ring  $R[X, Y]$ .

An obvious necessary condition for  $F$  to be a coordinate in  $R[X, Y]$  is that  $F$  should be a *residual coordinate* in  $R[X, Y]$ , defined as follows:

**Definition** An element  $F \in R[X, Y]$  is called a *residual coordinate* (also called *residual variable*) in  $R[X, Y]$  if, at every prime ideal  $P$  of  $R$ ,  $k(P)[X, Y] = k(P)[\bar{F}]^{[1]}$ , where  $\bar{F}$  denotes the image of  $F$  in  $k(P)[X, Y]$ .

The concept of residual coordinate is closely related to that of  $\mathbb{A}^1$ -fibration. Indeed, as an application of Asanuma's structure theorem (Theorem 2.3), Bhatwadekar and the first author obtained the following result on residual coordinates ([17, Theorems 3.1 and 3.2]; most of the implications in this result occurred implicitly in Bhatwadekar's paper [16]):

**Theorem 2.9** *Let  $R$  be a Noetherian domain. Then the following statements are equivalent for any  $F \in R[X, Y](= R^{[2]})$ .*

- (i)  $F$  is a residual coordinate in  $R[X, Y]$ .
- (ii)  $R[X, Y]$  is an  $\mathbb{A}^1$ -fibration over  $R[F]$ .
- (iii)  $R[X, Y]^{[m]} = R[F]^{[m+1]}$  for some  $m$ .
- (iv)  $R[F] \subset R[X, Y] \subset B$ , for some  $B = R[F]^{[1]}$ .

Moreover, if either  $R$  contains  $\mathbb{Q}$  or  $R$  is seminormal, then  $R[X, Y] = R[F]^{[1]}$ .

Although (ii) is not written explicitly in the statement in [17, Theorem 3.1], the proof of (i)  $\implies$  (iii) involves first showing (i)  $\implies$  (ii). Analogous results hold for a system of  $n - 1$  algebraically independent elements  $F_1, F_2, \dots, F_{n-1}$  in the polynomial ring  $R[X_1, \dots, X_n]$  (cf. [17, Remark 3.4] and [35]). For a neat application, see the short paper [76] of Animesh Lahiri.

The concept of residual coordinate can be defined for any polynomial ring  $R^{[n]}$ . This concept has been extended to that of any  $\mathbb{A}^n$ -fibration over  $R$  by M.E. Kahoui in [66]. Results on the concept have been obtained by Prosenjit Das and the first author in [35], by Animesh Lahiri and the first author in [45], by M.E. Kahoui and M. Ouali in [68], [69] and [70] (with N. Essamaoui) and by Janaki Raman Babu and Prosenjit Das in [13].

We now return to Question 2.2. For  $n > 1$ , the only affirmative solution known so far is by A. Sathaye when  $n = 2$  and  $\dim R = 1$  ([84]). In fact, by a result of Asanuma ([5, Theorem 3.1]), the hypothesis “ $A$  is finitely generated” can be dropped so that we have the following result:

**Theorem 2.10** *Let  $R$  be a discrete valuation ring containing  $\mathbb{Q}$  with field of fractions  $K$  and residue field  $k$ . Let  $A$  be an integral domain containing  $R$  such that  $A \otimes_R K = K^{[2]}$  and  $A \otimes_R k = k^{[2]}$ . Then  $A = R^{[2]}$ .*

Consequently, using Theorem 2.1, we have the following theorem over a PID (Principal Ideal Domain):

**Theorem 2.11** *Let  $R$  be a PID containing  $\mathbb{Q}$  and  $A$  be an integral domain containing  $R$  such that  $A$  is an  $\mathbb{A}^2$ -fibration over  $R$ . Then  $A = R^{[2]}$ .*

An important example of Asanuma (Example 3.5 in the next section) shows that Theorem 2.11 does not extend to PID’s not containing  $\mathbb{Q}$ . However, as we shall see in Theorem 5.14 in Section 5, Parnashree Ghosh and the second author have shown that under the additional hypothesis that  $A$  is an  $\mathbb{A}^2$ -form over the PID  $R$  with respect to a field  $k$  (of arbitrary characteristic), an  $\mathbb{A}^2$ -fibration over  $R$  will indeed be  $R^{[2]}$ .

Asanuma and Bhatwadekar investigated the structure of  $\mathbb{A}^2$ -fibrations over arbitrary one-dimensional Noetherian domains containing  $\mathbb{Q}$  in [7]. We quote below one of their results ([7, Theorem 3.8]).

**Theorem 2.12** *Let  $R$  be a one-dimensional Noetherian  $\mathbb{Q}$ -domain. Let  $A$  be an  $\mathbb{A}^2$ -fibration over  $R$ . Then there exists  $H \in A$  such that  $A$  is an  $\mathbb{A}^1$ -fibration over  $R[H]$ .*

In response to questions of D.L. Costa as to the structures of retracts of polynomial rings ([28, Section 4]), Sagnik Chakraborty, Nikhilesh Dasgupta and the authors have established the following results, where retracts have turned out to be affine fibrations ([27, Theorems 5.4, 5.9]):

**Theorem 2.13** *Let  $R$  be a Noetherian domain and  $A$  an  $R$ -algebra retract of  $R^{[m]}$  for some integer  $m$ . Then the following hold:*

- (I) *If  $\text{tr. deg}_R A = 1$ , then  $A$  is an  $\mathbb{A}^1$ -fibration over  $R$ .*
- (II) *If  $\text{tr. deg}_R A = 2$  and  $R$  contains  $\mathbb{Q}$ , then  $A$  is an  $\mathbb{A}^2$ -fibration over  $R$ .*

$\mathbb{A}^2$ -fibrations also arise as kernels of locally nilpotent  $R$ -derivations of  $R[X, Y, Z]$  having slices. G. Freudenburg has proved ([51, Theorem 1.1]):

**Theorem 2.14** *Let  $k$  be a field of characteristic zero,  $R$  a  $k$ -domain,  $D$  a locally nilpotent  $R$ -derivation of  $R[X, Y, Z]$  which admits a slice. Then the kernel of  $D$  is an  $\mathbb{A}^2$ -fibration over  $R$ .*

Till the question of triviality of  $\mathbb{A}^2$ -fibrations over regular local rings containing  $\mathbb{Q}$  gets settled, one tries to obtain results establishing the triviality of such  $\mathbb{A}^2$ -fibrations at least under additional hypotheses. We quote below one such result due to Kahoui-Ouali ([67, Corollary 3.5]) addressing a question of Freudenburg in [51, Question 2].

**Theorem 2.15** *Let  $R$  be a regular UFD containing  $\mathbb{Q}$  and  $A$  an  $\mathbb{A}^2$ -fibration over  $R$  having a fixed point free locally nilpotent  $R$ -derivation. Then  $A = R^{[2]}$ .*

Over a general Noetherian  $\mathbb{Q}$ -domain, the following partial result has been proved by J.R. Babu and P. Das in [12, Theorem 4.4]:

**Theorem 2.16** *Let  $R$  be a Noetherian domain containing  $\mathbb{Q}$  and  $A$  an  $\mathbb{A}^2$ -fibration over  $R$ . Let  $D : A \rightarrow A$  be a fixed point free  $R$ -linear locally nilpotent derivation on  $A$ . Then,  $\text{Ker}(D)$  is an  $\mathbb{A}^1$ -fibration over  $R$  and  $D$  has a slice, i.e.,  $A = \text{Ker}(D)^{[1]}$ . In particular, if  $R$  is a normal domain, then  $A \cong \text{Sym}_R(I)^{[1]}$  for some invertible ideal  $I$  of  $R$ .*

Since it is not known whether any  $\mathbb{A}^2$ -fibration over an integral domain containing  $\mathbb{Q}$  is essentially trivial, Theorem 2.16 may be regarded as a generalisation of the corresponding result for  $A = R^{[2]}$  that had been proved by Bhatwadekar and the first author ([21, Theorem 4.7]) and A. van den Essen ([49]). It may also be seen as a partial generalisation of Theorem 2.12 to higher dimensions.

Theorem 2.16 was earlier shown by Kahoui-Ouali ([69, Theorems 3.1, 2.4 and Corollary 3.2]) under the additional hypothesis that  $A$  is a stably polynomial ring.

During his investigations on the possible generalisations of D. Wright's result on cancellation of variables of the form  $bT^n - a$  ([91]), P. Das had discovered the following result ([32, Corollary 3.7]):

**Theorem 2.17** *Let  $R$  be a Noetherian UFD containing  $\mathbb{Q}$  and  $A$  an  $\mathbb{A}^2$ -fibration over  $R$ . Let  $n \geq 2$ . Suppose there exist  $a, b \in A$  such that  $B = A[T]/(bT^n - a)$  satisfies  $B \otimes_R k(P) = k(P)^{[2]}$  for all  $P \in \text{Spec } R$ . Then  $A = R[a]^{[1]} = R^{[2]}$  and  $A[T] = R[bT^n - a, T]^{[1]}$ .*

The next section contains some examples which are considered potential counterexamples to the Affine Fibration Problem.

### 3 Relationships with other problems on Affine Spaces

The Affine Fibration Problem has had a close symbiotic relationship with other fundamental problems in Affine Algebraic Geometry, like the Epimorphism (or Embedding) Problem and the Zariski Cancellation Problem. We recall the statements of the two longstanding and challenging problems.

Throughout the section  $k$  will denote a field. For  $n \geq 1$ , the affine  $n$ -space, denoted by  $\mathbb{A}_k^n$  is the set  $k^n$  with Zariski topology and the standard sheaf structure.

**Definition** A  $k$ -algebra epimorphism  $\phi : k[X_1, \dots, X_n](= k^{[n]}) \rightarrow k[Y_1, \dots, Y_m](= k^{[m]})$  is said to be *rectifiable* if there exist  $F_1, \dots, F_n \in k[X_1, \dots, X_n]$  such that  $k[F_1, \dots, F_n] = k[X_1, \dots, X_n]$  and  $\text{Ker } \phi = (F_{m+1}, \dots, F_n)$ .

The Epimorphism Problem (or the Embedding Problem) is concerned with the rectifiability of any  $k$ -algebra epimorphism  $k^{[n]} \rightarrow k^{[m]}$ . When  $m = n - 1$ , the Epimorphism Problem over a field  $k$  takes the form:

**Question 3.1** Suppose that  $k[X_1, \dots, X_n]/(F) = k^{[n-1]}$ . Is  $k[X_1, \dots, X_n] = k[F]^{[n-1]}$ ?

For a  $k$ -algebra  $D$  of dimension  $n$ , the Zariski Cancellation Problem (ZCP in short) asks:

**Question 3.2** If  $D^{[1]} = k^{[n+1]}$ , does it necessarily follow that  $D = k^{[n]}$ ?

We first state the celebrated Epimorphism Theorem of Abhyankar-Moh ([4]) and Suzuki ([87]), addressing Question 3.1 for  $n = 2$ :

**Theorem 3.3** *Let  $k$  be a field of characteristic zero. Let  $F \in k[X, Y]$  satisfy  $k[X, Y]/(F) = k^{[1]}$ . Then  $k[X, Y] = k[F]^{[1]}$ .*

There are examples due to Segre ([85]) and Nagata ([77]) to show that Theorem 3.3 does not hold when  $k$  is a field of positive characteristic  $p$ . We quote below one.

**Example 3.4** Let  $k$  be a field of characteristic  $p (> 0)$  and  $g(Z, T) = Z^{p^e} + T + T^{sp} \in k[Z, T]$ , where  $e, s \in \mathbb{N}$  are such that  $p^e \not\mid sp, sp \not\mid p^e$ . Then  $k[Z, T]/(g(Z, T)) = k^{[1]}$  but  $k[Z, T] \neq k[g(Z, T)]^{[1]}$ .

The Abhyankar-Sathaye Conjecture expects an affirmative answer to Question 3.1 when the characteristic of  $k$  is zero. In view of Example 3.4, Question 3.1 can be asked in general only over fields of characteristic zero. However, when  $F$  is given to be of some specified type, the question can be asked even in arbitrary characteristic.

An essential tool in the Abhyankar-Moh proof of the Epimorphism Theorem was their theory of “expansion techniques”. Sathaye’s breakthrough result on  $\mathbb{A}^2$ -fibrations (Theorem 2.10) involved a brilliant application of a generalisation of Abhyankar-Moh techniques that he had formulated in [83].

**Definition** Any  $f \in k[Z, T]$  is called a line if  $k[Z, T]/(f) = k^{[1]}$ . A line  $f \in k[Z, T]$  is called non-trivial if  $k[Z, T] \neq k[f]^{[1]}$ .

In [5, Theorem 5.1], Asanuma used the non-trivial line in Example 3.4 to construct the first counter-example to the  $\mathbb{A}^2$ -fibration problem over a PID in positive characteristic. We quote below a slightly modified version of the example that Asanuma described in [5].

**Example 3.5** Let  $k$  be a field of characteristic  $p (> 0)$  and let  $A = k[X, Y, Z, T]/(X^r Y + g(Z, T))$ , where  $g(Z, T)$  is as defined in Example 3.4. Let  $x$  denote the image of  $X$  in  $A$  and  $R = k[x]$ . Then  $R \subset A$  and the following properties are satisfied by the threefold  $A$ :

- (1)  $A \otimes_R k(P) = k(P)^{[2]}$  for all  $P \in \text{Spec } R$ .
- (2)  $A \neq R^{[2]}$ .
- (3)  $A^{[1]} = R^{[3]}$ .

In the original example in [5], the ring  $R$  was taken to be a discrete valuation with parameter  $\pi$  such that the residue field  $k = R/\pi R$  is of positive characteristic  $p$ ; and  $p$  played the role of  $x$ . For instance, one could take  $R = \mathbb{Z}_{(p\mathbb{Z})}$  for a positive prime  $p$ , a discrete valuation ring of characteristic zero not containing  $\mathbb{Q}$ .

In Example 3.5, (1) and (2) show that  $A$  is a non-trivial  $\mathbb{A}^2$ -fibration over  $R = k[x]$ . The ring  $A$  was soon to acquire a wider significance. In a subsequent paper ([6, Theorem 2.2]), using the ring  $A$ , Asanuma constructed non-linearizable algebraic torus actions on  $\mathbb{A}_k^n$  over any infinite field  $k$  of positive characteristic when  $n \geq 4$ .

While the answer to the Zariski Cancellation Problem (Question 3.2) was shown to be positive for  $n < 3$  during the 1970’s, the higher-dimensional case  $n \geq 3$  continued to intrigue mathematicians. Asanuma’s example of non-trivial  $\mathbb{A}^2$ -fibration (Example 3.5) opened up a possibility for solving the ZCP in positive characteristic. In this example, the property (3) of the ring  $A$  shows that

$$A^{[1]} = k[x]^{[3]} = k^{[4]}.$$

Asanuma asked about the ring  $A$  in Example 3.5:

**Question 3.6** Is  $A = k^{[3]}$ ?

He explained the significance of his question as follows ([6, Remark 2.3]): If  $A = k^{[3]}$ , then  $A$  will give an example of a non-linearizable torus action on  $\mathbb{A}_k^3$  in positive characteristic; on the other hand if  $A \neq k^{[3]}$ , then  $A$  will clearly give a counter-example to the ZCP.

Thus either way one would answer a major problem in Affine Algebraic Geometry. This dichotomy had been popularized by P. Russell as “Asanuma’s Dilemma” (cf. [52, Problem 2, p. 9]).

In [59], the second author showed that  $A \neq k^{[3]}$  when  $r \geq 2$ , thus establishing that the Zariski Cancellation Problem does not have an affirmative answer in positive characteristic. She used the methods developed by L. Makar-Limanov and A. Crachiola for studying an affine domain  $A$  through  $\mathbb{G}_a$ -actions on  $A$ , especially their ring of invariants known as the Derksen invariant  $\text{DK}(A)$ . However, Question 3.6 remains open for  $r = 1$ .

In a subsequent paper [60], the second author made further investigations on a more general version of the Asanuma ring defined in Example 3.5, bringing out conditions for such a ring to be a polynomial ring and conditions for the polynomial defining the ring to be a coordinate. The main theorem quoted below ([60, Theorem 3.11]) connects the Epimorphism and Cancellation Problems with the Affine Fibration Problem and the triviality of the Derksen invariant.

**Theorem 3.7** *Let  $k$  be a field and*

$$A = k[X, Y, Z, T]/(X^r Y - F(X, Z, T)), \text{ where } r > 1.$$

*Let  $x, y, z$  and  $t$  denote, respectively, the images of  $X, Y, Z$  and  $T$  in  $A$ . Set  $f(Z, T) := F(0, Z, T)$  and  $G := X^m Y - F(X, Z, T)$ . Then the following statements are equivalent:*

- (i)  $k[X, Y, Z, T] = k[X, G]^{[2]}$ .
- (ii)  $k[X, Y, Z, T] = k[G]^{[3]}$ .
- (iii)  $A = k[x]^{[2]}$ .
- (iv)  $A = k^{[3]}$ .
- (v)  $A^{[\ell]} \cong_k k^{[\ell+3]}$  for some integer  $\ell \geq 0$  and  $\text{DK}(A) \neq k[x, z, t]$ .
- (vi)  $A$  is an  $\mathbb{A}^2$ -fibration over  $k[x]$  and  $\text{DK}(A) \neq k[x, z, t]$ .
- (vii)  $A$  is geometrically factorial over  $k$ ,  $\text{DK}(A) \neq k[x, z, t]$  and the canonical map  $k^* \rightarrow \mathbf{K}_1(A)$  (induced by the inclusion  $k \hookrightarrow A$ ) is an isomorphism.
- (viii)  $A$  is geometrically factorial over  $k$ ,  $\text{DK}(A) \neq k[x, z, t]$  and  $(A/xA)^* = k^*$ .
- (ix)  $k[Z, T] = k[f]^{[1]}$ .
- (x)  $k[Z, T]/(f) = k^{[1]}$  and  $\text{DK}(A) \neq k[x, z, t]$ .

Analogous results in higher dimension have been obtained recently by the second author with Parnashree Ghosh and Ananya Pal (for instance, [54, Theorem 3.10], [56, Theorem 4.6], [57, Theorem 5.29]).

While Asanuma’s ring (Example 3.5) gives a non-trivial  $\mathbb{A}^2$ -fibration over  $k^{[1]}$ , the second author generalised the construction to generate counterexamples to the ZCP in every higher dimension ([61, Theorem 3.7]):

**Theorem 3.8** *Let  $k$  be a field of any characteristic and  $A$  an integral domain defined by*

$$A = k[X_1, \dots, X_m, Y, Z, T]/(X_1^{r_1} \cdots X_m^{r_m} Y - f(Z, T)),$$

*where  $r_i \geq 2$  for each  $i$ . Suppose that  $k[Z, T]/(f) = k^{[1]}$ . Then*

$$A^{[1]} = k[X_1, \dots, X_m]^{[3]} = k^{[m+3]}.$$

*Moreover, if  $(\text{ch } k > 0)$  and  $k[Z, T] \neq k[f]^{[1]}$ , then*

$$A \neq k^{[m+2]}.$$

*In particular,  $A \neq k[X_1, \dots, X_m]^{[2]}$ .*

The result shows how to use an example of the type Example 3.4 to generate a non-trivial  $\mathbb{A}^2$ -fibration over  $k^{[m]}$  for any  $m$  ([42, Corollary 3.9]):

**Corollary 3.9** *Let  $R = k[X_1, \dots, X_m]$  and  $A$  be as in Theorem 3.8. Then  $A$  is an  $\mathbb{A}^2$ -fibration over  $R$ . Moreover, if  $\text{ch } k > 0$  and  $f(Z, T)$  is a non-trivial line in  $k[Z, T]$ , then  $A$  is a non-trivial  $\mathbb{A}^2$ -fibration over  $R (= k^{[m]})$ .*

We now discuss potential non-trivial  $\mathbb{A}^3$ -fibrations. A.R. Shastri ([86]) had shown that there are non-rectifiable epimorphisms  $\mathbb{R}^{[3]} \twoheadrightarrow \mathbb{R}^{[1]}$ . He presented the following two explicit examples.

**Example 3.10** Let  $\phi, \psi : \mathbb{R}[X, Y, Z] \twoheadrightarrow \mathbb{R}[T]$  be  $k$ -algebra epimorphisms defined by

$$\begin{aligned} \phi(X) &= T^3 - 3T, & \phi(Y) &= T^4 - 4T^2, & \phi(Z) &= T^5 - 10T; \text{ and} \\ \psi(X) &= T^3 - 3T, & \psi(Y) &= T(T^2 - 1)(T^2 - 4), & \psi(Z) &= T^7 - 42T. \end{aligned}$$

Then neither  $\phi$  nor  $\psi$  is rectifiable over  $\mathbb{R}$ .

It is not known whether every  $\mathbb{A}^n$ -fibration over a discrete valuation ring containing  $\mathbb{Q}$  is trivial when  $n \geq 3$ . In [8], Asanuma generalised his example (Example 3.5) to develop a theory based on non-rectifiable embeddings of  $\mathbb{A}_k^m$  in  $\mathbb{A}_k^n$  (if and when they exist) for constructing rings which could be possible counter-examples to the  $\mathbb{A}^n$ -fibration problem for  $n \geq 2$ . His construction can be formulated as follows.

**Proposition 3.11** *Let  $D = k^{[m]}$  and  $\phi : k[X_1, \dots, X_n] \twoheadrightarrow D$  be a  $k$ -algebra epimorphism with  $\text{Ker } \phi = (F_1, \dots, F_{n-m})$ . Let  $t$  be an indeterminate over  $k$  and*

$$A = k[t][X_1, \dots, X_n][Y_1, \dots, Y_{n-m}] / (t^\ell Y_1 - F_1, \dots, t^\ell Y_{n-m} - F_{n-m}), \text{ where } \ell \geq 1.$$

Then

- (i)  $A[1/t] = k[t, 1/t]^{[n]}$ .
- (ii)  $A/tA = k^{[n]}$ .
- (iii)  $A$  is an  $\mathbb{A}^n$ -fibration over  $k[t]$ .
- (iv)  $A^{[m]} = k[t]^{[n+m]}$ .

(i) and (ii) are easy to see, (iii) follows from (i) and (ii), and (iv) is proved in [8, Corollary 4.2]. If  $\phi$  is rectifiable, then one can see that  $A = k[t]^{[n]}$ . Note that Example 3.5 was a special case of Proposition 3.11 by taking  $m = 1, n = 2$ . Asanuma asked ([8, Remark 7.8]):

**Question 3.12** Let  $A$  be the ring obtained from Proposition 3.11 by taking  $m = 1, n = 3, k = \mathbb{R}$  and  $\phi$  be a non-rectifiable epimorphism  $\mathbb{R}^{[3]} \twoheadrightarrow \mathbb{R}^{[1]}$  defined in Example 3.10. Is  $A = \mathbb{R}[t]^{[3]}$ ?

This is again an ‘‘Asanuma’s Dilemma’’: the ring  $A$  in Question 3.12 is a counterexample to some important problem, but we do not know which one! If  $A \neq \mathbb{R}[t]^{[3]}$  then, by (iii), it would be a counter-example to the  $\mathbb{A}^3$ -fibration Problem over the PID  $\mathbb{R}[t]$  (and also, by (iv), a counterexample to the Cancellation Problem over  $\mathbb{R}[t]$ ). If  $A = \mathbb{R}[t]^{[3]}$  (in fact, if  $A = \mathbb{R}^{[4]}$ ), then it would give rise to a counterexample to the Linearisation Problem for  $\mathbb{R}^{[4]}$ .

Some of the affirmative answers to Question 3.1 over fields could be partially generalised to corresponding epimorphism results over Noetherian domains with suitable hypotheses, making use of the theory of Affine Fibrations, especially residual coordinates, along with traditional commutative algebra. Bhatwadekar is a pioneer in this regard. His paper [16] was one of the earliest to make effective use of Asanuma’s structure theorem (Theorem 2.3) to generalised epimorphism theorems over rings ([16, Theorems 3.7, 3.9]):

**Theorem 3.13** *Let  $R$  be an integral domain of characteristic zero such that either  $\mathbb{Q} \subseteq R$  or  $R$  is Noetherian seminormal. Suppose that  $F \in R[X, Y]$  is such that  $R[X, Y]/(F) = R^{[1]}$ . Then  $R[X, Y] = R[F]^{[1]}$ .*

It has been shown in [11] by Asanuma and the first author that if  $k$  is an infinite field of positive characteristic  $p > 2$  and  $R$  the non-seminormal Noetherian local ring  $R = k[[t^2, t^3]]$ , then one can construct a polynomial  $F$  in  $R[X, Y]$  such that  $R[X, Y]/(F) = R^{[1]}$  and  $F$  is a residual coordinate in  $R[X, Y]$ , and yet  $F$  is not a coordinate in  $R[X, Y]$ . A modification of the above example (for instance, taking  $R = \mathbb{Z}_{(3\mathbb{Z})}[3\sqrt{3}]$ ) shows that Bhatwadekar's theorem (Theorem 3.13) cannot be extended to an arbitrary Noetherian domain of characteristic zero.

Now, consider Question 3.1 for the case  $n = 3$ , i.e., for the ring  $k[X, Y, Z]$ . It had been shown by A. Sathaye (in [82] for  $\text{ch } k = 0$ ) and P. Russell (in [78] for any field  $k$ ) that if  $F = aZ - b$  (where  $a, b \in k[X, Y]$ ) satisfies  $k[X, Y, Z]/(F) = k^{[2]}$ , then  $k[X, Y, Z] = k[F]^{[2]}$ . Under certain assumptions, D. Wright proved such a result in [91], when  $F = aZ^n - b$  for  $n \geq 2$ . Bhatwadekar and the first author generalised the Sathaye-Russell theorem as follows ([18, Proposition 3.4, Theorem 3.5]):

**Theorem 3.14** *Let  $(R, \pi)$  be a discrete valuation ring with field of fractions  $K$  and residue field  $k$ . Let  $F = aZ - b \in R[X, Y, Z]$ , where  $a (\neq 0), b \in R[X, Y]$  and either  $\pi \nmid a$  or  $a = \pi^n$ . Suppose that  $R[X, Y, Z]/(F) = R^{[2]}$ . Then  $R[X, Y, Z] = R[F]^{[2]}$ .*

In [34, Theorem 6.2], P. Das and the first author obtained generalisation of Wright's theorem for  $F = aZ^n - b \in R[X, Y, Z]$  ( $a, b \in R[X, Y]$ ), when  $R$  is a Noetherian domain containing  $\mathbb{Q}$  and several other cases.

Again, using the theory of affine fibrations, some of the equivalences in Theorem 3.7 can be readily generalised to the case of a Noetherian domain  $R$  such that either  $R$  contains  $\mathbb{Q}$  or  $R$  is seminormal ([42, Theorem 4.9]).

Apart from the intrinsic curiosity value, investigations on generalised epimorphism problems have generated significant inputs and potential counterexamples to the central problems on Affine Spaces. The epimorphism problem over a discrete valuation ring (DVR), only partially solved in Theorem 3.14 above, is closely related to the problem of  $\mathbb{A}^2$ -fibrations over two dimensional affine spots containing  $\mathbb{Q}$  ([19, Section 4]). The following example of Bhatwadekar and the first author, which emerged during the investigations leading to Theorem 3.14, and which has remained open for more than three decades, will perhaps indicate the close connections.

**Example 3.15** Let  $R = \mathbb{C}[[t]]$  or  $\mathbb{C}[t]_{(t)}$  with  $K = R[1/t]$  (or  $R = \mathbb{C}[t]$  and  $K = \mathbb{C}(t)$ ), where  $t$  is an indeterminate over  $\mathbb{C}$ . Let

$$F = tX^2Z + X + tX(Y + Y^2) + t^2Y \in R[X, Y, Z].$$

Let  $\bar{F}$  denote the image of  $F$  in  $(R/tR)[X, Y, Z] = \mathbb{C}[X, Y, Z]$ . Then

- (i)  $K[X, Y, Z] = K[X, F]^{[1]}$ .
- (ii)  $\mathbb{C}[X, Y, Z] = \mathbb{C}[\bar{F}]^{[2]}$ .
- (iii)  $R[X, Y, Z, 1/F] = R[F, 1/F]^{[2]}$ .
- (iv)  $R[X, Y, Z]/(F) = R^{[2]}$ .
- (v)  $R[X, Y, Z]$  is an  $\mathbb{A}^2$ -fibration over  $R[F]$ .
- (vi)  $R[X, Y, Z]^{[1]} = R[F]^{[3]}$ .

A version of the above example first appeared in [19, Example 4.13] and then in [36, Example 2.8]. It has been discussed in [50], [29], [42] and [63]. The proofs of the above statements appear in these articles. The elusive question about this example:

**Question 3.16** In Example 3.15:

- (i) Is  $R[X, Y, Z] = R[F]^{[2]}$ ?
- (ii) For the case  $R = \mathbb{C}[t]$ , is  $\mathbb{C}[t, X, Y, Z] = \mathbb{C}[F]^{[3]}$ ?

A negative solution to Question 3.16 will not only show that Theorem 3.14 cannot be extended over all discrete valuation rings, but more significantly, lead to counterexamples to the following problems:

- (1)  $\mathbb{A}^2$ -fibration problem over the two-dimensional regular ring  $\mathbb{C}^{[2]}$ .
- (2) Cancellation problem over  $\mathbb{C}^{[1]}$  (taking  $R = \mathbb{C}[t]$ ).
- (3) Abhyankar-Sathaye Conjecture (cf. Q. 3.1) for  $n = 4$  (taking  $R = \mathbb{C}[t]$ ).

### 4 On Generic and Codimension-one $\mathbb{A}^1$ -fibrations

As mentioned earlier, a striking feature of many of the results on  $\mathbb{A}^1$ -fibrations discovered from the 1990’s is that conditions on merely the generic and codimension one fibres very often determine an  $\mathbb{A}^1$ -fibration! In this section, we shall recall a few of them. We shall use the following terminology:

**Definition** An  $R$ -algebra  $A$  is said to be  $\mathbb{A}^1$  if  $A = R^{[1]}$  and *locally*  $\mathbb{A}^1$  if  $A_m (= A \otimes_R R_m) = R_m^{[1]}$  for every maximal ideal  $m$  of  $R$ ; its fibre ring at the prime ideal  $P$  will be called  $\mathbb{A}^1$  if  $A \otimes_R k(P) = k(P)^{[1]}$ .

We first record the special case  $n = 1$  of Theorem 2.1 which was first proved by P. Eakin and W. Heinzer ([47, Lemma 3.4]):

**Theorem 4.1** *Let  $R$  be a Noetherian domain and  $A$  a finitely generated algebra which is locally  $\mathbb{A}^1$ . Then  $A$  is  $R$ -isomorphic to the Rees algebra  $R[IT]$  for some invertible ideal  $I$  of  $R$ .*

The necessity of the finite generation hypothesis can be seen from the following well-known example:

**Example 4.2** Let  $R = \mathbb{Z}$  and  $A = \mathbb{Z}[\{\frac{X}{p} \mid p \text{ prime}\}] = \mathbb{Z}[\frac{X}{2}, \frac{X}{3}, \dots, \frac{X}{p}, \dots]$ , a non-finitely generated  $\mathbb{Z}$ -algebra. However, for each maximal ideal  $P = p\mathbb{Z} \in \text{Spec } \mathbb{Z}$ ,  $A_P \cong \mathbb{Z}_P[X]$ .

By Asanuma’s structure theorem (Theorem 2.3), an  $\mathbb{A}^1$ -fibration over a Noetherian domain  $R$  is necessarily isomorphic to an  $R$ -subalgebra of a polynomial algebra. This leads to a question:

**Question 4.3** Let  $R$  be a Noetherian domain and  $B = R^{[n]}$ . What fibre conditions ensure that an  $R$ -subalgebra  $A$  of  $B$  is an  $\mathbb{A}^1$ -fibration?

As discussed in detail in [39], ring-theoretic analogues of the “Lüroth Theorem” in Field Theory, quoted below, played a vital role in the discovery of such conditions. The first of the two results is well-known ([3, Theorem 2.6]):

**Theorem 4.4** *Let  $k$  be a field,  $B = k^{[n]}$  and  $A$  a one-dimensional  $k$ -subalgebra of  $B$ . If  $A$  is normal, then  $A = k^{[1]}$ .*

Theorem 4.4 has the following interesting and useful generalisation by Abhyankar-Eakin-Heinzer ([3, Theorem 4.1]) to any factorial domain (UFD):

**Theorem 4.5** *Let  $R$  be a UFD,  $B = R^{[n]}$  and  $A$  an  $R$ -subalgebra of  $B$ . If  $A$  is a UFD of transcendence degree one over  $R$ , then  $A = R^{[1]}$ .*

Now suppose  $R$  is a Noetherian UFD and  $B = R^{[n]}$  and  $A$  an  $R$ -subalgebra of  $B$ . Then, using Nagata’s criterion for UFD and the fact that over a Noetherian UFD, height one prime ideals are principal, the hypothesis that “ $A$  is a UFD of transcendence degree one over  $R$ ” becomes equivalent to the conditions

- (i)  $S^{-1}A$  is a UFD of dimension one, where  $S = R \setminus \{0\}$ .
- (ii)  $PA$  is a prime ideal in  $A$  for every height one prime ideal  $P$  in  $R$ .

Thus, using Theorem 4.4, Theorem 4.5 leads to the following result on fibre conditions, addressing Question 4.3 for UFD's.

**Corollary 4.6** *Let  $R$  be a Noetherian UFD with field of fractions  $K$ ,  $\Delta = \{P \in \text{Spec } R \mid \text{ht}(P) = 1\}$ ,  $B = R^{[n]}$  and  $A$  an  $R$ -subalgebra of  $B$  such that*

- (i)  $A \otimes_R K = K^{[1]}$ .
- (ii)  $A/PA$  is an integral domain for every  $P \in \Delta$ .

Then  $A = R^{[1]}$ .

A striking feature of this reformulation is the sufficiency of conditions on prime ideals in  $R$  of height  $\leq 1$ . Theorem 4.5 led to a series of results addressing Question 4.3 for flat  $R$ -algebras  $A$  over more general rings  $R$  demonstrating the sufficiency of generic and codimension one fibre conditions for  $A$  to be an  $\mathbb{A}^1$ -fibration. Note that when  $A$  is  $R$ -flat, the condition (ii) above becomes equivalent to the condition that the fibre ring “ $A \otimes_R k(P)$  is an integral domain for every prime ideal  $P$  in  $\Delta$ ”.

We first quote below some of the initial results obtained by Bhatwadekar and the first author, as a generalisation of Theorem 4.5, addressing Question 4.3 ([20, Theorems 3.10, 3.12, 3.16]):

**Theorem 4.7** *Let  $R$  be a Noetherian domain with field of fractions  $K$ ,  $B = R^{[n]}$ ,  $\Delta = \{P \in \text{Spec } R \mid \text{ht}(P) = 1\}$ , and  $A$  an  $R$ -subalgebra of  $B$  such that  $A$  is flat over  $R$  and the following fibre conditions hold:*

- (i)  $A \otimes_R K = K^{[1]}$ .
- (ii)  $A \otimes_R k(P)$  is an integral domain for every  $P \in \Delta$ .

Then we have the following results:

- (I) If  $R$  is normal, then  $A \cong R[IT]$  for an invertible ideal  $I$  of  $R$ .
- (II) If  $R$  contains  $\mathbb{Q}$ , then  $A$  is an  $\mathbb{A}^1$ -fibration over  $R$ .
- (III) If  $R$  is seminormal and contains  $\mathbb{Q}$ , then  $A \cong R[IT]$  for an invertible ideal  $I$  of  $R$ .

Note that, in the above result, since  $A$  was assumed to be  $R$ -flat and an  $R$ -subalgebra of  $R^{[n]}$ , it was automatically faithfully flat over  $R$ . The result inspired explorations of analogous results for faithfully flat  $R$ -algebras which are not, a priori, known to be subalgebras of polynomial algebras, and which would then drastically generalise the earlier results of Kambayashi, Miyanishi and Wright that  $\mathbb{A}^1$ -fibrations over Noetherian normal local domains are polynomial algebras. The following results were then obtained by the first author ([37, Theorems 3.4, 3.5]):

**Theorem 4.8** *Let  $R$  be a Noetherian domain with field of fractions  $K$ ,  $\Delta = \{P \in \text{Spec } R \mid \text{ht}(P) = 1\}$  and  $A$  a faithfully flat finitely generated  $R$ -algebra such that*

- (i)  $A \otimes_R K = K^{[1]}$ .
- (ii)  $A \otimes_R k(P)$  is geometrically integral for every prime ideal  $P \in \Delta$ .

Then we have:

- (I) If  $R$  is normal, then  $A \cong R[IT]$  for an invertible ideal  $I$  of  $R$ .

- (II) If  $R$  contains  $\mathbb{Q}$ , then  $A$  is an  $\mathbb{A}^1$ -fibration over  $R$ .
- (III) If  $R$  is seminormal and contains  $\mathbb{Q}$ , then  $A \cong R[IT]$  for an invertible ideal  $I$  of  $R$ .

Another variant of Theorem 4.7 was obtained by Prosenjit Das and the first author in [33, Theorems 3.9, 3.13]. Here, the condition that “ $A$  is a subalgebra of a polynomial algebra” is replaced by “ $A$  is Noetherian with a retraction to  $R$ ”.

We now compare Theorem 4.7 (I) with Theorem 4.8 (I). In the former, the hypothesis that  $A$  is finitely generated was not needed — it turned out to be a consequence. However, the following example of Bhatwadekar, quoted in [37, Example 4.1], would appear to indicate the necessity of the hypothesis “ $A$  is finitely generated” in the latter; and that it may not suffice to assume the weaker hypothesis “ $A$  is a subalgebra of a finitely generated algebra”:

**Example 4.9** Let  $k$  be a field and  $R = k[x](= k^{[1]})$ . Fix  $y \in k[[x]]$  such that  $y$  is transcendental over  $k(x)$ . Let  $A = k[[x]] \cap k[x, x^{-1}, y]$ , an  $R$ -subalgebra of the finitely generated  $R$ -algebra  $R[x^{-1}, y]$ . Then  $A$  is a Noetherian factorial domain such that  $A$  is faithfully flat over  $R$ , the generic fibre  $A \otimes_R K$  is  $K^{[1]}$ , and all codimension one fibres (i.e., closed fibres) are geometrically integral. But  $A$  is not an  $\mathbb{A}^1$ -fibration over  $R$ ;  $A$  is not even finitely generated.

However, in the above example, the codimension one fibres are not all of the correct (i.e., one) dimension. Here,  $A/(x - \lambda)A = k^{[1]}$  for every non-zero  $\lambda$  in  $k$  but  $A/xA = k$ . The following result of the first author with N. Onoda shows that this degeneracy is the only obstruction for Theorem 4.8 (I) to be extended to subalgebras of finitely generated algebras ([46, Theorem 3.5]):

**Theorem 4.10** Let  $R$  be a Noetherian normal domain with field of fractions  $K$  and  $\Delta = \{P \in \text{Spec } R \mid \text{ht}(P) = 1\}$ . Let  $A$  be a faithfully flat  $R$ -algebra such that  $A$  is an  $R$ -subalgebra of a finitely generated  $R$ -algebra  $B$ . Suppose that  $A$  satisfies the fibre conditions:

- (i)  $A \otimes_R K = K^{[1]}$ .
- (ii) For every  $P \in \Delta$ ,  $A \otimes_R k(P)$  is an integral domain with  $\text{tr. deg}_{k(P)}(A \otimes_R k(P)) > 0$  and  $k(P)$  is algebraically closed in  $A \otimes_R k(P)$ .

Then  $A \cong R[IX]$  for an invertible ideal  $I$  of  $R$ .

Theorems 4.7, 4.8, 4.10 and related results led to the isolation of the following concept:

**Definition** An  $R$ -algebra  $A$  for which  $A_P = R_P^{[1]}$  for every prime ideal  $P \in \Delta = \{P \in \text{Spec } R \mid \text{ht}(P) = 1\}$  will be called *locally  $\mathbb{A}^1$  in codimension-one*.

For convenience, we recall the following elementary result (cf. [33, Corollary 2.6]):

**Lemma 4.11** Let  $(R, \pi)$  be a discrete valuation ring with field of fractions  $K = R[1/\pi]$  and residue field  $k = R/\pi R$ , and  $A$  an integral domain containing  $R$  such that  $A \otimes_R K = K^{[1]}$  and  $A \otimes_R k = k^{[1]}$ . Then  $A = R^{[1]}$ .

Over a Noetherian normal domain  $R$ , Theorem 4.10 gives a common generalisation of Theorems 4.7 (I) and 4.8 (I) ([36, Remark 3.8]). In particular, it shows that over a Noetherian normal domain  $R$ , any faithfully flat  $R$ -algebra  $A$ , which is an  $R$ -subalgebra of a finitely generated  $R$ -algebra, and whose generic and codimension fibres are  $\mathbb{A}^1$ , is isomorphic to the symmetric algebra of an invertible ideal of  $R$ . By Lemma 4.11, the condition of generic and codimension-one fibres being  $\mathbb{A}^1$  becomes equivalent to the condition of being locally  $\mathbb{A}^1$  in codimension-one. For the rest of this section, we shall focus on this concept.

Theorem 4.10 yields:

**Corollary 4.12** *Let  $R$  be a Noetherian normal domain and  $A$  a faithfully flat  $R$ -algebra which is an  $R$ -subalgebra of a finitely generated  $R$ -algebra. If  $A$  is locally  $\mathbb{A}^1$  in codimension-one, then  $A$  is isomorphic to the Rees algebra of an invertible ideal of  $R$ .*

The necessity of the hypothesis “ $A$  is an  $R$ -subalgebra of a finitely generated  $R$ -algebra” in the above result can be seen from the well-known example of the  $\mathbb{Z}$ -algebra  $A = \mathbb{Z}[\{\frac{X}{p} \mid p \text{ a prime in } \mathbb{Z}\}]$ . But can this hypothesis be dropped in the case of a nice local domain  $R$ , say, when  $R$  is a regular local ring?

Over a discrete valuation ring  $(R, \pi)$ , any faithfully flat  $R$ -algebra  $A$  satisfying  $A[1/\pi] = R[1/\pi]^{[1]}$  is automatically a subalgebra of the finitely generated  $R$ -algebra  $A[1/\pi]$ . But [46, Example 1.4], quoted below, shows that even over a regular local ring  $R$  of dimension two, there could exist a faithfully flat algebra  $A$  which is locally  $\mathbb{A}^1$  in codimension-one, but which is not finitely generated over  $R$ .

**Example 4.13** Let  $k$  be an infinite field and  $R = k[[t_1, t_2]]$ , where  $t_1, t_2$  are algebraically independent over  $k$ . Let  $A = R[\{\frac{X}{q} \mid q \text{ a square-free non-unit in } R\}]$ . For every height one prime ideal  $P$  of the factorial domain  $R$ ,  $P = pR$  for some prime element  $p$ ; thus  $A_P = R_P[\frac{X}{p}] = R_P^{[1]}$ . However,  $A/(t_1, t_2)A = k$ . Note that  $A$ , being a direct limit of the polynomial rings  $R[\frac{X}{q}]$ , is flat over  $R$  and hence faithfully flat over  $R$  (since  $(t_1, t_2)A \neq A$ ).  $A$  is not finitely generated.

It was shown by Onoda and the first author that Example 4.13 is really a prototype of the general structure of a faithfully flat algebra  $A$  over a UFD  $R$  which is locally  $\mathbb{A}^1$  in codimension-one — any such algebra occurs as a direct limit of polynomial algebras ([46, Theorem 4.6]):

**Theorem 4.14** *Let  $R$  be a UFD and  $A$  a faithfully flat  $R$ -algebra which is locally  $\mathbb{A}^1$  in codimension-one. Then  $A$  is a direct limit of polynomial algebras in one indeterminate over  $R$ .*

The precise structure theorem [46, Theorem 4.6] has several consequences. We quote one below ([46, Theorem 4.12]).

**Theorem 4.15** *Let  $(R, m)$  be a local UFD and  $A$  a faithfully flat  $R$ -algebra which is locally  $\mathbb{A}^1$  in codimension-one. If  $\text{tr. deg}_{R/m} A/mA > 0$  (or if  $\dim A/mA > 0$ ), then  $A = R^{[1]}$ .*

Thus, in Example 4.13, the closed fibre  $A/(t_1, t_2)A (= k)$  not having the correct dimension turns out to be the only obstacle for  $A$  to be a polynomial algebra.

An example of Eakin-Silver ([48, (3.15)]) shows that Theorem 4.15 cannot be extended, by any local-global principle, even to any regular UFD if it has infinitely many maximal ideals. In the example,  $R = \mathbb{Z}$  and  $A$  a non finitely generated Noetherian UFD satisfying  $A_P = R_P^{[1]}$  for every prime ideal  $P$  of  $R$ .

Now, let  $R$  be a Noetherian normal local domain and  $A$  a faithfully flat  $R$ -algebra which is locally  $\mathbb{A}^1$  in codimension-one. In [24], an extensive study of the structure and properties of such an algebra was undertaken by Bhatwadekar, Onoda and the first author. It was shown that  $A$  can be expressed as a direct limit of a family of symbolic Rees algebras of divisorial ideals of  $R$  ([24, Theorem 2.3]). Thus, unlike Theorem 4.14, the general structure turned out to be quite technical. Intricate examples in Section 6 of [24] (especially Example 6.2) show that the technicality cannot be avoided — even when  $A$  is Noetherian normal,  $A$  need not be the direct limit of polynomial algebras over  $R$ . However, the structure theorem [24, Theorem 2.3] has several elegant consequences. We quote two. The first result appears in [24, Theorem 3.7].

**Theorem 4.16** *Let  $R$  be a Noetherian complete normal local domain and  $A$  a Noetherian faithfully flat  $R$ -algebra which is locally  $\mathbb{A}^1$  in codimension-one. Then  $A = R^{[1]}$ .*

The result cannot be extended over non-complete domains, not even when  $R$  is a UFD ([24, Example 6.4]). The next result was originally proved in [24, Theorem 3.10] under the additional hypothesis that  $R$  is analytically irreducible; Bhatwadekar and the second author proved the following version in [25, Corollary 3.11] without the additional hypothesis.

**Theorem 4.17** *Let  $(R, m)$  be a Noetherian normal local domain and  $A$  a faithfully flat  $R$ -algebra which is locally  $\mathbb{A}^1$  in codimension-one. Then either  $A = R^{[1]}$  or  $A/mA = R/mR$ .*

Some of the results in this section, stated over a Noetherian normal domain, hold even when  $R$  is a Krull domain. In some of the results, the hypothesis “ $A$  is faithfully flat” is used only to ensure two consequences of faithful flatness:

- (i)  $A = \bigcap_{P \in \Delta} A_P$ , where  $\Delta = \{P \in \text{Spec } R \mid \text{ht}(P) = 1\}$  (cf. [46, Lemma 2.8]).
- (ii)  $(a, b)A \cap R = (a, b)R$  for any two elements  $a$  and  $b$  of  $R$ .

The term “semi-faithfully flat” was coined to describe an  $R$ -algebra  $A$  satisfying (i) and (ii). Discussions on semi-faithfully flat algebras have been presented in Section 7 of [24], Section 3 of [25] and Section 3 of [62].

We now describe the concept of “ $\mathbb{A}^1$ -patch”, results on which are the cornerstones of the findings narrated in this section. Note that for  $a \in R$ ,  $R_a$  will denote the ring of fractions  $R[1/a]$ .

**Definition.** Let  $R \subset A$  be integral domains and  $x, y$  be elements of  $R$  such that  $R = R_x \cap R_y$ . An  $R$ -algebra  $A$  will be called an  $\mathbb{A}^1$ -patch (over  $R$ ) by  $x$  and  $y$  if it satisfies the following conditions:

- (i)  $A_x = R_x^{[1]}$ .
- (ii)  $A_y = R_y^{[1]}$ .
- (iii)  $A = A_x \cap A_y$ .

Note that the condition  $R = R_x \cap R_y$  is satisfied if and only if either  $x, y$  form a regular sequence or  $x, y$  are comaximal.

Results on the structure of an  $\mathbb{A}^1$ -patch arose out of, and have played a crucial role in, the study of affine fibrations and locally nilpotent derivations made in [20], [37], [21], [46], [33], [27], among others.

It was observed in [44, Lemma 4.3] that flat  $R$ -algebras which arise as  $\mathbb{A}^1$ -patch are finitely generated:

**Lemma 4.18** *Let  $R \subset A$  be integral domains and  $x, y$  be elements of  $R$  such that  $R = R_x \cap R_y$ ,  $A_x = R_x^{[1]}$  and  $A_y = R_y^{[1]}$ . If  $R$  is Noetherian and  $A$  is flat over  $R$ , then  $A$  is finitely generated over  $R$ .*

One of the earliest results on flat  $R$ -algebras which arise as  $\mathbb{A}^1$ -patch is the following lemma ([20, Lemma 3.9]):

**Lemma 4.19** *Let  $R$  be a Noetherian local domain,  $B = R^{[n]}$  and  $A$  a flat  $R$ -algebra such that  $R \subset A \subset B$ . Suppose that there exist  $x, y$  in  $R$  such that  $R = R_x \cap R_y$  and  $A$  is an  $\mathbb{A}^1$ -patch over  $R$  by  $x$  and  $y$ . Then  $A = R^{[1]}$ .*

This lemma was a crucial ingredient in the proof of Theorem 4.7. The next such result, Theorem 4.8, was a consequence of the following result on  $\mathbb{A}^1$ -patch:

**Lemma 4.20** *Let  $R$  be a Noetherian local domain and  $A$  a faithfully flat  $R$ -algebra. Suppose that there exist  $x, y$  in  $R$  such that  $R = R_x \cap R_y$  and  $A$  is an  $\mathbb{A}^1$ -patch over  $R$  by  $x$  and  $y$ . Then  $A = R^{[1]}$ .*

This result was first stated in [37, Lemma 3.3] with “finite generation” of  $A$  as a part of the hypotheses; but this condition now turns out to be superfluous in view of Lemma 4.18.

Thus, the apparently surprising discovery in [37] that any finitely generated faithfully flat algebra over a Noetherian normal domain  $R$ , whose generic and codimension-one fibres are  $\mathbb{A}^1$ , must itself be  $\mathbb{A}^1$  over  $R$  (quoted in Theorem 4.8 above) turns out, in retrospect, to be a simple application of the fact that any faithfully flat  $\mathbb{A}^1$ -patch over a Noetherian normal local domain is  $\mathbb{A}^1$  (Lemma 4.20).

It is thus clear that the study of  $\mathbb{A}^1$ -patch deserves to be studied as a concept in its own right. Along with Onoda, the authors had initiated such a study in [44] when  $R$  is a two-dimensional UFD containing a regular sequence  $x, y$  which generates a maximal ideal of  $R$  (for instance, when  $R = k[x, y]$  over a field  $k$ ). The structure of any  $\mathbb{A}^1$ -patch over  $R$  was described in the paper. We hope that follow-up investigations on this concept would be undertaken by researchers in the area. The first author had written an expository article on the concept of  $\mathbb{A}^1$ -patch through examples, known results and applications ([41]).

We end this section by quoting a patching result [27, Lemma 3.4] which was discovered after the publication of the survey article [41]. This result was crucial to the investigations by Sagnik Chakraborty, Nikhilesh Dasgupta and the authors on retracts of polynomial rings and their relationship with kernels of locally nilpotent derivations and affine fibrations.

**Lemma 4.21** *Let  $R$  be a Noetherian domain and  $B$  be a faithfully flat  $R$ -algebra such that  $R$  is factorially closed in  $B$ . Let  $A$  be an  $R$ -subalgebra of  $B$  such that  $A$  is an  $R$ -algebra retract of  $B$ . Suppose there exist non-zero elements  $x, y \in R$  such that*

- (i)  $y$  is an  $(R/xR)$ -regular element.
- (ii)  $B_x = A_x^{[1]}$ .
- (iii)  $B_y = A_y^{[1]}$ .

*Then  $B \cong \text{Sym}_A(IA)$  for some invertible ideal  $I$  of  $R$ . In particular,  $B$  is faithfully flat over  $A$ .*

Finally, in [10, Theorem 6.3], Asanuma, Bhatwadekar and Onoda described the following structure theorem for the reduced closed fibres of any finitely generated integrally closed generic fibrations over a DVR:

**Theorem 4.22** *Let  $(R, \pi)$  be a DVR with field of fractions  $K$  and residue field  $k$ . Let  $A$  be a finitely generated integrally closed  $R$ -domain such that  $A[1/\pi] = K[X] = K^{[1]}$ . Then there exist a finite number of finite algebraic field extensions  $k_1, \dots, k_n$  of  $k$  such that  $A/\sqrt{\pi}A = (k_1 \times \dots \times k_n)^{[1]}$ .*

## 5 Affine Forms

As before,  $k$  will always denote a field and  $A$  a  $k$ -algebra.

Recall that  $A$  is an  $\mathbb{A}^n$ -form over  $k$  if there exists a field extension  $L$  of  $k$  such that  $A \otimes_k L = L^{[n]}$ .  $A$  is said to be a separable  $\mathbb{A}^n$ -form if  $L$  can be taken to be a separable extension over  $k$ .

Note that any  $\mathbb{A}^n$ -form is a finitely generated  $k$  algebra. Further, if  $A$  is an  $\mathbb{A}^n$ -form, then we may consider  $L$  to be a finite extension of  $k$  and if moreover,  $A$  is a separable  $\mathbb{A}^n$ -form, then  $L$  can be taken to be a finite Galois extension of  $k$ .

Clearly if  $A = k^{[n]}$ , then  $A$  is an  $\mathbb{A}^n$ -form over  $k$ . It is well known that separable  $\mathbb{A}^1$ -forms over  $k$  are necessarily polynomial rings. For a simple proof of this, one can see [38, Lemma 5]. However, there are plenty of examples of non-trivial  $\mathbb{A}^1$ -forms (i.e.,  $\mathbb{A}^1$ -forms which are not polynomial rings) over non-perfect fields (see [73] and [9]). We present one below:

**Example 5.1** Let  $k$  be a non-perfect field of characteristic  $p$ . Let  $\lambda \in k \setminus k^p$  and

$$A = \frac{k[X, Y]}{(X^p - \lambda Y^p + Y)}.$$

Let  $\beta \in \bar{k}$  be such that  $\beta^p = \lambda$  and  $L := k(\beta)$ . Then

$$k[X, Y] \otimes_k L = L[X, Y] = L[X - \beta Y, Y] = L[X - \beta Y, Y + (X - \beta Y)^p].$$

Hence,  $A \otimes_k L = L^{[1]}$ . However,  $A \neq k^{[1]}$ . To see this, let  $x, y$  denote the images of  $X$  and  $Y$  in  $A$ . Then the ring  $V = k[1/x]_{(1/x)}[y/x]$  is a valuation ring of the field of fractions of  $A$  and  $V$  does not contain  $A$ , but the residue field of  $V$  is isomorphic to  $L (\neq k)$ .

Now, let  $A$  be a non-trivial  $\mathbb{A}^1$ -form over a field  $k$ . Then the  $k$ -algebra  $B = A^{[n]}$  is an  $\mathbb{A}^{n+1}$ -form over  $k$  but if  $B = k^{[n+1]}$ , then it would imply that  $A = k^{[1]}$  by Theorem 2.6, which is not the case. Thus, for every  $n \geq 1$ , there are examples of non-trivial  $\mathbb{A}^n$ -forms, whenever  $k$  is a non-perfect field.

In [9, Theorem 8.1], Asanuma has given a complete description of the structure of non-trivial  $\mathbb{A}^1$ -forms over non-perfect fields when  $\text{ch } k = p > 2$ .

There are thus two directions for investigations. The following question has been intriguing mathematicians for a long time:

**Question 5.2** Let  $A$  be a separable  $\mathbb{A}^n$ -form over  $k$ . Does it necessarily follow that  $A = k^{[n]}$ ?

Inspired by Asanuma’s findings in [9], the following question is also being explored in relatively recent times.

**Question 5.3** Let  $k$  be a field of arbitrary characteristic and  $A$  an  $\mathbb{A}^n$ -form over  $k$ . What additional conditions will ensure  $A = k^{[n]}$ ?

For  $n = 1$ , the following response to Question 5.3 follows from a result of Asanuma [9, Theorem 4.8]:

**Theorem 5.4** Let  $A$  be an  $\mathbb{A}^1$ -form over a field  $k$  of arbitrary characteristic such that:

- (i)  $A$  is a UFD.
- (ii)  $A$  has a  $k$ -rational point (i.e., there exists a maximal ideal  $m$  of  $A$  such that  $A/m = k$ ).

Then  $A = k^{[1]}$ .

The above consequence was mentioned by Asanuma during a series of lectures by him on inseparable affine forms. In [31], P. Das has presented a simple short self-contained proof of a more general result (Theorem 5.11 below).

Very recently, the following response to Question 5.3 has been obtained by Debojyoti Saha for  $n = 2$  ([81]):

**Theorem 5.5** Let  $A$  be an  $\mathbb{A}^2$ -form over a field  $k$  of arbitrary characteristic such that  $A$  is non-rigid (i.e.,  $A$  admits a non-trivial  $\mathbb{G}_a$ -action). Then  $A = B^{[1]}$ , where  $B$  is an  $\mathbb{A}^1$ -form over  $k$ . Moreover if  $A$  is normal and is contained in  $k^{[n]}$  for some  $n$ , then  $A = k^{[2]}$ .

Examples given in [81] show the necessity of each of the hypotheses.

We now discuss separable  $\mathbb{A}^n$ -forms for  $n \geq 2$  over a field  $k$ . In this regard, T. Kambayashi has proved the following non-trivial result, solving Question 5.2 for  $n = 2$  ([71, Theorem 3]):

**Theorem 5.6** *Let  $k$  be a field and  $A$  be a separable  $\mathbb{A}^2$ -form over  $k$ . Then  $A = k^{[2]}$ .*

Kambayashi's proof depends heavily on the structure of the automorphism group of the  $k$ -algebra  $k^{[2]}$ . Unfortunately, the structure of the automorphism group of the  $k$ -algebra  $k^{[3]}$  is not known.

For  $n > 2$ , it is not known whether there exist non-trivial separable  $\mathbb{A}^n$ -forms. Some results on the triviality of separable  $\mathbb{A}^3$ -forms have been obtained under various additional hypotheses. We summarise below (Theorem 5.7) the partial affirmative answers to Question 5.3 for  $n = 3$ .

**Theorem 5.7** *Let  $k$  be a field of characteristic zero with algebraic closure  $\bar{k}$  and  $A$  an  $\mathbb{A}^3$ -form over  $k$ , i.e.,  $A \otimes_k \bar{k} = \bar{k}^{[3]}$ . Then  $A = k^{[3]}$ , if any one of the following conditions is satisfied:*

- (I)  *$A$  admits a fixed point free locally nilpotent derivation  $D$  (D. Daigle and S. Kaliman in [30, Corollary 3.3]).*
- (II)  *$A$  contains an element  $f$  which is a coordinate of  $A \otimes_k \bar{k}$  (Daigle and Kaliman in [30, Proposition 4.9]).*
- (III)  *$A$  admits an effective action of a reductive algebraic  $k$ -group of positive dimension (M. Koras and P. Russell in [75, Theorem C]).*
- (IV)  *$A$  admits a non-confluent action of a unipotent group of dimension two (R.V. Gurjar, K. Masuda and M. Miyanishi in [64]).*
- (V)  *$A$  admits a locally nilpotent derivation  $D$  such that rank of the derivation  $D \otimes_k 1_{\bar{k}}$  on  $A \otimes_k \bar{k}$  is at most two, i.e.,  $\text{rk}(D \otimes 1_{\bar{k}}) \leq 2$  (A.K. Dutta, N. Gupta and A. Lahiri in [43, Theorem 3.2]).*

We now consider separable  $\mathbb{A}^n$ -forms over rings. Let  $B$  be a ring containing a field  $k$  and  $A$  be a  $B$ -algebra. As mentioned earlier, we shall say that  $A$  is an  $\mathbb{A}^n$ -form over  $B$  with respect to  $k$ , if  $A \otimes_k L \cong (B \otimes_k L)^{[n]}$ , for some field extension  $L$  of  $k$ . If, in addition,  $L$  is a separable field extension of  $k$ , then we call  $A$  a separable  $\mathbb{A}^n$ -form over  $B$  with respect to  $k$ .

We have already seen that over non-perfect fields, there are examples of non-trivial  $\mathbb{A}^n$ -forms. But even over a Dedekind domain  $B$  containing a perfect field  $k$ , one can see that an  $\mathbb{A}^1$ -form over  $B$  with respect to  $k$  need not be exactly a polynomial ring. Consider the following example.

**Example 5.8** Let  $B = \mathbb{R}[X, Y]/(X^2 + Y^2 - 1)$  and  $A = \text{Sym}_B(I)$ , where  $I$  is a non-principal invertible ideal of  $B$ . Hence  $A \neq B^{[1]}$  ([47, Lemma 1.3]). Now we know that  $B \otimes_{\mathbb{R}} \mathbb{C}$  is a PID and hence,  $A \otimes_{\mathbb{R}} \mathbb{C} = \text{Sym}_{B \otimes_{\mathbb{R}} \mathbb{C}}(I) = (B \otimes_{\mathbb{R}} \mathbb{C})^{[1]}$ .

However, in [38, Theorem 7], the first author established a structure theorem for separable  $\mathbb{A}^1$ -forms over rings which shows that they are, in essence, of the above type (Example 5.8). More precisely, he showed:

**Theorem 5.9** *Let  $B$  be a ring containing a field  $k$  and  $A$  be a  $B$ -algebra. Suppose  $L$  is a separable field extension of  $k$  and  $A \otimes_k L$  is isomorphic to  $\text{Sym}_{(B \otimes_k L)}(J)$  for some invertible ideal  $J$  of  $B \otimes_k L$ . Then  $A \cong \text{Sym}_B(I)$  for some invertible ideal  $I$  of  $B$ .*

*In particular, if  $A \otimes_k L \cong (B \otimes_k L)^{[1]}$ , then  $A \cong \text{Sym}_B(I)$  for some invertible ideal  $I$  of  $B$ .*

The genesis of the studies on separable forms on rings lies in the investigations made in [20]. As a step to one of the main results in the paper, it was shown that when  $B$  is a Noetherian normal domain containing a perfect field  $k$  and  $A$  an  $\mathbb{A}^1$ -form over  $B$  with respect to  $k$ , then  $A$  is  $B$ -isomorphic to the symmetric algebra of an invertible ideal of  $B$  ([20, Lemma 3.7]). Subsequently, the general picture was investigated in [38].

In the spirit of Question 5.3, one may like to explore possible generalisations of Theorem 5.4 for UFDs; for instance:

**Question 5.10** Let  $R$  be a UFD containing a field  $k$  of arbitrary characteristic and  $A$  an UFD which is an  $\mathbb{A}^1$ -form over  $R$  with respect to  $k$ . What additional conditions will ensure  $A = R^{[1]}$ ?

Addressing the above, P. Das obtained the following criterion ([31, Corollary 4]):

**Theorem 5.11** *Let  $R$  be a  $k$ -algebra and  $A$  an  $R$ -algebra such that*

- (i)  $A$  is a UFD.
- (ii)  $R$  is a retract of  $A$ .
- (iii)  $A$  is an  $\mathbb{A}^1$ -form over  $R$  with respect to  $k$ .

*Then  $A = R^{[1]}$ .*

The above result has been shown to be a consequence of a result on “faithfully flat descent” ([31, Theorem 3]).

We now discuss  $\mathbb{A}^2$ -forms over rings. Using Theorems 5.6 and 2.11, the first author has proved the following result:

**Theorem 5.12** *Let  $k$  be a field of characteristic zero and let  $R$  be a PID containing  $k$ . Suppose that  $A$  is an  $\mathbb{A}^2$ -form over  $B$  with respect to  $k$ . Then  $A = B^{[2]}$ .*

However, if  $k$  is a non-perfect field, then using non-trivial  $\mathbb{A}^1$ -forms over  $k$ , one can construct examples of non-trivial  $\mathbb{A}^2$ -forms over a PID. Therefore, one is led to the following analogue of Question 5.10:

**Question 5.13** *Let  $R$  be a PID containing a field  $k$  of arbitrary characteristic and  $A$  an  $\mathbb{A}^2$ -form over  $R$  with respect to  $k$ . What additional conditions will ensure  $A = R^{[2]}$ ?*

Addressing the above question, Parnashree Ghosh and the second author proved the following two generalisations of Theorems 5.6 and 5.12 in [55, Theorems 2.8, 2.9]:

**Theorem 5.14** *Let  $k$  be a field and  $B$  be a PID which contains  $k$ . Suppose that  $A$  is an  $\mathbb{A}^2$ -fibration over  $B$  as well as an  $\mathbb{A}^2$ -form over  $B$  with respect to  $k$ . Then  $A = B^{[2]}$ .*

Theorem 5.14 plays a crucial role in detecting rectifiability of embeddings of certain linear hyperplanes in  $\mathbb{A}_k^4$  that has been investigated by the second author along with Parnashree Ghosh and Ananya Pal in [56].

**Theorem 5.15** *Let  $k$  be a field and  $R$  be a PID containing  $k$ . Suppose that  $A$  is a separable  $\mathbb{A}^2$ -form over  $R$  with respect to  $k$ . Then  $A = R^{[2]}$ .*

In another direction, Animesh Lahiri and the authors have proved the following four results in [43] (Lemma 3.6, Proposition 3.1, Theorems 3.7 and 3.8 respectively):

**Lemma 5.16** *Let  $k$  be a field of characteristic zero,  $R$  be a  $k$ -algebra and  $A$  be an  $R$ -algebra. Let  $A$  be an  $\mathbb{A}^2$ -form over  $R$  with respect to  $k$ . Then  $A$  is an  $\mathbb{A}^2$ -fibration over  $R$ .*

**Proposition 5.17** *Let  $k$  be a field of characteristic zero with algebraic closure  $\bar{k}$ ,  $R$  a PID containing  $k$  and  $A$  be an  $\mathbb{A}^3$ -form over  $R$  with respect to  $k$ . Suppose that there exists an  $R$ -linear locally nilpotent derivation  $D$  on  $A$  such that  $\text{rk}(D \otimes 1_{\bar{k}}) = 1$ . Then  $A = R^{[3]}$ .*

**Theorem 5.18** *Let  $k$  be a field of characteristic zero and  $B$  a one-dimensional Noetherian  $k$ -algebra. If  $A$  is an  $\mathbb{A}^2$ -form over  $B$  with respect to  $k$ , then there exists a finitely generated rank one projective  $B$ -module  $Q$  such that  $A \cong (\text{Sym}_B(Q))^{[1]}$ .*

**Theorem 5.19** *Let  $k$  be a field of characteristic zero,  $B$  a ring containing  $k$  and  $A$  be an  $\mathbb{A}^2$ -form over  $B$  with respect to  $k$ . Suppose  $A$  has a fixed point free locally nilpotent  $B$ -derivation. Then there exists a finitely generated rank one projective  $B$ -module  $Q$  such that  $A \cong (\text{Sym}_B(Q))^{[1]}$ .*

It may be noted that while the concept of separable forms over rings may appear, at first sight, to be technical and of limited interest, results on it have helped establish new results on the more traditional concept of separable forms over fields which have been of wide interest. For instance, the proof of the result on the triviality  $\mathbb{A}^3$ -forms in Theorem 5.7 under hypothesis (V) used Theorem 5.12 and Proposition 5.17.

## 6 Laurent Polynomial Fibrations

In this section we record analogues of results in Section 4 for Laurent polynomial fibrations. The investigations began in the mid-1990's when, after the discovery of results like Theorem 4.8 showing the necessity of generic and codimension-one fibres for  $\mathbb{A}^1$ -fibrations, M. Miyanishi had suggested to the first author to explore whether similar phenomena hold for  $\mathbb{A}^*$ -fibrations as well. The initial findings are recorded in [22] and [23]. After a deeper probe was made on locally  $\mathbb{A}^1$ -algebras in codimension-one in [46] and [24], a fresh series of investigations were launched on Laurent polynomial algebras in [25], [26], [62], [1] and [2]. We first clarify the terminology.

**Definition.** Let  $R$  be an integral domain and  $A$  an  $R$ -algebra.

$A$  is said to be a “Laurent polynomial algebra” in  $n$  indeterminates over  $R$  if  $A = R[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ , where  $X_1, X_2, \dots, X_n$  are transcendental over  $R$ .

$A$  is called a “locally Laurent polynomial algebra” in  $n$  indeterminates over  $R$  if  $A \otimes_R R_m$  is a Laurent polynomial algebra in  $n$  indeterminates over the local ring  $R_m$  for every maximal ideal  $m$  of  $R$ .

A Laurent polynomial algebra in one indeterminate over  $R$  is called  $\mathbb{A}^*$  over  $R$ .

As an analogue of the Eakin-Heinzer Theorem (Theorem 4.1), Bhatwadekar and the first author had established the following local-global theorem in [22, Theorem 3.4]:

**Theorem 6.1** *Let  $A$  be a finitely generated algebra over a Noetherian normal domain  $R$  such that for each maximal ideal  $m$  of  $R$ ,  $A_m \cong R_m[T, T^{-1}]$ . Then there exists an invertible ideal  $I$  in  $R$  such that  $A$  is isomorphic to the  $\mathbb{Z}$ -graded  $R$ -algebra  $\bigoplus_{n \in \mathbb{Z}} I^n T^n = R[IT, I^{-1}T^{-1}]$ .*

Subsequently, in [26, Theorem 2.3] quoted below (Theorem 6.2), S.M. Bhatwadekar and the second author described the structure of a locally Laurent polynomial algebra in  $n$  indeterminates over  $R$ , an analogue of the local-global theorem for locally polynomial algebras due to Bass-Connell-Wright and Suslin (Theorem 2.1).

Note that the polynomial algebra  $B = R[X_1, \dots, X_n]$  may be envisaged as

$$B = R[X_1, \dots, X_n] = \text{Sym}_R(Q),$$

where  $Q = RX_1 \oplus \dots \oplus RX_n$  is a direct sum of  $n$  finitely generated projective modules of rank one. The Laurent polynomial algebra may be envisaged as

$$A = R[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] = R[X_1, \dots, X_n] \left[ \frac{1}{X_1 \cdots X_n} \right].$$

Setting  $J = (X_1 \cdots X_n)B$ , an invertible ideal of  $B = \text{Sym}_R(Q)$ , we then have

$$A = B[J^{-1}] = \text{Sym}_R(Q)[J^{-1}].$$

Bhatwadekar and the second author show that locally Laurent polynomial algebras too have a similar structure:

**Theorem 6.2** *Let  $R$  be an integral domain and  $A$  be a locally Laurent polynomial algebra in  $n$  indeterminates over  $R$ . Then there exist  $n$  finitely generated rank one projective modules  $L_i$ ,  $1 \leq i \leq n$ , such that  $A$  is isomorphic to an  $R$ -algebra of the form*

$$(\text{Sym}_R(Q))[I^{-1}],$$

where  $Q = L_1 \oplus \cdots \oplus L_n$  and  $I$  is an invertible ideal of  $\text{Sym}_R(Q)$  generated by the image of  $L_1 \otimes \cdots \otimes L_n$ . In particular,  $A$  is finitely presented over  $R$ .

If  $\text{Pic}(R) = (0)$  (that is the case when  $R$  is a UFD), then  $A$  is a Laurent polynomial algebra over  $R$ .

This structure theorem shows that a locally Laurent polynomial algebra  $A$  over an integral domain  $R$  is necessarily finitely presented. This is in contrast to the locally polynomial algebra situation where finite presentation hypothesis is necessary even for the case  $n = 1$  (Example 4.2).

Moreover, the following result of Bhatwadekar and the second author is the Laurent polynomial analogue of Corollary 4.12, proved for any  $n \geq 1$  ([26, Proposition 2.7]):

**Theorem 6.3** *Let  $R$  be an integral domain which is either Noetherian or Krull domain and let  $A$  be a faithfully flat  $R$ -algebra such that  $A \otimes_R R_P$  is a Laurent polynomial algebra in  $n$  indeterminates over  $R_P$  for every prime ideal  $P$  of  $R$  for which  $\text{depth } R_P \leq 1$ . Then  $A$  is a locally Laurent polynomial algebra in  $n$  indeterminates over  $R$ .*

Thus, when  $R$  is normal, it is enough to assume the local condition on prime ideals of height at most one. Over a normal domain, this result was proved by Bhatwadekar and the first author in [25, Theorem 4.8] for the case  $n = 1$ .

Unlike an affine fibration, the structure of any Laurent polynomial fibration over a Noetherian normal domain is now completely known. Over a DVR, Bhatwadekar and the second author proved the following result ([26, Theorem 3.5]), reminiscent of Sathaye’s theorem (Theorem 2.10) on  $\mathbb{A}^2$ -fibrations:

**Theorem 6.4** *Let  $(R, t)$  be a discrete valuation ring with a regular parameter  $t$ , field of fractions  $K$  and residue field  $k$ . Let  $A$  be an integral domain containing  $R$  such that*

- (i)  $A[1/t]$  is a Laurent polynomial algebra in  $n$  indeterminates over  $K$ .
- (ii)  $A/tA$  is a Laurent polynomial algebra in  $n$  indeterminates over  $k$ .

Then  $A$  is a Laurent polynomial algebra in  $n$  indeterminates over  $R$ .

For  $n = 1$ , Bhatwadekar and the first author had described the structure of all finitely generated algebras over a discrete valuation ring which are integral domains with generic fibre  $\mathbb{A}^*$  and closed fibre geometrically integral ([22, Proposition 3.7]):

**Theorem 6.5** *Let  $(R, t)$  be a discrete valuation ring with a regular parameter  $t$ , field of fractions  $K$  and residue field  $k$ . Let  $A$  be a finitely generated integral domain containing  $R$  such that*

- (i)  $A[1/t] \cong K[X, X^{-1}]$  for some  $X$  transcendental over  $K$ .
- (ii)  $A/tA$  is geometrically integral over  $k$ .

Then there are precisely two possibilities:

- (a) If  $(A/tA)^* \neq k^*$ , then  $A \cong R[X, X^{-1}]$  for some  $X$  transcendental over  $R$ .

(b) If  $(A/tA)^* = k^*$ , then  $A \cong R[X, Y]/(t^m XY + \alpha X + \beta Y + \gamma)$  for some  $\alpha, \beta \in R^*$ ,  $\gamma \in R$  and positive integer  $m$ .

Note that the ring  $R[X, Y]/(t^m XY + \alpha X + \beta Y + \gamma)$  is equal to  $R[X, 1/(t^m X + \beta)]$ . The above result was a motivation for research on the concept of “quasi  $A^*$  algebra” defined later in this section.

Over a Noetherian normal domain  $R$ , Bhatwadekar and the second author proved ([26, Theorem 3.6]):

**Theorem 6.6** *Let  $R$  be a Noetherian normal domain with field of fractions  $K$  and  $A$  be a faithfully flat  $R$ -algebra such that*

- (i) *The generic fibre  $A \otimes_R K$  is a Laurent polynomial algebra in  $n$  indeterminates over  $K$ .*
- (ii) *For each height one prime ideal  $P$  in  $R$ ,  $A \otimes_R k(P)$  is a Laurent polynomial algebra in  $n$  indeterminates over  $k(P)$ .*

*Then  $A$  is a locally Laurent polynomial algebra in  $n$  indeterminates over  $R$ .*

For the case  $n = 1$ , this result was proved earlier by Bhatwadekar and the first author in [22, Theorem 3.11] under the additional hypothesis that  $A$  is finitely generated, an analogue of Theorem 4.8.

Bhatwadekar and the first author had constructed an example ([23, Example 3.9]) to show that even for  $n = 1$ , Theorem 6.6 cannot be extended to non-normal domains. Further, they described a structure theorem ([23, Theorem 3.4]) for an  $\mathbb{A}^*$ -fibration over a one-dimensional Noetherian seminormal semilocal domain  $R$ . As a consequence, if 2 is a unit in  $R$ , they gave the following result ([23, Corollary 3.5]):

**Theorem 6.7** *Let  $(R, m)$  be a one-dimensional Noetherian seminormal local domain with field of fractions  $K$  and residue field  $k$  such that  $1/2 \in R$ . Suppose that  $A$  is a finitely generated faithfully flat  $R$ -algebra such that  $A \otimes K \cong K[T, T^{-1}]$  and  $A \otimes k \cong k[T, T^{-1}]$ . Then there exist  $\lambda \in R^*$  and  $\mu \in R^* \cap (K^*)^2$  with the image of  $\mu$  in  $k$  is in  $(k^*)^2$  such that  $A \cong R[X, Y]/(X^2 - \mu Y^2 - \lambda)$ . Moreover,  $A \cong R[T, T^{-1}]$  if and only if  $\mu \in (R^*)^2$ .*

In [26, Theorem 4.4], Bhatwadekar and the second author gave the following necessary and sufficient condition for extending Theorem 6.6 to an arbitrary Noetherian domain:

**Theorem 6.8** *Let  $R$  be a Noetherian domain with field of fractions  $K$  and let  $A$  be a faithfully flat  $R$ -algebra such that*

- (i)  $A \otimes_R K = K[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ , where  $X_1, \dots, X_n$  are transcendental over  $R$ .
- (ii) *For each height one prime ideal  $P$  in  $R$ ,  $A \otimes_R k(P)$  is a Laurent polynomial algebra in  $n$  indeterminates over  $k(P)$ .*
- (iii)  $L_i := A \cap K X_i$  is a finitely generated projective  $R$ -module of rank one,  $1 \leq i \leq n$ .

*Then  $A$  is a locally Laurent polynomial algebra in  $n$  indeterminates over  $R$ .*

Now suppose that  $(V, \pi)$  is a DVR with field of fractions  $K$  and  $A$  a faithfully flat finitely generated  $V$ -algebra such that

- (i)  $\pi A \in \text{Spec } A$ .
- (ii)  $V/\pi V$  is algebraically closed in  $A/\pi A$ .

Under these hypotheses, it can be shown by using the Russell-Sathaye criterion [79, Theorem 2.3.1] that if  $A[1/\pi] = K[Y]$  for some  $Y \in A$  transcendental over  $V$ , then  $A = V[X]$  for some  $X \in A$  transcendental over  $V$ .

However, under the above hypotheses, if  $A[1/\pi] = K[Y, Y^{-1}]$  for some  $Y \in A$  transcendental over  $V$ , then, by Theorem 6.5,  $A = V[X, (\alpha X + \beta)^{-1}]$ , for some  $X \in A$  transcendental over  $V$ ,  $0 \neq \alpha, \beta \in V$  and  $(\alpha, \beta)V = V$ . This motivates the following generalisation of  $\mathbb{A}^*$ -algebras:

**Definition** An  $R$ -algebra  $A$  is said to be *quasi  $\mathbb{A}^*$*  if  $A = R[X, (aX + b)^{-1}]$  for some  $X$  transcendental over  $R$ ,  $a \in R \setminus 0$  and  $b \in R$  with  $(a, b)R = R$ .

Note that if  $a \in R^*$ , then  $A$  is  $\mathbb{A}^*$  over  $R$ . We have seen how this notion of quasi  $\mathbb{A}^*$  arises naturally in the study of algebras whose generic fibres are  $\mathbb{A}^*$ .

In [25], Bhatwadekar and the second author considered the notion of a quasi  $\mathbb{A}^*$ -fibration over a Noetherian domain  $R$ , i.e., an algebra whose fibre over a prime ideal  $P$  of  $R$  is of the form  $k(P)[X, (aX + b)^{-1}]$ , for some  $X$  transcendental over  $k(P)$ ,  $a \in k(P) \setminus 0$  and  $b \in k(P)$ .

Generalising the notion of “locally  $\mathbb{A}^1$ -algebras in codimension one”, we say:

**Definition** An  $R$ -algebra  $A$  is said to be *locally quasi  $\mathbb{A}^*$  in codimension-one* if, for every height one prime ideal  $P$  of  $R$ ,  $A_P (= A \otimes_R R_P)$  is quasi  $\mathbb{A}^*$  over  $R_P$ .

In [25, Theorems 4.6 and 5.2], Bhatwadekar and the second author showed that a faithfully flat locally quasi  $\mathbb{A}^*$  algebra  $A$  in codimension-one over a Noetherian normal domain has an  $R$ -subalgebra  $B$  which is faithfully flat and locally  $\mathbb{A}^1$  in codimension-one over  $R$  such that  $A = B[I^{-1}]$  for some invertible ideal  $I$  of  $B$ . As a consequence of this structure theorem, if  $R$  is factorial then the following analogue of Theorem 4.14 holds:  $A$  is a direct limit of quasi  $\mathbb{A}^*$  algebras over  $R$  and hence, if further  $A$  is finitely generated over  $R$ , then  $A$  is quasi  $\mathbb{A}^*$  over  $R$ . In [25, Theorems 5.9 and 5.10], they have also described some minimal sufficient conditions for such an algebra to be finitely generated. Later in [62], the second author considered fibrations which may miss more than one point, i.e., algebras whose generic fibre is of the form  $K[X, \frac{1}{(X-\lambda_1)\cdots(X-\lambda_n)}]$  or more generally  $K[X, \frac{1}{f(X)}]$ . More precisely, she proved ([62, Theorem 5.2]):

**Theorem 6.9** *Let  $R$  be a Noetherian normal domain with field of fractions  $K$  and  $A$  a faithfully flat  $R$ -algebra such that:*

- (i)  $A \otimes_R K = K[X, \frac{1}{f(X)}]$  for some  $f(X) \in K[X] \setminus K$  and  $X$  transcendental over  $K$ .
- (ii) For each prime ideal  $P$  in  $R$  of height one,  $A_P \cong R_P[X, \frac{1}{f_P(X)}]$  for some  $f_P(X) \in R_P[X]$ .

*Let  $B := A \cap K[X]$ . Then  $B$  is a faithfully flat  $R$ -algebra which is locally  $\mathbb{A}^1$  in codimension-one over  $R$  and there exists an invertible ideal  $I$  of  $B$  such that  $A = B[I^{-1}]$ . Moreover,  $A$  is Noetherian (respectively, finitely generated over  $R$ ) if and only if  $B$  is Noetherian (respectively, finitely generated over  $R$ ).*

Theorem 6.9 reduces the study of the  $R$ -algebra  $A$  to the study of an  $R$ -algebra  $B$  which is faithfully flat and locally  $\mathbb{A}^1$  in codimension-one over  $R$ . As already mentioned, such algebras have been studied extensively by Bhatwadekar, Dutta and Onoda in [24]. For instance, Theorem 4.8 of the first author already shows that  $B$  is finitely generated over  $R$  if and only if  $B \cong \text{Sym}_R(Q)$  for some invertible ideal  $Q$  of  $R$ . From Theorem 6.9 and an analogous result for the ring  $B$  (Theorem 4.17), the second author established the following result when  $R$  is local ([62, Theorem 5.3]):

**Theorem 6.10** *Let  $R$  be a Noetherian normal local domain with maximal ideal  $m$  and  $A$  be as in Theorem 6.9. Then the following are equivalent:*

- (I)  $A \cong R[X, \frac{1}{g(X)}]$  for some  $g(X) \in R[X] \setminus R$ .
- (II)  $A$  is a finitely generated  $R$ -algebra.

- (III)  $A[1/a]$  is a finitely generated  $R[1/a]$ -algebra for some  $a(\neq 0) \in R$ .  
 (IV) The image of  $f_P(X)$  (as in (ii) of Theorem A) in  $k(P)[X]$  is transcendental over  $k(P)$  for all but finitely many prime ideals  $P$  in  $R$  of height one.  
 (V)  $R/m \subsetneq A/m$ .

Moreover, if  $R$  is complete and  $A$  is Noetherian then the above equivalent conditions hold.

Theorem 6.6 on fibrations with punctured lines has been generalised by the second author as follows ([62, Theorem 5.4]):

**Theorem 6.11** Let  $R$  be a Noetherian normal domain and  $A$  be a faithfully flat  $R$ -algebra such that for each prime ideal  $P$  in  $R$  of  $\text{ht}(P) \leq 1$ ,  $A \otimes_R k(P) \cong k(P)[X, \frac{1}{f_P(X)}]$  for some  $f_P(X) \in k(P)[X]$  with  $\deg_X f_P(X) \geq 1$ . Then  $A \cong B[I^{-1}]$ , where  $B \cong \text{Sym}_R(Q)$  for some invertible ideal  $Q$  of  $R$  and  $I$  is an invertible ideal of  $B$ . In particular,  $A$  is finitely generated over  $R$  and, for every maximal ideal  $m$  of  $R$ ,  $A_m \cong R_m[X, \frac{1}{g_m(X)}]$  for some  $g_m(X) \in R_m[X]$ .

As a step towards the proof of the above theorem, the second author proved the following result, which may be of independent interest, relating faithful flatness, finite generation and Noetherian property of an  $R$ -algebra  $B$  with the corresponding property of a localisation  $B[f^{-1}]$  for some  $f \in B \setminus R$  ([62, Theorem 3.4]):

**Theorem 6.12** Let  $R$  be a Noetherian normal domain. Let  $\Delta$  denote the set of all height one prime ideals of  $R$ . Let  $B$  be an  $R$ -algebra such that

- (i)  $B_P \cong R_P[X] \forall P \in \Delta$ ,
- (ii)  $B = \bigcap_{P \in \Delta} B_P$ ,
- (iii)  $JB \cap R = J$  for every ideal  $J$  of  $R$  and
- (iv)  $B_f$  is faithfully flat over  $R$  for some  $f \in B \setminus R$ .

Then  $B$  is faithfully flat over  $R$ . Moreover, if  $B_f$  is Noetherian (respectively, finitely generated over  $R$ ) then  $B$  is Noetherian (respectively, finitely generated over  $R$ ).

In [1], A.M. Abhyankar and S.M. Bhatwadekar considered the Laurent polynomial analogue of quasi  $\mathbb{A}^*$ -algebras.

**Definition.** Let  $(R, \pi)$  be a DVR with field of fractions  $K$  and residue field  $k$ . An  $R$ -algebra  $A$  is called a *quasi Laurent polynomial algebra* in  $n$  indeterminates over  $R$ , if

$$A = R[X_1, \dots, X_n, 1/(a_1X_1 + b_1), \dots, 1/(a_nX_n + b_n)]$$

for some elements  $X_1, \dots, X_n$  in  $A$  algebraically independent over  $R$  and some  $a_1, \dots, a_n, b_1, \dots, b_n \in R$  with  $(a_i, b_i)R = R$  for each  $i, 1 \leq i \leq n$ .

Thus, the following properties hold for a quasi Laurent polynomial algebra in  $n$  indeterminates over the DVR  $R$ :

- (i)  $A \otimes_R K$  is a Laurent polynomial algebra in  $n$  indeterminates over  $R$ .
- (ii)  $A \otimes_R k$  is of the form  $k[X_1, \dots, X_s, Y_1, Y_1^{-1}, \dots, Y_r, Y_r^{-1}]$  where  $r + s = n$ .

**Definition** Over a DVR  $R$  with field of fractions  $K$  and residue field  $k$ , an algebra  $A$  which is faithfully flat, finitely generated algebra and which satisfies (i) and (ii) above is called a *mixed  $(s, n - s)$  LP-algebra*.

When  $s = 0$ , by Theorem 6.4, it follows that  $A$  is a Laurent polynomial algebra.

In [1], Asawari Abhyankar and Bhatwadekar investigated mixed  $(s, 2-s)$  LP-algebras for  $s \geq 1$  and constructed an example to show that in general such algebras need not be quasi Laurent polynomial algebras.

In [2], generalising the notion of locally quasi  $\mathbb{A}^*$ -algebras, Asawari Abhyankar and Bhatwadekar considered finitely generated faithfully flat algebras which are locally quasi Laurent polynomial algebra in  $n$  ( $n \geq 2$ ) indeterminates, but which are not locally Laurent polynomial algebras. They constructed examples ([2, Examples 4.1, 4.5 and 4.6]) to show that in general, even over a PID, such algebras are not necessarily quasi Laurent polynomial algebras. Further, in [2, Proposition 3.3], they proved that if  $R$  is a factorial domain and  $A$  is a faithfully flat  $R$ -algebra such that  $A_P$  is quasi Laurent polynomial in  $n$  indeterminates over  $R_P$  for every height one prime ideal  $P$  of  $R$  and  $A[1/\pi]$  is a Laurent polynomial algebra in  $n$  indeterminates over  $R[1/\pi]$  for some prime element  $\pi$  of  $R$ , then  $A$  is quasi Laurent polynomial in  $n$  indeterminates over  $R$ .

Analogous to the results on generic  $\mathbb{A}^1$ -fibration (4.22), Asanuma, Bhatwadekar and Onoda established the following result on generic  $\mathbb{A}^*$ -fibration ([10, Theorem 6.6]):

**Theorem 6.13** *Let  $(R, \pi)$  be a DVR with field of fractions  $K$  and residue field  $k$ . Let  $A$  be a finitely generated integrally closed  $R$ -domain such that  $A[1/\pi] = K[T, T^{-1}]$  for some  $T$  transcendental over  $K$ . Then  $A/\sqrt{\pi}A$  is  $k$ -isomorphic to exactly one of the following:*

- (i)  $T := (k_1 \times \cdots \times k_n)^{[1]}$ ,
- (ii)  $L = k[X, Y]/(XY - 1)$ ,
- (iii)  $T \times L$ ,
- (iv)  $U := k[X, Y]/(XY)$ ,
- (v)  $T \times U$ ,

where  $n$  is a positive integer and each  $k_i$  is a finite algebraic extension of  $k$ .

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