ORIGINAL RESEARCH

The semi-stable Local Langlands Correspondence

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Abstract
We start with background that goes into an Iwahori-theoretic reformulation of the mod p Local Langlands Correspondence (§2). We then explain some classical p-adic functional analytic results (§3) that go into defining the p-adic Banach space (§4) attached to a two-dimensional semi-stable representation $V_{k,\mathcal{L}}$ of the Galois group of $\mathbb{Q}_p$ of weight k and $\mathcal{L}$-invariant $\mathcal{L}$ under the p-adic Local Langlands correspondence. We then sketch how to compute the reduction of a lattice in this Banach space, which along with the Iwahori mod p LLC, allows one to completely determine the mod p reduction of $V_{k,\mathcal{L}}$ for all weights $3 \leq k \leq p+1$ and all $\mathcal{L}$ for $p \geq 5$ (§5). These notes are a summary of our joint work with Anand Chitrao [21]. Emphasis is placed on motivation and background rather than completeness.

1 Introduction

Let p be a prime. It has been over 10 years now since I started working on mathematics connected with the p-adic Local Langlands correspondence. The theory was initiated by Breuil about 20 years ago, and caused a mini-revolution in the field of number theory. I remember him speaking about his vision at the ICM in Hyderabad in 2010 [13]. A few years later Colmez made the next quantum leap forward by making Breuil's correspondences functorial [22]. I was fortunate to have a ringside view of these developments especially since some of them were made during the years 2007-2010 when we ran a joint Indo-French CEFIPRA project. Meanwhile many other prominent mathematicians such as Berger, Dospinescu, Paškūnas, to name just a few, contributed deep results along the way.

I entered the subject for the following reason. About 13 years ago, I was trying to show that certain mod p Galois representations attached to modular forms had large image. Indeed, I was trying to generalize Serre's famous conjecture that the global mod p Galois representation attached to a non-CM rational elliptic curve has full image for all primes p larger than an absolute constant to the setting of modular forms. It turns out that the naïve generalization of this statement is necessarily false but Pierre Parent and I could state a variant of this conjecture and make some mild progress on it for weight 2 forms [29].

One way to tackle this problem is to show that the corresponding restricted local mod p Galois representation has large image. Nothing like showing a group is large by showing that it has a large subgroup. This approach works to some extent but not entirely, not least because it turns out that the mod p reductions of local Galois representations attached to modular forms have not yet been written down in all cases. It was very disconcerting to me that there was such a glaring gap in the subject. I decided to devote a good chunk of my future research time in making some headway with this question.

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It turns out the p-adic Local Langlands program is ideally suited to studying the mod p images of local modular Galois representations. Indeed, Breuil invented his theory to tackle exactly this problem. In the beginning he restricted to the so called crystalline case where p does not divide the level of the modular form. Let us from now on always restrict to odd primes p (though in some results in this Introduction we assume without warning that $p \geq 5$). Breuil showed that one could compute the local mod p reductions for all forms of weights $k \leq 2p+1$ and positive slope [10], [11]. Here the slope of a form is the p-adic valuation of its p-th Fourier coefficient, where the valuation is normalized so that the valuation of p is 1. (The case of slope 0 and all weights is classical and is due to Deligne.) This generalized the work of his advisor Fontaine, although the details were worked out in [24], who had earlier computed the reductions for weights $k \leq p+1$ and positive slope.

One obvious restriction in these theorems is that the weight is bounded above, but there is no restriction on the slope. In an orthogonal direction, in a short but influential paper, Buzzard and Gee [17] used the p-adic Local Langlands correspondence to compute the mod p reductions of crystalline Galois representations of slopes in the small range $(0, 1)$, but for all weights. Some difficulties encountered at slope $1/2$ were only cleared up in a second paper [18]. (The case of $p = 2$ for slopes in this range is being treated in a forthcoming thesis of Arathy Venugopal.) For historical completeness, let us mention that earlier and using a different method, Berger, Li and Zhu [6] had treated the case of slopes which are large compared to the weight, namely slopes which are larger than $\lfloor \frac{k-2}{p-1} \rfloor$ (an interesting variant of this results was recently proved by Bergdall-Levin [4] who treated the case of slope larger than $\lfloor \frac{k-1}{p} \rfloor$).

This is where I entered the problem. In a series of papers with my coauthors (students, postdocs, colleagues), I first extended the result of Buzzard-Gee to all slopes in $(0, 2)$. This does not seem like much, but this marginal gain of a unit interval's worth of slopes was to consume me for the better part of the first half of the following decade. At first, the answers we got for the reduction seemed unpredictable, almost as if one was entering a fractal. There were no general guidelines (other than a folklore conjecture of Breuil, Buzzard and Emerton which said that for fractional slopes and even weights the reduction should be irreducible). Murphy's law (if things can go wrong, they will) seemed to rule the roost - if a particular exception to a general rule for the shape of the reduction was in principle possible, then it always wound up occurring.

Some initial headway for the case of slopes in $(1, 2)$ was made with Abhik Ganguli for bounded weights [25], and then in a very nice paper with Shalini Bhattacharya for all weights [7] but again we were only able to partially treat the case of slope $3/2$. The missing case of slope 1 was then treated with Shalini Bhattacharya and Sandra Rozensztajn [8]. The complete picture for slope $3/2$ was finally provided only a few years ago with Vivek Rai, though the paper [30] appeared just this year. (Beyond this range, I also wish to mention forthcoming work of Sudipta Majumder for slope 2, some partial results of Nagell-Pande and Arsovski for slopes in $(2, 3)$, and a forthcoming project [9] with Shalini Bhattacharya and Ravitheja Vangala which aims to treat all fractional slopes in the range $(0, p)$ building on the foundations laid in [31].) All these papers use the functoriality of the p-adic Local Langlands Correspondence with respect to reduction (established by Berger if one is willing to work up to semi-simplification - as we mostly were - and in general by Colmez), to reduce the question of studying the reductions of local crystalline Galois representations to studying the reduction of the standard lattice in a certain p-adic Banach space. The slope 1 paper [8] also computes the reductions of several other lattices, and in particular establishes criteria to distinguish between *peu* and *très ramifiée* cases.

Part of the problem encountered at the half-integral slopes $1/2, 1, 3/2, \ldots$ was that the reduction seemed to behave even more erratically than usual at the so called exceptional weights k (these are weights which are congruent to two more than twice the slope modulo $(p-1)$). Based on the results in slope $1/2$ and 1, and some cautionary computations of Rozensztajn, I eventually wound up making a conjecture which I called the *zig-zag conjecture* which described the behaviour of the reduction for all positive half-integral slopes less than or equal to $\frac{p-1}{2}$ and all sufficiently large exceptional weights. Roughly, the conjecture predicted that the reductions varied through an alternating sequence of irreducible and reducible representations depending on the (relative) sizes of two parameters. The statement appeared in a proceedings of an annual number theory conference at RIMS in Kyoto, Japan [27], where I was thrown to the wolves by my kind host Shinichi Kobayashi as the opening speaker

(I thank him for this honor). At the time it had become my mission to settle the case of slope $3/2$ if only to prove that the conjecture had some merit. However, over the years there was an uncomfortable truth that began to emerge. The paper [18] for slope $1/2$ was about 10 pages long, the one [8] for slope 1 was about 50 pages long (though not all of it dealt with zig-zag), and the (unabridged arXiv version of the) one [30] for slope $3/2$ was just under 80 pages long (its entire focus was zig-zag at $3/2$). So clearly another approach would be required to prove the conjecture in general. To my complete surprise this was to surface a few years later.

To explain this, let us return to Breuil's early foundational work. Apart from treating the crystalline case, Breuil also wrote some important papers in the so called semi-stable case. This case occurs for modular forms for which p exactly divides the level of the form (and does not divide the conductor of the nebentypus character). In an important work with Mézard dating to more than 20 years ago, Breuil computed the mod p reductions of all semi-stable Galois representations of *even* weight $k \leq p-1$ using techniques from integral p-adic Hodge theory [15]. The case of *odd* weights in this range was completed much later (about 6 years ago) in another *tour de force* by Guerberoff and Park [32] (and Lee and Park [34]), though several constants remained to be determined completely. For completeness, we also mention that an alternate algebro-geometric approach to computing the reduction involving the global sections of certain bundles on the p-adic upper half-plane was developed by Breuil-Mézard in some cases [16], though this approach does indeed seem to require that the weight k be even since it involved $k/2$-powers of certain line bundles.

In any case, some time in 2022, I realized that all of these works in the parallel universe of semi-stable representations could be used to give a proof of the zig-zag conjecture in the crystalline world, at least for most slopes and on the inertia group. This was perhaps one of the more important observations that I have made in the past 10 years. Let us explain how it came about. A bit earlier than this, Anand Chitrao, Seidai Yasuda and I had been trying to use the above mentioned works in the crystalline world to try and deduce results about $V_{k,\mathcal{L}}$ in the semi-stable world, using a limiting argument in Colmez's blow-up space of non-split rank 2 trianguline (φ, Γ)-modules. We wrote a nice paper about this which appeared this year [20] and which allowed one, for instance, to predict the exact shape of some of the above mentioned missing constants for small odd weights (e.g., for $k = 5$ from the slope $3/2$ paper), and, in general, to recover the work of Breuil-Mézard and Guerberoff-Park on inertia *assuming* my zig-zag conjecture. The breakthrough came when I realized that one could reverse the entire argument and instead deduce information about crystalline representations - in particular, a large portion of the zig-zag phenomenon - from the literature in the semi-stable case. In fact, after this realization I could immediately prove zig-zag up to slope $\frac{p-3}{2}$ (though in the first instance I could only prove it on the inertia group since, as already mentioned, the constants in the semi-stable world had not yet been completely determined for odd weights). The missing cases of slope $\frac{p-2}{2}$ and $\frac{p-1}{2}$ would require extending the work of Guerberoff-Park [32] to the odd weight $k = p$ and the classical work of Breuil-Mézard [15] to the case of the even weight $k = p + 1$.

The possible extension to these two weights was more than just a technicality. There was a theoretical obstruction. It turned out that the strongly divisible modules occurring in integral p-adic Hodge theory were either not as well behaved ($k = p$, see [26]) or not even available (for $k \geq p + 1$) (although since then a theory of Breuil-Kisin modules has become available which works for all weights k). So an entirely new perspective was required. Based on my experience with computing the reduction using the functoriality of the p-adic Local Langlands Correspondences in the crystalline world (a method initiated by Breuil and Buzzard-Gee), I wondered whether Anand Chitrao and I might be able to tackle the reduction problem in the semi-stable case in a similar manner. We were given to understand that one might have to wait for a very long time for this hope to be realized.

However, not ones to shy away from a challenge, Chitrao and I started work on this ambitious project. Initial gains were few and far between. But, to make a long story short, we were eventually able to compute the mod p reductions of all semi-stable representations for weights $k \leq p + 1$, including the cases of weight $k = p$ and $k = p + 1$. We were also able to provide a complete and uniform treatment of all the constants involved. The goal of this expository paper is to explain this result and to expand on some of the mathematical background that goes into its proof. But before I go further, let me record that this work allowed one to complete the proof of my

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zig-zag conjecture (i.e., to extend the initial proof up to slope $\frac{p-1}{2}$, and to determine all the constants that occur in the unramified characters on the decomposition group in the reduction, see [28]).

Already, in the early days, Breuil had written two important papers describing the Banach space attached to a semi-stable representation $V_{k,\mathcal{L}}$ of weight k and $\mathcal{L}$-invariant $\mathcal{L}$ under the p-adic Local Langlands Correspondence. The first description, denoted by $B(k, \mathcal{L})$ in [12], involved some work of Schneider and Teitelbaum and used Morita duality and seemed a bit abstract to us. But the second description, denoted by $\tilde{B}(k, \mathcal{L})$ in [14], 'only' used p-adic functional analysis (a beautiful summary of which can be found in [23], see §3) and this definition seemed much more amenable to computation. In this model, the Banach space $\tilde{B}(k, \mathcal{L})$ was nothing but the space of differentiable functions on $\mathbb{Q}_p$ of order $r/2$ where $r = k - 2$ (more precisely of type $\mathscr{C}^{r/2}$, these notions are slightly different), with a similar differentiability condition at ∞, modulo polynomial functions of degree at most r and certain finite sums of polynomial times logarithmic (poly·log!) functions (with the polynomial part having degree less than $r/2$). This description was something that you could explain to a clever high school student learning calculus. Using it, we began to search for an integral structure (lattice) on this Banach space.

It was not immediately clear how to proceed, but we soon realized that one could uniformize this Banach space (much as in the crystalline world) by the compactly induced space $\mathrm{ind}_{IZ}^G \mathrm{Sym}^{k-2} E^2$ of certain rational polynomial valued functions allowing us to define the (standard) lattice in the Banach space as the (closure of the) image of the space $\mathrm{ind}_{IZ}^G \mathrm{Sym}^{k-2} \mathcal{O}_E^2$ of integral polynomial valued functions under this uniformization. Here I is the Iwahori subgroup of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and Z is the center of G. It then 'remained' to compute the reduction of this lattice. In hindsight, this required several new creative computations. After much effort we were able to compute all the Jordan-Hölder (JH) factors in the reduction in terms of certain mod p compactly induced spaces modulo the images of certain Iwahori-Hecke operators. Applying a reformulation of Breuil's mod p Local Langlands correspondence worked out by Chitrao, which is called the Iwahori mod p Local Langlands correspondence in [19], we could then return to the Galois side and compute the reductions of the semi-stable Galois representations $V_{k,\mathcal{L}}$. Our target to treat all weights in the range $3 \leq k \leq p+1$ was achieved in the paper [21], though we remark that our method can be used to treat all weights $k \geq 3$ at least in principle (unlike the initial approach with strongly divisible modules). Indeed, Anand and I have just begun to see whether our approach to computing the reduction of semi-stable representations using p-adic Langlands can be used to recover very recent work of Bergdall, Levin, and Liu [5] which uses Breuil-Kisin modules to study the reduction of $V_{k,\mathcal{L}}$ with $\mathcal{L}$-invariant having very negative p-adic valuation.

Let us state the main result in the paper [21] with Anand Chitrao. Let $p \geq 5$ be a prime and E be a finite extension of $\mathbb{Q}_p$ containing $\sqrt{p}$. We describe completely the semi-simplification of the reduction mod p of the irreducible two-dimensional semi-stable representation $V_{k,\mathcal{L}}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over E with Hodge-Tate weights $(0, k-1)$ for $k \in [3, p+1]$ and $\mathcal{L}$-invariant $\mathcal{L} \in E$. To do this we need a little bit more notation. Let as usual $r = k - 2$ so that $1 \leq r \leq p - 1$.

Let v_- and v_+ be the largest and smallest integers, respectively, such that $v_- < r/2 < v_+$ for $r \geq 1$. For $n \geq 1$, let $H_n = \sum_{i=1}^{n} \frac{1}{i}$ be the n-th partial harmonic sum. Set $H_0 = 0$ and $H_\pm = H_{v_\pm}$. Let v_p be the p-adic valuation on $\overline{\mathbb{Q}}_p$ normalized such that $v_p(p) = 1$. Let

$$\nu = v_p(\mathcal{L} - H_- - H_+)$$

be the p-adic valuation of $\mathcal{L}$ shifted by the partial harmonic sums H_- and H_+. Everything hinges on the size of the parameter ν. Let ω be the fundamental character of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of level 1. Similarly, for an integer c (with $p+1 \nmid c$), let ω_2^c be an extension from the inertia subgroup $I_{\mathbb{Q}_p}$ to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^2})$ of the c-th power of the fundamental character ω_2 of level 2 chosen so that the (irreducible) representation ind (ω_2^c) obtained by inducing this character from $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^2})$ to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ has determinant ω^c. Let μ_λ be the unramified character of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ sending geometric Frobenius to $\lambda \in \overline{\mathbb{F}}_p^*$. Our main theorem is the following result.

Theorem 1.1 *(Chitrao-Ghate [21]) For $k \in [3, p+1]$ and for primes $p \geq 5$, the semi-simplification of the reduction mod p of the semi-stable representation $V_{k,\mathcal{L}}$ on $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is given by an alternating sequence of irreducible and reducible representations:*

$$\overline{V}_{k,\mathcal{L}} \sim \begin{cases} \mathrm{ind}\,(\omega_2^{r+1+(i-1)(p-1)}), & \text{if } (i-1) - r/2 < \nu < i - r/2 \\ \mu_{\lambda_i}\omega^{r+1-i} \oplus \mu_{\lambda_i^{-1}}\omega^i, & \text{if } \nu = i - r/2, \end{cases}$$

where

$$\begin{cases} 1 \leq i \leq (r+1)/2 & \text{if } r \text{ is odd} \\ 1 \leq i \leq (r+2)/2 & \text{if } r \text{ is even,} \end{cases}$$

and the mod p constants λ_i are determined by the formulas:

$$\lambda_i = \overline{(-1)^i\, i \binom{r+1-i}{i} \frac{(\mathcal{L} - H_- - H_+)}{p^{i-r/2}}}, \quad \text{if } 1 \leq i < \frac{r+1}{2}$$
$$\lambda_i + \lambda_i^{-1} = \overline{(-1)^i\, i \binom{r+1-i}{i} \frac{(\mathcal{L} - H_- - H_+)}{p^{i-r/2}}}, \quad \text{if } i = \frac{r+1}{2} \text{ and } r \text{ is odd.}$$

We remark that we have adopted the following conventions in the statement of the theorem:

- The first interval $-r/2 < \nu < 1 - r/2$ is interpreted as $\nu < 1 - r/2$.
- If r is odd, then the last case $\nu = 1/2$ should be interpreted as $\nu \geq 1/2$. If r is even, then the interval $0 < \nu < 1$ should be interpreted as $\nu > 0$ and we drop the case $\nu = 1$.

In any case, the theorem says that the reduction $\overline{V}_{k,\mathcal{L}}$ varies through an alternating sequence of irreducible and reducible mod p representations as ν varies through finitely many marked points.

1.1 Idea of proof of Theorem 1.1

A picture is worth a thousand words, and so we draw one to explain the proof. Let $B_{k,\mathcal{L}} = \tilde{B}(k, \mathcal{L})$. The following diagram commutes:

$$\begin{array}{ccc} V_{k,\mathcal{L}} & \xmapsto{p-\text{adic LLC}} & B_{k,\mathcal{L}} \\ \downarrow & & \downarrow \\ \overline{V}_{k,\mathcal{L}} & \xmapsto{\text{Iwahori mod } p \text{ LLC}} & \overline{B}_{k,\mathcal{L}} \end{array}$$

where the vertical maps are (semi-simplifications of) reductions of lattices in the corresponding spaces in the top row. One is trying to compute the left vertical map. But one computes instead the right vertical map since the bottom map is injective.

We now give a broad outline of the remaining text in terms of this picture. The paper is broken into four further sections. The bottom map is explained in §2. The top right corner is explained in §4 based on the foundational material described in §3. The computation of the right vertical map is then explained in §5.

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1.2 Notation

- $p \geq 5$ is a prime.
- E is a finite extension of $\mathbb{Q}_p$ containing $\sqrt{p}$ and $\mathcal{L}$. $\mathcal{O}_E$ is the ring of integers in E with a uniformizer $\pi = \pi_E$ and residue field $\mathbb{F}_q$. Note $\sqrt{p} \equiv 0 \mod \pi$.
- k denotes the weight of a semi-stable representation and $r = k - 2$.
- v_-, v_+ are the largest, smallest integers, respectively, such that $v_- < r/2 < v_+$ for $r \geq 1$.
- For $n \geq 1$, $H_n = \sum_{i=1}^{n} \frac{1}{i}$ and $H_0 = 0$, $H_\pm = H_{v_\pm}$.
- v_p is the p-adic valuation normalized such that $v_p(p) = 1$.
- $\nu = v_p(\mathcal{L} - H_- - H_+)$ is the valuation of $\mathcal{L}$ shifted by the partial harmonic sums H_- and H_+.
- $I_{\mathbb{Q}_p}$ is the inertia subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.
- ω is the fundamental character of $I_{\mathbb{Q}_p}$ of level 1; it has a canonical extension to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.
- ω_2 is the fundamental character of $I_{\mathbb{Q}_p}$ of level 2; for an integer c with $p+1 \nmid c$, choose an extension of ω_2^c to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^2})$ so that the irreducible representation $\mathrm{ind}(\omega_2^c)$ obtained by inducing this extension from $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^2})$ to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ has determinant ω^c.
- G is the group $\mathrm{GL}_2(\mathbb{Q}_p)$.
- K is the maximal compact subgroup $\mathrm{GL}_2(\mathbb{Z}_p)$ of G.
- I is the Iwahori subgroup of G consisting of matrices in K that are upper triangular mod p.
- B is the Borel subgroup of G consisting of upper triangular matrices.
- $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, $\beta = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ and $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Note that $\beta = \alpha w$.
- $I_n = \{[a_0] + [a_1]p + \cdots + [a_{n-1}]p^{n-1} \mid a_i \in \mathbb{F}_p\}$ for $n \geq 1$, where $[\ \ ]$ denotes Teichmüller representative. $I_0 = \{0\}$.
- $V_r = \mathrm{Sym}^r \mathbb{F}_q^2$ and $\underline{\mathrm{Sym}}^{k-2} E^2 := |\det|^{\frac{k-2}{2}} \otimes \mathrm{Sym}^{k-2} E^2$ for $k \geq 2$.
- d^r for an integer r denotes the character $IZ \to \mathbb{F}_p^*$ which sends $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I$ to $d^r \bmod p$ and which is trivial on the scalar matrix p.
- For a representation (ρ, V) of IZ over E or $\mathbb{F}_q$, let $\mathrm{ind}_{IZ}^G \rho$ be the vector space of functions $f : G \to V$ that are compactly supported modulo IZ such that $f(hg) = \rho(h) \cdot f(g)$, for all $h \in IZ$ and $g \in G$. The vector space $\mathrm{ind}_{IZ}^G \rho$ has a G action: $g \cdot f(g') = f(g'g)$, for all $g, g' \in G$ and $f \in \mathrm{ind}_{IZ}^G \rho$. For $g \in G$ and $v \in V$, define the function $[\![g, v]\!] \in \mathrm{ind}_{IZ}^G \rho$ by

$$[\![g, v]\!](g') = \begin{cases} \rho(g'g) \cdot v, & \text{if } g'g \in IZ \\ 0, & \text{otherwise.} \end{cases}$$

- Let $(\mathrm{ind}_B^G E)^{\mathrm{smooth}}$ be the E-vector space of locally constant functions from G to E, with the action of G given by $g \cdot f(g') = f(g'g)$ for any $g, g' \in G$ and $f \in (\mathrm{ind}_B^G E)^{\mathrm{smooth}}$.
- Let St be the Steinberg representation of G over E, i.e., St is the vector space of all locally constant functions $H : \mathbb{P}^1(\mathbb{Q}_p) \to E$ modulo constant functions with the action of G given by $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot H\right)(z) = H\left(\frac{az+c}{bz+d}\right)$.
- $[a] \in \{0, \ldots, p-2\}$ denotes the class of $a \bmod p-1$.
- $\delta_{a,b} = 1$ if $a = b$ and is 0 otherwise.

2 Iwahori mod p Local Langlands Correspondence

2.1 Bruhat-Tits tree

We start with the definition of the famous Bruhat-Tits tree.

A lattice $L \subset \mathbb{Q}_p^2$ is a free $\mathbb{Z}_p$-module of rank 2. Two lattices L and L' are said to be homothetic if there is a non-zero scalar $z \in \mathbb{Q}_p^*$ such that $L' = zL$. The vertices of the tree are homothety classes $[L]$ of lattices $L \subset \mathbb{Q}_p^2$.

Two vertices represented by lattices L and L' are joined by an edge if $pL \subsetneq L' \subsetneq L$ (or equivalently $[L : L'] = p$). Note that this condition is symmetric in $[L]$ and $[L']$ since $p(\frac{1}{p}L') \subsetneq L \subsetneq \frac{1}{p}L'$ and $\frac{1}{p}L'$ is homothetic to L'. The tree is a regular graph of valency $p+1$ since working mod p there are $p+1$ subgroups of index p in $\mathbb{Z}_p^2/p\mathbb{Z}_p^2 = (\mathbb{Z}/p\mathbb{Z})^2$.

An oriented edge is an edge with a direction. The orientation can be specified by ordering the tuple $([L],[L'])$ so that $[L]$ is the origin or source of the edge and $[L']$ is the target or sink of the edge.

Lemma 2.1 *Let* $G = \mathrm{GL}_2(\mathbb{Q}_p) \supset K = \mathrm{GL}_2(\mathbb{Z}_p) \supset I = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod p\} =$ *the Iwahori subgroup.*

1. *The vertices of the tree are in one-to-one correspondence with* G/KZ.
2. *The oriented edges of the tree are in one-to-one correspondence with* G/IZ.

Proof We claim that G acts transitively on the vertices of the tree via

$$g \cdot [L] = [gL]$$

for $g \in G$ and $L \subset \mathbb{Q}_p^2$ a lattice. Note that gL is indeed a lattice in $\mathbb{Q}_p^2$. Also if L' is homothetic to L, then clearly gL' is homothetic to gL. So the action is well defined. The transitivity of the action follows from the fact that given any two basis vectors $v_1 = (a,c)$ and $v_2 = (b,d)$ of $\mathbb{Q}_p^2$ which span a particular lattice, the invertible matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of G takes the standard lattice L_0 spanned by $e_1 = (1,0)$ and $e_2 = (0,1)$ to v_1 and v_2 respectively. Finally note that KZ is the stabilizer of the standard lattice L_0. Indeed KZ clearly stabilizes it. If $g \in G$ stabilizes $[L_0]$, then there exists a scalar $z \in \mathbb{Q}_p^*$ such that $g\mathbb{Z}_p^2 = z\mathbb{Z}_p^2$. This means that $z^{-1}g \in K$. Thus $g \in KZ$.

We similarly claim that G acts transitively on the oriented edges of the tree via

$$g \cdot ([L],[L']) = ([gL],[gL'])$$

for g in G and $[L],[L']$ adjacent vertices in the tree. Note that $g \cdot ([zL],[z'L']) = ([gzL],[gz'L']) = ([zgL],[z'gL']) = ([gL],[gL'])$ for $g \in G$, $z, z' \in Z$. Also, if say $L' \subsetneq L$ has index p, then $gL' \subsetneq gL$ also has index p. So the action is well defined. For the transitivity, assume that $L' \subsetneq L$ has index p. By the 'matrix game' from Prof. Murray Schacher's first year graduate algebra course at UCLA in 1991, there is a $\mathbb{Z}_p$-basis v_1, v_2 of L with respect to which L' is spanned by v_1 and pv_2. Then the matrix g in the previous paragraph takes the standard edge $([L_0],[\alpha L_0])$ where $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ to $([L],[L'])$. Finally note that IZ is the stabilizer of $([L_0],[\alpha L_0])$. Indeed a small check shows that

$$I = K \cap \alpha K \alpha^{-1}$$

so clearly I stabilizes both L_0 and αL_0, so IZ stabilizes the standard edge. Conversely if $g \in G$ stabilizes the standard edge, then $([gL_0],[g\alpha L_0]) = ([L_0],[\alpha L_0])$ so by the previous paragraph, looking at the first coordinate we get $g \in KZ$, and looking at the second coordinate we get $\alpha^{-1}g\alpha \in KZ$, so that $g \in KZ \cap \alpha KZ\alpha^{-1} = IZ$, as desired. Note that in the last equality, the containment '$\supset$' is obvious by the display just above; the reverse containment '$\subset$' follows by a short computation: if $g = kz = \alpha k'z'\alpha^{-1}$ for $k, k' \in K$, $z, z' \in Z$, then taking determinants we see that $2v_p(z) = 2v_p(z')$, so we may cancel a like power of p on both sides and assume z and z' are units, in which case $k \in K \cap \alpha K \alpha^{-1} = I$ and so $g = kz \in IZ$. ◻

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2.2 Hecke algebras

We now recall the definitions of the spherical Hecke algebra and the Iwahori-Hecke algebra. The development proceeds in parallel. Consider the following irreducible mod p **representations**:

- Let $V_r = \mathrm{Sym}^r \mathbb{F}_q^2 = \{\text{homogeneous polynomials } v(X,Y) \text{ of degree } r \text{ over } \mathbb{F}_q\}$ for $0 \le r \le p-1$. Then K acts on V_r noting $K \twoheadrightarrow \mathrm{GL}_2(\mathbb{F}_q)$ which in turn acts on V_r via
$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot v(X,Y) = v(aX + cY, bX + dY).$$
Extend the action to KZ by making $p \in Z$ act trivially.
- Let d^r be the one-dimensional space over $\mathbb{F}_q$ for $0 \le r \le p-1$. Then I acts on d^r noting $I \twoheadrightarrow B(\mathbb{F}_p)$ for the Borel subgroup $B(\mathbb{F}_p)$ of upper-triangular matrices which in turn acts on the space d^r via the character d^r given by
$$\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \cdot v = d^r v.$$
Again, extend the action to IZ by making $p \in Z$ act trivially.

We may **compactly induce** these representations to G to obtain function spaces:

- Let $\mathrm{ind}_{KZ}^G V_r = \{f : G \to V_r \mid f(kg) = k \cdot f(g), \forall k \in KZ, g \in G\}$. Here we only consider those f which are compactly supported mod KZ, i.e., those f supported on only finitely many right cosets of KZ in G. For instance, for $g \in G$, $v \in V_r$, we may consider the elementary function $[g, v]$ supported on the coset KZg^{-1} defined by
$$[g, v](g') = \begin{cases} g'g \cdot v(X,Y) & \text{if } g' \in KZg^{-1} \\ 0 & \text{otherwise.} \end{cases}$$
- Similarly, let $\mathrm{ind}_{IZ}^G d^r = \{f : G \to \mathbb{F}_q(d^r) \mid f(ig) = d^r(i) f(g), \forall i \in IZ, g \in G\}$. Again we only consider those f which are compactly supported mod IZ, i.e., supported on only finitely many right cosets of IZ in G. Again, for $g \in G$, $v \in d^r$, we may consider the elementary function $[\![g, v]\!]$ supported on the coset IZg^{-1} defined by
$$[\![g, v]\!](g') = \begin{cases} g'g \cdot v & \text{if } g' \in IZg^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Consider now the structure of the corresponding **Hecke algebras**. These algebras are by definition the algebra of G-equivariant endomorphisms of the above compactly induced spaces:

- (Spherical Hecke algebra)
$$\mathrm{End}_G(\mathrm{ind}_{KZ}^G V_r) = \mathbb{F}_q[T],$$
where the action of the Hecke operator T on an elementary function is given by
$$T[g, v] = \sum_{\lambda \in \mathbb{F}_p} [g \left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix}\right), v(X, -[\lambda]X + pY)] + [g\alpha, v(pX, Y)]. \tag{1}$$
- (Iwahori-Hecke algebra)
$$\mathrm{End}_G(\mathrm{ind}_{IZ}^G d^r) = \begin{cases} \mathbb{F}_q[T_{-1,0}, T_{1,2}] \text{ with } T_{-1,0}T_{1,2} = 0 = T_{1,2}T_{-1,0} & \text{if } r \ne 0, p-1, \\ \mathbb{F}_q[T_{1,0}, T_{1,2}] \text{ with } T_{1,2}T_{1,0}T_{1,2} = -T_{1,2},\ T_{1,0}^2 = 1 & \text{if } r = 0, p-1, \end{cases}$$

where the action of the Iwahori-Hecke operators $T_{1,0}$, $T_{-1,0}$ and $T_{1,2}$ are given by the formulas

$$\begin{aligned} T_{1,0}[\![g, v]\!] &= [\![g\beta, v]\!] \\ T_{-1,0}[\![g, v]\!] &= \sum_{\lambda \in \mathbb{F}_p} [\![g \left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix}\right), v]\!] \\ T_{1,2}[\![g, v]\!] &= \sum_{\lambda \in \mathbb{F}_p} [\![g \left(\begin{smallmatrix} 1 & 0 \\ p[\lambda] & p \end{smallmatrix}\right), v]\!], \end{aligned} \tag{2}$$

where $\beta = \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$.

Remark 2.2 We will later see that when $r = 0, p - 1$, the operator $T_{-1,0}$ satisfies

$$T_{-1,0} = T_{1,0}T_{1,2}T_{1,0} \tag{3}$$

and so lies in the Hecke algebra, but since it is generated by the other two generators, it has been dropped from the list of generators. This relation also shows that in this case the Iwahori-Hecke algebra is *non-commutative* since otherwise $T_{1,0}$ would commute past the $T_{1,2}$ to give $T_{-1,0} = T_{1,2}$ since the square of $T_{1,0}$ is the identity, which as we shall see is a contradiction. The Iwahori-Hecke algebra is clearly commutative when $r \neq 0, p - 1$, as we see from the relations between the generators above.

We now **reinterpret the above formulas for the Hecke operators** in terms of some classical operators on the vertices and edges in graph theory (in the case $r = 0$). We make use of the fact that left coset representatives can be taken to be the inverses of right coset representatives.

- We have $\operatorname{ind}_{KZ}^{G} V_0 = \{f : KZ\backslash G \to \mathbb{F}_q\} = \{f' : G/KZ \to \mathbb{F}_q\}$ where the first equality is by definition (we always implicitly assume the compactly supported condition) and the second equality is obtained by setting $f'(gKZ) = f(KZg^{-1})$. Now, this last space of functions can be thought of as functions on the vertices of the tree by Lemma 2.1. Under this identification we have

$$[g, 1] \leftrightarrow f' = \text{ characteristic function of the vertex } [gL_0],$$

since

$$\begin{aligned} f'(gKZ) &= [g, 1](KZg^{-1}) = g^{-1}g \cdot 1 = 1, \\ f'(g'KZ) &= [g, 1](KZg'^{-1}) = 0 \text{ if } g'KZ \neq gKZ. \end{aligned}$$

Using this observation, and identifying the characteristic function of a vertex of the tree with the vertex itself, we see that the formula for T in (1) is nothing but the usual formula for the *sum-of-neighbours operator* on vertices of the tree from classical graph theory. Indeed, when $g = 1$, we have

$$T[1, 1] = \sum_{\lambda \in \mathbb{F}_p} [\left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix}\right), 1] + [\alpha, 1].$$

But $[1, 1]$ corresponds to the standard lattice L_0, and as is well known, the lattices $\left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix}\right) L_0$, for $\lambda \in \mathbb{F}_p$, and αL_0 form a complete set of sublattices of L_0 of index p.
- Similarly, we have $\operatorname{ind}_{IZ}^{G} d^0 = \{f : IZ\backslash G \to \mathbb{F}_q\} = \{f' : G/IZ \to \mathbb{F}_q\}$ where the first equality is again by definition (again the compactly supported condition is implicit) and the second equality is obtained by setting

$f'(gIZ) = f(IZg^{-1})$. This time, the last space of functions can be thought of as functions on the oriented edges of the tree by Lemma 2.1. Under this identification we have

$$[\![g, 1]\!] \leftrightarrow f' = \text{ characteristic function of the edge } ([gL_0], [g\alpha L_0]),$$

since

$$f'(gIZ) = [\![g, 1]\!](IZg^{-1}) = g^{-1}g \cdot 1 = 1,$$
$$f'(g'IZ) = [\![g, 1]\!](IZg'^{-1}) = 0 \text{ if } g'IZ \neq gIZ.$$

Using this and identifying the characteristic function of an edge of the tree with the edge itself, we see that the formulas for $T_{1,0}$, $T_{-1,0}$ and $T_{1,2}$ in (2) are nothing but the *flip*, *source* and *sink* operators on the oriented edges of the tree. Indeed, when $g = 1$, we have

$$T_{1,0}[\![1, 1]\!] = [\![\beta, 1]\!].$$

But $[\![1, 1]\!]$ corresponds to the standard edge $([L_0], [\alpha L_0])$ and $[\![\beta, 1]\!]$ corresponds to the edge $([\beta L_0], [\beta\alpha L_0])$ $= ([\alpha L_0], [pL_0])$ (since $\beta = \alpha w$ and $\beta\alpha = pw$) which is the standard edge with flipped orientation (since $[pL_0] = [L_0]$). Similarly, we have

$$T_{-1,0}[\![1, 1]\!] = \sum_{\lambda\in\mathbb{F}_p} [\![\left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix}\right), 1]\!].$$

This time the functions on the right correspond to the edges $([\left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix}\right) L_0], [\left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix}\right) \alpha L_0]) = ([\left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix}\right) L_0], [L_0])$ (since $\left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix}\right) \alpha = p \left(\begin{smallmatrix} 1 & [\lambda] \\ 0 & 1 \end{smallmatrix}\right)$ and $[p \left(\begin{smallmatrix} 1 & [\lambda] \\ 0 & 1 \end{smallmatrix}\right) L_0] = [L_0]$) which are exactly p of the edges with target the standard vertex $[L_0]$, which is the source of the standard edge $([L_0], [\alpha L_0])$. Thus the formula for $T_{-1,0}$ reduces to that of the classical source operator on oriented edges in graph theory. We remark that the subscripts in this source operator $T_{-1,0}$ also hint at its definition. Label the vertices corresponding to the lattice spanned by $p^{-n}e_1, e_2$ by n for $n \in \mathbb{Z}$. Then the formula for $T_{-1,0}$ takes the standard oriented edge $(0, 1)$ to all the oriented edges with target the source vertex 0 - including the edge $(-1, 0)$ corresponding to $\lambda = 0$! Finally, we have

$$T_{1,2}[\![1, 1]\!] = \sum_{\lambda\in\mathbb{F}_p} [\![\left(\begin{smallmatrix} 1 & 0 \\ p[\lambda] & p \end{smallmatrix}\right), 1]\!].$$

This time the functions on the right correspond to the edges $([\left(\begin{smallmatrix} 1 & 0 \\ p[\lambda] & p \end{smallmatrix}\right) L_0], [\left(\begin{smallmatrix} 1 & 0 \\ p[\lambda] & p \end{smallmatrix}\right) \alpha L_0]) = ([\alpha L_0], [L_\lambda])$ (since $\left(\begin{smallmatrix} 1 & 0 \\ p[\lambda] & p \end{smallmatrix}\right) = \alpha \left(\begin{smallmatrix} 1 & 0 \\ [\lambda] & 1 \end{smallmatrix}\right)$ and so $[\alpha \left(\begin{smallmatrix} 1 & 0 \\ [\lambda] & 1 \end{smallmatrix}\right) L_0] = [\alpha L_0]$) which we claim are exactly p of the edges emanating from the sink of the standard edge $([L_0], [\alpha L_0])$. Moreover, none is the flip of the standard edge, else we would have $[L_\lambda] = [L_0]$, which implies that $\begin{pmatrix} 1 & 0 \\ p[\lambda] & p^2 \end{pmatrix} = kz$ for some $k \in K$ and $z \in Z$ which by taking determinants implies $z = p$ which implies that $\begin{pmatrix} p^{-1} & 0 \\ [\lambda] & p \end{pmatrix} \in K$, a contradiction. Thus $T_{1,2}$ is the sink operator from classical graph theory. With respect to the labeling of the vertices mentioned above, we see that $T_{1,2}$ takes the standard oriented edge $(0, 1)$ to all edges emanating from its sink - including the oriented edge $(1, 2)$! Thus again the subscripts allow one to recall the definition of $T_{1,2}$.

Using this interpretation of the Hecke operators $T_{1,0}, T_{-1,0}, T_{1,2}$ as the flip, source and sink operators, it is now possible to check all the relations satisfied by these operators. To start with, we see $T_{-1,0} \neq T_{1,2}$ by their graph theoretic interpretations. Moreover:

– the relation $T_{1,0}^2 = 1$ is now self-evident since flipping orientation twice does nothing.

- the relation $T_{1,2}T_{1,0}T_{1,2} = -T_{1,2}$ follows, since on the left the standard edge (0, 1) first maps to the p edges emanating from 1 such as (1, 2), then $T_{1,0}$ flips them, then $T_{1,2}$ maps each flipped edge such as (2, 1), to the other edges coming out of 1, namely (1, 0) and the other $p-1$ edges coming out of 1. Summing the action of $T_{1,2}$ of each flipped edge, we see that (1, 0) gets counted p times, whereas the other edges coming out of 1 get counted $p-1$ times. But $p = 0$ in the mod p Hecke algebra, so we obtain the formula for $-T_{1,2}$!
- finally the relation (3) follows since the operations of flipping an edge, then taking the edges coming out of the new sink and finally flipping back is exactly the same thing as taking the edges coming into the source of the original edge!

For further relations involving the Iwahori-Hecke operators see the work of Barthel and Livné [2], [3].

2.3 The ABC theorem

Using the above constructions we may now define some **basic mod p representations of G**. Let

- $0 \leq r \leq p-1$,
- $\eta : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \overline{\mathbb{F}}_p^*$ be a smooth character (also thought of as a character of $\mathbb{Q}_p^*$ by pre-composing with the Artin map $\mathbb{Q}_p^* \to \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^{\mathrm{ab}}$, and even a character of G by further pre-composing with the determinant $G \twoheadrightarrow \mathbb{Q}_p^*$), and,
- $\lambda \in \overline{\mathbb{F}}_p$.

With this notation we define the basic mod p representation $\pi(r, \lambda, \eta)$ of G in two ways:

1. Let
$$\pi(r, \lambda, \eta) = \frac{\mathrm{ind}_{KZ}^G V_r}{T - \lambda} \otimes \eta.$$
2. Let
$$\pi(r, \lambda, \eta) = \frac{\mathrm{ind}_{IZ}^G d^r}{(T_{-1,0} + \delta_{r,p-1} T_{1,0}) + (T_{1,2} + \delta_{r,0} T_{1,0} - \lambda)} \otimes \eta$$
where $\delta_{i,j} = 1$ if $i = j$ and is 0 otherwise.

Every subject should have an ABC theorem, and so does this one:

Theorem 2.3 *(*Anandavardhanan, Borisagar, Chitrao*) The two definitions of $\pi(r, \lambda, \eta)$ given in the spherical and Iwahori cases above coincide.*

Proof When $0 < r < p-1$, the statement is simpler. One needs to show for $\lambda \in \overline{\mathbb{F}}_p$ (and $\eta = 1$) that

$$\frac{\mathrm{ind}_{IZ}^G d^r}{T_{-1,0} + (T_{1,2} - \lambda)} \simeq \frac{\mathrm{ind}_{KZ}^G V_r}{T - \lambda}.$$

For this, it suffices to show that there is an isomorphism

$$\frac{\mathrm{ind}_{IZ}^G d^r}{T_{-1,0}} \simeq \mathrm{ind}_{KZ}^G V_r$$

with the action of $T_{1,2}$ on the left corresponding to the action of T on the right. This was proved in [1] and we do not comment on it further.

The case $r = 0, p-1$ was only proved more recently in [19]. As above, it suffices to show that there is an isomorphism

$$\frac{\mathrm{ind}_{IZ}^{G} 1}{T_{-1,0} + \delta_{r,p-1} T_{1,0}} \simeq \mathrm{ind}_{KZ}^{G} V_r$$

with the action of $T_{1,2} + \delta_{r,0} T_{1,0}$ on the left corresponding to the action of T on the right. The argument is a bit more delicate since it involves the non-commutative Iwahori-Hecke algebra above. We sketch the main points here - for details see [19].

Extend the definition of V_r above to any degree $r \geq 0$. Let $V_r^* \subset V_r$ be the subspace of polynomials which are divisible by the Dickson polynomial $\theta = X^p Y - X Y^p$. Then

$$\frac{V_{2p-2}}{V_{2p-2}^*} = V_0 \oplus V_{p-1}$$

and the generators of these two spaces on the right are given by the polynomials $X^{2p-2} - X^{p-1}Y^{p-1} + Y^{2p-2}$ and X^{2p-2} respectively. It is not hard to see that

$$\frac{V_{2p-2}}{V_{2p-2}^*} \simeq \mathrm{ind}_{IZ}^{KZ} 1$$

under the usual 'evaluation of a polynomial at the lower row of the matrix' map, with $Y^{2p-2} - X^{p-1}Y^{p-1}$ corresponding to the function on KZ that takes the coset IZ to 1 and the other cosets to 0. Inducing both sides from KZ to G we obtain an isomorphism

$$\begin{aligned} \mathrm{ind}_{IZ}^{G} 1 &\to \mathrm{ind}_{KZ}^{G} \frac{V_{2p-2}}{V_{2p-2}^*} \\ [\![\mathrm{id}, 1]\!] &\mapsto [\mathrm{id}, Y^{2p-2} - X^{p-1}Y^{p-1}]. \end{aligned}$$

Now the relation $T_{1,2}T_{1,0}T_{1,2} = -T_{1,2}$ in the Iwahori-Hecke algebra shows that $-T_{1,2}T_{1,0}$ is an idempotent, so the left hand side decomposes as $\mathrm{Im}\,(T_{1,2}T_{1,0}) \oplus \mathrm{Im}\,(1 + T_{1,2}T_{1,0})$. The right hand side decomposes as $\mathrm{ind}_{KZ}^{G} V_0 \oplus \mathrm{ind}_{KZ}^{G} V_{p-1}$. The main technical result in [19] is that the above isomorphism takes $\mathrm{Im}\,(T_{1,2}T_{1,0})$ to $\mathrm{ind}_{KZ}^{G} V_{p-1}$ (since one checks $T_{1,2}T_{1,0}[\![\mathrm{id}, 1]\!] \mapsto [1, X^{2p-2}]$) and $\mathrm{Im}\,(1 + T_{1,2}T_{1,0})$ to $\mathrm{ind}_{KZ}^{G} V_0$ (since now $(1 + T_{1,2}T_{1,0})[\![\mathrm{id}, 1]\!] \mapsto [1, Y^{2p-2} - X^{p-1}Y^{p-1} + X^{2p-2}]$), and so induces isomorphisms:

$$\frac{\mathrm{ind}_{IZ}^{G} 1}{T_{1,2}T_{1,0}} \simeq \mathrm{ind}_{KZ}^{G} V_0 \text{ and } \frac{\mathrm{ind}_{IZ}^{G} 1}{1 + T_{1,2}T_{1,0}} \simeq \mathrm{ind}_{KZ}^{G} V_{p-1}, \tag{4}$$

where the action of $T_{-1,0} + T_{1,0}$ (respectively $T_{-1,0}$) on the left corresponds to the action of T on the right. Here we have used the small checks that $\mathrm{Im}\,(T_{1,2}T_{1,0}) \subset \ker(T_{-1,0} + T_{1,0})$ and similarly $\mathrm{Im}\,(1 + T_{1,2}T_{1,0}) \subset \ker(T_{-1,0})$ so that these operators do indeed act on the quotients on the left in (4).

Now clearly the flip involution $T_{1,0}$ induces an isomorphism

$$\mathrm{ind}_{IZ}^{G} 1 \simeq \mathrm{ind}_{IZ}^{G} 1$$

where the roles of the source operator $T_{-1,0}$ and the sink operator $T_{1,2}$ get interchanged. This induces an isomorphism (where we use (3))

$$\frac{\mathrm{ind}_{IZ}^{G} 1}{T_{1,2}T_{1,0}} \simeq \frac{\mathrm{ind}_{IZ}^{G} 1}{T_{1,0}(T_{1,2}T_{1,0}) = T_{-1,0} + \delta_{0,p-1} T_{1,0}}$$

where $T_{-1,0}+T_{1,0}$ on the left corresponds to the conjugate $T_{1,0}(T_{-1,0}+T_{1,0})T_{1,0}=T_{1,2}+\delta_{0,0}T_{1,0}$ on the right. Similarly, we get an induced isomorphism (again by (3))

$$\frac{\operatorname{ind}_{IZ}^{G}1}{1+T_{1,2}T_{1,0}}\simeq\frac{\operatorname{ind}_{IZ}^{G}1}{T_{1,0}(1+T_{1,2}T_{1,0})=T_{-1,0}+\delta_{p-1,p-1}T_{1,0}}$$

where $T_{-1,0}$ on the left corresponds to the conjugate $T_{1,0}(T_{-1,0})T_{1,0}=T_{1,2}+\delta_{p-1,0}T_{1,0}$ on the right. Combining these two isomorphisms with the two in (4) proves the theorem for $r=0,\ p-1$. □

We remark that the mod p representation $\pi(r,\lambda,\eta)$ of G is mostly irreducible, e.g., it always is when $(r,\lambda)\neq(0,\pm1),(p-1,\pm1)$.

2.4 mod *p* Galois representations

We will be brief here. Let $G_{\mathbb{Q}_p}=\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ be the Galois group of $\mathbb{Q}_p$ and $I_{\mathbb{Q}_p}$ be the inertia subgroup of $G_{\mathbb{Q}_p}$.

Lemma 2.4 *Every n-dimensional irreducible representation $\bar\rho$ of $G_{\mathbb{Q}_p}$ over $\overline{\mathbb{F}}_p$ is of the form $\operatorname{ind}_{G_F}^{G_{\mathbb{Q}_p}}\chi$ for F the unramified extension of $\mathbb{Q}_p$ of degree n and some character $\chi:G_F\to\overline{\mathbb{F}}_p^*$.*

Proof This is perhaps well-known, but it is difficult to find a proof (for general n) in the literature so we give a proof. As for $n=2$, the proof starts by noting that the wild inertia subgroup $I^{\mathrm{w}}_{\mathbb{Q}_p}$ of $I_{\mathbb{Q}_p}$ acts trivially. Indeed, $\bar\rho$ has a non-zero $I^{\mathrm{w}}_{\mathbb{Q}_p}$-invariant vector since the subgroup is a pro-p group and we are working mod p. Moreover, $I^{\mathrm{w}}_{\mathbb{Q}_p}$ is normalized by $G_{\mathbb{Q}_p}$ so the invariant vectors are $G_{\mathbb{Q}_p}$-stable, and hence everything, by irreducibility. So $\bar\rho|_{I_{\mathbb{Q}_p}}$ factors through the tame inertia group $I^{\mathrm{t}}_{\mathbb{Q}_p}=I_{\mathbb{Q}_p}/I^{\mathrm{w}}_{\mathbb{Q}_p}$ which is abelian and prime to p, so the restricted representation $\bar\rho|_{I_{\mathbb{Q}_p}}$ is a direct sum of mod p characters of $I_{\mathbb{Q}_p}$. Let χ_0 be one such, and say the χ_0-isotypic component of $\bar\rho|_{I_{\mathbb{Q}_p}}$ has dimension d. A Frobenius element Fr_p of $G_{\mathbb{Q}_p}$ acts on χ_0 by inner conjugation. Say that $\chi_i(g)=\chi(\mathrm{Fr}_p^i g\mathrm{Fr}_p^{-i})$ for $i=0,\dots,n/d-1$ are the n/d distinct characters so obtained so that $\chi_{n/d}=\chi_0$ and

$$\bar\rho|_{I_{\mathbb{Q}_p}}=\chi_0^{\oplus d}\oplus\chi_1^{\oplus d}\oplus\cdots\oplus\chi_{n/d-1}^{\oplus d}.$$

Now $\mathrm{Fr}_p^{n/d}$ preserves χ_0^d, so has an eigen-vector. This will also be an eigen-vector under the action of $G_F=\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ where F is the maximal unramified extension of $\mathbb{Q}_p$ of degree n/d (since G_F is generated by $I_{\mathbb{Q}_p}$ and $\mathrm{Fr}_p^{n/d}$), say with eigen-character χ.

Now $\chi\hookrightarrow\bar\rho|_{G_F}$ implies by Frobenius reciprocity that there is a non-zero map $\operatorname{ind}_{G_F}^{G_{\mathbb{Q}_p}}\chi\to\bar\rho$. This map must be surjective by irreducibility of the target and therefore an isomorphism for dimension reasons since the LHS has dimension n/d. So we must have $d=1$, and the lemma follows. □

We return to the case of $n=2$. Recall that $\omega:G_{\mathbb{Q}_p}\to\mathbb{F}_p^*$ is the fundamental character of level 1 (the mod p cyclotomic character). Let $F=\mathbb{Q}_{p^2}$ be the unramified quadratic extension of $\mathbb{Q}_p$ and $G_{\mathbb{Q}_{p^2}}=\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^2})$. Then recall that $\omega_2^c:I_{\mathbb{Q}_p}\to\mathbb{F}_{p^2}^*$ is the c-th power (for $p+1\nmid c$) of the fundamental character of level 2, extended to $G_{\mathbb{Q}_{p^2}}$ so that $\det\left(\operatorname{ind}\omega_2^c\right)=\omega^c$, where the induction is from $G_{\mathbb{Q}_{p^2}}$ to $G_{\mathbb{Q}_p}$. Finally, recall $\mu_\lambda:G_{\mathbb{Q}_p}\to\overline{\mathbb{F}}_p^*$ is the unramified character taking Fr_p^{-1} (geometric Frobenius) to $\lambda\in\overline{\mathbb{F}}_p^*$.

By the lemma, every two-dimensional (semi-simple) mod p representation of $G_{\mathbb{Q}_p}$ is of the form

1. $\mu_\lambda\omega^a\oplus\mu_{\lambda'}\omega^b$ (reducible case)

2. ind $\omega_2^c \otimes \mu_{\lambda''}$ (irreducible case),

for some integers a, b, c (with $p+1 \nmid c$), and some $\lambda, \lambda', \lambda'' \in \overline{\mathbb{F}}_p^*$.

2.5 Iwahori mod *p* LLC

We can now state an Iwahori theoretic version of Breuil's semi-simple mod p LLC for $\mathbb{Q}_p$. Recall that for $0 \le r \le p-1$, $\lambda \in \overline{\mathbb{F}}_p$ and $\eta : \mathbb{Q}_p^* \to \overline{\mathbb{F}}_p^*$ a smooth character, we had defined the following smooth mod p representation of G

$$\pi(r, \lambda, \eta) := \frac{\mathrm{ind}_{IZ}^G\, d^r}{(T_{-1,0} + \delta_{r,p-1} T_{1,0}) + (T_{1,2} + \delta_{r,0} T_{1,0} - \lambda)} \otimes \eta,$$

where $\delta_{a,b} = 1$ if $a = b$ and is 0 otherwise.

Theorem 2.5 *(Iwahori mod p LLC) For $r \in \{0, \ldots, p-1\}$, $\lambda \in \overline{\mathbb{F}}_p$ and $\eta : \mathbb{Q}_p^* \to \overline{\mathbb{F}}_p^*$ a smooth character, we have the following correspondence between mod p representations of $G_{\mathbb{Q}_p}$ and certain smooth mod p representations of $G = \mathrm{GL}_2(\mathbb{Q}_p)$.*

- *If $\lambda = 0$:*
$$(\mathrm{ind}\, \omega_2^{r+1}) \otimes \eta \quad \longmapsto \quad \pi(r, 0, \eta)$$
- *If $\lambda \neq 0$:*
$$(\mu_\lambda \omega^{r+1} \oplus \mu_{\lambda^{-1}}) \otimes \eta \longmapsto \pi(r, \lambda, \eta)^{\mathrm{ss}} \oplus \pi([p-3-r], \lambda^{-1}, \eta\omega^{r+1})^{\mathrm{ss}},$$

where $[a] \in \{0, \ldots, p-2\}$ represents the class of a modulo $(p-1)$.

Proof This follows immediately from the ABC theorem and an identical statement (the definition of the semi-simple mod p LLC) due to Breuil [11] but where the $\pi(r, \lambda, \eta)$ are defined using spherical induction. In fact, the end goal of [19] was to be able to state such an Iwahori theoretic version of the mod p LLC. □

3 Functions of a *p*-adic variable

In this section, we recall some functional analysis for functions of one variable on $\mathbb{Q}_p$. Our main reference is [23] which is an excellent introduction to the topic.

3.1 *p*-adic Banach spaces

3.1.1 *E*-Banach spaces

Let $E \subset \mathbb{C}_p$ be a subfield, usually taken to be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_E$ its valuation ring, $\mathfrak{m}_E$ its maximal ideal, and $k_E = \mathcal{O}_E/\mathfrak{m}_E$ its residue field. Let v_p be the valuation on $\mathbb{C}_p$ normalized such that $v_p(p) = 1$ and let $|x|_p = p^{-v_p(x)}$ be the corresponding norm on $\mathbb{C}_p$.

Let B be an E-vector space. A *valuation* on B is a function $v_B : B \to \mathbb{R} \cup \{\infty\}$ such that

i) $v_B(x) = \infty \iff x = 0$

ii) $v_B(x+y) \geq \inf(v_B(x), v_B(y))$ for all $x, y \in B$
iii) $v_B(\lambda x) = v_p(\lambda) + v_B(x)$ for all $\lambda \in E$, $x \in B$.

An *E-Banach space* is a topological vector space over E with topology given by a valuation v_B with respect to which the topology is complete. A *map* $f : B_1 \to B_2$ of E-Banach spaces is a linear map that is continuous. A map $f : B_1 \to B_2$ is an isometry if it is bijective and $v_{B_2}(f(x)) = v_{B_1}(x)$ for all $x \in B$ (this last condition implies that f must be injective).

For example, if I is an indexing set (assumed in these notes to be $\mathbb{N}_{\geq 0}$, mostly to avoid some extra language), then the following are E-Banach spaces:

1. (bounded sequences)
$$l_\infty(I, E) = \{(a_i)_{i \in I} \mid |a_i|_p \text{ is bounded above}\}$$
with valuation $v_{l_\infty}((a_i)) = \inf_{i \in I} v_p(a_i)$.
2. (null sequences) the subspace
$$l_\infty^0(I, E) = \{(a_i)_{i \in I} \mid |a_i|_p \to 0\}$$
with the same valuation.

The space $l_\infty^0(I, E)$ is the closure in $l_\infty(I, E)$ of the subspace consisting of sequences with only finitely many non-zero terms.

Proposition 3.1 *1. (Open mapping theorem) If $f : B_1 \to B_2$ is a map of E-Banach spaces with f bijective, then f^{-1} is continuous.*
2. (Banach-Steinhaus theorem) A limit of (linear, continuous) maps between Banach spaces is continuous.

3.1.2 ONB of a *p*-adic Banach space

A family $(e_i)_{i \in I}$ of elements of an E-Banach space B is an *orthonormal basis* (ONB) of B if the map

$$\begin{array}{ccc} l_\infty^0(I, E) & \to & B \\ (a_i)_{i \in I} & \mapsto & \sum_{i \in I} a_i e_i \end{array} \tag{5}$$

is an isometry of E-Banach spaces. That is,

i) every element $x \in B$ can be written uniquely as a convergent series $x = \sum_{i \in I} a_i e_i$ with $|a_i|_p \to 0$, and
ii) $v_B(x) = \inf_{i \in I} v_p(a_i)$.

A bit less stringently, a family $(e_i)_{i \in I}$ of elements in B is (only) a *Banach basis* if the map displayed above is 'only' an isomorphism of Banach spaces, that is, i) holds but ii) is replaced by the weaker condition:

ii)$'$ There exists a constant $C \geq 0$ such that $-C + \inf_{i \in I} v_p(a_i) \leq v_B(x) \leq C + \inf_{i \in I} v_p(a_i)$.

For example, if I is a set and if $\delta_i = (a_j)$ with $a_j = \delta_{i,j}$, then $(\delta_i)_{i \in I}$ forms an ONB of $l_\infty^0(I, E)$.

Proposition 3.2 *Say $v_p(E)$ is discrete and π_E is a uniformizer. Then*

i) Every E-Banach space has a Banach basis.

ii) An E-Banach space possesses an ONB $\iff$ $v_B(B) = v_p(E)$. Moreover, under this hypothesis if $B^0 = \{x \in B \mid v_B(x) \geq 0\}$, then a family of elements $(e_i)_{i\in I}$ in B^0 is an ONB of B $\iff$ $(\bar{e}_i)_{i\in I}$ is an algebraic basis of the k_E-vector space $\bar{B} := B^0/\pi_E B^0$.

Proof This proof is adapted from the proof of [23, Proposition I.1.5] but reminds one of the proof given in Serre's foundational article [35] which is also a good general reference.

First note that we may assume $v_B(B) = v_p(E)$ by replacing v_B by the equivalent valuation v'_B defined by $v'_B(x) = v_p(\pi_E)\cdot\lfloor\frac{v_B(x)}{v_p(\pi_E)}\rfloor \in v_p(\pi_E)\cdot\mathbb{Z}$ which is clearly $v_p(E)$-valued. If one proves ii), so that there is an ONB for v'_B, then this basis will be a Banach basis for v_B (whose values differ from those of v'_B by at most the constant $v_p(\pi_E)$), so i) will follow.

Also note that the forward implication in the first statement in ii) is clear since v_p is discrete on E and so infimums are attained. So to prove ii), it suffices to show that if $v_B(B) = v_p(E)$, then for a family (e_i) in B^0

$$(e_i) \text{ is an ONB of } B \iff (\bar{e}_i) \text{ is an algebraic basis of } \bar{B}.$$

($\Rightarrow$) Suppose (e_i) is an ONB of B. If $\bar{x} \in \bar{B}$ for $x \in B^0$, then $v_B(x) \geq 0$ so $x = \sum a_i e_i$ with $a_i \in \mathcal{O}_E$ and $a_i \to 0$. Therefore $\bar{a}_i = 0$ for $i \gg 0$. So $\bar{x} = \sum \bar{a}_i\bar{e}_i$ is a finite linear combination of the $\bar{e}_i$. So the $(\bar{e}_i)$ form a spanning set of $\bar{B}$.

If $\sum \bar{a}_i\bar{e}_i = 0$, with $\bar{a}_i \in k_E$, lifting to say $a_i \in \mathcal{O}_E$, then $x = \sum a_i e_i$ satisfies $\bar{x} = 0$, so $v_B(x) > 0$. But $v_B(x) = \inf v_p(a_i)$. So $v_p(a_i) > 0$ for all i. So $\bar{a}_i = 0$ for all i. So the $(\bar{e}_i)$ are also linearly independent.

($\Leftarrow$) Suppose $(\bar{e}_i)$ is an algebraic basis of $\bar{B}$. Let $s : k_E \to S \subset \mathcal{O}_E$ be a system of representatives of k_E with $s(0) = 0$. If $x \in B^0$ and $\bar{x} = \sum \bar{a}_i\bar{e}_i \in \bar{B}$ with $\bar{a}_i \in k_E$ almost all 0, set $s(x) = \sum s(\bar{a}_i)e_i$. So $x - s(x) \in \pi_E B^0$. Construct by induction a sequence of elements x_n of B^0 for $n \geq 0$ by setting $x_0 = x$ and $x_{n+1} = \frac{1}{\pi_E}(x_n - s(x_n))$ for $n \geq 0$. Then

$$x = \sum_{n=0}^{k} s(x_n)\pi_E^n + x_{k+1}\pi_E^{k+1}$$

for all $k \geq 0$. Write $s(x_n) = \sum_{i\in I} s_{n,i}e_i$ with $s_{n,i} \in S$ (almost all 0). Set $a_i = \sum_{n=0}^{\infty} s_{n,i}\pi_E^n$. Clearly $a_i \to 0$ since for i sufficiently large compared to n the coefficient $s_{m,i}$ is equal to 0 for all $m \leq n$. Note that $\sum_{i\in I} a_i e_i = x$. Thus the map (5) is surjective onto B^0, and therefore onto B (by multiplying and later dividing by a sufficiently large power of π_E).

Now suppose that $v_{l_\infty}(a_i) = 0$, i.e., $\inf v_p(a_i) = 0$. Then at least one a_i is a unit, so that $\sum a_i e_i \neq 0 \mod \pi_E B^0$ since some $\bar{a}_i \neq 0$ and the $\bar{e}_i$ form a basis of $\bar{B}$. This implies that $0 \leq v_B(\sum a_i e_i) \lneq v_p(\pi_E)$. But $v_B(B) = v_p(E)$. So we must have $v_B(\sum a_i e_i) = 0$. So (5) is an isometry (and is injective). □

3.1.3 Dual of an *E*-Banach

We recall the notion of the dual space B^* of an E-Banach space B: it is the E-vector space of linear forms $f : B \to E$ with (the strong) topology defined by the valuation

$$v_{B^*}(f) = \inf_{x\in B\setminus\{0\}} \big(v_p(f(x)) - v_B(x)\big)$$

with respect to which it is complete.

Proposition 3.3 *Let I $(= \mathbb{N}_{\geq 0})$ be an indexing set.*

i) If $a = (a_i)_{i\in I} \in l^0_\infty(I, E)$, $b = (b_i)_{i\in I} \in l_\infty(I, E)$, then the series

$$f_b(a) = \sum_{i\in I} a_i b_i$$

converges in E.

ii) The map

$$l_\infty(I,E) \to l^0_\infty(I,E)^*$$
$$b \mapsto f_b$$

is an isometry.

Proof i) is obvious since $a_i \to 0$ and (b_i) bounded implies $a_i b_i \to 0$. We remark that if $a_j = (a_{i,j}) \to 0$ as $j \to 0$, then clearly $f_b(a_j) \to 0$ as well so f_b is continuous and the map in ii) is well defined. The map in ii) is also continuous. Indeed

$$v_{l^0_\infty(I,E)^*}(f_b) = \inf_{(a_i)\in l^0_\infty(I,E)\setminus\{0\}} \big(v_p(f_b(a_i)) - v_{l_\infty}(a_i)\big) \geq \inf_{(a_i)} \left(\inf_i (v_p(b_i) + v_p(a_i)) - \inf_i v_p(a_i)\right)$$
$$\geq \inf_{(a_i)} \left(\inf_i v_p(b_i) + \inf_i v_p(a_i) - \inf_i v_p(a_i)\right) = v_{l_\infty}(b).$$

Injectivity of the map in ii) is also obvious (if $f_b = 0$, then $b_i = f_b(\delta_i) = 0$ for all $i \in I$, so $b = 0$). For surjectivity, let $f \in l^0_\infty(I,E)^*$. If $b_i := f(\delta_i)$, then $v_p(b_i) = v_p(b_i) - v_{l_\infty}(\delta_i) \geq v_{l^0_\infty(I,E)^*}(f)$ so $b = (b_i) \in l_\infty(I,E)$ is bounded. Moreover $(f - f_b)(\delta_i) = b_i - b_i = 0$. But the space generated by the δ_i is dense in $l^0_\infty(I,E)$, and f continuous implies $f - f_b$ is continuous, so must be identically 0. So $f = f_b$ and the map is surjective. It follows from Proposition 3.1, part i) that the map in ii) is open. Finally, the map in ii) is an isometry. Indeed, if $v_{l_\infty}(b) = 0$, then by the continuity proved above $v_{l^0_\infty(I,E)^*}(f_b) \geq 0$. But also, there is a b_i which must be a unit. Then $v_p(f_b(\delta_i)) = v_p(b_i) = 0$ and since $v_{l_\infty}(\delta_i) = 0$ the infimum $v_{l^0_\infty(I,E)^*}(f_b)$ of the difference over all non-zero (a_i) is at most 0, hence is 0. □

3.2 Continuous functions on $\mathbb{Z}_p$

Let $\mathscr{C}^0(\mathbb{Z}_p, E) = \{g : \mathbb{Z}_p \to E \mid g \text{ is continuous}\}$. Since $\mathbb{Z}_p$ is compact, $g(\mathbb{Z}_p)$ is bounded if $g \in \mathscr{C}^0(\mathbb{Z}_p, E)$, so $v_{\mathscr{C}^0}(g) = \inf_{x\in\mathbb{Z}_p} v_p(g(x))$ makes sense and makes $\mathscr{C}^0(\mathbb{Z}_p, E)$ into an E-Banach space.

3.2.1 Binomial polynomials

For $n \in \mathbb{N}_{\geq 0}$, let

$$\binom{x}{n} = \begin{cases} 1 & \text{if } n = 0, \\ \dfrac{x(x-1)\cdots(x-n+1)}{n!} & \text{if } n \geq 1. \end{cases}$$

Proposition 3.4 $v_{\mathscr{C}^0}(\binom{x}{n}) = 0$ *for all* $n \geq 0$.

Proof The polynomial $\binom{x}{n}$ maps $\mathbb{Z}$ to $\mathbb{Z}$ and is continuous (it is a polynomial!), so takes $\mathbb{Z}_p$ to $\mathbb{Z}_p$, so certainly $v_{\mathscr{C}^0}(\binom{x}{n}) \geq 0$. But $\binom{n}{n} = 1$, so $v_{\mathscr{C}^0}(\binom{x}{n}) = 0$. □

3.2.2 Mahler coefficients of continuous functions

For $g \in \mathscr{C}^0(\mathbb{Z}_p, E)$, define $g^{[0]} = g$ and $g^{[k+1]}(x) = g^{[k]}(x+1) - g^{[k]}(x)$ for $k \geq 0$. An easy check shows that

$$\binom{x}{n}^{[k]} = \begin{cases} \binom{x}{n-k} & \text{for } 0 \leq k \leq n, \\ 0 & \text{for } k \geq n. \end{cases} \tag{6}$$

 https://doi.org/10.1007/s13226-025-00897-0

Also define the *n-th Mahler coefficient* of g to be

$$a_n(g) = g^{[n]}(0).$$

Then one may check

$$g^{[n]}(x) = \sum_{i=0}^{n}(-1)^i\binom{n}{i}g(x+n-i), \text{ so} \tag{7}$$

$$a_n(g) = \sum_{i=0}^{n}(-1)^i\binom{n}{i}g(n-i). \tag{8}$$

Lemma 3.5 $g \in \mathscr{C}^0(\mathbb{Z}_p, E) \implies v_{\mathscr{C}^0}(g^{[p^k]}) \geq v_{\mathscr{C}^0}(g) + 1$ *for some* $k \in \mathbb{N}_{\geq 0}$.

Proof Since $\mathbb{Z}_p$ is compact, g is uniformly continuous. So there exists $k \geq 0$ such that $v_p(g(x+p^k) - g(x)) \geq v_{\mathscr{C}^0}(g) + 1$ for all $x \in \mathbb{Z}_p$. Fix $x \in \mathbb{Z}_p$. By (7) and adding and subtracting $g(x)$ we can write

$$g^{[p^k]}(x) = \Big(g(x+p^k) - g(x)\Big) + \left(\sum_{i=1}^{p^k-1}(-1)^i\binom{p^k}{i}g(x+p^k-i)\right) + \Big((1+(-1)^{p^k})g(x)\Big).$$

Now the valuation v_p of each of the terms in parentheses on the right is at least

$$v_{\mathscr{C}^0}(g) + 1,$$

the first by what we just deduced above, the second since $p|\binom{p^k}{i}$ for $i \neq 0,\ p^k$, and the last if $p = 2$ (though this term vanishes if p is odd). This proves that the valuation v_p of the term on the left is also bounded below by the above quantity, from which we deduce the lemma by varying over all $x \in \mathbb{Z}_p$. □

Theorem 3.6 *i) If* $g \in \mathscr{C}^0(\mathbb{Z}_p, E)$, *then*

a) $a_n(g) \to 0$, *and*
b) $\sum_{n=0}^{\infty} a_n(g)\binom{x}{n} = g(x)$, *for all* $x \in \mathbb{Z}_p$.

ii) The map

$$\begin{aligned}\mathscr{C}^0(\mathbb{Z}_p, E) &\to l_\infty^0(\mathbb{N}_{\geq 0}, E)\\ g &\mapsto (a_n(g))_{n\geq 0}\end{aligned}$$

is an isometry.

Proof The proof here adapted from the proof of [22, Theorem I.2.3]; another good exposition can be found in Hida's text book [33].

Let $g \in \mathscr{C}^0(\mathbb{Z}_p, E)$. By repeated use of the lemma, and the obvious fact that $g^{[l+k]} = (g^{[l]})^{[k]}$ for $l, k \geq 0$, we see that there are $k_1, k_2, \dots, k_m$ such that $v_{\mathscr{C}^0}(g^{[p^{k_1}+\dots+p^{k_m}]}) \geq v_{\mathscr{C}^0}(g) + m$. Taking m sufficiently large, we see that given $C \geq 0$, there exists N such that $v_{\mathscr{C}^0}(g^{[N]}) \geq C$. Then by (8) we have $v_p(a_n(g)) = v_p(a_{n-N}(g^{[N]})) \geq v_{\mathscr{C}^0}(g^{[N]}) \geq C$ for $n \geq N$. We deduce that $a_n(g) \to 0$. This proves i) a).

This shows that the map in ii)

$$\mathscr{C}^0(\mathbb{Z}_p, E) \to l_\infty(\mathbb{N}_{\geq 0}, E)$$

$$g \mapsto a(g) := (a_n(g))_{n\geq 0}$$

is well defined. It is also clearly injective since if the Mahler coefficients $a_n(g)$ of g vanish, then by a recursive use of (8) the continuous function g vanishes on the dense set $\mathbb{N}_{\geq 0} \subset \mathbb{Z}_p$ and hence $g = 0$. The map is also continuous since again by (8)

$$v_{l_\infty}(a(g)) = \inf_{n\geq 0} v_p(a_n(g)) \geq \inf_{n\geq 0, 0\leq i\leq n} v_p(g(n-i)) \geq \inf_{x\in\mathbb{Z}_p} v_p(g(x)) = v_{\mathscr{C}^0}(g). \tag{9}$$

We prove the surjectivity of the above map onto $l^0_\infty(\mathbb{N}_{\geq 0}, E)$. We first claim that if $a = (a_n)_{n\geq 0} \in l^0_\infty(\mathbb{N}_{\geq 0}, E)$, then

$$s_\infty(x) = \sum_{k=0}^{\infty} a_k \binom{x}{k}$$

is a continuous function of $x \in \mathbb{Z}_p$. First note that the sequence of partial sums $s_n(x)$ converges uniformly to $s_\infty(x)$ on $\mathbb{Z}_p$. Indeed, given $C \geq 0$, there exists N_C such that

$$v_p(s_n(x) - s_\infty(x)) = v_p\left(\sum_{k=n+1}^{\infty} a_k \binom{x}{k}\right) \geq \inf_{k\geq n+1}(v_p(a_k) + v_{\mathscr{C}^0}(\binom{x}{k})) \geq \inf_{k\geq n+1} v_p(a_k) \geq C$$

for all $n \geq N_C$, since $a_k \to 0$, and where we have used Proposition 3.4. A uniform limit of continuous functions is continuous, so $g_a := s_\infty : \mathbb{Z}_p \to E$ is continuous. Now, by (6), $g_a^{[n]}(0) = a_n$, so under the above map $g_a \mapsto a$, and so the above map is surjective onto $l^0_\infty(\mathbb{N}_{\geq 0}, E)$.

Now if $g \in \mathscr{C}^0(\mathbb{Z}_p, E)$, then g and $g_{a(g)}$ have the same Mahler coefficients so must be equal since the map above is injective. Thus g satisfies i) b).

Note that

$$v_{\mathscr{C}^0}(g_a) \geq \inf_{n\geq 0} v_{\mathscr{C}^0}(a_n \binom{x}{n}) \geq \inf_{n\geq 0} v_p(a_n) = v_{l_\infty}(a) \tag{10}$$

by Proposition 3.4. So if $g \in \mathscr{C}^0(\mathbb{Z}_p, E)$, then by (9) and (10), we have

$$v_{l_\infty}(a(g)) \geq v_{\mathscr{C}^0}(g) = v_{\mathscr{C}^0}(g_{a(g)}) \geq v_{l_\infty}(a(g)).$$

So $v_{l_\infty}(a(g)) = v_{\mathscr{C}^0}(g)$ and the map in ii) is an isometry.

□

Corollary 3.7 *The* $\binom{x}{n}$ *for* $n \geq 0$ *form an ONB of* $\mathscr{C}^0(\mathbb{Z}_p, E)$.

3.3 Wavelet decompositions of continuous functions

3.3.1 Locally constant functions

Let

$$\mathrm{LC}_h(\mathbb{Z}_p, E) = \{g : \mathbb{Z}_p \to E \mid g|_{a+p^h\mathbb{Z}_p} \text{ is constant for all } a \in \mathbb{Z}_p\}.$$
$$\mathrm{LC}(\mathbb{Z}_p, E) = \bigcup_{h\geq 0} \mathrm{LC}_h(\mathbb{Z}_p, E).$$

We start with the following well-known fact.

Lemma 3.8 $\mathrm{LC}(\mathbb{Z}_p, E) \subset \mathscr{C}^0(\mathbb{Z}_p, E)$ *is dense.*

Proof Let $g \in \mathscr{C}^0(\mathbb{Z}_p, E)$. Then g continuous, $\mathbb{Z}_p$ compact $\implies$ g is uniformly continuous, so for all $C \geq 0$, there exists an $m \geq 0$ such that $v_p(x-y) \geq m \implies v_p(g(x)-g(y)) \geq C$. Let

$$g_m(x) = \sum_{i=0}^{p^m-1} g(i)\mathbb{1}_{i+p^m\mathbb{Z}_p}(x).$$

If $x \in \mathbb{Z}_p$, then there exists a unique $0 \leq i \leq p^m - 1$ such that $x \in i + p^m\mathbb{Z}_p$, so $v_p(g(x) - g_m(x)) = v_p(g(x) - g(i)) \geq C$. Then $v_{\mathscr{C}^0}(g - g_m) = \inf_{x\in\mathbb{Z}_p} v_p(g(x) - g_m(x)) \geq C$. ◻

If $i \in \mathbb{N}_{\geq 0}$, set $l(i)$ to be the smallest $n \geq 0$ such that $p^n > i$. So $l(0) = 0$ and $l(i) = \lfloor \frac{\log i}{\log p} \rfloor + 1$.

Proposition 3.9 *We have*

1. *The* $\mathbb{1}_{i+p^{l(i)}\mathbb{Z}_p}$ *for* $0 \leq i \leq p^h - 1$ *form a basis of* $\mathrm{LC}_h(\mathbb{Z}_p, E)$.
2. *The* $\mathbb{1}_{i+p^{l(i)}\mathbb{Z}_p}$ *for* $i \geq 0$ *form a basis of* $\mathrm{LC}(\mathbb{Z}_p, E)$.
3. *The* $\mathbb{1}_{i+p^{l(i)}\mathbb{Z}_p}$ *for* $i \geq 0$ *form an ONB of* $\mathscr{C}^0(\mathbb{Z}_p, E)$.

Proof By definition, the $\mathbb{1}_{i+p^h\mathbb{Z}_p}$ form a basis of $\mathrm{LC}_h(\mathbb{Z}_p, E)$. Also, for $i \leq p^h - 1$

$$\mathbb{1}_{i+p^{l(i)}\mathbb{Z}_p} = \sum_{j=0}^{p^{h-l(i)}-1} \mathbb{1}_{i+jp^{l(i)}+p^h\mathbb{Z}_p}.$$

One checks that the change of basis matrix (of size $p^h \times p^h$) is lower-triangular with 1's on the diagonal, so is invertible, so i), ii) follow.

Consider the isometry

$$\{\text{a. e. zero sequences}\} \to \mathrm{LC}(\mathbb{Z}_p, E)$$
$$(a_i)_{i\geq 0} \mapsto \sum_{i\geq 0} a_i \mathbb{1}_{i+p^{l(i)}\mathbb{Z}_p}.$$

Since the LHS is dense in $l^0_\infty(\mathbb{N}_{\geq 0}, E)$, the above map extends to an isometry

$$l^0_\infty(\mathbb{N}_{\geq 0}, E) \to \overline{\mathrm{LC}(\mathbb{Z}_p, E)} = \mathscr{C}^0(\mathbb{Z}_p, E)$$

where we have used Lemma 3.8. ◻

The ONB of $\mathscr{C}^0(\mathbb{Z}_p, E)$ consisting of $\mathbb{1}_{i+p^{l(i)}\mathbb{Z}_p}$ is called the *basis of wavelets*. If $g \in \mathscr{C}^0(\mathbb{Z}_p, E)$ satisfies

$$g = \sum_{i\geq 0} b_i(g)\mathbb{1}_{i+p^{l(i)}\mathbb{Z}_p},$$

then the $b_i(g) \in E$ are called the *amplitude coefficients* of g.

3.3.2 Mahler coefficients of locally constant functions

Notation: If $z \in E$ and $v_p(z-1) > 0$, then

$$g_z(x) = \sum_{n=0}^{\infty} \binom{x}{n} (z-1)^n$$

converges uniformly (one uses Proposition 3.4 again, see the proof of Theorem 3.6) so is a continuous function of $x \in \mathbb{Z}_p$. Also $g_z(k) = \sum_{n=0}^{\infty} \binom{k}{n}(z-1)^n = (z-1+1)^k = z^k$, so we may speak of $x \mapsto z^x$ as a function on $\mathbb{Z}_p$. If $z = \zeta_{p^n} \in \mu_{p^n}$ is a p^n-th root of unity, then $z^{p^n} = 1 \implies z^{x+p^n\mathbb{Z}_p} = z^x$ for $x \in \mathbb{Z}_p$, so z^x is a locally constant function on $\mathbb{Z}_p$.

Proposition 3.10 *1. Say $\mu_{p^h} \subset E$. Then $\zeta_{p^h}^x$ for $\zeta \in \mu_{p^h}$ form a basis of* $\mathrm{LC}_h(\mathbb{Z}_p, E)$.
2. Say $\mu_{p^\infty} \subset E$. Then ζ^x for $\zeta \in \mu_{p^\infty}$ form a basis of $\mathrm{LC}(\mathbb{Z}_p, E)$.

Proof For i), note that for $x \in \mathbb{Z}_p$, we have $\mathbb{1}_{a+p^h\mathbb{Z}_p}(x) = \frac{1}{p^h}\sum_{\zeta \in \mu_{p^h}} \zeta^{x-a}$ since

$$\sum_{\zeta \in \mu_{p^h}} \zeta^x = \begin{cases} p^h & \text{if } x \in p^h\mathbb{Z}_p \\ 0 & \text{otherwise.} \end{cases}$$

Thus the functions ζ^x for $\zeta \in \mu_{p^h}$ generate $\mathrm{LC}_h(\mathbb{Z}_p, E)$ and even form a basis of $\mathrm{LC}_h(\mathbb{Z}_p, E)$ since their number is p^h.

Statement ii) follows from i). □

Remark 3.11 For all $i, j \geq 0$, set

$$\alpha_{i,j} = \frac{1}{p^{l(i)}} \sum_{\zeta \in \mu_{p^{l(i)}}} \zeta^{-i}(\zeta-1)^j$$

Using some algebraic number theoretic arguments (see [23, Lemma I.3.5]), it can be shown that

$$\begin{cases} \alpha_{i,j} = 0 & \text{if } j < i \\ \alpha_{i,j} = 1 & \text{if } j = i \\ v_p(\alpha_{i,j}) \geq \left\lfloor \dfrac{j - p^{l(i)-1}}{p^{l(i)} - p^{l(i)-1}} \right\rfloor & \text{if } j > i. \end{cases}$$

Now

$$\mathbb{1}_{i+p^{l(i)}\mathbb{Z}_p}(x) = \frac{1}{p^{l(i)}} \sum_{\zeta \in \mu_{p^{l(i)}}} \zeta^{x-i} = \sum_{n=0}^{\infty} \binom{x}{n} \sum_{\zeta \in \mu_{p^{l(i)}}} \frac{1}{p^{l(i)}} \zeta^{-i}(\zeta-1)^n$$

so its Mahler coefficients $a_n(\mathbb{1}_{i+p^{l(i)}\mathbb{Z}_p})$ equal $\alpha_{i,n}$, which clearly tend to 0 as $n \to \infty$. This gives another proof of the surjectivity of the map in Theorem 3.6, ii). Indeed considering that map to be taking values in the bigger space $l_\infty(\mathbb{N}_{\geq 0}, E)$, we note that the pre-image B of the closed subspace $l_\infty^0(\mathbb{N}_{\geq 0}, E)$ is closed and, by the above remarks and Proposition 3.10, part ii), contains $\mathrm{LC}(\mathbb{Z}_p, E)$. Then $B = \overline{B} \supset \overline{\mathrm{LC}(\mathbb{Z}_p, E)} = \mathscr{C}^0(\mathbb{Z}_p, E)$, by Lemma 3.8. So $B = \mathscr{C}^0(\mathbb{Z}_p, E)$.

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3.4 Locally analytic functions

3.4.1 Locally analytic functions on a closed disk

We will be brief. For $a \in E$ and $r \in \mathbb{R}$, let

$$B(a, r) = \{x \in \mathbb{C}_p \mid v_p(x - a) \geq r\}$$

be the ball of radius p^{-r} centered at a in $\mathbb{C}_p$.

A function $g : B(a, r) \to \mathbb{C}_p$ is *E-analytic* if there exists a sequence $a_k(g, a)$ of elements of E for $k \geq 0$ such that $v_p(a_k(g, a) + kr) \to \infty$ as $k \to \infty$ and

$$g(x) = \sum_{k=0}^{\infty} a_k(g, a)(x - a)^k \tag{11}$$

for all $x \in B(a, r)$. Let

$$\mathrm{An}(B(a, r), E) = \{E\text{-analytic functions on } B(a, r)\}.$$

This is an E-Banach space under

$$v_{B(a,r)}(g) = \inf_{k \geq 0} v_p(a_k(g, a) + kr).$$

Proposition 3.12 *If $g_1, g_2 \in \mathrm{An}(B(a, r), E)$, then $g_1 g_2 \in \mathrm{An}(B(a, r), E)$.*

Proof See [23, Proposition I.4.2]. □

Proposition 3.13 *If $g \in \mathrm{An}(B(a, r), E)$, then $v_{B(a,r)}(g) = \inf_{x \in B(a,r)} v_p(g(x))$.*

Proof See [23, Proposition I.4.3]. □

3.4.2 Locally analytic functions on $\mathbb{Z}_p$

Define for $h \in \mathbb{N}_{\geq 0}$

$$\mathrm{LA}_h(\mathbb{Z}_p, E) = \{g : \mathbb{Z}_p \to E \mid \text{for } a \in \mathbb{Z}_p,\ g\,|_{a + p^h \mathbb{Z}_p} = g_{a,h} \text{ for some } g_{a,h} \in \mathrm{An}(B(a, h), E)\}.$$

This is an E-Banach space under

$$v_{\mathrm{LA}_h}(g) = \inf_{a \in \mathbb{Z}_p} v_{B(a,h)}(g_{a,h}).$$

Remark 3.14 There is an alternative description to $\mathrm{LA}_h(g)$. For each $a \in \mathbb{Z}_p$, write g as

$$g(x) = \sum_{k=0}^{\infty} \tilde{a}_k(g, a) \left(\frac{x - a}{p^h} \right)^k. \tag{12}$$

Since the coefficients in (11) and (12) are related by $a_k(g, a) = \tilde{a}_k(g, a) p^{-hk}$, we see that $v_{B(a,h)}(g) = \inf_{k \geq 0} v_p(\tilde{a}_k(g, a))$. Thus

$$v_{\mathrm{LA}_h}(g) = \inf_{a \in \mathbb{Z}_p} \inf_{k \geq 0} v_p(\tilde{a}_k(g, a)).$$

Note that both here and just before the remark, we may restrict the infimum over $a \in \mathbb{Z}_p$ to a (finite) set of representatives a of $\mathbb{Z}_p/p^h\mathbb{Z}_p$.

Let

$$\mathrm{LA}(\mathbb{Z}_p, E) = \bigcup_{h \in \mathbb{N}_{\geq 0}} \mathrm{LA}_h(\mathbb{Z}_p, E).$$

3.4.3 Mahler coefficients of locally analytic functions

We only state the following interesting results as background since we will not use them.

Theorem 3.15 *The* $\lfloor \frac{n}{p^h} \rfloor ! \binom{x}{n}$ *for* $n \in \mathbb{N}_{\geq 0}$ *form an ONB of* $\mathrm{LA}_h(\mathbb{Z}_p, E)$.

Proof See [23, Theorem I.4.7]. □

Corollary 3.16 *Say* $g \in \mathscr{C}^0(\mathbb{Z}_p, E)$. *Then*

$$g \in \mathrm{LA}(\mathbb{Z}_p, E) \iff \liminf_{n \geq 0} \left(\frac{v_p(a_n(g))}{n} \right) > 0.$$

Proof See [23, Corollary I.4.8]. □

3.5 Functions of class $\mathscr{C}^r$

3.5.1 Differentiable functions

A function $g : \mathbb{Z}_p \to E$ is *differentiable* at $x_0 \in \mathbb{Z}_p$ if

$$\lim_{h \to 0} \frac{g(x_0 + h) - g(x_0)}{h}$$

exists. If the limit exists, then it is denoted as in the real world (pun intended) by $g'(x_0)$ or $g^{(1)}(x_0)$. We say $g : \mathbb{Z}_p \to E$ is *differentiable of order* 1 if g is differentiable at each $x_0 \in \mathbb{Z}_p$. As usual such a function is continuous on $\mathbb{Z}_p$ since for all $x_0 \in \mathbb{Z}_p$, we have $\lim_{h\to 0} g(x_0 + h) - g(x_0) = g'(x_0) \lim_{h \to 0} h = 0$. More generally, say that $g : \mathbb{Z}_p \to E$ is *differentiable of order* k with the k-th derivative denoted by $g^{(k)}$ if g is differentiable of order $k - 1$ and $g^{(k-1)}$ is differentiable of order 1.

We now come to one of the most important definition of these notes. Let $r \geq 0$ be a real number. We say that a function $g : \mathbb{Z}_p \to E$ is of *class* $\mathscr{C}^r$ if there exist functions

$$g^{(j)} : \mathbb{Z}_p \to E$$

for $0 \leq j \leq \lfloor r \rfloor$ such that if $\varepsilon_{g,r} : \mathbb{Z}_p \times \mathbb{Z}_p \to E$ is given by

$$\varepsilon_{g,r}(x, y) = g(x + y) - \sum_{j=0}^{\lfloor r \rfloor} g^{(j)}(x) \frac{y^j}{j!} \tag{13}$$

and $C_{g,r} : \mathbb{N}_{\geq 0} \to \mathbb{R} \cup \{\infty\}$ is given by

$$C_{g,r}(h) = \inf_{\substack{x \in \mathbb{Z}_p \\ y \in p^h \mathbb{Z}_p}} (v_p(\varepsilon_{g,r}(x, y)) - rh),$$

then $C_{g,r}(h) \to \infty$ as $h \to \infty$.

Remark 3.17
- Taking $y = 0$, we have $\varepsilon_{g,r}(x, 0) = g(x) - g^{(0)}(x)$, so
$$C_{g,r}(h) \leq v_p(g(x) - g^{(0)}(x)) - rh.$$
Since the LHS goes to ∞ as $h \to \infty$, so must the RHS, and we deduce $g^{(0)}(x) = g(x)$ for all $x \in \mathbb{Z}_p$.
- One may check that more generally the $g^{(j)}$ in the definition are the j-th derivatives of g for $0 \leq j \leq \lfloor r \rfloor$ (see Remark 3.21).
- Fix $x \in \mathbb{Z}_p$. If follows from (13) that for $h \in \mathbb{N}_{\geq 0}$ and $y \in p^h\mathbb{Z}_p$
$$v_p(\varepsilon_{g,r}(x, y)) \geq \min\left(v_p(g(x + y) - g(x)), \min_{j=1,\dots,\lfloor r \rfloor} v_p(\frac{g^{(j)}(x)}{j!} y^j)\right).$$
Now the LHS goes to ∞ as $h \to \infty$ (even after subtracting rh from it). But each of the terms except the first on the RHS also goes to ∞ as $h \to \infty$ since $v_p(\frac{g^{(j)}(x)}{j!} y^j) \geq v_p(\frac{g^{(j)}(x)}{j!}) + jh$ and x is fixed. Thus the first term on the RHS must also go to ∞ as $h \to \infty$ (else the inequality in the display above would be an equality and this would lead to a contradiction) and we deduce that g is continuous at x.
- Finally if $r = 0$, then (13) gives
$$v_p(\varepsilon_{g,r}(x, y)) - rh = v_p(g(x + y) - g(x)).$$
Since the infimum of the LHS over all $x \in \mathbb{Z}_p$ and $y \in p^h\mathbb{Z}_p$ goes to ∞ as $h \to \infty$, so must the RHS, and we deduce that g is uniformly continuous on $\mathbb{Z}_p$. This of course already follows from the previous bullet point which shows g is continuous on the compact domain $\mathbb{Z}_p$.

We let
$$\mathscr{C}^r(\mathbb{Z}_p, E) = \{g : \mathbb{Z}_p \to E \mid g \text{ is of class } \mathscr{C}^r\}.$$
This is an E-Banach space under
$$v'_{\mathscr{C}^r}(g) = \min\left(\inf_{\substack{0\leq j\leq\lfloor r\rfloor \\ x\in\mathbb{Z}_p}} v_p(\frac{g^{(j)}(x)}{j!}), \inf_{\substack{x\in\mathbb{Z}_p \\ y\in\mathbb{Z}_p}} (v_p(\varepsilon_{g,r}(x, y)) - rv_p(y))\right). \tag{14}$$

The following remark [23, Remark I.5.2] is worth expanding on a bit:

Remark 3.18 If $g : \mathbb{Z}_p \to E$ is differentiable of order r, with the r-th derivative $g^{(r)}$ continuous, then g does not necessarily lie in $\mathscr{C}^r(\mathbb{Z}_p, E)$. Consider the function $g : \mathbb{Z}_p \to \mathbb{Z}_p$ given by
$$g : \sum_{n=0}^{\infty} a_n p^n \mapsto \sum_{n=0}^{\infty} a_n p^{2n}.$$
Then one checks that $v_p(g(x) - g(y)) \geq 2v_p(x - y)$ for all $x, y \in \mathbb{Z}_p$. For instance, take $y = 0$ and note $g(0) = 0$. Take $x \neq 0$ and let n_0 be the first index in the base p expansion of x such that $a_{n_0} \neq 0$. Then the first non-zero coefficient in the base p expansion of $g(x)$ is the coefficient of p^{2n_0}, so we see $v_p(g(x)) = 2n_0 = 2v_p(x)$. In any case, we have
$$v_p\left(\frac{g(x) - g(y)}{x - y}\right) \geq v_p(x - y),$$

so g is differentiable at all $x \in \mathbb{Z}_p$ with $g'(x) = 0$. Hence $g''(x)$ is identically 0 (as are all higher derivatives). But $g \notin \mathscr{C}^2(\mathbb{Z}_p, E)$. Indeed, $v_p(\varepsilon_{g,2}(0, p^h)) - 2h = v_p(g(0 + p^h) - g(0)) - 2h = 2h - 2h = 0$, so the infimum of $\varepsilon_{g,2}(x, y) - 2h$ over all $x \in \mathbb{Z}_p$ and $y \in p^h\mathbb{Z}_p$ is at most 0 and cannot escape to ∞. (It turns out that $g \in \mathscr{C}^1(\mathbb{Z}_p, \mathbb{Q}_p)$, however.)

3.5.2 Local properties of $\mathscr{C}^r$-functions

Lemma 3.19 *Let $C(N) = \sum_{n=1}^N v_p(n!)$ and $a_j \in E$ for $0 \le j \le N$. Then for all $h \in \mathbb{Z}$*

$$\min_{0 \le j \le N} (v_p(a_j) + jh) \ge \left(\min_{z \in p^h\mathbb{Z}_p} v_p(\sum_{j=0}^N a_j \frac{z^j}{j!}) \right) - C(N).$$

Proof See [23, Lemma I.5.3]. □

Proposition 3.20 *If $r \ge 1$ and $g \in \mathscr{C}^r(\mathbb{Z}_p, E)$, then g is differentiable at each $x \in \mathbb{Z}_p$. Moreover*

a) $g' \in \mathscr{C}^{r-1}(\mathbb{Z}_p, E)$
b) $(g')^{(j)} = g^{(j+1)}$ if $j \le r - 1$.

Proof Say $g \in \mathscr{C}^r(\mathbb{Z}_p, E)$ with $r \ge 1$. Let $x \in \mathbb{Z}_p$ and $y \in p^h\mathbb{Z}_p \setminus p^{h+1}\mathbb{Z}_p$. Then

$$\frac{g(x+y) - g(x)}{y} = \sum_{j=1}^{\lfloor r \rfloor} \frac{g^{(j)}(x) y^{j-1}}{j!} + \frac{\varepsilon_{g,r}(x, y)}{y}.$$

The last term on the right tends to 0 as $h \to \infty$ since $v_p(\varepsilon_{g,r}(x, y)) - v_p(y) \ge v_p(\varepsilon_{g,r}(x, y)) - rh \ge C_{g,r}(h) \to \infty$. On the other hand as $y \to 0$, the LHS tends to $g'(x)$, and the first term on the right tends to $g^{(1)}(x)$. This proves that g is differentiable at each $x \in \mathbb{Z}_p$ and $g' = g^{(1)}$.

Also

$$\begin{aligned}\varepsilon_{g,r}(x, y+z) - \varepsilon_{g,r}(x+y, z) &= g(x+y+z) - \sum_{j=0}^{\lfloor r \rfloor} \frac{g^{(j)}(x)(y+z)^j}{j!} - \left(g(x+y+z) - \sum_{j=0}^{\lfloor r \rfloor} \frac{g^{(j)}(x+y)z^j}{j!} \right) \\ &= \sum_{j=0}^{\lfloor r \rfloor} \frac{g^{(j)}(x+y)z^j}{j!} - \sum_{j=0}^{\lfloor r \rfloor} g^{(j)}(x) \sum_{k=0}^{j} \frac{y^{j-k}z^k}{(j-k)!k!} \\ &= \sum_{j=0}^{\lfloor r \rfloor} \frac{z^j}{j!} \left(g^{(j)}(x+y) - \sum_{k=0}^{\lfloor r \rfloor - j} g^{(j+k)}(x) \frac{y^k}{k!} \right) \end{aligned} \tag{15}$$

since the second sum in the second line is (substitute $u = j - k$ and then relabel):

$$\sum_{k=0}^{\lfloor r \rfloor} \sum_{j=k}^{\lfloor r \rfloor} g^{(j)}(x) \frac{y^{j-k}z^k}{(j-k)!k!} = \sum_{k=0}^{\lfloor r \rfloor} \sum_{u=0}^{\lfloor r \rfloor - k} g^{(u+k)}(x) \frac{y^u z^k}{u!k!} = \sum_{j=0}^{\lfloor r \rfloor} \frac{z^j}{j!} \sum_{k=0}^{\lfloor r \rfloor - j} g^{(j+k)}(x) \frac{y^k}{k!}.$$

If $v_p(y), v_p(z) \geq h$, then the LHS, hence the RHS, of (15) is bounded below by $C_{g,r}(h)+rh$. Applying Lemma 3.19 with $N = \lfloor r \rfloor$ and a_j the quantity in the large parentheses above we obtain that for each $0 \leq j \leq \lfloor r \rfloor$

$$v_p\left(g^{(j)}(x+y) - \sum_{k=0}^{\lfloor r \rfloor - j} g^{(j+k)}(x)\frac{y^k}{k!}\right) - (r-j)h \geq C_{g,r}(h) - C(\lfloor r \rfloor). \tag{16}$$

This shows that for each $0 \leq j \leq \lfloor r \rfloor$, we have $g^{(j)} \in \mathscr{C}^{r-j}(\mathbb{Z}_p, E)$. It also shows that

$$(g^{(j)})^{(k)} = g^{(j+k)}$$

if $j + k \leq \lfloor r \rfloor$. This proves a) and b). □

Remark 3.21 It follows that if $g \in \mathscr{C}^r(\mathbb{Z}_p, E)$, then g is differentiable of order $\lfloor r \rfloor$ and $g^{(j)}$ is the j-th derivative of g for $j \leq \lfloor r \rfloor$.

Proposition 3.22 *If $g_1 : \mathbb{Z}_p \to \mathbb{Z}_p$ if of class $\mathscr{C}^r$ and $g_2 \in \mathscr{C}^r(\mathbb{Z}_p, E)$, then $g_2 \circ g_1 \in \mathscr{C}^r(\mathbb{Z}_p, E)$.*

Proof See [23, Proposition I.5.6]. □

3.5.3 Locally analytic functions and functions of class $\mathscr{C}^r$

Proposition 3.23 *If $h \in \mathbb{N}_{\geq 0}$ and $r \geq 0$, then $\mathrm{LA}_h(\mathbb{Z}_p, E) \subset \mathscr{C}^r(\mathbb{Z}_p, E)$. Moreover, if $g \in \mathrm{LA}_h(\mathbb{Z}_p, E)$, then $v'_{\mathscr{C}^r}(g) \geq v_{\mathrm{LA}_h}(g) - rh$.*

Proof Let $g \in \mathrm{LA}_h(\mathbb{Z}_p, E)$. On $a + p^h\mathbb{Z}_p$, we have $g(x) = \sum_{k=0}^{\infty} \tilde{a}_k(g, a)\left(\frac{x-a}{p^h}\right)^k$ so that $\frac{g^{(j)}(a)}{j!} = \frac{\tilde{a}_j(g,a)}{p^{hj}}$, hence we have (see Remark 3.14)

$$v_p(\frac{g^{(j)}(x)}{j!}) \geq v_{\mathrm{LA}_h}(g) - jh \tag{17}$$

for all $j \geq 0$ and all $x \in \mathbb{Z}_p$. On the other hand

$$\varepsilon_{g,r}(x, y) = \begin{cases} \sum_{j>r} \frac{g^{(j)}(x)}{j!} y^j & \text{if } v_p(y) \geq h \quad (\text{since } g \text{ is analytic on } x + p^h\mathbb{Z}_p) \\ g(x+y) - \sum_{j=0}^{\lfloor r \rfloor} \frac{g^{(j)}(x)}{j!} y^j & \text{if } v_p(y) < h \quad (\text{by definition}). \end{cases}$$

So if $v_p(y) \geq h$, then by (17)

$$v_p(\varepsilon_{g,r}(x, y)) \geq v_{\mathrm{LA}_h}(g) - jh + jv_p(y) \geq v_{\mathrm{LA}_h}(g) + (\lfloor r \rfloor + 1)(v_p(y) - h) \tag{18}$$

since $j \geq \lfloor r \rfloor + 1$, so that for all $h' \in \mathbb{N}_{\geq 0}$ and for $v_p(y) \geq h'$, we have

$$v_p(\varepsilon_{g,r}(x, y)) - rh' \geq v_p(\varepsilon_{g,r}(x, y)) - rv_p(y) \geq v_{\mathrm{LA}_h}(g) + (\lfloor r \rfloor + 1 - r)v_p(y) - h(\lfloor r \rfloor + 1).$$

Letting $v_p(y) \to \infty$, the second term on the right tends to ∞ since $\lfloor r \rfloor + 1 - r > 0$ (the last term is constant since h is fixed). Thus the infimum over $y \in p^{h'}\mathbb{Z}_p$ of the term on the left is ∞. Letting $h' \to \infty$, we see $g \in \mathscr{C}^r(\mathbb{Z}_p, E)$.

We now show that $v'_{\mathscr{C}^r}(g) \geq v_{\mathrm{LA}_h}(g) - hr$. The first term(s) in (14) are bounded below by $v_{\mathrm{LA}_h}(g) - hr$ by (17) since $-j \geq -r$. We estimate the second term in (14). If $v_p(y) \geq h$, then by (18), we have

$$v_p(\varepsilon_{g,r}(x, y)) - rv_p(y) \geq v_{\mathrm{LA}_h}(g) + (\lfloor r \rfloor + 1)(v_p(y) - h) - rv_p(y) \geq v_{\mathrm{LA}_h}(g) - hr$$

since $\lfloor r \rfloor + 1 > r$. If $v_p(y) < h$, then adding and subtracting rh and noting that by the second formula for $\varepsilon_{g,r}(x, y)$ above that only $g(x + y)$ and $\frac{g^{(j)}(x)}{j!} y^j$ appear so that we may use (17) above, we have

$$v_p(\varepsilon_{g,r}(x, y)) - rv_p(y) \geq v_{\mathrm{LA}_h}(g) - jh - rv_p(y) + jv_p(y) - rh + rh = v_{\mathrm{LA}_h}(g) + (h - v_p(y))(r - j) - rh$$

which is again at least $v_{\mathrm{LA}_h}(g) - rh$ since the second term on the extreme right is non-negative. taking the infimum over all $x, y \in \mathbb{Z}_p$, we see that the second term in (14) is also bounded below by $v_{\mathrm{LA}_h}(g) - rh$, as desired. □

3.5.4 Amplitude coefficients of locally polynomial functions

Let $I \subset \mathbb{N}_{\geq 0}$. A function $g : \mathbb{Z}_p \to E$ is said to be *locally polynomial* (with degrees of the monomials in I) if there is an $h \in \mathbb{N}_{\geq 0}$ such that $g|_{a+p^h\mathbb{Z}_p}(x) = \sum_{i \in I} a_i(g, a)x^i$ for all $a \in \mathbb{Z}_p$ and some coefficients $a_i(g, a) \in E$. Set

$$\begin{aligned} \mathrm{LP}^I_h(\mathbb{Z}_p, E) &= \{g : \mathbb{Z}_p \to E \mid g \text{ is locally polynomial (for } h)\} \\ \mathrm{LP}^I(\mathbb{Z}_p, E) &= \bigcup_{h \geq 0} \mathrm{LP}^I_h(\mathbb{Z}_p, E). \end{aligned}$$

We extend the definition to $I \subset \mathbb{R}$, by considering only indices $i \in I \cap \mathbb{N}_{\geq 0}$.

Proposition 3.24 *Let $r \geq 0$.*

i) The $\mathbb{1}_{i+p^{l(i)}\mathbb{Z}_p}(x) \cdot \left(\frac{x-i}{p^{l(i)}}\right)^k$ for $0 \leq i \leq p^h - 1$ and $0 \leq k \leq \lfloor r \rfloor$ form a basis of $\mathrm{LP}^{[0,r]}_h(\mathbb{Z}_p, E)$.
ii) The $\mathbb{1}_{i+p^{l(i)}\mathbb{Z}_p}(x) \cdot \left(\frac{x-i}{p^{l(i)}}\right)^k$ for $i \in \mathbb{N}_{\geq 0}$ and $0 \leq k \leq \lfloor r \rfloor$ form a basis of $\mathrm{LP}^{[0,r]}(\mathbb{Z}_p, E)$.

Proof The proof is similar to the proof of Proposition 3.10. □

We define for $i, k \in \mathbb{N}_{\geq 0}$ the following rescaled locally polynomial functions

$$e_{i,k,r} := p^{\lfloor l(i)r \rfloor} \cdot \mathbb{1}_{i+p^{l(i)}\mathbb{Z}_p}(x) \cdot \left(\frac{x-i}{p^{l(i)}}\right)^k.$$

The following lemma is immediate.

Lemma 3.25 *Let $r \geq 0$.*

i) The $e_{i,k,r}$ for $0 \leq i \leq p^h - 1$ and $0 \leq k \leq \lfloor r \rfloor$ form a basis of $\mathrm{LP}^{[0,r]}_h(\mathbb{Z}_p, E)$.
ii) The $e_{i,k,r}$ for $i \in \mathbb{N}_{\geq 0}$ and $0 \leq k \leq \lfloor r \rfloor$ form a basis of $\mathrm{LP}^{[0,r]}(\mathbb{Z}_p, E)$.

We denote by $b_{i,k}(g)$, the *amplitude coefficients* of g, the coefficients of $g \in \mathrm{LP}^{[0,r]}(\mathbb{Z}_p, E)$ in the basis $e_{i,k,r}$ for $i \in \mathbb{N}_{\geq 0}$ and $0 \leq k \leq \lfloor r \rfloor$.

3.5.5 Decomposition of $\mathscr{C}^r$ functions in wavelets

The following result allows us to approximate functions of type $\mathscr{C}^r$ by locally polynomial functions. It is the starting point of many of the computations in [21].

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Proposition 3.26 *Let $r \geq 0$. If $g \in \mathscr{C}^r(\mathbb{Z}_p, E)$, set for $h \in \mathbb{N}_{\geq 0}$*

$$\tilde{g}_h(x) := \sum_{i=0}^{p^h-1} \mathbb{1}_{i+p^h\mathbb{Z}_p}(x) \cdot \sum_{k=0}^{\lfloor r \rfloor} \frac{g^{(k)}(i)}{k!}(x-i)^k.$$

Then

i) $v_{\mathrm{LA}_{h+1}}(\tilde{g}_{h+1} - \tilde{g}_h) \geq rh + C_{g,r}(h) - C(\lfloor r \rfloor)$
ii) $\tilde{g}_h \to g$ *in* $\mathscr{C}^r(\mathbb{Z}_p, E)$ *as* $h \to \infty$.

Proof We prove i). Write

$$\tilde{g}_{h+1} - \tilde{g}_h = \sum_{i=0}^{p^h-1} \sum_{a=0}^{p-1} \mathbb{1}_{i+ap^h+p^{h+1}\mathbb{Z}_p}(x) \cdot \left(\sum_{k=0}^{\lfloor r \rfloor} \frac{g^{(k)}(i+ap^h)}{k!}(x-i-ap^h)^k - \sum_{k=0}^{\lfloor r \rfloor} \frac{g^{(k)}(i)}{k!}(x-i)^k \right). \tag{19}$$

Now subtracting and adding ap^h we may write the last sum as

$$\begin{aligned}\sum_{k=0}^{\lfloor r \rfloor} \sum_{j=0}^{k} \frac{g^{(k)}(i)}{j!(k-j)!}(x-i-ap^h)^j(ap^h)^{k-j} &= \sum_{j=0}^{\lfloor r \rfloor} \sum_{k=j}^{\lfloor r \rfloor} \frac{g^{(k)}(i)}{j!(k-j)!}(x-i-ap^h)^j(ap^h)^{k-j} \\ &= \sum_{j=0}^{\lfloor r \rfloor} \sum_{l=0}^{\lfloor r \rfloor - j} \frac{g^{(l+j)}(i)}{j!l!}(x-i-ap^h)^j(ap^h)^l\end{aligned}$$

by substituting $l = k - j$. Thus, substituting for this sum and summing over j instead of k in the other inner sum in (19), we may rewrite the inner term on the RHS of (19) as

$$\sum_{j=0}^{\lfloor r \rfloor} \frac{p^{j(h+1)}}{j!} \left(g^{(j)}(i+ap^h) - \sum_{l=0}^{\lfloor r \rfloor - j} \frac{g^{(l+j)}(i)}{l!}(ap^h)^l \right) \left(\frac{x-i-ap^h}{p^{h+1}} \right)^j.$$

By (16), the term in the large parentheses has v_p at least $(r-j)h + C_{g,r}(h) - C(\lfloor r \rfloor)$. Thus by (19) and Remark 3.14, we have

$$\begin{aligned}v_{\mathrm{LA}_{h+1}}(\tilde{g}_{h+1} - \tilde{g}_h) &\geq \inf_{j \leq r} \left(v_p\left(\frac{p^{j(h+1)}}{j!}\right) + (r-j)h + C_{g,r}(h) - C(\lfloor r \rfloor) \right) \\ &= \inf_{j \leq r} (j - v_p(j!) + rh + C_{g,r}(h) - C(\lfloor r \rfloor)) = rh + C_{g,r}(h) - C(\lfloor r \rfloor)\end{aligned}$$

since $j - v_p(j!) \geq 0$, so the infimum is attained at the $j = 0$ term. This proves i).

We now turn to ii). By Proposition 3.23, we have

$$v'_{\mathscr{C}^r}(\tilde{g}_{h+1} - \tilde{g}_h) \geq v_{\mathrm{LA}_{h+1}}(\tilde{g}_{h+1} - \tilde{g}_h) - r(h+1) \geq C_{g,r}(h) - C(\lfloor r \rfloor)) - r$$

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by i). Since the RHS goes to ∞ as $h \to \infty$, we see that $\{\tilde{g}_h\}$ is a Cauchy sequence. But $\mathscr{C}^r(\mathbb{Z}_p, E)$ is complete (it's an E-Banach!) so there is a function $\tilde{g} \in \mathscr{C}^r(\mathbb{Z}_p, E)$ such that $\tilde{g}_h \to \tilde{g}$. But

$$\tilde{g}_h(i) = g^{(0)}(i) = g(i) \quad \text{for all } 0 \le i \le p^h - 1.$$

So $\tilde{g}(i) = g(i)$ for all such i. Letting $h \to \infty$ gives $\tilde{g} = g$ on $\mathbb{N}_{\ge 0}$. By continuity, we get $\tilde{g} = g$, proving ii). □

Theorem 3.27 *The family $e_{i,k,r}$ for $i \in \mathbb{N}_{\ge 0}$ and $0 \le k \le \lfloor r \rfloor$ is a Banach basis of $\mathscr{C}^r(\mathbb{Z}_p, E)$*

Proof This can be proved using the above proposition (and some auxiliary results). For the details, see [23, Theorem I.5.14]. □

The basis $e_{i,k,r}$ is called a *basis of wavelets* of $\mathscr{C}^r(\mathbb{Z}_p, E)$. The coefficients $b_{i,k}(g)$ of $g \in \mathscr{C}^r(\mathbb{Z}_p, E)$ in this basis are called the *amplitude coefficients* of g.

3.5.6 Mahler coefficients of functions of class $\mathscr{C}^r$

Finally, we state an important theorem which allows one to show that the topologies defined by two different valuations on $\mathscr{C}^r(\mathbb{Z}_p, E)$ are the same. The proof is a bit long so we omit the proof.

Theorem 3.28 *Let $r \ge 0$ and let $g \in \mathscr{C}^r(\mathbb{Z}_p, E)$. If $g(x) = \sum_{n=0}^{\infty} a_n(g)\binom{x}{n}$ is the Mahler expansion of g, then $v_p(a_n(g)) - rl(n) \to \infty$ as $n \to \infty$. If*

$$v_{\mathscr{C}^r}(g) = \inf_{n \ge 0}(v_p(a_n(g)) - rl(n))$$

then $v_{\mathscr{C}^r} \sim v'_{\mathscr{C}^r}$ differ at most by a finite constant.

Proof See the proof of [23, Theorem I.5.17]. □

Corollary 3.29 *The $p^{\lfloor rl(n) \rfloor}\binom{x}{n}$ form a Banach basis of $\mathscr{C}^r(\mathbb{Z}_p, E)$.*

3.5.7 Poly·log functions are $\mathscr{C}^r$

We now show that certain (finite sums of) polynomial times logarithmic (poly·log) functions are $\mathscr{C}^r$. These functions are of key importance in the entire argument.

Let $\mathcal{L} \in E$. We define the corresponding branch of the logarithmic function

$$\begin{aligned} \log_{\mathcal{L}} : \mathbb{Q}_p^* &\longrightarrow E \\ \zeta &\mapsto 0 \quad \text{for } \zeta \in \mu_{p-1} \\ 1 + px &\mapsto \log(1+px) = px - \frac{(px)^2}{2!} + \frac{(px)^3}{3!} - \cdots \quad \text{for } x \in \mathbb{Z}_p \\ p &\mapsto \mathcal{L} \end{aligned}$$

and such that $\log_{\mathcal{L}}(xy) = \log_{\mathcal{L}}(x) + \log_{\mathcal{L}}(y)$ for all $x, y \in \mathbb{Q}_p^*$. Most number theorists leave this planet encountering only the Iwasawa p-adic logarithmic function which is the case $\mathcal{L} = 0$. But the case of general $\mathcal{L} \in E$ will be key to all that follows.

We make several elementary but enlightening remarks about the continuity and differentiability of this function and it's extensions to $\mathbb{Q}_p$. First note that $\log_{\mathcal{L}}$ is continuous and even infinitely differentiable on $\mathbb{Q}_p^*$. Indeed, on the open set $\{z \in \mathbb{Q}_p^* \mid v_p(z) = i\}$ for $i \in \mathbb{Z}$, writing $z = p^i \cdot \frac{z}{p^i}$ we see that $\log_{\mathcal{L}}(z) = i\mathcal{L} + \log(\frac{z}{p^i})$ and log is locally analytic on $\mathbb{Z}_p^*$.

However, $\log_{\mathcal{L}}$ does not extend to a continuous function on $\mathbb{Q}_p$. Indeed, the limits of two different sequences tending to 0 can be different. For instance, $\log_{\mathcal{L}}(p^{1+p^n}) = (1+p^n)\mathcal{L}$ and $\log_{\mathcal{L}}(p^{p^n}) = p^n\mathcal{L}$ tend to $\mathcal{L}$ and 0, respectively, which may be distinct.

However, the function $z^n \log_{\mathcal{L}}(z)$ extends to a continuous function on $\mathbb{Q}_p$ for any integer $n \geq 1$. Indeed, for any $z \in \mathbb{Q}_p^*$, the valuation of $\log_{\mathcal{L}}(z)$ is bounded below by $\min\{v_p(\mathcal{L}), 1\}$. Therefore the limit of $z^n \log_{\mathcal{L}}(z)$ as $z \to 0$ is equal to 0. So $z^n \log_{\mathcal{L}}(z)$ can be extended to a continuous function on $\mathbb{Q}_p$ if we set its value at 0 to be 0.

We now discuss the differentiability of $z^n \log_{\mathcal{L}}(z)$ for $n \geq 1$. This function is clearly differentiable on $\mathbb{Q}_p^*$. Indeed for $n = 1$, $z \log_{\mathcal{L}}(z)$ has derivative $\log_{\mathcal{L}}(z) + 1$ on $\mathbb{Q}_p^*$. Similarly the function $z^2 \log_{\mathcal{L}}(z)$ is differentiable on $\mathbb{Q}_p^*$ with first derivative given by $2z \log_{\mathcal{L}}(z) + z$ and second derivative given by $2\log_{\mathcal{L}}(z) + 3$. In fact in general it is not hard to show that $g(z) = z^n \log_{\mathcal{L}}(z)$ is differentiable on $\mathbb{Q}_p^*$ with j-th derivative for $0 \leq j \leq n$ given by

$$g^{(j)}(z) = \frac{n!}{(n-j)!} z^{n-j} \log_{\mathcal{L}}(z) + t_j z^{n-j} \tag{20}$$

for some $t_j \in \mathbb{Z}_p$. In fact, there is a more precise formula for the derivatives of the function $z^n \log_{\mathcal{L}}(z)$ in terms of certain partial harmonic sums $H_i = 1 + \frac{1}{2} + \cdots + \frac{1}{i}$. Indeed, for $z \neq 0$, we have

$$\frac{d^j}{dz^j}(z^n \log_{\mathcal{L}}(z)) = \frac{n!}{(n-j)!}\left[z^{n-j} \log_{\mathcal{L}}(z) + (H_n - H_{n-j}) z^{n-j}\right], \tag{21}$$

for $0 \leq j \leq n$. We remind the reader of the convention that $H_0 = 0$.

We now check the differentiability of $z^n \log_{\mathcal{L}}(z)$ for $n \geq 1$ at $z = 0$. By definition, the derivative at $z = 0$ is given by the limit

$$\lim_{z \to 0} \frac{z^n \log_{\mathcal{L}}(z) - 0}{z}.$$

We have seen above that this limit does not exist for $n = 1$. Therefore $z \log_{\mathcal{L}}(z)$ is only differentiable for $z \neq 0$. For $n = 2$ however, one sees that the limit exists. Therefore, the function $z^2 \log_{\mathcal{L}}(z)$ is differentiable everywhere on $\mathbb{Q}_p$ with derivative given by $2z \log_{\mathcal{L}}(z) + z$. Since the derivative involves $z \log_{\mathcal{L}}(z)$, we see that $z^2 \log_{\mathcal{L}}(z)$ is differentiable everywhere only once. Extending this, we see that the function $z^n \log_{\mathcal{L}}(z)$ is differentiable everywhere only $n-1$ times. Moreover, its $(n-1)^{\text{th}}$ derivative is continuous on $\mathbb{Q}_p^*$.

As has been pointed out in Remark 3.18, while $\mathscr{C}^r$ functions are differentiable of order $\lfloor r \rfloor$, the converse is not necessarily true. However, we prove that the converse is indeed true for poly·log functions. We have

Proposition 3.30 *For $1 \leq n \leq p-1$, the function $z^n \log_{\mathcal{L}}(z)$ belongs to $\mathscr{C}^s(\mathbb{Z}_p, E)$ for any $0 \leq s < n$.*

Proof The argument is taken from [21]. Let $g(x) = x^n \log_{\mathcal{L}}(x)$. We need to check that for

$$\varepsilon_{g,s}(x, y) = g(x+y) - \sum_{j=0}^{\lfloor s \rfloor} g^{(j)}(x) \frac{y^j}{j!},$$

we have

$$\inf_{\substack{x \in \mathbb{Z}_p \\ y \in p^h \mathbb{Z}_p}} (v_p(\varepsilon_{g,s}(x, y)) - sh) \to \infty.$$

as $h \to \infty$. Fix h. Let $x \in \mathbb{Z}_p$. There are two cases to consider:

- First assume that $h > v_p(x)$. Therefore $x \neq 0$ and $h \geq 1$. For $y \in p^h\mathbb{Z}_p$, we have a Taylor expansion for $\log_{\mathcal{L}}(1+y/x)$ which is analytic. So for such x and y

$$\varepsilon_{g,s}(x,y) = \sum_{j=\lfloor s\rfloor+1}^{\infty} g^{(j)}(x)\frac{y^j}{j!}.$$

Say $j \leq n$. Then by (20), we have $g^{(j)}(x) = \frac{n!}{(n-j)!}x^{n-j}\log_{\mathcal{L}}(x) + t_j x^{n-j}$ for some $t_j \in \mathbb{Z}_p$. We have seen that the valuation of $\log_{\mathcal{L}}(x)$ is bounded below by $\min\{v_p(\mathcal{L}),1\} \geq \min\{v_p(\mathcal{L}),0\}$. Therefore the valuation of the j^{th} summand above is bounded below by $\min\{v_p(\mathcal{L}),0\} + jh - v_p(j!)$. Using the well-known formula

$$v_p(j!) = \frac{j-\sigma_p(j)}{p-1} \leq \frac{j}{p-1},$$

where $\sigma_p(j)$ is the sum of the p-adic digits of j, we see that

$$v_p\left(g^{(j)}(x)\frac{y^j}{j!}\right) \geq \min\{v_p(\mathcal{L}),0\} + j\left(h - \frac{1}{p-1}\right) \geq \min\{v_p(\mathcal{L}),0\} + (\lfloor s\rfloor+1)\left(h-\frac{1}{p-1}\right).$$

Next say $j > n$. Then since $x \neq 0$, we have

$$g^{(j)}(x) = n!\frac{(-1)^{j-n-1}(j-n-1)!}{x^{j-n}}.$$

So similarly

$$\begin{aligned} v_p\left(g^{(j)}(x)\frac{y^j}{j!}\right) &\geq -(j-n)v_p(x) + jh - \frac{j}{p-1} \\ &= n\left(h-\frac{1}{p-1}\right) + (j-n)\left(h - \frac{1}{p-1} - v_p(x)\right). \\ &\geq n\left(h-\frac{1}{p-1}\right) \ \geq \ (\lfloor s\rfloor+1)\left(h-\frac{1}{p-1}\right) \end{aligned}$$

since $h - v_p(x) \geq 1$. Putting these cases together, we see that the valuation of the j^{th} term in $\varepsilon_{g,s}(x,y)$ is greater than or equal to

$$\min\{v_p(\mathcal{L}),0\} + (\lfloor s\rfloor+1)\left(h-\frac{1}{p-1}\right).$$

Hence

$$v_p(\varepsilon_{g,s}(x,y)) \geq \min\{v_p(\mathcal{L}),0\} + (\lfloor s\rfloor+1)\left(h-\frac{1}{p-1}\right).$$

- Now assume that $h \leq v_p(x)$. Say $y \in p^h\mathbb{Z}_p$. Then $v_p(x+y) \geq h$. Now $g(x+y) = (x+y)^n\log_{\mathcal{L}}(x+y)$ implies

$$v_p(g(x+y)) \geq nh + \min\{v_p(\mathcal{L}),1\} \geq nh + \min\{v_p(\mathcal{L}),0\}.$$

Now using (20) we have for all $0 \leq j \leq \lfloor s\rfloor$

$$v_p(\frac{g^{(j)}(x)y^j}{j!}) \geq (n-j)h + \min\{v_p(\mathcal{L}),0\} + jh - v_p(j!) \geq nh + \min\{v_p(\mathcal{L}),0\} - v_p(\lfloor s\rfloor!).$$

Since $n \geq \lfloor s \rfloor + 1$, it follows that

$$v_p(\varepsilon_{g,s}(x, y)) \geq \min\{v_p(\mathcal{L}), 0\} + (\lfloor s \rfloor + 1)h - v_p(\lfloor s \rfloor !).$$

Both of these estimates then show that

$$\inf_{x \in \mathbb{Z}_p, y \in p^h \mathbb{Z}_p} (v_p(\varepsilon_{g,s}(x, y)) - sh) \to \infty$$

as $h \to \infty$, since $\lfloor s \rfloor + 1 - s > 0$. Therefore, the function $z^n \log_{\mathcal{L}}(z)$ belongs to $\mathscr{C}^s(\mathbb{Z}_p, E)$ for $0 \leq s < n$. □

Corollary 3.31 *E-valued functions of $z \in \mathbb{Q}_p$ of the form*

$$g(z) = (z - z_i)^{n_i} \log_{\mathcal{L}}(z - z_i),$$

where $z_i \in \mathbb{Z}_p$ and $n_i \geq 1$ are $\mathscr{C}^s$ for $0 \leq s < n_i$.

Proof By Proposition 3.22, the composition of $\mathscr{C}^s$ functions is $\mathscr{C}^s$. Therefore we are done by Proposition 3.30. □

4 $\mathbf{GL_2}(\mathbb{Q}_p)$-Banach spaces

4.1 The Banach space $B_{k,\mathcal{L}}$

In [12, Section 4.2], Breuil defined the Local Langlands correspondent $B(k, \mathcal{L})$ of the semi-stable representation $V_{k,\mathcal{L}}$ for $k \geq 3$. Let $r = k - 2$. In this section, we recall the alternative definition $\tilde{B}(k, \mathcal{L})$ of $B(k, \mathcal{L})$ given in [14, Corollary 3.3.4] which uses the notion of functions of type $\mathscr{C}^s$ for $s = \frac{r}{2}$.

By Proposition 3.28, for a real number $s \geq 0$, a continuous function $g : \mathbb{Z}_p \to E$ belongs to the space $\mathscr{C}^s(\mathbb{Z}_p, E)$ if in its Mahler's expansion $g(z) = \sum_{n=0}^{\infty} a_n(g)\binom{z}{n}$, the coefficients $a_n(g)$ satisfy $n^s|a_n(g)| \to 0$ as $n \to \infty$. The space $\mathscr{C}^s(\mathbb{Z}_p, E)$ is a Banach space with the norm $||g||_s = \sup_n (n+1)^s |a_n(g)|$.

Let $D(k)$ be the E-vector space of functions $g : \mathbb{Q}_p \to E$ such that

- $g_1 : z \mapsto g(z)$ for $z \in \mathbb{Z}_p$ belongs to $\mathscr{C}^{\frac{r}{2}}(\mathbb{Z}_p, E)$, and
- $g_2 : z \mapsto z^r g(1/z)$ for $z \in \mathbb{Z}_p \setminus \{0\}$ extends to a function on $\mathbb{Z}_p$ belonging to $\mathscr{C}^{\frac{r}{2}}(\mathbb{Z}_p, E)$.

This space is a Banach space under the norm to be

$$||g|| = \max(||g_1||_{\frac{r}{2}}, ||g_2||_{\frac{r}{2}}).$$

We define an action of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ on $D(k)$ by:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot g\right)(z) = |ad - bc|^{\frac{r}{2}} (bz + d)^r g\left(\frac{az + c}{bz + d}\right). \tag{22}$$

Let $P(k)$ be the space of polynomials of degree less than or equal to $r = k - 2$. Note that $P(k) \subset D(k)$. Indeed, if g is a polynomial of degree less than or equal to r, then both g_1 and g_2 are polynomials and therefore belong to $\mathscr{C}^{\frac{r}{2}}(\mathbb{Z}_p, E)$. Moreover, the space of such polynomials is clearly stable under the G-action defined above.

Set $\tilde{B}(k)$ to be the quotient of $D(k)$ by $P(k)$.

Now we define $\tilde{B}(k,\mathcal{L})$ as a quotient of $\tilde{B}(k)$. Let $L(k,\mathcal{L})$ be the subspace of $D(k)$ generated by $P(k)$ and finite sums of poly·log functions of the form

$$g(z) = \sum_{i \in I} \lambda_i (z - z_i)^{n_i} \log_{\mathcal{L}}(z - z_i), \tag{23}$$

where

- I is a finite (indexing) set
- $\lambda_i \in E$
- $z_i \in \mathbb{Q}_p$
- $n_i \in \{\lfloor \frac{r}{2} \rfloor + 1, \ldots, r\}$ and

$$\deg\left(\sum_{i \in I} \lambda_i (z - z_i)^{n_i}\right) < \frac{r}{2}. \tag{24}$$

The subspace $L(k,\mathcal{L})$ is G-stable (see [14, Lemma 3.3.2]). We make some comments on why $L(k,\mathcal{L}) \subset D(k)$. Assume that the $z_i \in \mathbb{Z}_p$. We already saw in Corollary 3.31 that the function $g(z)$ in (23) - being a (a finite sum of scalar multiples of) poly·log function(s) - is $\mathscr{C}^{\frac{r}{2}}$ because each $n_i > r/2$. What is less clear is that $z^r g(1/z)$ is also $\mathscr{C}^{\frac{r}{2}}$. The somewhat mysterious degree condition (24) is made precisely to ensure this. To see, let us assume that all the n_i are equal to some n ($> \frac{r}{2}$) for simplicity. Then

$$z^r g(1/z) = \sum_{i \in I} \lambda_i z^{r-n} (1 - z z_i)^n \log_{\mathcal{L}}(1 - z z_i) - \sum_{i \in I} \lambda_i z^{r-n} (1 - z z_i)^n \log_{\mathcal{L}}(z).$$

Now the first term on the RHS is $\mathscr{C}^{\frac{r}{2}}$ by Corollary 3.31 and Proposition 3.22 ($z \mapsto 1 - z z_i$ is $\mathscr{C}^{\frac{r}{2}}$) and multiplication by the monomial z^{r-n} preserves the property of being $\mathscr{C}^{\frac{r}{2}}$. Write the second term as

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\sum_{i \in I} \lambda_i z_i^k\right) z^{r-n+k} \log_{\mathcal{L}}(z).$$

Now an easy check using (24) shows that $\sum_{i \in I} \lambda_i z_i^k = 0$ for $k \leq n - r/2$. This forces the only powers $r - n + k$ of z to survive are those which are strictly bigger than $r/2$. So again Corollary 3.31 applies to show that the second term on the RHS is $\mathscr{C}^{\frac{r}{2}}$!

Define $\tilde{B}(k,\mathcal{L})$ to be the quotient of $D(k)$ by the closure of $L(k,\mathcal{L})$ in $D(k)$.

It turns out that, $\tilde{B}(k,\mathcal{L})$ is isomorphic to $B(k,\mathcal{L})$, the Local Langlands correspondent of $V_{k,\mathcal{L}}$ (cf. [14, Corollary 3.3.4]).

For convenience, we introduce a third notation and set

$$B_{k,\mathcal{L}} := \tilde{B}(k,\mathcal{L}).$$

Thus $B_{k,\mathcal{L}}$ is a unitary G-Banach space which corresponds to $V_{k,\mathcal{L}}$ under the p-adic Local Langlands correspondence.

4.2 Uniformizing $B_{k,\mathcal{L}}$

Recall that

$$Z = \mathbb{Q}_p^* \subset B = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\} \subset G = \mathrm{GL}_2(\mathbb{Q}_p) \supset K = \mathrm{GL}_2(\mathbb{Z}_p) \supset I = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid c \equiv 0 \mod p\}$$

where I is the Iwahori subgroup.

To compute the reduction of $B_{k,\mathcal{L}}$ we need to define an integral structure on $B_{k,\mathcal{L}}$. It is not clear how to proceed. However, if one is able to uniformize (a dense subspace of) $B_{k,\mathcal{L}}$ by a compactly induced space of rational valued polynomial functions on G, then one can define a lattice in $B_{k,\mathcal{L}}$ by taking the (closure of the) image of the integral valued polynomial functions on G (much as the standard lattice is defined in the corresponding Banach space attached to a crystalline representation).

Recall that $\underline{\mathrm{Sym}}^{k-2}E^2$ is the $(k-2)$-th symmetric power representation of KZ on E^2 (which is modelled on homogeneous polynomials of degree $k-2$ in two variables X and Y over E) twisted by the character $|\det|^{\frac{k-2}{2}}$ so that $p \in Z$ acts trivially.

We prove (see [21, Section 5] for the details):

Proposition 4.1 *There are maps:*

$$\mathrm{ind}_{IZ}^G \underline{\mathrm{Sym}}^{k-2}E^2 \twoheadrightarrow \underline{\mathrm{Sym}}^{k-2}E^2 \otimes \mathrm{ind}_{IZ}^G \mathbb{1}_E \to \underline{\mathrm{Sym}}^{k-2}E^2 \otimes (\mathrm{ind}_B^G\, E)^{\mathrm{smooth}} \hookrightarrow D(k).$$

Under the composition of all these maps $[\![1, X^iY^{r-i}]\!] \mapsto z^i \mathbb{1}_{p\mathbb{Z}_p}$.

Proof We recall the definitions of each of the maps in order:

1. $[\![g, P]\!] \mapsto gP \otimes [\![g, 1]\!]$.
2. $P \otimes [\![\mathrm{id}, 1]\!] \mapsto P \otimes f_{\mathrm{id}}$ where $f_{\mathrm{id}} \in (\mathrm{ind}_B^G\, E)^{\mathrm{smooth}}$ is defined by

$$f_{\mathrm{id}}(g) = \begin{cases} 1 & \text{if } g \in BIZ \\ 0 & \text{otherwise.} \end{cases}$$

   One checks that $f_{\mathrm{id}}\left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right) = \mathbb{1}_{p\mathbb{Z}_p}$.
3. $P(X,Y) \otimes f \mapsto P(z,1) f\left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)$.

An easy check shows that under these maps

$$[\![\mathrm{id}, X^iY^{r-i}]\!] \mapsto X^iY^{r-i} \otimes [\![\mathrm{id}, 1]\!] \mapsto X^iY^{r-i} \otimes f_{\mathrm{id}} \mapsto z^i \mathbb{1}_{p\mathbb{Z}_p}$$

as claimed. □

We make two remarks

- The map 2. above induces an isomorphism

$$\frac{\mathrm{ind}_{IZ}^G \mathbb{1}_E}{(T_{1,0}+1) + (T_{1,2}-1)} \simeq \frac{(\mathrm{ind}_B^G\, E)^{\mathrm{smooth}}}{\mathbb{1}_E} = \mathrm{St}.$$

- The map 3. factors as

$$\frac{\underline{\mathrm{Sym}}^{k-2}E^2 \otimes (\mathrm{ind}_B^G\, E)^{\mathrm{smooth}}}{\underline{\mathrm{Sym}}^{k-2}E^2 \otimes \mathbb{1}_E} = \underline{\mathrm{Sym}}^{k-2}E^2 \otimes \mathrm{St} \hookrightarrow \frac{D(k)}{P(k)} = \tilde{B}(k) \twoheadrightarrow B_{k,\mathcal{L}}.$$

The image of this map consists of the *locally algebraic vectors* in $B_{k,\mathcal{L}}$ of which it is the completion.

The second remark and Proposition 4.1 together say we have a uniformization (the image is dense)

$$\mathrm{ind}_{IZ}^G \underline{\mathrm{Sym}}^{k-2}E^2 \to B_{k,\mathcal{L}}. \tag{25}$$

4.3 The lattice $\tilde{\Theta}(k, \mathcal{L})$

We can now define the standard lattice $\tilde{\Theta}(k, \mathcal{L})$ in the Banach space $\tilde{B}(k, \mathcal{L})$ introduced above. We let

$$\Theta_{k,\mathcal{L}} := \tilde{\Theta}(k, \mathcal{L}) = \text{(closure of the) image} \left(\mathrm{ind}_{IZ}^G \underline{\mathrm{Sym}}^{k-2}\mathcal{O}_E^2 \to B_{k,\mathcal{L}}\right)$$

under the map in (25).

We now describe some explicit elements of this lattice. In this context, we are reminded of a warning by Prof. Murray Schacher who while teaching an algebraic number theory course at UCLA in 1993 mentioned that graduate students experience 'considerable giddiness' when trying to write down integral elements in a number field. A similar warning applies when trying to identify explicit elements in $\tilde{\Theta}(k, \mathcal{L})$. We now mention two important lemmas which help identify some (integral) elements in this lattice.

Lemma 4.2 *Let $r = k - 2 \geq 1$, $0 \leq j \leq r$, $h \in \mathbb{Z}$ and $z_0 \in \mathbb{Q}_p$. Then the function*

$$p^{(h-1)(r/2-j)}(z - z_0)^j \mathbb{1}_{z_0 + p^h\mathbb{Z}_p} \in \tilde{\Theta}(k, \mathcal{L})$$

is integral.

Proof By the G-action defined in (22)

$$\begin{pmatrix} 1 & 0 \\ -z_0 & p^{h-1} \end{pmatrix} \cdot z^j \mathbb{1}_{p\mathbb{Z}_p} = p^{(h-1)(r/2-j)}(z - z_0)^j \mathbb{1}_{z_0 + p^h\mathbb{Z}_p}.$$

The lemma follows since $z^j \mathbb{1}_{p\mathbb{Z}_p} \in \tilde{\Theta}(k, \mathcal{L})$ and $\tilde{\Theta}(k, \mathcal{L})$ is G-stable. □

The following stronger result applies if $j \geq r/2$. Note that $(h - 1)(r/2 - j) \geq h(r/2 - j)$.

Lemma 4.3 *Let $r = k - 2 \geq 1$, $r/2 \leq j \leq r$, $h \in \mathbb{Z}$ and $z_0 \in \mathbb{Q}_p$. Then the function*

$$p^{h(r/2-j)}(z - z_0)^j \mathbb{1}_{z_0 + p^h\mathbb{Z}_p} \in \tilde{\Theta}(k, \mathcal{L})$$

is integral.

Proof Again, by (22),

$$-\begin{pmatrix} 0 & 1 \\ p^h & -z_0 \end{pmatrix} \cdot z^{r-j} \mathbb{1}_{p\mathbb{Z}_p} = -p^{h(r/2-j)}(z - z_0)^j \mathbb{1}_{p\mathbb{Z}_p}\left(\frac{p^h}{z - z_0}\right).$$

Now

$$\mathbb{1}_{p\mathbb{Z}_p}\left(\frac{p^h}{z-z_0}\right)=1 \iff \frac{p^h}{z-z_0}\in p\mathbb{Z}_p \iff v_p(z-z_0)<h \iff \mathbb{1}_{\mathbb{Q}_p\setminus(z_0+p^h\mathbb{Z}_p)}(z)=1.$$

Using the fact that polynomials of degree less than or equal to r are equal to 0 in $\tilde{B}(k,\mathcal{L})$, we see that

$$-(z-z_0)^j\mathbb{1}_{p\mathbb{Z}_p}\left(\frac{p^h}{z-z_0}\right)=-(z-z_0)^j\left(1-\mathbb{1}_{z_0+p^h\mathbb{Z}_p}(z)\right)=(z-z_0)^j\mathbb{1}_{z_0+p^h\mathbb{Z}_p}.$$

The lemma again follows since $z^{r-j}\mathbb{1}_{p\mathbb{Z}_p}\in\tilde{\Theta}(k,\mathcal{L})$ and $\tilde{\Theta}(k,\mathcal{L})$ is G-stable. □

4.4 Uniformizing the reduced lattice and its subquotients

Let $\mathbb{F}_q=\mathcal{O}_E/\pi$ be the residue field of E. Let $\mathrm{Sym}^{k-2}\mathbb{F}_q^2$ be the symmetric power representation of $\mathrm{GL}_2(\mathbb{F}_q)$ modelled again on all homogeneous polynomials of degree $r=k-2$ over $\mathbb{F}_q$ in X and Y and thought of as a representation of IZ by noting that $I\subset K\twoheadrightarrow \mathrm{GL}_2(\mathbb{F}_q)$ and letting $p\in Z$ act trivially.

Recall that by definition there is a map $\mathrm{ind}_{IZ}^G\,\underline{\mathrm{Sym}}^{k-2}\mathcal{O}_E^2\to\tilde{\Theta}(k,\mathcal{L})$ with dense image. Tensoring this map over $\mathcal{O}_E$ with $\mathbb{F}_q$ gives us a surjection

$$\mathrm{ind}_{IZ}^G\,\mathrm{Sym}^{k-2}\mathbb{F}_q^2\twoheadrightarrow\overline{\tilde{\Theta}(k,\mathcal{L})}=\overline{\Theta_{k,\mathcal{L}}}. \tag{26}$$

Now $\mathrm{Sym}^{k-2}\mathbb{F}_q^2$ has an IZ-stable filtration

$$\langle X^r, X^{r-1}Y,\ldots,XY^{r-1},Y^r\rangle\supset\langle X^r,X^{r-1}Y,\ldots,XY^{r-1}\rangle\supset\cdots\supset\langle X^r\rangle$$

where the successive quotients are given by the characters $d^r, ad^{r-1},\ldots,a^r$ of IZ. Thus the module $\mathrm{ind}_{IZ}^G\,\mathrm{Sym}^{k-2}\mathbb{F}_q^2$ inherits a filtration where the successive subquotients are given by

$$\mathrm{ind}_{IZ}^G\,d^r,\ \mathrm{ind}_{IZ}^G\,ad^{r-1},\ldots,\ \mathrm{ind}_{IZ}^G\,a^r.$$

Pushing this filtration forward, we obtain a filtration on $\overline{\tilde{\Theta}(k,\mathcal{L})}$. Define $F_{2l,2l+1}$ to be the subquotient of $\overline{\tilde{\Theta}(k,\mathcal{L})}$ corresponding to $\mathrm{ind}_{IZ}^G\,a^ld^{r-l}$ for $0\le l\le r$. Then, there is a surjection

$$\mathrm{ind}_{IZ}^G\,a^ld^{r-l}\twoheadrightarrow F_{2l,\,2l+1},$$

for $0\le l\le r$.

As mentioned in the introduction, by the Iwahori mod p LLC (Theorem 2.5), to determine the reduction $\overline{V}_{k,\mathcal{L}}$, it suffices to determine the reduction $\overline{\tilde{\Theta}(k,\mathcal{L})}$ of $\tilde{\Theta}(k,\mathcal{L})$. By what we have just said, it therefore suffices to determine the subquotients

$$F_{0,1},\ F_{2,3},\ \ldots,\ F_{2r,2r+1}.$$

Theorem 1.1 is proved by a detailed analysis of these subquotients!

5 Reductions of semi-stable Galois representations

5.1 Some results concerning functions in $\mathscr{C}^{r/2}(\mathbb{Z}_p, E)$

We now describe some further generalities on functions in the space $\mathscr{C}^s(\mathbb{Z}_p, E)$ that are needed in our computations of the reduction of the lattice $\overline{\tilde{\Theta}(k, \mathcal{L})}$. Much of the material in this section is taken verbatim from [21, Section 8].

Recall that Proposition 3.26 says that if $g \in \mathscr{C}^s(\mathbb{Z}_p, E)$ for some $s \geq 0$, then

$$\tilde{g}_h(z) := \sum_{m=0}^{p^h-1} \left[\sum_{j=0}^{\lfloor s \rfloor} \frac{g^{(j)}(m)}{j!}(z-m)^j \right] \mathbb{1}_{m+p^h\mathbb{Z}_p} \to g \text{ as } h \to \infty. \tag{27}$$

We now modify $\tilde{g}_h$ slightly by fake-adding some terms which vanish mod $\pi\tilde{\Theta}(k, \mathcal{L})$ to get a new function g_h which is more amenable to further computation. More precisely, we have the following general lemma.

Lemma 5.1 *Suppose $g \in \mathscr{C}^{r/2}(\mathbb{Z}_p, E)$ is a function that has continuous derivatives of order $\lfloor r/2 \rfloor + 1, \ldots, t$ for an integer $r/2 < t \leq r$. Define*

$$g_h(z) := \sum_{m=0}^{p^h-1} \left[\sum_{j=0}^{t} \frac{g^{(j)}(m)}{j!}(z-m)^j \right] \mathbb{1}_{m+p^h\mathbb{Z}_p}.$$

Then for large h, we have

$$\tilde{g}_h(z) \equiv g_h(z) \mod \pi\tilde{\Theta}(k, \mathcal{L}).$$

Proof The lemma follows by noting that for large h, $\lfloor r/2 \rfloor + 1 \leq j \leq t$ and $0 \leq m \leq p^h - 1$, we have

$$\frac{g^{(j)}(m)}{j!}(z-m)^j \mathbb{1}_{m+p^h\mathbb{Z}_p} \equiv 0 \mod \pi\tilde{\Theta}(k, \mathcal{L}). \tag{28}$$

Indeed, by the continuity of the derivatives we see that the valuation of $\frac{g^{(j)}(m)}{j!}$ is bounded below by some finite rational number M for all $m \in \mathbb{Z}_p$ and all $\lfloor r/2 \rfloor + 1 \leq j \leq t$. We choose h large enough so that $M > (h-1)(r/2 - j)$, noting $r/2 - j$ is negative. Then (28) follows by Lemma 4.2. □

Lemma 5.2 *Let $k/2 \leq n \leq r \leq p-1$ and $z_0 \in \mathbb{Z}_p$. Fix $x \in \mathbb{Q}$ such that $x \geq -1$ and $x + v_p(\mathcal{L}) \geq r/2 - n$. Let*

$$g(z) = p^x (z - z_0)^n \log_{\mathcal{L}}(z - z_0) \mathbb{1}_{\mathbb{Z}_p}.$$

Then for $h \geq 3$, $0 \leq a \leq p^{h-1} - 1$, $0 \leq \alpha \leq p-1$ and $0 \leq j \leq n-1$, we have

$$g^{(j)}(a + \alpha p^{h-1}) \equiv g^{(j)}(a) + \alpha p^{h-1} g^{(j+1)}(a) + \cdots + \frac{(\alpha p^{h-1})^{n-1-j}}{(n-1-j)!} g^{(n-1)}(a) \mod (p^{h-1})^{r/2-j}\pi.$$

Moreover, we have

$$\begin{aligned} &g^{(j)}(a + \alpha p^{h-1})(z - a - \alpha p^{h-1})^j \mathbb{1}_{a+\alpha p^{h-1}+p^h\mathbb{Z}_p} \\ &\equiv \left[g^{(j)}(a) + \alpha p^{h-1} g^{(j+1)}(a) + \cdots + \frac{(\alpha p^{h-1})^{n-1-j}}{(n-1-j)!} g^{(n-1)}(a) \right] (z - a - \alpha p^{h-1})^j \mathbb{1}_{a+\alpha p^{h-1}+p^h\mathbb{Z}_p} \end{aligned}$$

modulo $\pi\tilde{\Theta}(k, \mathcal{L})$.

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Proof We prove the first congruence in the conclusion of the lemma.

The proof is similar to the proof that $g(z) \in \mathscr{C}^{n-1}(\mathbb{Z}_p, E)$. The difference is that we need to keep track of the estimates. We consider two cases.

1. $v_p(a - z_0) < h - 1$: Since g is analytic in the neighborhood $a + p^{h-1}\mathbb{Z}_p$, using (21), we write

$$\begin{aligned} g^{(j)}(a + \alpha p^{h-1}) = g^{(j)}(a) + \cdots + \frac{(\alpha p^{h-1})^{n-1-j}}{(n-1-j)!} g^{(n-1)}(a) \\ + \frac{(\alpha p^{h-1})^{n-j}}{(n-j)!} n! p^x [\log_{\mathcal{L}}(a - z_0) + H_n] \\ + \sum_{l \geq 1} \frac{(\alpha p^{h-1})^{n-j+l}}{(n-j+l)!} n! p^x \frac{(-1)^{l-1}}{(a-z_0)^l} (l-1)!. \end{aligned} \tag{29}$$

Since $x + v_p(\mathcal{L}) \geq r/2 - n$ and $x \geq -1 \geq r/2 - n$, we see that the valuation of the term in the second line above is at least $(n-j)(h-1) + r/2 - n$. Now

$$(n-j)(h-1) + r/2 - n > (h-1)(r/2 - j) \iff h > 2. \tag{30}$$

Therefore for $h \geq 3$, the term in the second line in equation (29) is 0 modulo $(p^{h-1})^{r/2-j}\pi$. Next, write the general term in the last sum in equation (29) as

$$p^x (\alpha p^{h-1})^{n-j} \frac{n!}{(n-j)!} \left(\frac{\alpha p^{h-1}}{a - z_0} \right)^l (-1)^{l-1} \frac{1}{l \binom{n-j+l}{l}}.$$

Using the estimate $v_p\binom{n-j+l}{l} \leq \lfloor \log_p(n-j+l) \rfloor - v_p(l)$ obtained using Kummer's theorem, the valuation of this term is greater than or equal to

$$-1 + (n-j)(h-1) + l - \log_p(n-j+l).$$

Moreover,

$$\begin{aligned} -1 + (n-j)(h-1) + l - \log_p(n-j+l) > (r/2 - j)(h-1) \\ \overset{h \geq 3}{\iff} 2n - r - 1 + l > \log_p(n-j+l), \end{aligned}$$

which is true for all $l \geq 1$. Therefore we have proved the first congruence in the lemma when $v_p(a - z_0) < h - 1$.

2. $v_p(a - z_0) \geq h - 1$: Recall the derivative formula (20):

$$g^{(j)}(z) = \left[\frac{n!}{(n-j)!} p^x (z - z_0)^{n-j} \log_{\mathcal{L}}(z - z_0) + p^x t_j (z - z_0)^{n-j} \right] \mathbb{1}_{\mathbb{Z}_p}, \tag{31}$$

where $t_j \in \mathbb{Z}_p$. The valuations of the terms $g^{(j)}(a + \alpha p^{h-1})$, $g^{(j)}(a)$, $\alpha p^{h-1} g^{(j+1)}(a)$, $\ldots$, $(\alpha p^{h-1})^{n-1-j}$ $g^{(n-1)}(a)$ are greater than or equal to $(n-j)(h-1) + r/2 - n$. So equation (30) implies that these terms are congruent to 0 modulo $(p^{h-1})^{r/2-j}\pi$. This proves the first congruence in the lemma when $v_p(a - z_0) \geq h - 1$.

The second congruence in the conclusion of the lemma follows from the first using Lemma 4.2. □

Lemma 5.3 *Let $k/2 \le n \le r \le p-1$ and $z_0 \in \mathbb{Z}_p$. Fix $x \in \mathbb{Q}$ such that $x \ge -1$ and $x + v_p(\mathcal{L}) \ge r/2 - n$. If*

$$g(z) = p^x (z - z_0)^n \log_{\mathcal{L}}(z - z_0) \mathbb{1}_{\mathbb{Z}_p},$$

then we have $g_h(z) \equiv g_2(z) \mod \pi\tilde{\Theta}(k, \mathcal{L})$ for $h \ge 3$.

Proof Making the substitution $m = a + \alpha p^{h-1}$, we write

$$g_h(z) = \sum_{a=0}^{p^{h-1}-1} \sum_{\alpha=0}^{p-1} \left[\sum_{j=0}^{n-1} \frac{g^{(j)}(a + \alpha p^{h-1})}{j!} (z - a - \alpha p^{h-1})^j \right] \mathbb{1}_{a+\alpha p^{h-1}+p^h\mathbb{Z}_p}.$$

By Lemma 5.2

$$\begin{aligned} g_h(z) \equiv \sum_{a=0}^{p^{h-1}-1} \sum_{\alpha=0}^{p-1} \Bigg\{ & \left[\frac{1}{0!\,0!} g(a) + \frac{\alpha p^{h-1}}{0!\,1!} g^{(1)}(a) + \cdots + \frac{(\alpha p^{h-1})^{n-2}}{0!\,(n-2)!} g^{(n-2)}(a) + \frac{(\alpha p^{h-1})^{n-1}}{0!\,(n-1)!} g^{(n-1)}(a) \right] \\ & + \left[\frac{1}{1!\,0!} g^{(1)}(a) + \frac{\alpha p^{h-1}}{1!\,1!} g^{(2)}(a) + \cdots + \frac{(\alpha p^{h-1})^{n-2}}{1!\,(n-2)!} g^{(n-1)}(a) \right] (z - a - \alpha p^{h-1}) \\ & \quad \vdots \qquad\qquad \vdots \\ & + \left[\frac{1}{(n-1)!\,0!} g^{(n-1)}(a) \right] (z - a - \alpha p^{h-1})^{n-1} \Bigg\} \mathbb{1}_{a+\alpha p^{h-1}+p^h\mathbb{Z}_p} \mod \pi\tilde{\Theta}(k, \mathcal{L}). \end{aligned}$$

Expanding $[(z-a) - \alpha p^{h-1}]^j$ for $0 \le j \le n-1$ and collecting like terms of $(z-a)$ diagonally, the terms involving α disappear

$$\begin{aligned} g_h(z) &\equiv \sum_{a=0}^{p^{h-1}-1} \sum_{\alpha=0}^{p-1} \left[\sum_{j=0}^{n-1} \frac{g^{(j)}(a)}{j!} (z-a)^j \right] \mathbb{1}_{a+\alpha p^{h-1}+p^h\mathbb{Z}_p} \\ &\equiv \sum_{a=0}^{p^{h-1}-1} \left[\sum_{j=0}^{n-1} \frac{g^{(j)}(a)}{j!} (z-a)^j \right] \mathbb{1}_{a+p^{h-1}\mathbb{Z}_p} \equiv g_{h-1}(z) \mod \pi\tilde{\Theta}(k, \mathcal{L}). \end{aligned}$$

Iterating this process, we get $g_h(z) \equiv g_2(z) \mod \pi\tilde{\Theta}(k, \mathcal{L})$.

Therefore the claim follows. □

Lemma 5.4 *Let $k/2 \le n \le r \le p-1$. For i in a finite indexing set I, let $\lambda_i \in E$ and $z_i \in \mathbb{Z}_p$ be such that $\sum_{i \in I} \lambda_i z_i^j = 0$ for all $0 \le j \le n$.*

Fix $x \in \mathbb{Q}$ such that $x \ge -1$. Then for

$$f(z) = \sum_{i \in I} p^x \lambda_i z^{r-n} (1 - z z_i)^n \log_{\mathcal{L}}(1 - z z_i) \mathbb{1}_{p\mathbb{Z}_p},$$

we have $f(z) \equiv 0 \mod \pi\tilde{\Theta}(k, \mathcal{L})$.

We refer the reader to [21, Lemma 8.5] since giving the proof here will take us too far away from the goal of these notes. We do however give an example of a choice of λ_i and z_i which make the sums in the above lemma vanish.

Lemma 5.5 *If*
$I = \{0, 1, \ldots, n, p\}$ for some $r/2 < n \leq r \leq p-1$, and $z_i = i$ for $i \in I$, then there are $\lambda_i \in \mathbb{Z}_p$ not all zero such that $\sum_{i \in I} \lambda_i z_i^j = 0$ for $0 \leq j \leq n$. Moreover, these λ_i satisfy $\lambda_0 = 1 \bmod p$, $\lambda_p = -1$ and $\lambda_i = 0 \bmod p$ for $1 \leq i \leq n$.

Proof Taking $\lambda_p = -1$ and separating the $i = p$ summand from the equations $\sum_{i \in I} \lambda_i i^j = 0$, we see that for $0 \leq j \leq n$, we have

$$\sum_{i=0}^{n} \lambda_i i^j = p^j.$$

Since $i \not\equiv i' \bmod p$ for $0 \leq i, i' \leq n \leq p-1$, there is a unique solution to the system of equations above with $\lambda_i \in \mathbb{Z}_p$. Moreover, reducing the equations in the display above modulo p, we get

$$\sum_{i=0}^{n} \overline{\lambda_i}\, \bar{i}^j = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } 1 \leq j \leq n. \end{cases}$$

Since $\overline{\lambda_0} = 1$ and $\overline{\lambda_i} = 0$ for $1 \leq j \leq n$ is the unique solution to the mod p system of equations in the display above, we obtain the lemma. □

5.1.1 Strategy

We now give an outline of the strategy we used to establish congruences using the generalities above. These congruences allow us to determine the JH factors in the reduction of $\tilde{\Theta}(k, \mathcal{L})$.

We first choose in an intelligent way a poly·log function

$$g : \mathbb{Q}_p \to E$$

as in equation (23). The importance of choosing this function carefully cannot be overemphasized. Note that $g(z) \in D(k)$. Write

$$g(z) = g(z)\mathbb{1}_{\mathbb{Z}_p} + g(z)\mathbb{1}_{\mathbb{Q}_p \setminus \mathbb{Z}_p}.$$

We claim that $g(z)\mathbb{1}_{\mathbb{Z}_p} \in D(k)$. Indeed, the function

$$z^r g(1/z)\mathbb{1}_{\mathbb{Z}_p}(1/z) = z^r g(1/z)\mathbb{1}_{\mathbb{Z}_p \setminus p\mathbb{Z}_p}(z) = z^r g(1/z) - z^r g(1/z)\mathbb{1}_{p\mathbb{Z}_p}(z)$$

on $\mathbb{Z}_p \setminus 0$ extends to a function in $\mathscr{C}^{r/2}(\mathbb{Z}_p, E)$, since $z^r g(1/z)$ does by definition of $D(k)$, and an easy further check shows that multiplication by $\mathbb{1}_{p\mathbb{Z}_p}$ preserves the space $\mathscr{C}^{r/2}(\mathbb{Z}_p, E)$. It follows that $g(z)\mathbb{1}_{\mathbb{Q}_p \setminus \mathbb{Z}_p} \in D(k)$. Now:

- For large h we have $g(z)\mathbb{1}_{\mathbb{Z}_p} \equiv \tilde{g}_h(z) \mod \pi\tilde{\Theta}(k, \mathcal{L})$. Indeed by (27), we have that $\tilde{g}_h$ is close to $g(z)\mathbb{1}_{\mathbb{Z}_p}$ in the $v'_{\mathscr{C}^{r/2}}$-adic topology. Recall that by Theorem 3.27, the locally polynomial functions $e_{i,j,r/2}$ for $i \geq 0$ and $0 \leq j \leq \lfloor r/2 \rfloor$ form a Banach basis of $\mathscr{C}^{r/2}(\mathbb{Z}_p, E)$. So for large h there are constants $a_{i,j}$ with $v_p(a_{i,j}) > 0$ such that
$$g(z)\mathbb{1}_{\mathbb{Z}_p} - \tilde{g}_h(z) = \sum_{i,j} a_{i,j} e_{i,j,r/2}.$$

We can check that this is an equality of functions in $D(k)$. Since

$$\lfloor l(i)(r/2)\rfloor - l(i)j \geq (l(i)-1)(r/2-j),$$

we deduce by Lemma 4.2 that the $e_{i,j,r/2}$ are in the lattice $\tilde{\Theta}(k,\mathcal{L})$.

- Then using Lemma 5.1, we see that $g(z)\mathbb{1}_{\mathbb{Z}_p} \equiv g_h(z) \mod \pi\tilde{\Theta}(k,\mathcal{L})$ for large h.
- Then using Lemma 5.2 and Lemma 5.3 we may assume that we can descend from large h to $h=2$ (though as we see in [21] for some exceptional g we can only descend to $h=3$.)
- We also prove that $g(z)\mathbb{1}_{\mathbb{Q}_p\setminus\mathbb{Z}_p} \equiv 0 \mod \pi\tilde{\Theta}(k,\mathcal{L})$ using Lemma 5.4 (though as we see in [21] in an exceptional case this function also contributes to the argument).

Since $g(z)$ is equal to 0 in $\tilde{B}(k,\mathcal{L})$, we get a congruence

$$g_2(z) \equiv 0 \mod \pi\tilde{\Theta}(k,\mathcal{L})$$

(or $g_3(z) \equiv 0 \mod \pi\tilde{\Theta}(k,\mathcal{L})$ for some functions g).

These congruences allow us to show that the subquotients $F_{2l,2l+1}$ for $0 \leq l \leq r$ of $\overline{\tilde{\Theta}(k,\mathcal{L})}$ are quotients of cokernels of linear expressions involving Iwahori-Hecke operators acting on compactly induced representations from IZ to G. Finally, we are able to deduce Theorem 1.1 by applying the Iwahori mod p LLC stated in Theorem 2.5.

5.2 Analysis of $\overline{\tilde{\Theta}(k,\mathcal{L})}$ around all points but the last

Section 9 of [21] gives a uniform treatment of all the subquotients around the marked points on the ν line appearing in Theorem 1.1 except for the last marked point. The goal of this section is to describe how this is done in a simplified example.

Note that the last marked point $\nu = \frac{1}{2}$, respectively $\nu = 0$, for r odd, respectively r even, is particularly tricky to deal with and requires much additional work (see [21, Sections 10, 11]). In fact, the former case requires establishing a formula for the constant λ_i for $i = \frac{r+1}{2}$ with r odd which requires a much more elaborate treatment. The latter case requires working in a non-commutative Hecke algebra. We do not make any further comments about the behaviour of the subquotients about these last marked points in these notes.

In this section, we give a sketch of the proof of the fact that if $\nu = i - r/2$ for $i = 1, 2, \ldots, \lceil r/2\rceil - 1$, then the map $\mathrm{ind}_{IZ}^G\, a^i d^{r-i} \twoheadrightarrow F_{2i,\,2i+1}$ factors as

$$\mathrm{ind}_{IZ}^G\, a^i d^{r-i} \twoheadrightarrow \frac{\mathrm{ind}_{IZ}^G\, a^i d^{r-i}}{\mathrm{Im}\,(T_{1,2}-\lambda_i)} \twoheadrightarrow F_{2i,\,2i+1},$$

where

$$\lambda_i = (-1)^i i \binom{r-i+1}{i} p^{r/2-i}\mathcal{L}.$$

Moreover, we prove that the second map in the display above induces a surjection

$$\pi(r-2i, \lambda_i, \omega^i) \twoheadrightarrow F_{2i,\,2i+1}.$$

Further details may be found in [21, Section 9.2].

Here is the key technical proposition. It starts with an intelligently chosen function $g : \mathbb{Q}_p \to E$ as in (23).

Proposition 5.6 *[21, Proposition 9.6] Let $k/2 < n \leq k-2 \leq p-1$. For $\nu \leq 1 + r/2 - n$, set*

$$g(z) = p^x \left[\sum_{i \in I} \lambda_i (z-i)^n \log_{\mathcal{L}}(z-i) \right],$$

where $x \in \mathbb{Q}$ with $x + \nu = r/2 - n$, $I = \{0, 1, \ldots, n, p\}$ and the $\lambda_i \in \mathbb{Z}_p$ are as in Lemma 5.5.

Then, we have

$$g(z) \equiv p^{1+x} \sum_{a=1}^{p-1} a^{-1} z^n \mathbb{1}_{a+p\mathbb{Z}_p} + \sum_{j=\lceil r/2 \rceil}^{n-1} (-1)^{n-j+1} \binom{n}{j} p^{x+n-j} \mathcal{L} z^j \mathbb{1}_{p\mathbb{Z}_p} \mod \pi \tilde{\Theta}(k, \mathcal{L}).$$

Proof Write $g(z) = g(z)\mathbb{1}_{\mathbb{Z}_p} + g(z)\mathbb{1}_{\mathbb{Q}_p \setminus \mathbb{Z}_p}$. By Lemma 5.5, we have

$$\sum_{i \in I} \lambda_i i^j = 0, \text{ for } 0 \leq j \leq n, \ \lambda_0 \equiv 1 \mod p, \ \lambda_i \equiv 0 \mod p \text{ for } 1 \leq i \leq n \text{ and } \lambda_p = -1. \tag{32}$$

Since the summation identity in (32) is also true for $j = n$, the $\log_{\mathcal{L}}(z)$ term dies and so

$$w \cdot g(z)\mathbb{1}_{\mathbb{Q}_p \setminus \mathbb{Z}_p} = \sum_{i \in I} p^x \lambda_i z^{r-n} (1 - zi)^n \log_{\mathcal{L}}(1 - zi) \mathbb{1}_{p\mathbb{Z}_p}.$$

Lemma 5.4 implies that $w \cdot g(z)\mathbb{1}_{\mathbb{Q}_p \setminus \mathbb{Z}_p} \equiv 0 \mod \pi \tilde{\Theta}(k, \mathcal{L})$.

Next, using Lemmas 5.1, 5.2 and 5.3 we see that $g(z)\mathbb{1}_{\mathbb{Z}_p} \equiv g_2(z) \mod \pi \tilde{\Theta}(k, \mathcal{L})$. We next claim that $g_2(z) - g_1(z) \equiv 0 \mod \pi \tilde{\Theta}(k, \mathcal{L})$. We see that the coefficient of $(z - a - \alpha p)^j \mathbb{1}_{a+\alpha p+p^2\mathbb{Z}_p}$ in $g_2(z) - g_1(z)$ is

$$\binom{n}{j} \sum_{i \in I} p^x \lambda_i \Big[(a + \alpha p - i)^{n-j} \log_{\mathcal{L}}(a + \alpha p - i) - (a-i)^{n-j} \log_{\mathcal{L}}(a-i) \tag{33}$$
$$- \binom{n-j}{1} (\alpha p)(a-i)^{n-j-1} \log_{\mathcal{L}}(a-i) - \cdots - \binom{n-j}{n-j-1} (\alpha p)^{n-j-1} (a-i) \log_{\mathcal{L}}(a-i) \Big].$$

By Lemma 4.2, to prove the claim it suffices to show that this coefficient is congruent to 0 modulo $p^{r/2-j}\pi$. This can be done but we omit the details.

This shows that $g(z) \equiv g_1(z) \mod \pi \tilde{\Theta}(k, \mathcal{L})$. Next, we simplify $g_1(z)$. Recall that

$$g_1(z) = \sum_{a=0}^{p-1} \left[\sum_{j=0}^{n-1} \frac{g^{(j)}(a)}{j!} (z-a)^j \right] \mathbb{1}_{a+p\mathbb{Z}_p}, \tag{34}$$

where

$$\frac{g^j(a)}{j!} = \sum_{i \in I} \binom{n}{j} p^x \lambda_i (a-i)^{n-j} \log_{\mathcal{L}}(a-i). \tag{35}$$

First assume that $a \neq 0$. The $i \neq 0, a, p$ summands on the right side of equation (35) are congruent to 0 modulo π since $a \not\equiv i \mod p$ and $p \mid \lambda_i$ for $i \neq 0, p$. The $i = a$ term is equal to 0. The sum of the $i = 0$ and $i = p$ terms in (35) is congruent to

$$\binom{n}{j} p^x \left[a^{n-j} \log_{\mathcal{L}}(a) - (a-p)^{n-j} \log_{\mathcal{L}}(a-p)\right] \mod \pi$$

since, by (32), $\lambda_0 = 1 \mod p$ and $\lambda_p = -1$. Expanding $(a-p)^{n-j}$ and dropping the terms divisible by p since $x \geq -1$ and $p \mid \log_{\mathcal{L}}(a-p)$, this equals

$$\binom{n}{j} p^x (-a^{n-j}) \log_{\mathcal{L}}(1 - a^{-1}p) \mod \pi.$$

Expanding $\log_{\mathcal{L}}(1 - a^{-1}p)$ using the usual Taylor series and dropping the terms that are congruent to 0 mod π, we see that the sum of the $i = 0$ and p summands in (35) is

$$\binom{n}{j} p^{1+x} a^{n-j-1} \mod \pi.$$

Therefore for $a \neq 0$, we get

$$\frac{g^{(j)}(a)}{j!} \equiv \binom{n}{j} p^{1+x} a^{n-j-1} \mod \pi. \tag{36}$$

Next, assume that $a = 0$. Then again the $i \neq 0,\ p$ summands in (35) are congruent to 0 modulo π by the same reasoning as in the $a \neq 0$ case above. The $i = 0$ summand is 0. The $i = p$ summand is

$$-\binom{n}{j} p^x (-p)^{n-j} \mathcal{L}.$$

Therefore for $a = 0$, we have

$$\frac{g^{(j)}(0)}{j!} \equiv (-1)^{n-j+1} \binom{n}{j} p^{x+n-j} \mathcal{L} \mod \pi. \tag{37}$$

Putting equations (36) and (37) in equation (34) and using Lemma 4.2, we get

$$\begin{aligned} g_1(z) &\equiv \sum_{a=1}^{p-1} \left[\sum_{j=0}^{n-1} \binom{n}{j} p^{1+x} a^{n-j-1} (z-a)^j\right] \mathbb{1}_{a+p\mathbb{Z}_p} + \left[\sum_{j=0}^{n-1} (-1)^{n-j+1} \binom{n}{j} p^{x+n-j} \mathcal{L} z^j\right] \mathbb{1}_{p\mathbb{Z}_p} \\ &\equiv \sum_{a=1}^{p-1} p^{1+x} a^{-1} \left[z^n - (z-a)^n\right] \mathbb{1}_{a+p\mathbb{Z}_p} + \left[\sum_{j=0}^{n-1} (-1)^{n-j+1} \binom{n}{j} p^{x+n-j} \mathcal{L} z^j\right] \mathbb{1}_{p\mathbb{Z}_p} \\ &\qquad \mod \pi \tilde{\Theta}(k, \mathcal{L}). \end{aligned}$$

Using Lemma 4.3 for the first sum and Lemma 4.2 for the second sum above, we get

$$g_1(z) \equiv \sum_{a=1}^{p-1} p^{1+x} a^{-1} z^n \mathbb{1}_{a+p\mathbb{Z}_p} + \left[\sum_{j=\lceil r/2 \rceil}^{n-1} (-1)^{n-j+1} \binom{n}{j} p^{x+n-j} \mathcal{L} z^j\right] \mathbb{1}_{p\mathbb{Z}_p} \mod \pi \tilde{\Theta}(k, \mathcal{L}).$$

This proves the proposition since $g(z) \equiv g_1(z) \mod \pi \tilde{\Theta}(k, \mathcal{L})$. □

Theorem 5.7 *[21, Theorem 9.7] Let $i = 1, 2, \ldots, \lceil r/2 \rceil - 1$. If $\nu = i - r/2$, then the map $\mathrm{ind}_{IZ}^{G} a^i d^{r-i} \twoheadrightarrow F_{2i,\,2i+1}$ factors as*

$$\mathrm{ind}_{IZ}^{G}\, a^i d^{r-i} \twoheadrightarrow \frac{\mathrm{ind}_{IZ}^{G}\, a^i d^{r-i}}{\mathrm{Im}\,(T_{1,2} - \lambda_i)} \twoheadrightarrow F_{2i,\,2i+1},$$

where

$$\lambda_i = (-1)^i i \binom{r-i+1}{i} p^{r/2-i} \mathcal{L}.$$

Moreover, the second map in the display above induces a surjection

$$\pi(r-2i, \lambda_i, \omega^i) \twoheadrightarrow F_{2i,\,2i+1}.$$

Proof All congruences in this proof are in the space $\overline{\tilde{\Theta}(k, \mathcal{L})}$ modulo the image of the subspace $\mathrm{ind}_{IZ}^{G} \oplus_{j<r-i} \mathbb{F}_q X^{r-j} Y^j$ under $\mathrm{ind}_{IZ}^{G} \mathrm{Sym}^{k-2}\mathbb{F}_q^2 \twoheadrightarrow \overline{\tilde{\Theta}(k, \mathcal{L})}$.

Fix an $i = 1, 2, \ldots, \lceil r/2 \rceil - 1$. Applying Proposition 5.6 with $n = r - i + 1$ and the remark above, we get

$$0 \equiv p^{1+x} \sum_{a=1}^{p-1} a^{-1} z^{r-i+1} \mathbb{1}_{a+p\mathbb{Z}_p} + (r-i+1) p^{x+1} \mathcal{L} z^{r-i} \mathbb{1}_{p\mathbb{Z}_p}. \qquad (38)$$

Since $\nu = i - r/2$, we see that $x = -1$.
After much massaging of this equation using some inductive steps and some matrix computations (several pages of work in [21]), we get

$$0 \equiv \sum_{a=0}^{p-1} p^{r/2-i} (z - ap)^i \mathbb{1}_{ap+p^2\mathbb{Z}_p} - (-1)^i i \binom{r-i+1}{i} p^{r/2-i} \mathcal{L} z^i \mathbb{1}_{p\mathbb{Z}_p}.$$

This equation is the image of $(T_{1,2} - \lambda_i)[\![\mathrm{id}, X^i Y^{r-i}]\!]$ under $\mathrm{ind}_{IZ}^{G}\, a^i d^{r-i} \twoheadrightarrow F_{2i,\,2i+1}$. Indeed,

$$T_{1,2}[\![\mathrm{id}, X^i Y^{r-i}]\!] = \sum_{\lambda \in I_1} [\![\begin{pmatrix} 1 & 0 \\ -p\lambda & p \end{pmatrix}, X^i Y^{r-i}]\!] \mapsto \sum_{\lambda \in I_1} p^{r/2-i} (z - \lambda p)^i \mathbb{1}_{\lambda p + p^2\mathbb{Z}_p}.$$

Therefore we have proved that the surjection $\mathrm{ind}_{IZ}^{G}\, a^i d^{r-i} \twoheadrightarrow F_{2i,\,2i+1}$ factors as

$$\mathrm{ind}_{IZ}^{G}\, a^i d^{r-i} \twoheadrightarrow \frac{\mathrm{ind}_{IZ}^{G}\, a^i d^{r-i}}{\mathrm{Im}\,(T_{1,2} - \lambda_i)} \twoheadrightarrow F_{2i,\,2i+1}.$$

Now consider the following exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Im}\, T_{1,2} & \longrightarrow & \mathrm{ind}_{IZ}^{G}\, a^i d^{r-i} & \longrightarrow & \dfrac{\mathrm{ind}_{IZ}^{G}\, a^i d^{r-i}}{\mathrm{Im}\, T_{1,2}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S & \longrightarrow & F_{2i,\,2i+1} & \longrightarrow & Q & \longrightarrow & 0, \end{array}$$

where S is the image of $\mathrm{Im}\, T_{1,2}$ under the surjection $\mathrm{ind}_{IZ}^{G}\, a^i d^{r-i} \twoheadrightarrow F_{2i,\,2i+1}$ and $Q = F_{2i,\,2i+1}/S$. Since $\mathrm{Im}\,(T_{1,2} - \lambda_i)$ maps to 0 under the middle vertical map in the diagram above, we see that the right vertical map is

0. Therefore we have a surjection $\mathrm{Im}\, T_{1,2} \twoheadrightarrow F_{2i,\,2i+1}$. By the first isomorphism theorem and [1, Proposition 3.1], we get a surjection

$$\frac{\mathrm{ind}_{IZ}^{G}\, a^i d^{r-i}}{\mathrm{Im}\, T_{-1,0}} \twoheadrightarrow F_{2i,\,2i+1},$$

which factors through

$$\frac{\mathrm{ind}_{IZ}^{G}\, a^i d^{r-i}}{\mathrm{Im}\, T_{-1,0} + \mathrm{Im}\,(T_{1,2} - \lambda_i)} \twoheadrightarrow F_{2i,\,2i+1}.$$

The space on the left is

$$\frac{\mathrm{ind}_{IZ}^{G}\, a^i d^{r-i}}{\mathrm{Im}\, T_{-1,0} + \mathrm{Im}\,(T_{1,2} - \lambda_i)} = \pi(r-2i, \lambda_i, \omega^i).$$

This completes the sketch of the proof of the theorem. □

5.3 Reduction mod p of $\tilde{\Theta}(k, \mathcal{L})$

In this section, we summarize all the results proved in [21] (such as Theorem 5.7 above) and mention how they are used to prove Theorem 1.1.

Recall that

$$\nu = v_p(\mathcal{L} - H_- - H_+).$$

Theorem 5.8 *[21, Theorem 12.1] For $3 \le k \le p+1$ and $p \ge 5$, the semi-simplification of the reduction $\overline{V}_{k,\mathcal{L}}$ of the semi-stable representation $V_{k,\mathcal{L}}$ of $G_{\mathbb{Q}_p}$ of Hodge-Tate weights $(0, k-1)$ and $\mathcal{L}$-invariant $\mathcal{L}$ satisfies:*

$$\overline{V}_{k,\mathcal{L}} \sim \begin{cases} \mathrm{ind}\,(\omega_2^{r+1+(i-1)(p-1)}), & \text{if } (i-1) - r/2 < \nu < i - r/2 \\ \mu_{\lambda_i}\omega^{r+1-i} \oplus \mu_{\lambda_i^{-1}}\omega^i, & \text{if } \nu = i - r/2, \end{cases}$$

where $1 \le i \le \dfrac{r+1}{2}$ if r is odd and $1 \le i \le \dfrac{r+2}{2}$ if r is even. The constants λ_i are determined by

$$\lambda_i = \overline{(-1)^i\, i \binom{r+1-i}{i} p^{r/2-i}(\mathcal{L} - H_- - H_+)}, \quad \text{if } 1 \le i < \frac{r+1}{2}$$

$$\lambda_i + \lambda_i^{-1} = \overline{(-1)^i\, i \binom{r+1-i}{i} p^{r/2-i}(\mathcal{L} - H_- - H_+)}, \quad \text{if } i = \frac{r+1}{2} \text{ and } r \text{ is odd.}$$

We follow the conventions stated in the Introduction.

Proof We collect the necessary results proved in [21] here. We first state the common results for odd and even weights (the fourth of which was sketched in Theorem 5.7):

1. For $i = 1, 2, \ldots, \lceil r/2 \rceil - 1$ if $\nu > i - r/2$, then $F_{2i-2,\,2i-1} = 0$.
2. For $i = 1, 2, \ldots, \lceil r/2 \rceil - 1$ if $\nu = i - r/2$, then $\pi([2i-2-r], \lambda_i^{-1}, \omega^{r-i+1}) \twoheadrightarrow F_{2i-2,\,2i-1}$.
3. For $i = 1, 2, \ldots, \lceil r/2 \rceil - 1$ if $\nu < i - r/2$, then $F_{2i,\,2i+1} = 0$.
4. For $i = 1, 2, \ldots, \lceil r/2 \rceil - 1$ if $\nu = i - r/2$, then $\pi(r-2i, \lambda_i, \omega^i) \twoheadrightarrow F_{2i,\,2i+1}$.

Next, we state the extra results for odd weights around the point $\nu = \frac{1}{2}$.

5. If $\nu \ge 0.5$, then $\pi(p-2, \lambda_{\frac{r+1}{2}}, \omega^{\frac{r+1}{2}}) \oplus \pi(p-2, \lambda_{\frac{r+1}{2}}^{-1}, \omega^{\frac{r+1}{2}}) \twoheadrightarrow F_{r-1,\,r}$.

6. If $-0.5 < \nu < 0.5$, then $\dfrac{\mathrm{ind}_{IZ}^{G}\, a^{\frac{r-1}{2}} d^{\frac{r+1}{2}}}{\mathrm{Im}\, T_{-1,0}} \twoheadrightarrow F_{r-1,\, r}$.

Finally, we state the extra results for even weights around the point $\nu = 0$.

5. If $\nu > 0$, then $F_{r-2,\, r-1} = 0$.
6. If $\nu = 0$, then $\pi(p-3, \lambda_{r/2}^{-1}, \omega^{\frac{r+2}{2}}) \twoheadrightarrow F_{r-2,\, r-1}$.
7. If $\nu < 0$, then $F_{r,\, r+1} = 0$.
8. If $\nu = 0$, then $\pi(p-1, \lambda_{r/2}, \omega^{r/2}) \twoheadrightarrow F_r$ and F_{r+1} is not isomorphic to $\pi(0, 0, \omega^{r/2})$.

The proof of the theorem is then a standard application of the compatibility with respect to mod p reduction between the p-adic and the Iwahori mod p LLC (stated in Theorem 2.5). □

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Declaration

Conflict of Interest The author declares that he has no conflict of interest.

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