ORIGINAL RESEARCH

Existence, regularity, and qualitative properties in choquard-type equations: a contemporary review

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Received: 11 November 2025 / Accepted: 13 November 2025 / Published online: 27 November 2025

Abstract
This article presents a contemporary survey of mathematical advances in Choquard-type equations, a class of nonlinear, nonlocal partial differential equations with profound applications in mathematical physics and analysis. The survey synthesizes foundational results and recent progress regarding existence, regularity, and qualitative properties of solutions, including ground states, multiplicity, and topological effects in bounded domains. It highlights developments for equations with fractional Laplacians, singular and doubly nonlocal nonlinearities, and Kirchhoff-type operators. Alongside technical advances, the article outlines open research problems and underscores both the analytical challenges and rich structures inherent to nonlocal elliptic equations, serving as a comprehensive resource for researchers in nonlinear analysis and mathematical physics.

Keywords Choquard equation · Fractional Laplacian · Regularity · Singular nonlinearity · Kirchhoff operator · Quasilinear operator

Mathematics Subject Classification 35J20 · 35R09 · 35B65 · 35B33

1 Introduction

The Choquard equation is a celebrated nonlinear, nonlocal partial differential equation with origins in mathematical physics, specifically in the description of the quantum theory of polarons. Its most classical formulation over $\mathbb{R}^N$ reads as follows.

$$-\Delta u + V(x)u = \big(I_\mu * |u|^p\big)|u|^{p-2}u,$$

where $u : \mathbb{R}^N \to \mathbb{R}$, $V(x)$ is a potential, $I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}}$ is the Riesz potential [86, 88] and $*$ denotes the convolution. The equation first arose in 1954 when S. I. Pekar studied the self-consistent field equations for a polaron at rest, proposing a nonlocal term to describe the electrostatic interaction in a quantum mechanical system. In 1976, P. Choquard applied a similar nonlocal equation to model an electron trapped in its own hole, in the context of Hartree–Fock theory for a plasma [21, 71, 91]. This equation is often referred to as the *Choquard–Pekar equation* or the *Schrödinger–Newton equation*. It emerges in different physical settings, including quantum mechanics, astrophysics, and the theory of self-gravitating matter [71].

A series of works by Vitaly Moroz and Jean Van Schaftingen has made significant contributions to the mathematical analysis of this class of equations. Their investigations consider the autonomous Choquard equation in the form:

Communicated by Jugal Verma

 https://doi.org/10.1007/s13226-025-00894-3

$$-\Delta u + u = \left(I_\alpha * |u|^p\right)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $u \in H^1(\mathbb{R}^N)$, $p > 1$, $\alpha \in (0, N)$, and $I_\alpha(x) = A_\alpha |x|^{-(N-\alpha)}$ is the Riesz potential. Here, the constant A_α is given by

$$A_\alpha = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{N/2}2^\alpha}.$$

The following Hardy-Littlewood-Sobolev inequality [72, Theorem 4.3] plays a central role in the mathematical analysis of Choqard equations.

Proposition 1.1 *Let $r, s > 1$ and $0 < \mu < N$ with $1/r + \mu/N + 1/s = 2$, $g \in L^r(\mathbb{R}^N)$ and $h \in L^s(\mathbb{R}^N)$. Then there exists a sharp constant $C(\mu, N, r, s)$, independent of g, h such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)h(y)}{|x-y|^\mu} dxdy \leq C(\mu, N, r, s)\|g\|_{L^r(\mathbb{R}^N)}\|h\|_{L^s(\mathbb{R}^N)}. \tag{1.2}$$

Using the Hardy-Littlewood-Sobolev inequality (1.2), for $s = r = t$, the integral is

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^t|u(y)|^t}{|x-y|^\mu} dxdy$$

is well-defined if $|u|^t \in L^q(\mathbb{R}^N)$ for some $q > 1$ satisfying $\frac{2}{q} + \frac{\mu}{N} = 2$.
For $u \in H^1(\mathbb{R}^N)$, by the Sobolev embedding theorem, we have $p \leq tq \leq \frac{2N}{N-2}$. Thus

$$\frac{(2N-\mu)}{N} \leq t \leq \frac{(2N-\mu)}{(N-2)}$$

In this sense, we call $2_{*\mu} = \frac{(2N-\mu)}{N}$ the lower critical exponent and $2^*_\mu = \frac{(2N-\mu)}{(N-2)}$ the upper critical exponent in the sense of the Hardy Littlewood-Sobolev inequality. We use $S_{H,L}$ to denote

$$S_{H,L} = \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu} dxdy\right)^{1/2^*_\mu}}. \tag{1.3}$$

We recall the following result from [37] for $S_{H,L}$.

Lemma 1.2 *The constant $S_{H,L}$ defined in* (1.3) *is achieved if and only if*

$$u = C\left(\frac{b}{b^2 + |x-a|^2}\right)^{\frac{N-2}{2}},$$

where $C > 0$ is a fixed constant, $a \in \mathbb{R}^N$ and $b \in (0, \infty)$ are parameters.

The following is the associated energy functional of the problem in (1.1)

$$\mathcal{A}(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 + |u|^2\right)dx - \frac{1}{2p}\int_{\mathbb{R}^N}\left(I_\alpha * |u|^p\right)|u|^p dx.$$

The functional $\mathcal{A}$ is Fréchet differentiable on $H^1(\mathbb{R}^N)$ when the exponent p satisfies:

$$\frac{N-2}{N+\alpha} < \frac{1}{p} < \frac{N}{N+\alpha}$$

reflecting the underlying Sobolev and Hardy–Littlewood–Sobolev critical thresholds.
In [85], Moroz and Schaftingen proved the existence of groundstate solutions, i.e., nontrivial minimizers of the energy functional using variational methods over the Nehari manifold,

$$\mathcal{N} = \left\{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : \langle \mathcal{A}'(u), u \rangle = 0 \right\}.$$

They proved that the Choquard equation admits a ground state which can be characterized via the Sobolev-type quotient:

$$Q(u) = \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) dx}{\left[\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx\right]^{1/p}}.$$

Definition 1.3 A function $u \in H^1(\mathbb{R}^N)$ is called a weak solution of the Choquard equation (1.1) if it satisfies

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla \varphi \, dx + \int_{\mathbb{R}^N} u\varphi \, dx = \int_{\mathbb{R}^N} \left(I_\alpha * |u|^p\right) |u|^{p-2} u \, \varphi \, dx \quad \forall \, \varphi \in H^1(\mathbb{R}^N).$$

Moroz and Van Schaftingen proved that any *weak* solution $u \in H^1(\mathbb{R}^N)$ is actually a *classical* solution under appropriate conditions for exponents. Specifically, by applying the classical bootstrap method to semilinear elliptic PDEs and carefully estimating Riesz potentials, they demonstrated that solutions are in C^2. Furthermore, if u is positive or if p is an even integer, the solution enjoys even higher regularity, being C^∞ [85]. This additional smoothness stems from the convolution structure and classical elliptic regularity theory. They also demonstrated a nonlocal counterpart of the Brezis-Kato estimates to show the regularity results when $p = \frac{N+\alpha}{N-2}$, the critical exponent for the nonlinearity of the convolution. Also, the singular behavior was analyzed near the origin, revealing criteria for the removability or classification of singularities depending on the exponents and dimensions. Groundstates minimizers of the Choquard energy functional were shown to have constant sign throughout $\mathbb{R}^N$ [86]. By leveraging the regularity properties and applying the strong maximum principle, they demonstrate that ground states cannot vanish and are either strictly positive or strictly negative everywhere.

In their beautiful survey paper [88], they also detail the symmetrization arguments and the radially symmetric and monotonicity of ground states. Here, they proved that the symmetric decreasing rearrangement u^* of a function u preserves or decreases the nonlocal Choquard energy by exploiting equality cases in the Riesz-Sobolev convolution inequality. Uniqueness for radial positive groundstates is particularly sharpened for the case $\alpha = 2$, $p = 2$. Also, it was shown that the solutions exhibit exponential decay in the regime $p > 2$ (or $p = 2$ with $\alpha > N - 1$)

$$u(x) \sim C|x|^{-(N-1)/2} e^{-|x|}, \quad |x| \to \infty,$$

and polynomial decay for $p \in (1 + \frac{\alpha}{N}, 2)$. For more results on this class of elliptic equations involving the convolution-type nonlinearities we cite [3, 19, 28, 53, 66, 67, 79, 80, 101] and references therein, with no attempt to provide the full list.

In this article, we systematically review the variational methods used for establishing the existence of ground states, the role of functional inequalities like Hardy-Littlewood-Sobolev, and the classification of critical exponents that distinguish between various regimes of solution behavior. Regularity theory is discussed in depth, including bootstrapping and classical elliptic regularity techniques. The survey transitions to bounded domains, stressing the influence of boundary geometry and topology. Inspired by classical results on Laplacian equations with critical growth (Brezis-Nirenberg, Ambrosetti-Cerami-Brezis), this review covers how the interplay between nonlocal nonlinearities and compactness leads to novel existence, nonexistence, and multiplicity results. Notably, star-shaped domains may rule out solutions, while domains with holes or nontrivial topology can support them, as shown in groundbreaking work by Bahri and Coron. Throughout, the survey emphasizes both classical and modern analytic techniques, providing a thorough reference framework for future research in nonlinear analysis and PDE theory.

The structure of the paper is as follows. Section 2 addresses the Choquard equation on bounded domains, while Section 3 focuses on its fractional Laplacian counterpart in bounded settings. Subsequent sections explore advanced topics including the Kirchhoff-Choquard problem, normalized solutions, and quasilinear Schrödinger equations, presenting recent developments and analytical techniques for each.

2 The choquard equation on bounded domains

The analysis of the Choquard equation in bounded domains has become a significant area of research and generated new insights, revealing behaviors that are not present when considering the entire space. Inspired by the seminal works of Brezis and Nirenberg [18] and Ambrosetti, Brezis, and Cerami [4] on the existence and multiplicity of positive solutions for the Laplacian equation with critical growth nonlinearities, the field has received significant attention. Here, we investigate the existence of positive solutions for Dirichlet problems featuring a Choquard nonlinearity. On a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary, the typical boundary value problem of this type reads:

$$\begin{cases} -\Delta u = \left(\int_{\Omega} \frac{|u(y)|^p}{|x-y|^{\mu}} dy \right) |u|^{p-2}u + \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where $\mu \in (0, N)$, $p > 1$, and λ is a real parameter.

Definition 2.1 A function $u \in H_0^1(\Omega)$ is called a *weak solution* of (2.1) satisfies

$$\int_{\Omega} \nabla u \nabla \varphi \, dx - \lambda \int_{\Omega} u\varphi \, dx = \int_{\Omega} \left(\int_{\Omega} \frac{|u(y)|^p}{|x-y|^{\mu}} dy \right) |u(x)|^{p-2} u(x)\varphi(x) dx, \ \forall \, \varphi \in H_0^1(\Omega).$$

The interaction between the nonlocal convolution term and the geometry of the domain Ω gives rise to complex mathematical issues related to compactness and concentration when the nonlinearity has Hardy-Littlewood-Sobolev upper critical growth $p = 2_{\mu}^*$. The compact nature of the lower critical growth on bounded domains permits the use of direct variational methods for analysis. Minbo Yang and Fashun Gao [37] established Brezis–Nirenberg type existence results for critical Choquard equations on bounded domains. They proved the existence of nontrivial solutions using variational techniques and critical point theory, adapting the concentration–compactness principle to the nonlocal regime. Their results include:

Theorem 2.2 *[37] Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with Lipschitz boundary. There exists $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0)$, problem* (2.1) *admits a nontrivial solution for $p = 2_{\mu}^*$.*

After that, several works have been published, including those with combined concave-convex type of Choquard nonlinearities, problems on magnetic Choquard, and systems involving Choquard nonlinearity, as found in [88, 99]. For a study on sign-changing weak solutions on the bounded domain, Liu and Zhou [73] proved the following result:

Theorem 2.3 *There exist infinitely many sign-changing solutions to* (2.1)*, and their energies can be classified by the number of nodal domains.*

The proof combines topological minimax methods, symmetry group actions, and a meticulous analysis of the nonlocal functional. Nodal solutions are constructed as minimizers on subspaces determined by the sign structure and genus theory, with strict inequalities controlling energy levels. Similar multiplicity results with least-energy nodal solutions are found in [54, 105], where perturbation and descending flows yield localized and multi-bump solutions. In [37], Gao and Yang discussed the thresholds necessary for the existence of solutions related to the geometry of Ω. They proved that if Ω is a star-shaped domain and $\lambda \leq 0$, then there does not exist any solution of the above problem. This motivates the study of solutions in relation to the domain's topology.

2.1 Effect of topology on solutions

In this section, we present the connection between domain topology and the existence of positive solutions to the Choquard equation. Consider the problem

$$-\Delta u = u^{2^*-1} \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{2.2}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 3)$ with smooth boundary. The distinguished work of Pohozaev [92] presents an inequality which is satisfied by any solution to problem (2.2). As a consequence of this inequality, one proves the non-existence of a solution when Ω is a star-shaped domain. On the contrary, Kazdan and Warner [60] proved that if Ω is an annulus, then (2.2) has a radial solution. Subsequently, without the symmetry condition, Coron [23] proved the existence of at least one positive solution to (2.2) provided Ω has a hole of sufficiently small size. Hence, the idea of exploiting the geometry of Ω to establish the existence of a solution. But the phenomenal result in this direction was given by Bahri and Coron [8]. They proved that (2.2) has at least one solution if the domain Ω has nontrivial topology in the sense that the homology of the domain, $H_d(\Omega, \mathbb{Z}_2) \neq 0$ ($H_d(\Omega, \mathbb{Z}_2)$ is the homology of dimension d of Ω with $\mathbb{Z}_2$ coefficients). In the case of $N = 3$, the authors in [7] established that this condition is equivalent to Ω being non-contractible.

In this section, we will discuss the details of the existence and multiplicity of positive solutions to the following Choquard equation

$$(P_f) \quad \begin{cases} -\Delta u = \left(\int_\Omega \frac{|u^+(y)|^{2^*_\mu}}{|x-y|^\mu} dy\right) |u^+|^{2^*_\mu - 2} u^+ + f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $f \in L^\infty(\Omega)$, $f \geq 0$ and $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain with a hole of sufficiently small size. Recently, in [37], Gao and Yang proved a nonexistence result for the following nonlocal equation

$$-\Delta u = \lambda u + \left(\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy\right) |u|^{2^*_\mu - 2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 3)$, $\lambda \leq 0$, $2^*_\mu = \frac{2N-\mu}{N-2}$, $0 < \mu < N$. It implies that if $\lambda = 0$ and Ω is star-shaped, then the above problem has no solution. This was the first result in the literature for the critical Choquard equation, which examines the effect of domain shape on the existence of a solution. Before discussing more about the problem (P_f), we first define the weak solution to problem (P_f).

Definition 2.4 A function $u \in H^1_0(\Omega)$ is called a weak solution of the problem (P_f) if for all $v \in H^1_0(\Omega)$ the following holds

$$\int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega \int_\Omega \frac{|u^+(x)|^{2^*_\mu} |u^+(y)|^{2^*_\mu - 1} v(y)}{|x-y|^\mu} \, dx dy - \int_\Omega f v \, dx = 0.$$

In [47, 51], authors explored the existence and multiplicity of solutions to problem (P_f) variationally under the following condition on the domain Ω

$$(A) \quad \{x \in \mathbb{R}^N;\ R_1 < |x - x_0| < R_2\} \subset \Omega,\ \{x \in \mathbb{R}^N;\ |x - x_0| < R_1\} \nsubseteq \overline{\Omega}, x_0 \in \mathbb{R}^N.$$

The energy functionals associated with problem (P_f), $\mathcal{J}_f : H_0^1(\Omega) \to \mathbb{R}$ and $\mathcal{J}_\infty : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ are defined as

$$\mathcal{J}_f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{1}{2.2^*_\mu}\int_\Omega\int_\Omega \frac{|u^+(x)|^{2^*_\mu}|u^+(y)|^{2^*_\mu}}{|x-y|^\mu}\, dxdy - \int_\Omega fu\, dx,$$

$$\mathcal{J}_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{1}{2.2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u^+(x)|^{2^*_\mu}|u^+(y)|^{2^*_\mu}}{|x-y|^\mu}\, dxdy.$$

It is not difficult to show that the functional $\mathcal{J}_f \in C^1(H_0^1(\Omega), \mathbb{R})$ and moreover, if $\mu < \min\{4, N\}$ then $\mathcal{J}_f \in C^2(H_0^1(\Omega), \mathbb{R})$. To explain the results regarding (P_f), we discuss the two cases, viz $f \equiv 0$ and $f \not\equiv 0$, separately.

2.2 $f \equiv 0$

The main result of this Section is the following existence result

Theorem 2.5 *[47] Assume $\mu < \min\{4, N\}$ and Ω is a bounded domain in $\mathbb{R}^N$ satisfying the condition (A). If $\frac{R_2}{R_1}$ is sufficiently large then problem (P_0) admits a positive high-energy solution.*

To prove this result, the authors established the following global compactness result, which helps to establish the Palais-Smale condition for $\mathcal{J}_0$.

Lemma 2.6 ***Global compactness lemma****: Let $\{u_n\}_{n\in\mathbb{N}} \subset H_0^1(\Omega)$ be such that $\mathcal{J}_0(u_n) \to c$, $\mathcal{J}_0'(u_n) \to 0$. Then passing if necessary to a subsequence, there exists a solution $v_0 \in H_0^1(\Omega)$ of*

$$-\Delta u = \left(\int_\Omega \frac{|u^+(y)|^{2^*_\mu}}{|x-y|^\mu} dy\right)|u^+|^{2^*_\mu - 1} \ in\ \Omega$$

and (possibly) non-trivial solutions $\{v_1, v_2, ..., v_k\}$, $k \in \mathbb{N} \cup \{0\}$, of

$$-\Delta u = (|x|^{-\mu} * |u^+|^{2^*_\mu})|u^+|^{2^*_\mu - 1} \ in\ \mathbb{R}^N \tag{2.3}$$

with $v_i \in D^{1,2}(\mathbb{R}^N)$ and k sequences $\{y_n^i\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and $\{\lambda_n^i\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ $i = 1, 2, \cdots k$, satisfying

$$\frac{1}{\lambda_n^i} dist(y_n^i, \partial\Omega) \to \infty,\ and\ \|u_n - v_0 - \sum_{i=1}^k (\lambda_n^i)^{\frac{2-N}{2}} v_i((. - y_n^i)/\lambda_n^i)\| \to 0,\ n \to \infty,$$

$$\|u_n\|^2_{D^{1,2}} \to \sum_{i=0}^k \|v_i\|^2_{D^{1,2}}, n \to \infty, \quad \mathcal{J}_0(v_0) + \sum_{i=1}^k \mathcal{J}_\infty(v_i) = c.$$

To prove the above Lemma, the key result is the classification of solutions to the problem (2.3). In [47], authors obtained the following symmetry result.

Theorem 2.7 *Every solution $u \in D^{1,2}(\mathbb{R}^N)$ of equation* (2.3) *is radially symmetric, monotone decreasing and of the form*

$$u(x) = \left(\frac{c_1}{c_2 + |x - x_0|^2}\right)^{\frac{N-2}{2}}.$$

for some constants $c_1, c_2 > 0$ *and some* $x_0 \in \mathbb{R}^N$.

Observe that if $u \in H_0^1(\Omega)$ is any solution of the problem

$$-\Delta u = \left(\int_\Omega \frac{|u^+(y)|^{2^*_\mu}}{|x-y|^\mu} dy\right) |u^+|^{2^*_\mu - 1} \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega .$$

Then

$$\mathcal{J}_0(u) \geq \frac{1}{2}\left(\frac{N-\mu+2}{2N-\mu}\right) S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} =: \zeta .$$

This, in combination with the global compactness Lemma, one obtains the following compactness lemma.

Lemma 2.8 *The functional* $\mathcal{J}_0$ *satisfies Palais-Smale condition for any* $c \in (\zeta, 2\zeta)$, *where*

$$\zeta = \frac{1}{2}\left(\frac{N-\mu+2}{2N-\mu}\right) S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}.$$

Finally, by applying the concentration-compactness principle in conjunction with the deformation lemma, it was demonstrated that a high-energy positive solution exists. For more details for the proof, we refer the reader to [47].

2.3 $f \not\equiv 0$

In this section, we will discuss the existence and multiplicity of solutions of problem (P_f) with $f \in \hat{F} := \{f : f \in L^\infty(\Omega),\ f \geq 0,\ f \not\equiv 0\}$. The main result in this case is as follows

Theorem 2.9 *[51] Assume* $\mu < \min\{4, N\}$, $f \in \hat{F}$, *and* Ω *be a bounded domain satisfying the condition (A). Then there exists* $e^* > 0$ *such that* (P_f) *has at least three positive solutions whenever* $\|f\|_{H^{-1}} < e^*$. *Moreover, if* $\frac{R_2}{R_1}$ *is sufficiently large then there exists* $e^{**} > 0$ *such that* (P_f) *has at least four positive solutions whenever* $0 < \|f\|_{H^{-1}} < e^{**}$.

The Nehari Manifold associated with the problem (P_f) is defined as

$$\mathcal{N}_f := \{u \in H_0^1(\Omega) \setminus \{0\} \mid u^+ \not\equiv 0 \text{ and } \langle \mathcal{J}_f'(u), u\rangle = 0\},$$

where $\langle , \rangle$ denotes the usual duality. Define

$$\Upsilon_f = \inf_{u \in \mathcal{N}_f} \mathcal{J}_f(u).$$

Notice that $\mathcal{N}_f$ contains every non-zero solution of (P_f), and observe that the Nehari manifold is closely related to the behavior of the fibering maps $\phi_u : \mathbb{R}^+ \to \mathbb{R}$ defined as $\phi_u(t) = \mathcal{J}_f(tu)$. It is easy to see that $tu \in \mathcal{N}_f$ if and only if $\phi_u'(t) = 0$ and elements of $\mathcal{N}_f$ correspond to stationary points of the fibering maps. It is natural to divide $\mathcal{N}_f$ into the following sets

$$\mathcal{N}_f^+ =: \{u \in \mathcal{N}_f | \phi_u''(1) > 0\},\ \mathcal{N}_f^- =: \{u \in \mathcal{N}_f | \phi_u''(1) < 0\},\ \ \mathcal{N}_f^0 =: \{u \in \mathcal{N}_f | \phi_u''(1) = 0\}.$$

Also denote the infimum over $\mathcal{N}_f^+$ and $\mathcal{N}_f^-$ as

$$\Upsilon_f^+ = \inf_{u \in \mathcal{N}_f^+} \mathcal{J}_f(u) \qquad \Upsilon_f^- = \inf_{u \in \mathcal{N}_f^-} \mathcal{J}_f(u).$$

Authors investigate the structure of the Nehari manifold $\mathcal{N}_f$ associated with (P_f) to prove the existence of the first solution (say u_1). Moreover, $u_1 \in \mathcal{N}_f^+$. With the help of minimizers of the best constant $S_{H,L}$, it is proved that the minima of the functional over $\mathcal{N}_f$ are below the first critical level, which is define as

$$\mathcal{J}_f(u_1) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}.$$

Moreover, $\mathcal{J}_f$ satisfies the Palais-Smale condition below the first critical level. Subsequently, to show the existence of the second and the third solution of (P_f) in $\mathcal{N}_f^-$ by using a well-known result of Ambrosetti. Then in [51] Goel and Sreenadh show that the functional $\mathcal{J}_f$ satisfies the Palais-Smale condition between the first and the second critical levels, where the second critical level is

$$\inf_{u \in \mathcal{N}_f^-} \mathcal{J}_f(u) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}.$$

Then, the existence of the fourth solution, a high-energy solution, can be demonstrated using min-max theorems. For more details, we refer the reader to [51]. For the convex type perturbation, Goel [44] studied the following problem

$$(\mathfrak{K}_\lambda) \begin{cases} -\Delta u = \lambda |u|^{q-2}u + \left(\displaystyle\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy \right) |u|^{2^*_\mu - 2} u \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases}$$

where Ω is an open bounded set with continuous boundary in $\mathbb{R}^N (N \geq 3)$, and $q \in [2, 2^*)$ where $2^* = \frac{2N}{N-2}$.

Since the geometry of the domain plays an essential role, here author proved that the topology of the domain yields a lower bound on the number of positive solutions. More precisely, the author in [44] established the following result

Theorem 2.10 *Let λ_1 be the first eigenvalue of $-\Delta$ with zero Dirichlet boundary data. Then there exists $0 < \Lambda^* < \lambda_1$ such that for all $\lambda \in (0, \Lambda^*)$ there exists at least $cat_\Omega(\Omega)$ positive solutions of $(\mathfrak{K}_\lambda)$ under the following conditions*

1. *$q \in [2, 2^*)$ and $N > 3$ or*
2. *$4 < q < 6$ and $N = 3$.*

3 Fractional-Choquard problems

The fractional Choquard equation is a nonlocal partial differential equation that generalizes the classical Choquard equation by incorporating the fractional Laplacian operator $(-\Delta)^s,\ 0 < s < 1$, which models anomalous diffusion or long-range interactions. The first major study of the fractional Choquard equation in the whole space was done in [25]. Here authors established key existence, symmetry, and decay properties for these nonlocal equations with a fractional Laplacian on $\mathbb{R}^N$. While in the bounded domain [89], Sreenadh with his collaborators , established the Brezis-Nirenberg type result. Their work is widely recognized as foundational for the analysis of fractional Choquard equations in the bounded domain. In this section, we will focus on the significant findings of Sreenadh's work on nonlocal elliptic equations [5, 38, 39] that discuss the weak solutions to the Choquard equation with Hardy-Littlewood-Sobolev critical nonlinearity. In this work, the authors undertake a rigorous analysis of elliptic equations involving both fractional Laplacians and convolution-type nonlinearities, revealing new regularity phenomena and mathematical structures in the theory of nonlocal partial differential equations.

$$(P)\ \begin{cases} (-\Delta)^s u = g(x,u) + \left(\int_\Omega \frac{F(u)(y)}{|x-y|^\mu} dy\right) f(u) \text{ in } \Omega, \\ u = 0 \ \text{ in } \mathbb{R}^N \setminus \Omega \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N > 2s$, $s \in (0,1)$, $\mu \in (0,N)$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ Carathédory function, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and F is the primitive of f. Assume that f satisfies the following growth conditions
($\mathfrak{F}$) $F \in C^1(\mathbb{R}, \mathbb{R})$, $F' = f$ and there exists $C > 0$ such that for all $t \in \mathbb{R}$,

$$|tf(t)| \le C(|t|^{\frac{2N-\mu}{N}} + |t|^{\frac{2N-\mu}{N-2s}}).$$

Consider the space

$$X_0 := \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e in } \mathbb{R}^N \setminus \Omega\}$$

equipped with the norm induced by the inner product

$$\langle u, v\rangle = \int_Q \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy$$

where $Q = \mathbb{R}^N \setminus (\Omega^c \times \Omega^c)$.

Definition 3.1 A function $u \in X_0$ is said to be a solution to (P) if it satisfies

$$\int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\, dxdy = \int_\Omega g(x,u)\phi\, dx + \iint_{\Omega\times\Omega} \frac{F(u)f(u)}{|x-y|^\mu}\phi\, dxdy,\ \forall\, \phi \in X_0.$$

Let $G(x,u) = \int_0^u g(x,\tau)\, d\tau$ then functional associated with problem (P) is defined as

$$J(u) = \frac{\|u\|_{X_0}^2}{2} - \int_\Omega G(x,u)\, dx - \frac{1}{2}\iint_{\Omega\times\Omega} \frac{F(u)F(u)}{|x-y|^\mu}\, dxdy, \text{ for all } u \in X_0.$$

The main results of the section are the following

Theorem 3.2 *Let $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying*

$$g(x,u) = O(|u|^{2_s^*-1}),\ \textit{if}\ |u| \to \infty$$

uniformly for all $x \in \overline{\Omega}$. Then any solution $u \in X_0$ of (P) belongs to $L^\infty(\mathbb{R}^N) \cap C^s(\mathbb{R}^N)$.

Following is the main technical Lemma which helps us to prove Theorem 3.2.

Lemma 3.3 *[38] For $a, b \in \mathbb{R}$, $r \ge 2$, $k \ge 0$, we have*

$$\frac{4(r-1)}{r^2}\left(|a_k|^{r/2} - |b_k|^{r/2}\right)^2 \le (a-b)(a_k|a_k|^{r-2} - b_k|b_k|^{r-2})$$

where

$$a_k = \max\{-k, \min\{a,k\}\} = \begin{cases} -k, & \text{if } a \le -k, \\ a, & \text{if } -k < a < k, \\ k, & \text{if } a \ge k. \end{cases}$$

Proof From [57, Lemma 3.1], we have

$$\frac{4(r-1)}{r^2}\left(a|a_k|^{\frac{r}{2}-1} - b|b_k|^{\frac{r}{2}-1}\right)^2 \le (a-b)(a_k|a_k|^{r-2} - b_k|b_k|^{r-2}). \tag{3.1}$$

By symmetry of the inequality, it is enough to show that the result holds for $a \le b$. For this, let $a = a_k$ and $b = b_k$ in (3.1), we have

$$\frac{4(r-1)}{r^2}\left(a_k|a_k|^{\frac{r}{2}-1} - b_k|b_k|^{\frac{r}{2}-1}\right)^2 \le (a_k-b_k)(a_k|a_k|^{r-2} - b_k|b_k|^{r-2}).$$

Case 1: $0 \le b < a$

Clearly $0 \le b_k < a_k$ and $a_k - b_k \le a - b$. This implies

$$(a_k-b_k)(a_k|a_k|^{r-2} - b_k|b_k|^{r-2}) \le (a-b)(a_k|a_k|^{r-2} - b_k|b_k|^{r-2}).$$

Case 2: $b \le 0 \le a$

Again notice that $b_k \le 0 \le a_k$, $a_k - b_k \le a - b$ and $a_k b_k \le |a_k b_k|$ we have

$$(a_k-b_k)(a_k|a_k|^{r-2} - b_k|b_k|^{r-2}) \le (a-b)(a_k|a_k|^{r-2} - b_k|b_k|^{r-2})$$

and

$$\left(|a_k|^{r/2} - |b_k|^{r/2}\right)^2 \le \left(a_k|a_k|^{\frac{r}{2}-1} - b_k|b_k|^{\frac{r}{2}-1}\right)^2.$$

Hence the proof. □

To achieve the intended goal in Theorem 3.2, authors used the nonlocal version of Brezis-Kato estimates (See [38, Lemma 3.3]), above Lemma, and the following Proposition.

Proposition 3.4 *Let $H, K \in L^{\frac{2N}{N-\mu+2s}}(\Omega) + L^{\frac{2N}{N-\mu}}(\Omega)$. Let $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $g(x,u) = O(|u|^{2_s^*-1})$, if $|u| \to \infty$ uniformly for all $x \in \overline{\Omega}$. Then any solution $u \in X_0$ of the following problem (P_1) belongs to $L^r(\Omega)$ where $r \in [2, \frac{2N^2}{(N-\mu)(N-2s)})$.*

$$(P_1)\ \begin{cases} (-\Delta)^s u = g(x,u) + \left(\int_\Omega \frac{H(y)u(y)}{|x-y|^\mu}dy\right)K(x) \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $H, K \in L^{\frac{2N}{N-\mu+2s}}(\Omega) + L^{\frac{2N}{N-\mu}}(\Omega)$.

To prove the above Proposition, authors defined a sequence of coercive and bilinear maps. This sequence enables one to construct a further sequence of functions u_n that converges weakly to u (the weak solution to (P)). Then adopting some classical techniques of Brezis-Kato [17, 86], they proved that $u_n \in L^p(\Omega)$ with $2_s^* < p < p_0$ for some p_0.

As an application of Theorem 3.2, in [38], authors studied Sobolev versus Hölder-weighted minimizers. That is, they prove that the local minima with respect to $C^0_d(\bar{\Omega})$ topology will also be local minima with respect to X_0 topology. Precisely, they prove the following theorem

Theorem 3.5 *Let $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying*

$$g(x,u) = O(|u|^{2^*_s - 1}), \text{ if } |u| \to \infty$$

uniformly for all $x \in \overline{\Omega}$. Let $v_0 \in X_0$. Then the following assertions are equivalent:

(i) there exists $\varepsilon > 0$ such that $J(v_0 + v) \geq J(v_0)$ for all $c \in X_0$, $\|v\|_{X_0} \leq \varepsilon$.
(ii) there exists $\rho > 0$ such that $J(v_0 + v) \geq J(v_0)$ for all $v \in X_0 \cap C^0_d(\overline{\Omega})$, $\|v\|_{0,d} \leq \rho$.

In variational problems, this result illustrates a significant role as it helps to prove that the solutions of constrained minimization problem emerge as solutions to unconstrained local minimization problem. This procedure of constrained minimization has an ample amount of applications, such as to prove the existence of second positive solution to elliptic problems.

3.1 Doubly nonlocal equation with singular nonlinearity

The elliptic equations with singular nonlinearities emerge in the modelling of chemical catalyst kinetics, chemical heterogeneous catalysts, boundary layer phenomena for viscous fluids, non-Newtonian fluids, and so forth. for these applications, we refer the reader to [29, 35, 62] and references therein. For nonlocal problems, Barrios et.al [11] started the work and proved the existence and multiplicity of solutions to the following problem

$$(-\Delta)^s u = \lambda \left(\frac{a(x)}{u^r} + f(x,u) \right) \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \tag{3.2}$$

where Ω is a smooth bounded domain, $N > 2s,\ 0 < s < 1,\ r, \lambda > 0,\ f(x,u) \sim u^p,\ 1 < p < 2^*_s$. In the spirit of [24], here authors first prove the existence of solution u_n to the equation with singular term $1/u^r$ replaced by $1/(u + 1/n)^r$, $n \in \mathbb{N}$, and use the uniform estimates on the sequence $\{u_n\}$ to finally prove the existence of solution to (3.2) by taking the limit $n \to \infty$.

In recent times, Adimurthi, Giacomoni, and Santra [1] studied the problem (3.2) with $f = 0$ and improved the results of [11] to some extent, in particular, the authors provide the boundary behaviour and Hölder regularity of the classical solution. Moreover, authors exploit the asymptotic behavior to obtain global bifurcation results in the framework of weighted spaces for the problem (3.2) with subcritical f. Regarding the critical case, Giacomoni, Mukherjee and Sreenadh [40] studied the problem (3.2) with $a = \frac{1}{\lambda},\ r > 0$, and $p = 2^*_s - 1$. Here, the authors extended the techniques of [56] in a fractional framework and proved the existence and multiplicity of solutions in $C^\alpha_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ for some $\alpha > 0$ but it has some gaps. Recently, authors [41] proved the global multiplicity result for (3.2) with $a = \frac{1}{\lambda}$, $p = 2^*_s - 1$ and $r(2s-1) < (1+2s)$ for finite energy solutions.

In this section, we will discuss the regularity, existence, and multiplicity of weak solutions to the following singular problem:

$$(P_\lambda) \begin{cases} (-\Delta)^s u = u^{-q} + \lambda \left(\displaystyle\int_\Omega \frac{|u|^{2^*_{\mu,s}}(y)}{|x-y|^\mu} dy \right) |u|^{2^*_{\mu,s} - 2} u, \quad u > 0 \text{ in } \Omega, \\ \qquad u = 0 \ \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

for all $q > 0$, $s \in (0,1)$, $N \geq 2s$, $\mu \in (0,N)$, $2^*_{\mu,s} = \frac{2N-\mu}{N-2s}$ and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary.

The energy functional associated to the problem (P_λ) is defined as

$$I(u)=\frac{1}{2}\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy-\frac{1}{1-q}\int_\Omega |u|^{1-q}\,dx-\frac{\lambda}{22^*_\mu}\iint_{\Omega\times\Omega}\frac{|u|^{2^*_\mu}|u|^{2^*_\mu}}{|x-y|^\mu}\,dxdy.$$

Now observe that I is not a differentiable functional, not even in the sense of Gâteaux derivative. However, the functional I is continuous on X_0 when $0<q<1$; if $q\geq 1$, the functional I is not well defined on the whole space. Due to doubly nonlocal nature of the problem (P_λ) and the lack of regularity of I one needs to use non-smooth analysis. In this authors introduced a quite general definition of weak solutions as follows.

Definition 3.6 [39] A function $u\in H^s_{\text{loc}}(\Omega)\cap L^{2^*_s}(\Omega)$ is a weak solution of (P_λ) if the following hold:

(i) there exists $m_K>0$ such that $u>m_K$ for any compact set $K\subset\Omega$.
(ii) For any $\phi\in C_c^\infty(\Omega)$,

$$\int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\,dxdy=\int_\Omega u^{-q}\phi\,dx+\lambda\iint_{\Omega\times\Omega}\frac{u^{2^*_\mu}(x)u^{2^*_\mu-1}(y)\phi(y)}{|x-y|^\mu}\,dxdy.$$

(iii) $u^\gamma\in X_0$ for $\gamma\geq 1$.

This definition of solution is more general as compared to the definition of (3.2) in [1]. Suppose $\overline{u}$ is a solution of the following problem

$$(-\Delta)^s u=u^{-q},\ \ u>0 \text{ in } \Omega, u=0 \ \text{ in } \mathbb{R}^N\setminus\Omega.$$

then one can show the following optimal regularity result

Lemma 3.7 *(a) If $q(2s-1)\geq(2s+1)$ then $\overline{u}^\gamma\in X_0$ if and only if $\gamma>\frac{(2s-1)(q+1)}{4s}$. Moreover the lower bound on γ is optimal in the sense that $u^\gamma\notin X_0$ if $\gamma\leq\frac{(2s-1)(q+1)}{4s}$.*
(b) $(\overline{u}-\varepsilon)^+\in X_0$ for all $\varepsilon>0$.

As a consequence of this regularity result, it can be shown that the definition of a weak solution is well-defined. In the direction of existence of solution to (P_λ), authors in [39] translate the problem (P_λ). The translated problem is as follows:

$$(\widetilde{P_\lambda})\ \begin{cases}(-\Delta)^s u+\overline{u}^{-q}-(u+\overline{u})^{-q}=\lambda\left(\displaystyle\int_\Omega\frac{(u+\overline{u})^{2^*_\mu}(y)}{|x-y|^\mu}dy\right)(u+\overline{u})^{2^*_\mu-1},\ \ u>0 \text{ in } \Omega,\\ u=0\ \text{ in } \mathbb{R}^N\setminus\Omega.\end{cases}$$

Observe that $u+\overline{u}$ is a solution to (P_λ) if and only if $u\in X_0$ is a solution to $(\widetilde{P_\lambda})$.

Lemma 3.8 *[39] Let $u\in X_0$ be a weak solution to $(\widetilde{P_\lambda})$. Then for any $v\in X_0$, we have*

$$\int_Q \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy+\int_\Omega g(x,u)v\,dx-\iint_{\Omega\times\Omega}\frac{(u+\overline{u})^{2^*_\mu}(u+\overline{u})^{2^*_\mu-1}v}{|x-y|^\mu}\,dxdy=0.$$

The main results are the following

Lemma 3.9 *Any nonnegative solution to $(\widetilde{P_\lambda})$ belongs to $L^\infty(\Omega)$.*

Theorem 3.10 *Let $\mu\leq\min\{4s,N\}$. There exists a $\Lambda>0$ such that*

1. *For every* $\lambda \in (0, \Lambda)$ *the problem* (P_λ) *admits two solutions in* $C^+_{\phi_q}(\Omega) \cap L^\infty(\Omega)$.
2. *For* $\lambda = \Lambda$ *there exists a solution in* $C^+_{\phi_q}(\Omega) \cap L^\infty(\Omega)$.
3. *For* $\lambda > \Lambda$, *there exists no solution.*

Moreover, solution belongs to X_0 *if and only if* $q(2s-1) < (2s+1)$ *and solution belongs to* $C^\gamma(\mathbb{R}^N)$ *where* γ *is defined as follows*

$$\gamma = \begin{cases} s & \text{if } q < 1, \\ s-\varepsilon & \text{if } q = 1, \ \text{for all } \varepsilon > 0 \text{ small enough}, \\ \frac{1}{q+1} & \text{if } q > 1. \end{cases}$$

The existence of a minimal solution is proven using a monotone iteration scheme by exploiting the very singular nonlinearity u^{-q}. Using variational methods and the geometry of the energy functional, in combination with Critical point theory from nonsmooth analysis, they obtained a second solution as a mountain pass critical point.

Observe that the Lemma 3.9 concludes that the solution of $(\widetilde{P_\lambda})$ is in $L^\infty(\Omega)$ which in turn gives solution of (P_λ) is in $L^\infty(\Omega)$ only when $\mu \le \min\{4s, N\}$ and for any q. While in [38], the authors established the regularity of the solution to the following problem for $0 < q < 1$ and $\mu \in (0, N)$

$$(P_q) \quad \begin{cases} (-\Delta)^s u = u^{-q} + \left(\int_\Omega \frac{F(u)(y)}{|x-y|^\mu} dy \right) f(u) \text{ in } \Omega, \\ u = 0 \ \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $F \in C^1(\mathbb{R}, \mathbb{R})$, $F' = f$ and $|tf(t)| \le C(|t|^{\frac{2N-\mu}{N}} + |t|^{\frac{2N-\mu}{N-2s}})$, for some $C > 0$ and for all $t \in \mathbb{R}$. Regarding the regularity of solutions when $0 < q < 1$ and $0 < \mu < N$, Giacomoni, Goel and Sreenadh [38] proved the following result

Theorem 3.11 *Let* $q \in (0, 1)$ *and* $g(x, u) = u^{-q}$. *Then any positive solution* $u \in X_0$ *of* (P_q) *belongs to* $L^\infty(\mathbb{R}^N) \cap C^s(\mathbb{R}^N)$.

For the combination of local and nonlocal operators, Anthal, Giacomoni, and Sreenadh [5] studied the following problem :

$$\begin{cases} -\Delta u + (-\Delta)^s u = u^{-q} + \lambda \left(\int_\Omega \frac{|u|^{2^*_{\mu,s}}(y)}{|x-y|^\mu} dy \right) |u|^{2^*_{\mu,s}-2} u, \quad u > 0 \text{ in } \Omega, \\ u = 0 \ \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

for all $q > 0$, and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary.

Theorem 3.12 *(Existence, Asymptotic Behaviour, and Multiplicity) Let* $q > 0$, $N \ge 3$, *then the following assertions hold:*

1. ***Existence/Nonexistence****: There exists* $\lambda^* > 0$ *such that for every* $0 < \lambda < \lambda^*$, *the problem admits at least one weak solution, and for* $\lambda > \lambda^*$, *it has no solution.*
2. ***Asymptotic behaviour and regularity****: Any weak solution u is bounded,* $u \in C^0_{loc}(\Omega)$, *and for some* $\alpha \in (0, 1)$, $u \in C^\alpha_{loc}(\Omega)$. *If* $q < 3$, *then* $u \in H^1_0(\Omega) \cap C^1_{loc}(\Omega)$.
3. ***Multiplicity****: If* $N > 2s + 6$, *then there exist at least two distinct weak solutions for any* $0 < \lambda < \lambda^*$.

To prove the above Theorem, the authors construct sub- and supersolutions to control the singularity and establish nontrivial bounds. Further, using the non-smooth (lower semicontinuous) variational framework, including the linking Theorem for appropriate functionals to establish their existence results. To prove the regularity, they used the Bootstrap and comparison principles to obtain higher regularity results.

4 Kirchhoff Choquard Problems

In this section, we present results on the Kirchhoff-Choquard problem, which models physical phenomena by combining a Kirchhoff term, which accounts for effects such as wave propagation in suspension bridges, with a Choquard term, a nonlocal interaction that depends on the average of the solution. Researchers are actively studying these equations, particularly the non-compact perturbations. The analysis and existence theory of such problems involves advanced variational methods and concentration-compactness principles to overcome the non-compact phenomena.

4.1 Quasilinear-Kirchhoff problem

In the first subsection, we present the existence of high energy solutions for a class of p-Laplace equations with a general Kirchhoff term $M(t) = a + bt^{\theta-1}$. Precisely, we will discuss the following class of Kirchhoff problems with the general Choquard term:

$$-M\left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right) \Delta_p u + V(x)|u|^{p-2}u = \left(\int_{\mathbb{R}^N} \frac{F(u)(y)}{|x-y|^\mu} dy\right) f(u), \quad u > 0 \text{ in } \mathbb{R}^N, \tag{4.1}$$

where $N > \max\{2, p\}$ with $1 < p < N$, a, b, θ are positive parameters and $0 < \mu < N$. Here, Δ_p is the p-Laplacian operator, defined as $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. The function $f \in C(\mathbb{R}, \mathbb{R})$ is such that $f(t) \equiv 0$ for $t \leq 0$ and F is the primitive of f. The general function f satisfies the following growth conditions:

($\mathcal{F}_1$) $\lim\limits_{t\to 0} \dfrac{f(t)}{t^{p_{*,\mu}-1}} = 0$;
($\mathcal{F}_2$) f is a monotonic function and for some positive constant C and $q \in \left(p_{*,\mu}, p_\mu^*\right)$

$$|f(t)t| \leq C|t|^{p_{*,\mu}} + C|t|^q;$$

where $p_{*,\mu} = \frac{p(2N-\mu)}{2N}$ is the lower critical exponent and $p_\mu^* = \frac{p(2N-\mu)}{2(N-p)}$ is the upper critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality. Moreover, Assume that the potential function $V \in C^2(\mathbb{R}^N, \mathbb{R})$ satisfies the following:

($\mathcal{V}_1$) $\lim\limits_{|x|\to\infty} V(x) := V_\infty > 0$;
($\mathcal{V}_2$) $\nabla V(x) \cdot x \leq 0$, for all $x \in \mathbb{R}^N$ with the strict inequality on a subset of positive Lebesgue measure;
($\mathcal{V}_3$) $NV(x) + \nabla V(x) \cdot x \geq NV_\infty$, for all $x \in \mathbb{R}^N$;
($\mathcal{V}_4$) $N\nabla V(x) \cdot x + x \cdot H(x) \cdot x \leq 0$, for all $x \in \mathbb{R}^N$ where H represents the Hessian matrix of the function V.

To successfully establish the existence of a solution to (4.1), authors in [50] first studied the following limiting problem

$$-M\left(\int_{\mathbb{R}^N} |\nabla u|^p\right) \Delta_p u + V_\infty|u|^{p-2}u = \left(\int_{\mathbb{R}^N} \frac{F(u)(y)}{|x-y|^\mu} dy\right) f(u) \quad \text{in } \mathbb{R}^N. \tag{4.2}$$

The energy functionals associated with the problems (4.1) and (4.2) are $\mathcal{T}, \mathcal{T}_\infty : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ defined as

$$\mathcal{T}(u) = \frac{a}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{b}{p\theta}\left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^\theta + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{F(u)F(u)}{|x-y|^\mu} dxdy,$$

$$\mathcal{T}_\infty(u) = \frac{a}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{b}{p\theta}\left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^\theta + \frac{V_\infty}{p}\int_{\mathbb{R}^N} |u|^p dx - \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{F(u)F(u)}{|x-y|^\mu} dxdy.$$

Observe that both the functionals $\mathcal{T}, \mathcal{T}_\infty \in C^1$, and for any $\phi \in C_0^\infty(\mathbb{R}^N)$

$$\begin{aligned}\langle \mathcal{T}'(u), \phi\rangle =& a\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, dx + b\left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^{\theta-1}\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, dx \\ &+ \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\phi \, dx - \iint_{\mathbb{R}^{2N}} \frac{F(u(y))f(u(x))\phi(x)}{|x-y|^\mu} dxdy,\end{aligned}$$

and

$$\begin{aligned}\langle \mathcal{T}_\infty'(u), \phi\rangle =& a\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, dx + b\left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^{\theta-1}\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, dx \\ &+ V_\infty\int_{\mathbb{R}^N} |u|^{p-2}u\phi \, dx - \iint_{\mathbb{R}^{2N}} \frac{F(u(y))f(u(x))\phi(x)}{|x-y|^\mu} dxdy.\end{aligned}$$

Define the following Pohožaev manifolds associated with problems (4.1) and (4.2)

$$\mathcal{P} = \{u \in W^{1,p}(\mathbb{R}^N)\backslash\{0\} : P(u) = 0\} \text{ and } \mathcal{P}_\infty = \{u \in W^{1,p}(\mathbb{R}^N)\backslash\{0\} : P_\infty(u) = 0\}.$$

Here, $P(u)$ and $P_\infty(u)$ are the Pohožaev identities associated with problems (4.1) and (4.2) respectively, (see [87, Theorem 3])

$$\begin{aligned}P(u) :=& \frac{(N-p)a}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{(N-p)b}{p}\left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^\theta + \frac{1}{p}\int_{\mathbb{R}^N} (NV(x) + \nabla V(x)\cdot x)\,|u|^p \\ &- \frac{2N-\mu}{2}\iint_{\mathbb{R}^{2N}} \frac{F(u)F(u)}{|x-y|^\mu} dxdy\end{aligned}$$

$$\begin{aligned}P_\infty(u) :=& \frac{(N-p)a}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{(N-p)b}{p}\left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^\theta + \frac{NV_\infty}{p}\int_{\mathbb{R}^N} |u|^p dx \\ &- \frac{2N-\mu}{2}\iint_{\mathbb{R}^{2N}} \frac{F(u)F(u)}{|x-y|^\mu} dxdy.\end{aligned}$$

The main results of this section are following

Theorem 4.1 *[50] Assume that f and V satisfy $(\mathcal{F}_1)-(\mathcal{F}_2)$ and $(\mathcal{V}_1)-(\mathcal{V}_4)$. Then, problem* (4.2) *has a positive ground state solution that is radially symmetric.*

Theorem 4.2 *[50] Assume that f and V satisfy $(\mathcal{F}_1)-(\mathcal{F}_2)$ and $(\mathcal{V}_1)-(\mathcal{V}_4)$ along with*

$(\mathcal{V}_5)$ *there exists a constant $T>1$ such that*

$$\sup_{x\in\mathbb{R}^N} V(x) \le V_\infty + \frac{p\mathcal{T}_\infty(\overline{u})}{T^N \int\limits_{\mathbb{R}^N} |\overline{u}|^p},$$

where $\overline{u}$ is the ground state solution of the functional $\mathcal{T}_\infty$ obtained in Theorem 4.1. Then, there exists at least one positive solution to problem (4.1)*, which is a high-energy solution.*

The function $\frac{f(x,t)}{|t|^3}$ is not strictly increasing for $t\in\mathbb{R}\backslash\{0\}$ which makes the classical method of the Nehari manifold inappropriate. But the functional $\mathcal{T}_\infty$ satisfies the mountain-pass geometry with the scaling function u_t defined as

$$u_t(x) := u(t^{-1}x),$$

for $u\in W^{1,p}(\mathbb{R}^N)\backslash\{0\}$ and $t>0$. Keeping this in mind, one takes the minimum over the Pohožaev manifold $\mathcal{P}_\infty$.

Using the assumptions on f and V_∞, Goel, Rawat and Sreenadh [50] concluded that

(1) $\mathcal{P}_\infty$ is a natural constraint, in the sense that $\mathcal{T}_\infty$ is coercive and bounded below on $\mathcal{P}_\infty$
(2) Every critical point of $\mathcal{T}_\infty$ restricted to $\mathcal{P}_\infty$ is a critical point of $\mathcal{T}_\infty$
(3) $\mathfrak{p}_\infty := \inf\limits_{\mathcal{P}_\infty} \mathcal{T}_\infty(u) = c_\infty$ where c_∞ is the mountain pass level.

Further, to construct a bounded Palais-Smale sequence for $\mathcal{T}_\infty$, they employed Jeanjean's [58] technique and proved the existence of a nontrivial critical point of the functional $\mathcal{T}_\infty$ by the concentration-compactness Lemma. This approach can be applied in the Choquard case to determine the existence of a solution to (4.1) in the Pohožaev manifold $\mathcal{P}$. However, authors encounter that the functional $\mathcal{T}$ cannot attain the mountain pass level, thus indicating the absence of a ground state solution for equation (4.1). To overcome this obstacle, they establish that $\mathcal{T}$ satisfies the Palais-Smale condition above the min-max level, leading to solutions with higher energy. The compactness issues are addressed by proving a global compactness lemma (refer to [50, Lemma 5.9]). Additionally, by utilizing the linking Theorem in conjunction with a Barycenter mapping, the authors of Theorem 4.2 established the radial nature of the solution, effectively leveraging the nonlocal nature of the nonlinearity. The analysis and arguments are based on polarizations and equality cases in polarization inequalities.

4.2 Convex-Concave type combined nonlinearities

In this subsection, we will discuss a class of Kirchhoff-type equations for the fractional Laplacian with Choquard nonlinearity. We consider the following problem

$$(F_\lambda)\quad \begin{cases} \left(a+b\|u\|_{X_0}^{2\theta-2}\right)(-\Delta)^s u = \lambda f(x)|u|^{q-2}u + \left(\int\limits_{\Omega} \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}\,dy\right)|u|^{2^*_{\mu,s}-2}u \text{ in } \Omega, \\ u>0 \text{ in } \Omega,\ u=0 \text{ in } \mathbb{R}^N\backslash\Omega, \end{cases}$$

where Ω is open bounded domain of $\mathbb{R}^N$ having smooth boundary, $N>2s$ with $s\in(0,1)$, a,b,θ,λ are positive parameters, where $2^*_{\mu,s}=\frac{2N-\mu}{N-2s}$, and function $f(x)$ is a continuous real valued sign changing function.

In the nineties, Azorero and Alonso [6] studied the following problem

$$-\Delta u=\lambda|u|^{q-2}u+|u|^{2_s^*-1} \text{ in } \Omega, \quad u=0 \text{ on } \partial\Omega \tag{4.3}$$

for $s=1$, and $2<q<2^*=\frac{2N}{N-2}$. They proved the existence of a nontrivial solution for large λ. While for $s\in(0,1)$, Barrios, Colorado, Servadei and Soria [10] studied the problem (4.3) for $1<q<2_s^*$. For the convex power case $2<q<2_s^*$, the existence of the solution is proved using the Mountain-pass theorem, for a suitable value of λ depending on the dimension N. On the other hand, in the concave case, the authors established the multiplicity of solutions for small λ. Such an existence result for problems involving the critical Choquard nonlinearity has been established by Gao and Yang [36]. Subsequently, for a similar type of results, the whole space, we cite [82, 98] and references therein.

Fiscella and Valdinoci [34] explored the Dirichlet problem for the Kirchhoff equation on the bounded domains of $\mathbb{R}^N$, which takes into account the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string. Li and Liao [65] studied the following problem on the whole domain

$$\left(a+b\|u\|^2\right)(-\Delta)^s u=\lambda k(x)|u|^{q-2}u+\mu|u|^{2_s^*-2}u \text{ in } \mathbb{R}^N$$

for $s=1$, $a,b>0$. Here authors established the existence of two positive solutions when $2<q<2^*$, using minimization argument and Mountain-pass theorem, for some $\mu\in(0,\mu^*)$ and for λ large enough, where $0\le k\in L^{\frac{2^*}{2^*-q}}$. A substantial body of literature exists that examines various aspects of the existence of solutions to problems involving the Kirchhoff operator. We refer the reader to some recent articles [33, 83, 93, 103, 104].

In [49], the authors proved the existence and multiplicity of solutions to the Kirchhoff-Choquard equation for $1<q<2_s^*,\ \theta\ge 1$. The energy functional associated with problem (P_λ) is $J_\lambda: X_0(\Omega)\to\mathbb{R}$ defined as

$$J_\lambda(u)=\frac{a}{2}\|u\|^2+\frac{b}{2\theta}\|u\|^{2\theta}-\frac{\lambda}{q}\int_\Omega f(x)(u^+(x))^q\,dx-\frac{1}{2\cdot 2^*_{\mu,s}}\iint_{\Omega\times\Omega}\frac{(u^+(y))^{2^*_{\mu,s}}(u^+(x))^{2^*_{\mu,s}}}{|x-y|^\mu}\,dxdy.$$

Using the Hardy-Littlewood-Sobolev inequality, we can show $J_\lambda\in C^1$. Indeed, for $\phi\in X_0(\Omega)$

$$\begin{aligned}\langle J_\lambda'(u),\phi\rangle=&\left(a+b\|u\|^{2(\theta-1)}\right)\int_{\mathbb{R}^{2N}}\frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\,dxdy\\&-\lambda\int_\Omega f(x)(u^+(x))^{q-1}\phi(x)dx-\iint_{\Omega\times\Omega}\frac{(u^+(y))^{2^*_{\mu,s}}(u^+(x))^{2^*_{\mu,s}-1}\phi(x)}{|x-y|^\mu}\,dxdy.\end{aligned}$$

For the case of $1\le\theta<2^*_{\mu,s}$, they proved the following results:

Theorem 4.3 *Let $1<q<2\theta$ then there exist $\Lambda^*>0$ such that for $N>4s$, (P_λ) has at least one positive solution, for all $\lambda\in(0,\Lambda^*)$.*

Theorem 4.4 *Let $2\theta\le q<2_s^*$, then there exists $\Lambda_*>0$ such that (P_λ) has at least one positive solution for all $\lambda\ge\Lambda_*$.*

It is worth noting that the results are contrasting for $2<q<2\theta$ and $2\theta\le q<2_s^*$. While for $\theta\ge 2^*_{\mu,s}$, there exists two positive solutions. The novelty of this result is that it proves the multiplicity of solutions for any $\mu\in(0,N)$, which is an open problem in [70].

Theorem 4.5 *Let $2<q<2_s^*$, there exists $\Lambda_{**}>0$ such that for $\lambda>\Lambda_{**}$*

(i) *when* $\theta = 2^*_{\mu,s}$, $a > 0$ *and* $b > (S_s^H)^{-2^*_{\mu,s}}$, (P_λ) *admits at least two positive solutions,*
(ii) *when* $\theta > 2^*_{\mu,s}$, *there exist* $\mathfrak{A}$, $\mathfrak{B}$ *such that either for* $a > 0$ *and* $b > \mathfrak{B}$ *or* $b > 0$ *and* $a > \mathfrak{A}$, (P_λ) *admits at least two positive solutions*

where $\mathfrak{A} := \frac{\theta-2^*_{\mu,s}}{\theta-1}\left[\frac{2^*_{\mu,s}-1}{b(\theta-1)}\right]^{\frac{2^*_{\mu,s}-1}{\theta-2^*_{\mu,s}}}(S_s^H)^{\frac{-2^*_{\mu,s}(\theta-1)}{\theta-2^*_{\mu,s}}}$ *and* $\mathfrak{B} := \frac{2^*_{\mu,s}-1}{\theta-1}\left[\frac{\theta-2^*_{\mu,s}}{a(\theta-1)}\right]^{\frac{\theta-2^*_{\mu,s}}{2^*_{\mu,s}-1}}(S_s^H)^{\frac{-2^*_{\mu,s}(\theta-1)}{2^*_{\mu,s}-1}}$.

To prove the above theorem, the authors used the Mountain Pass theorem, together with the concentration-compactness and Brezis-Lieb lemma. The positivity of the solution is obtained using Theorem 3.2 and the Maximum principle. After this, they also discuss the convex-concave behavior of nonlinearities, i.e. $1 < q < 2$ and $\theta \geq 1$. Employing the minimization approach and the Mountain Pass Theorem in place of the Nehari Manifold technique, they proved the following

Theorem 4.6 *Let* $1 \leq \theta < 2^*_{\mu,s}$ *and* $1 < q \leq 2$, *then there exist* Λ^{**}, $\tilde{\Lambda}^{**} > 0$ *such that*

(i) *If* $0 < \mu < \min\{4s, N\}$, *then for* $\lambda \in (0, \Lambda^{**})$ *and* $1 < q < 2$, (P_λ) *admits at least two positive solutions.*
(ii) *If* $4s \leq \mu < N$, *then for* $\lambda \in (0, \tilde{\Lambda}^{**})$ *and* $\frac{N}{N-2s} \leq q < 2$, (P_λ) *admits at least two positive solutions.*
(iii) *For* $q = 2$, (P_λ) *admits at least one positive solution.*

The above theorem is a generalization of the result of Goel and Sreenadh [52].

5 Solutions with a prescribed mass

Normalized solutions to nonlinear PDEs are the solutions that are constrained to have a specific *mass* or L^2 norm. The mass critical case is a threshold at which the nonlinear terms in the energy functional are non-compact, necessitating different mathematical strategies compared to unconstrained critical cases. To obtain the solutions, it is natural to use variational methods and study the critical points of the associated energy functional under the prescribed norm constraint. Here we present some recent results on these problems for the Choquard equation. We consider the following problem

$$-\Delta u + \lambda u = (I_\mu * F(u))f(u) \text{ in } \mathbb{R}^N,$$

where $\lambda \in \mathbb{R}$ and $F(t) = \int_{\mathbb{R}^N} f(s)ds$. The following version of the Gagliardo-Nirenberg inequality is employed to analyze normalized solutions to Choquard equations.

Proposition 5.1 *[106, Proposition 1.2] Let* $N \geq 1$, $\mu \in (0, N)$ *and* $r \in \left(\frac{2N-\mu}{N}, \frac{2N-\mu}{N-2}\right)$ *then there exist* $C_r > 0$*(dependent on* r*) such that*

$$\int_{\mathbb{R}^N}(I_\mu * |u|^r)|u|^r dx \leq C_r\|\nabla u\|_2^{2r\delta_r}\|u\|_2^{2r(1-\delta_r)},$$

where

$$\delta_r = \frac{N(r-2)+\mu}{2r}.$$

It implies that the exponent $2+\frac{2-\mu}{N}$ is the mass exponent for the above equation. Li-Ye [64] studied the Choquard equations for normalized solutions for the mass critical exponent. Li [68] also proved various existence results for the following problem

$$\begin{cases} -\Delta u + \lambda u = \alpha(I_\mu * |u|^p)|u|^{p-2}u + \beta|u|^{q-2}u \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N}|u|^2 dx \quad = c^2, \end{cases}$$

where p is upper Choquard critical exponent and $2 < q < \frac{2N+4}{N}$ and $c \in \mathbb{R}$ For recent works on mass critical solutions, we refer to [20, 48, 63, 64, 69, 102].

For the Kirchhoff Choquard problem for normalized solutions, Goel and Gupta recently studied the following problem

$$\begin{cases} -\left(a + b\|\nabla u\|_2^{2(\theta-1)}\right) \Delta u = \lambda u + \alpha (I_\mu * |u|^q)|u|^{q-2}u + (I_\mu * |u|^p)|u|^{p-2}u \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \quad = c^2, \end{cases} \tag{5.1}$$

where $N \geq 3$, $0 < \mu < N$, $a, b, c > 0$, $1 < \theta < \frac{2N-\mu}{N-2}$, $\frac{2N-\mu}{N} < q < p \leq \frac{2N-\mu}{N-2}$, $\alpha > 0$. Using the Gagliardo-Nirenberg inequality, $A^* = \frac{2-\mu}{N} + 2$ and $B^* = \frac{2\theta-\mu}{N} + 2$ is the L^2- critical exponent for the above equation with $\theta = 1$ and $\theta > 1$, respectively. In [45], Goel and Gupta proved the following result for the problem (5.1)

Theorem 5.2 *(1) Let* $2_{\mu,*} = \frac{2N-\mu}{N}$, $< q < A^*$, $p = 2^*_\mu$. *Then there exists* $\alpha_0 > 0$ *such that for* $0 < \alpha < \alpha_0$, (5.1) *has a positive, radially decreasing ground state solution of negative energy.*

(2) Let $B^* < q < p < 2^*_\mu$ *and* $\alpha > 0$. *Then* (5.1) *has a positive, radially decreasing ground state solution of positive energy.*

(3) Let $N = 3$, $\theta = 2$, $\mu = 2$, $\frac{10}{3} < q < p = 4$ *and* $\alpha > 0$. *If* $\frac{a^2}{4} - \frac{b^3 S_{HL}^4}{27} > 0$ *then* (5.1) *has a positive, radially decreasing ground state solution of positive energy* $\varsigma(c, \alpha)$ *where*

$$\varsigma(c, \alpha) \in \left(0, \left(\frac{b\Lambda^2 S_{HL}^2}{8} + \frac{3a\Lambda S_{HL}}{8}\right)\right),$$

where

$$\Lambda = \left(\frac{aS_{HL}}{2} + \sqrt{\frac{a^2 S_{HL}^2}{4} - \frac{b^3 S_{HL}^6}{27}}\right)^{\frac{1}{3}} + \left(\frac{aS_{HL}}{2} - \sqrt{\frac{a^2 S_{HL}^2}{4} - \frac{b^3 S_{HL}^6}{27}}\right)^{\frac{1}{3}}.$$

Using the standard methodology of minimization and mountain pass theorem, authors established the existence of solution for case (1) and (2). For details, one can refer [45] while the L^2-supercritical case is a bit involving. To prove the Palais-Smale condition for the associated energy functional, they found an explicit positive solution to the following algebraic equation:

$$x^{2^*_\mu - 1} - bS_{HL}^\theta x^{\theta-1} - aS_{HL} = 0$$

The above results partially solve this problem in the $N = 3$ case. Precisely, by using Cardano's formula, they found the exact solution to the above algebraic equation for $N = 3$, $\theta = 2$, $\mu = 2$, and $\frac{a^2}{4} - \frac{b^3 S_{HL}^4}{27} > 0$. Utilizing this, they prove the existence of a ground state solution for the case $\frac{10}{3} < q < p = 4$. This is the first article which discusses the Kirchhoff Choquard problem in the supercritical regime with pure power type nonlinearities. Subsequently, Goel, Gupta and Rai [46], studied the following Kirchhoff-Choquard system

$$\begin{cases} \left(M(\|(-\Delta)^s u\|_2^2)\right)(-\Delta)^s u = \lambda_1 u + (I_\mu * |v|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}-2}u + \alpha p (I_\mu * |v|^q)|u|^{p-2}u \text{ in } \mathbb{R}^N, \\ \left(M(\|(-\Delta)^s v\|_2^2)\right)(-\Delta)^s v = \lambda_2 v + (I_\mu * |u|^{2^*_{\mu,s}})|v|^{2^*_{\mu,s}-2}v + \alpha q (I_\mu * |u|^p)|v|^{q-2}v \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 = d_1^2, \quad \int_{\mathbb{R}^N} |v|^2 = d_2^2, \end{cases} \tag{F_d}$$

where $N > 2s,\ s \in (0,1),\ \mu \in (0,N),\ \alpha > 0,\ 2_{\mu,*} := \frac{2N-\mu}{N} < p,\ q < \frac{2N-\mu}{N-2s} =: 2^*_{\mu,s}$ and $\lambda_1,\ \lambda_2 \in \mathbb{R}^N$ and $M(t) := a + bt,\ a,\ b > 0$. Here, they proved the following result

Theorem 5.3 *(i) Let $22_{\mu,*} < p+q < 4 + \frac{4s-2\mu}{N}$. Then there exists $\mu_0 > 0$ such that $0 < \mu < \mu_0$ such that (F_d) has a positive and radially decreasing ground state solution of negative energy.*

(ii) Let $\alpha > 0$, $4 + \frac{8s-2\mu}{N} < p+q < 22^*_{\mu,s} = 6$. If one of the following conditions holds:

(i) $p > \frac{3}{2},\ q > \frac{3}{2}$ and either $4s < N < 6s$, $\alpha < \tilde{\alpha}$ or $\frac{22s-4(p+q)s}{6-(p+q)} < N \leq 4s$,
(ii) $p > \frac{3}{2},\ q < \frac{3}{2}$ and $\frac{16s-4ps}{6-(p+q)} < N < \frac{18s-4ps}{6-(p+q)}$,
(iii) $p < \frac{3}{2},\ q > \frac{3}{2}$ and $\frac{16s-4qs}{6-(p+q)} < N < \frac{18s-4qs}{6-(p+q)}$,
(iv) $p > \frac{3}{2}$, $q = \frac{3}{2}$ and $\frac{8s(4-p)}{9-2p} < N \leq 4s$,
(v) $p = \frac{3}{2}$, $q > \frac{3}{2}$ and $\frac{8s(4-q)}{9-2q} < N \leq 4s$.

Then (F_d) has a mountain pass solution of positive energy $\sigma_\alpha(d_1,d_2)$ with

$$\sigma_\alpha(d_1,d_2) \in \left(0,\ \frac{abS_{HL}^3}{2} + \frac{b^3S_{HL}^6}{12} + \frac{2}{3}\left(\frac{b^2S_{HL}^4}{4} + aS_{HL}\right)^{\frac{3}{2}}\right).$$

The existence of Normalized solutions for Choquard equations involving mixed local and nonlocal (fractional) operators, with progression from the linear case and general mixed operators, to problems at critical exponents, and finally to quasilinear (p-Laplacian and fractional p-Laplacian) frameworks has been recently established by Giacomoni, Nidhi, and Sreenadh [43]. Here, the authors establish the existence and regularity theory for normalized solutions to the following class of mixed local non-local principle operators and Choquard nonlinearity.

$$-\Delta u + (-\Delta)^s u = \lambda u + (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

with mass critical exponents. Concentration-compactness, variational characterizations, and a Pohozaev-type identity form the technical backbone of the analysis. Subsequently, for $g(u) \not\equiv= 0$ and $p = 2^*_\mu$, the authors [42] established the existence and regularity of normalized radially symmetric solutions, with careful asymptotics and scaling analysis to treat the noncompact variational structures.

For the quasilinear case, Nidhi and Sreenadh [90] studied the following problem

$$-\Delta_p u + (-\Delta)^s_p u = \lambda|u|^{p-2}u + |u|^{p^*-2}u + \beta(I_\alpha * |u|^q)|u|^{q-2}u \quad \text{in } \mathbb{R}^N,$$

subject to a fixed L^p-norm constraint and $\frac{p(N+\alpha)}{2N} < q < \frac{p(N+\alpha)}{2(N-p)}$ reaches the critical exponent. The authors establish the existence and regularity of normalized solutions, providing Hölder regularity of radial solutions. They employ an intricate minimization over the Pohozaev manifold and use sharp compactness results adapted to the quasilinear and nonlocal mixed framework. For the non-local Kirchhoff equations and Choquard nonlinearities, we refer the reader to [45, 78] and references therein.

6 Modified quasilinear Schrödinger equation

Modified Schrödinger elliptic equations are a class of nonlinear PDEs that model various physical phenomena but present significant mathematical challenges. A standard modified Schrödinger equation includes an operator $-\Delta_p u - \Delta_p(u^2)u$, with $1 < p \leq N$, which complicates its analysis. These equations have numerous applications in modeling physical phenomena, such as plasma physics and fluid mechanics [12], as well as dissipative quantum mechanics [55], among others. Solutions of such equations are related to the existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$iu_t = -\Delta u + V(x)u - h_1(|u|^2)u - C\Delta h_2(|u|^2)h_2'(|u|^2)u, \quad x \in \mathbb{R}^N, \tag{6.1}$$

where C is a real constant and V, h_1 and h_2 are some real valued functions. Many physical phenomena can be described mathematically by functions h_2. If $h_2(s) = s$ ([61]), (6.1) models the superfluid film equation in plasma physics, also, if $h_2 = \sqrt{1+s^2}$ ([94]), it is used in studying the self-channeling of a high-power ultra short laser in matter.

In this section, we present some recent advances related to the study of positiv solutions for this class of PDEs involving the Choquard and Stein-Weiss nonlinear terms.

6.1 Schrödinger equations with Stein-Weiss nonlinearity

In [16], the authors considered the following family of quasilinear Schrödinger equations with Stein-Weiss type convolution:

$$(P_*) \begin{cases} -\Delta_N u - \Delta_N(u^2)u + V(x)|u|^{N-2}u = \left(\displaystyle\int_{\mathbb{R}^N} \frac{F(y,u)}{|y|^\beta |x-y|^\mu}\, dy\right) \frac{f(x,u)}{|x|^\beta} \text{ in } \mathbb{R}^N, \end{cases}$$

where Δ_N is the N-Laplacian, $N \geq 2$, $0 < \mu < N$, $\beta \geq 0$, and $2\beta + \mu < N$.

The nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function with critical exponential growth in the sense of the Trudinger-Moser inequality, and $F(x,s) = \int_0^s f(x,t)dt$ is the primitive of f. The potential function $V : \mathbb{R}^N \to \mathbb{R}$ is assumed to satisfy the following:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ such that $\inf\limits_{x\in\mathbb{R}^N} V(x) = V_0 > 0$.

(V_2) $V(x) \leq \lim\limits_{|x|\to\infty} V(x) = V_\infty < \infty$, with $V \not\equiv V_\infty$.

The critical growth non-compact problems for quasilinear operator associated to exponential type nonlinearity in bounded domains are initially studied in [2, 30, 100]. For more development on this topic, we refer to some recent contemporary works [26, 27], where the authors studied the equations of type (6.1) for $N = 2$, without the convolution term $\int_{\mathbb{R}^N} F(y,u)|x-y|^{-\mu}\, dy$. In [13], the authors studied the existence of positive solution for (6.1) in bounded domain for $\beta = 0$.

Moreover, in the case of N-Laplacian, the critical growth is equivalent to $\exp(|u|^{N/(N-1)})$, but in the problem (6.1), due to the term $\Delta_N(u^2)u$, the nature of the critical exponential growth is of the form $\exp(|s|^{2N/(N-1)})$. Similarly, if $1 < p < N$, then the nonlinearity is of polynomial growth and the critical growth is equivalent to $|u|^{2p^*}$, where $p^* = Np/(N-p)$ (see [31, 84]). Observe that the term $\Delta_N(u^2)u$, present in problem (6.1), restrains the natural energy functional corresponding to the problem (6.1) to be well defined on the Sobolev space $W^{1,N}(\mathbb{R}^N)$. Hence, the variational methods can't be applied directly for such problems. To deal with this inconvenience, researchers have developed several methods and arguments, such as a constrained minimization technique (see for e.g., [74, 75, 95]), the perturbation method (see for e.g., [76, 77]) and a change of variables (see for e.g., [22, 30, 31, 59]).

Now, the natural energy functional associated to the problem (6.1) is given by

$$I(u) = \frac{1}{N}\int_{\mathbb{R}^N} \left[(1 + 2^{N-1}|u|^N)|\nabla u|^N dx + V|u|^N\right] dx - \frac{1}{2}\int_{\mathbb{R}^N} (F(.,u) * |x|^\mu)\frac{F(x,u(x))}{|x|^\beta} dx$$

where $F(.,u) * |x|^\mu = \int_{\mathbb{R}^N} \frac{F(y,u(y))}{|y|^\beta |x-y|^\mu} dy$. Due to the presence of the term $\displaystyle\int_{\mathbb{R}^N} u^N |\nabla u|^N dx$ the functional I is not well defined in $W^{1,N}(\mathbb{R}^N)$. To overcome this difficulty, the authors employ the following change of variables which was introduced in [22], namely, $w := h^{-1}(u)$, where h is defined by

$$\begin{cases} h'(s) = \dfrac{1}{\left(1 + 2^{N-1}|h(s)|^N\right)^{\frac{1}{N}}} \text{ in } [0, \infty), \\ h(s) = -h(-s) \text{ in } (-\infty, 0]. \end{cases}$$

After applying the change of variable $w = h^{-1}(u)$, the new functional $J : W^{1,N}(\mathbb{R}^N) \to \mathbb{R}$ is defined as

$$J(w) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla w|^N dx + \frac{1}{N}\int_{\mathbb{R}^N} V(x)|h(w)|^N dx - \frac{1}{2}\int_{\mathbb{R}^N} (F(.,h(w)) * |x|^{\mu})\frac{F(x,h(w)(x))}{|x|^{\beta}} dx.$$

Using the properties of f, h, one can show that the functional J is well defined and $J \in C^1\left(W^{1,N}(\mathbb{R}^N), \mathbb{R}\right)$. Moreover, if $w \in W^{1,N}(\mathbb{R}^N)$ is a critical point of J, then
w is a weak solution to the following problem:

$$-\Delta_N w + V(x)|h(w)|^{N-2}h(w)h'(w) = \left(\int_{\mathbb{R}^N} \frac{F(y,h(w))}{|y|^{\beta}|x-y|^{\mu}}\, dy\right)\frac{f(x,h(w))}{|x|^{\beta}} h'(w) \text{ in } \mathbb{R}^N. \quad (6.2)$$

It can be checked that the transformed problem (6.2) is equivalent to our problem (6.1), which takes $u = h(w)$ as its solutions. Thus, it is enough to show the existence of the solution to (6.2), and then apply the h^{-1} on that solution, which will serve as the solution to the problem (6.1). By assuming the following assumptions that describe the critical growth structure of the nonlinearity, authors in [16] studied the existence of positive solutions:

(f_1) $f(x,s) = 0$, if $s \le 0$ and $f(x,s) > 0$, if $s > 0$, $\lim\limits_{s\to 0} \dfrac{f(x,s)}{|s|^{N-2}s} = 0$ uniformly in $x \in \mathbb{R}^N$.

(f_2) There exists some $\alpha_0 > 0$ such that,

$$f(x,s)\exp\left(-\alpha|s|^{\frac{2N}{2N-1}}\right) = \begin{cases} 0, & \alpha > \alpha_0, \\ \infty, & \alpha < \alpha_0. \end{cases}$$

(f_3) There exist positive constants s_0, M_0 and m_0 such that

$$0 < s^{m_0}F(x,s) \le M_0 f(x,s), \text{ for all } (x,s) \in \mathbb{R}^N \times [s_0, +\infty).$$

(f_4) There exists $\ell > N$ such that $0 < \ell F(x,s) \le f(x,s)s$, for all $s > 0$.

(f_5) critical growth at ∞: $\lim\limits_{s\to+\infty} \dfrac{sf(x,s)F(x,s)}{\exp\left(2|s|^{\frac{2N}{N-1}}\right)} = \infty$, uniformly in $x \in \mathbb{R}^N$.

The main existence result in this article is.

Theorem 6.1 *Suppose the hypotheses (V_1)-(V_2) and (f_1)-(f_5) hold. Then the problem (P_*) has a positive weak solution.*

Further, in [81], the authors studied the existence of a positive solution of the following problem involving the Hardy potential and the Choquard type exponential nonlinearity with a parameter α

$$\begin{cases} -\Delta_N w - \Delta_N(|w|^{2\alpha})|w|^{2\alpha-2}w - \lambda\frac{|w|^{2\alpha N-2}w}{\left(|x|\log\left(\frac{R}{|x|}\right)\right)^N} = \left(\int_{\Omega}\frac{H(y,w(y))}{|x-y|^{\mu}}dy\right)h(x,w(x)) \text{ in } \Omega, \\ w > 0 \text{ in } \Omega\setminus\{0\}, \qquad w = 0 \text{ on } \partial\Omega, \end{cases}$$

where $N \geq 2, \alpha > \frac{1}{2}, 0 \leq \lambda < \left(\frac{N-1}{N}\right)^N, 0 < \mu < N, h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function with critical exponential growth in the sense of the Trudinger-Moser inequality and $H(x,t) = \int_0^t h(x,s)ds$ is the primitive of h.

6.2 Kirchooff-Schrödinger problem

The authors in [14] studied the following modified quasilinear critical Kirchhoff-Schrödinger problem involving Stein-Weiss type critical nonlinearity:

$$\mathcal{K}(u) = \lambda f(x)|u(x)|^{q-2}u(x) + \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2p^*_{\beta,\mu}}}{|x-y|^{\mu}|y|^{\beta}} dy\right) \frac{|u(x)|^{2p^*_{\beta,\mu}-2}u(x)}{|x|^{\beta}} \text{ in } \mathbb{R}^N \tag{6.3}$$

where $\mathcal{K}(u) = \left(a + b\int_{\mathbb{R}^N} |\nabla u|^p dx\right) \Delta_p u - au\Delta_p(u^2)$, $2 \leq p < N$, $a > 0$, $b \geq 0$, $\beta \geq 0$, $0 < \mu < N$, $0 < 2\beta + \mu \leq N$, $0 < p^*_{\beta,\mu} := \frac{p(2N-2\beta-\mu)}{2(N-p)}$, $N \geq 3$ and $\lambda > 0$ is a parameter. Here $2p < q < 2p^*$, $p^* := \frac{Np}{N-p}$ and $f(\geq 0) \in L^{\frac{2p^*}{2p^*-q}}(\mathbb{R}^N)$.

In this case when $p < N$, one uses the following change of variables, which was introduced in [22], namely, $w := g^{-1}(u)$, where g is defined by

$$\begin{cases} g'(s) = \dfrac{1}{\left(1 + 2^{p-1}|g(s)|^p\right)^{\frac{1}{p}}} \text{ in } [0,\infty), \\ g(s) = -g(-s) \text{ in } (-\infty, 0]. \end{cases}$$

After applying the change of variable $w = g^{-1}(u)$, the new functional $\mathcal{I}_\lambda : D^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ is

$$\begin{aligned}\mathcal{I}_\lambda(w) = \frac{a}{p}\int_{\mathbb{R}^N} |\nabla w|^p dx + \frac{b}{2p}\left(\int_{\mathbb{R}^N} |g'(w)|^p |\nabla w|^p dx\right)^2 \\ - \frac{\lambda}{q}\int_{\mathbb{R}^N} f(x)|g(w)|^q dx - \frac{1}{4p^*_{\beta,\mu}}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|g(w(y))|^{2p^*_{\beta,\mu}}}{|y|^{\beta}|x-y|^{\mu}} dy\right) \frac{|g(w(x))|^{2p^*_{\beta,\mu}}}{|x|^{\beta}} dx.\end{aligned}$$

Note that, if $w \in D^{1,p}(\mathbb{R}^N)$ is a critical point of the functional $\mathcal{I}_\lambda$, then for every $v \in D^{1,p}(\mathbb{R}^N)$, $\langle \mathcal{I}'_\lambda(w), v\rangle = 0$. That is,

w is a weak solution to the following problem:

$$\begin{aligned} &- a\Delta_p w - b\int_{\mathbb{R}^N} |g'(w)|^p |\nabla w|^p dx \cdot \left(|g'(w)|^{p-2}g'(w)g''(w)|\nabla w|^p + |g'(w)|^p div(|\nabla w|^{p-2}\nabla w)\right) \\ &= \lambda f(x)|g(w)|^{q-2}g(w)g'(w) + \left(\int_{\mathbb{R}^N} \frac{|g(w)|^{2p^*_{\beta,\mu}}}{|x-y|^{\mu}|y|^{\beta}} dy\right) \frac{|g(w)|^{2p^*_{\beta,\mu}-2}g(w)}{|x|^{\beta}} g'(w) \quad \text{in } \mathbb{R}^N. \end{aligned}$$

Then as in the previous case, $u = g(w)$ is a solution of (6.3). The authors first explored the concentration compactness principle and proved the following results:

Theorem 6.2 *Let* $0 < 2\beta + \mu < \min\{2p, N\}$. *Then*

1. *if* $2 < q < 2p$ *and assume that* $\Omega := \{x \in \mathbb{R}^N : f(x) > 0\}$ *is an open subset of* $\mathbb{R}^N$ *such that* $0 < meas(\Omega) < \infty$. *Then there exists* $\lambda^* > 0$ *such that for all* $\lambda \in (0, \lambda^*)$, (6.3) *admits a sequence of*

nontrivial weak solutions $\{u_k\}$ *in* $D^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *with* $\mathcal{I}_\lambda(u_k) < 0$ *and* $u_k \to 0$ *strongly in* $D^{1,p}(\mathbb{R}^N)$ *as* $k \to \infty$.

2. *if* $q = 2p$*. Then there exists positive constants* $\hat{a}$ *such that for all* $a > \hat{a}$ *and for all* $\lambda \in (0,\ aS\|f\|^{-1}_{\frac{p^*}{p^*-p}})$, (6.3) *has at least n pairs of nontrivial weak solutions in* $D^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.
3. *if* $2p < q < 2p^*$*. Then for all* $\lambda > 0$, (6.3) *has at least n pairs of nontrivial weak solutions in* $D^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

For modified quasilinear Schrodinger equations involving singular nonlinearity, there are a few results in the existing literature. The authors in [9, 32, 97] discussed the existence of multiple solutions of such equations with singular nonlinearity u^{-q}, $0 < q < 1$, in combination with some polynomial-type perturbation.

Very recently, in [96], the authors studied the Dirichlet problems on bounded domains and proved the global multiplicity result for the quasilinear Schrödinger equations with singular source terms. The nonlinearity here of the form $s^{-q} + b(x)|s|^{\frac{3N+2}{N-2}}$, where b is a sign-changing continuous function in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N > 2$. For $N = 2$, in [15] authors studied the global multiplicity result for a class of modified quasilinear singular equations involving the critical exponential growth:

$$\begin{cases} -\Delta u - \Delta(u^2)u = \lambda\left(\alpha(x)u^{-q} + f(x,u)\right)\ u > 0 \text{ in } \Omega, \\ \qquad\qquad\qquad u > 0 \text{ in } \Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$, $0 < q < 3$ and $\alpha : \Omega \to (0, +\infty)$ such that $\alpha \in L^\infty(\Omega)$. The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and enjoys a critical exponential growth of the Trudinger-Moser type. Using a sub-super solution method, they showed that there exists some $\Lambda^* > 0$ such that for all $\lambda \in (0, \Lambda^*)$ the problem has at least two positive solutions, for $\lambda = \Lambda^*$, the problem achieves at least one positive solution and for $\lambda > \Lambda^*$, the problem has no solutions.

7 Open problems

Several compelling open problems remain in the study of Choquard-type equations and related nonlocal nonlinear models. Following are some important open questions:

1 Critical exponent problems involving the p-fractional Laplacian operator, in conjunction with Choquard terms, present another challenging avenue. Determining minimizers for the relevant Sobolev constant $S_{H,L}$, investigating regularity properties of corresponding solutions, and establishing global multiplicity results in the presence of convex-concave nonlinearities are essential topics for future study.

2 The second one is the existence and classification of positive solutions in exterior domains. Although variational and topological methods have produced some results for Choquard equations on such domains, the complete picture of the existence, uniqueness, and qualitative behavior of solutions remains incomplete, especially for domains with complex geometry or nontrivial topology.

3 Third is the classification and multiplicity of nodal (sign-changing) solutions, which is another unresolved area. Many researchers have noted that transforming the minimization problem into a minimax framework yields new families of solutions. However, the complete characterization and enumeration of sign-changing solutions remain unsettled, particularly for nonlinearities at or near the critical exponent.

4 A further open problem arises in the context of Hardy-Sobolev operators and nonlocal frameworks. Doubly critical problems become especially intricate due to the simultaneous presence of two noncompact terms. For example, the Hardy-Sobolev operator, defined as $-\Delta_p u - \frac{\mu|u|^{p-2}u}{|x|^2}$, together with critical growth Choquard nonlinearities in the equation, raises new questions regarding the existence and multiplicity of solutions. The analysis presented here involves studying minimizers and obtaining precise asymptotic estimates to understand the compactness properties of minimizing sequences–a challenging task that warrants further investigation.

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 https://doi.org/10.1007/s13226-025-00894-3

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 Indian J Pure Appl Math (2026) 57:114–142