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ORIGINAL RESEARCH

The Narasimhan-Seshadri Theorem revisited

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Abstract
Let X be a compact Riemann surface. The famous Narasimhan-Seshadri theorem [15] of 1965 uses the Grothendieck construction [5] of 1956 that associates vector bundles $\mathbb{E}(\sigma)$ on X to representations σ of a certain Fuchsian group π. Narasimhan and Seshadri show that by taking the representations σ to be irreducible unitary of a certain kind, this exactly gives all stable vector bundles on X of a given rank and degree. In this note we reformulate the correspondence from representations to bundles, which leads to simpler statements and proofs. The Fuchsian group π is replaced by the punctured fundamental group $\pi_1(X-x)$ where $x \in X$. The Grothendieck bundles $\mathbb{E}(\sigma)$ then become Deligne's logarithmic extensions to X of bundles with connections on $X-x$ associated to representations of $\pi_1(X-x)$ with scalar local monodromy. We also report how some ideas from algebraic geometry (which were all in place by 1970) have simplified some aspects of the original proof over the decades. This simplified approach works equally well for all values of the genus g, removing the restriction $g \geq 2$ in the 1965 original. Finally, we comment that such a logarithmic reformulation extends to related kinds of bundles such as principal bundles with reductive structure groups or parabolic bundles.

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1 Introduction

The statement and the proof of the Narasimhan-Seshadri theorem in their famous 1965 paper [15] can be seen to consist of the following five main steps. We begin by describing these five steps in broad terms, and then say how they can be modified and simplified. The principal modification is to replace Grothendieck's 1956 construction of bundles by Deligne's 1970 logarithmic extensions of connections. We also mention some other modifications which have become commonplace over time, involving ideas such as Quot schemes, algebraic deformation theory, Geometric Invariant Theory, good quotients, projective moduli variety of semistable bundles. These ideas were all in place by 1970, but were not available to Narasimhan and Seshadri in 1965.
(**1**) **The group π.** (Poincaré.) Let X be a compact Riemann surface of genus $g \geq 1$. For any integer $N \geq 1$, Poincaré showed that there is a simply connected Riemann surface Y and a ramified Galois cover $p : Y \to X$, which is ramified at most over a single point $x \in X$, such that the inertia subgroup of $\pi = Aut(Y/X)$ at any point of the fiber $p^{-1}(x)$ is cyclic of order N. If $N = 1$ then p is an unramified covering, and π becomes $\pi_1(X)$.
(**2**) **Grothendieck construction.** Let $\sigma : \pi \to GL(n)$ be any representation with a certain prescribed behaviour on the inertia subgroups of π. Grothendieck considers in [5] the trivial vector bundle $\mathcal{O}_Y^n$ on Y together with a π-action via σ, lifting the π-action on Y. He then takes its invariant direct image sheaf $\mathbb{E}(\sigma)$ on X, which is a locally free coherent $\mathcal{O}_X$-module. Grothendieck showed that if we took N to be a multiple of n, and chose suitably the restriction of σ to any inertia subgroup then the bundle $\mathbb{E}(\sigma)$ will have a prescribed degree $0 \leq m < n$. The Grothendieck construction of $\mathbb{E}(\sigma)$ from σ was somehow implicit in the 1938 paper of Weil [20], which Grothendieck reformulated in bundle theoretic and sheaf theoretic terms in 1956.
(**3**) **Irreducible unitary representations and stability.** Narasimhan and Seshadri were intrigued by Weil's 1938 paper, in particular, by Weil's wondering about the significance of unitary representations. In [15], they apply the above Grothendieck construction to unitary representations of π. They showed that if σ is an irreducible unitary representation with prescribed behaviour on isotropies, then the bundle $\mathbb{E}(\sigma)$ is stable, and moreover, this gives an injective map from the set of all conjugacy classes of representations to the set of isomorphism classes of stable bundles. The proof of stability of $\mathbb{E}(\sigma)$ and injectivity is ultimately based on the classical theory of plurisubharmonic functions on open disks.
(**4**) **Moduli space of stable bundles.** We must remember that the work on [15] was done based on the mathematics of 1950's (or older), but without any input from the ongoing revolution in Algebraic Geometry at the hands of Grothendieck and his school. The only modern material they used was the definition of a stable bundle by Mumford, without any further input from the ongoing development of GIT. Instead, the paper [15] uses the results of Kodaira, Nirenberg and Spencer [6, 7] on analytic deformation theory to construct a holomorphic manifold $S_s(n, m)$ of stable vector bundles of rank n degree m on X. They also show that $S_s(n, m)$ is connected.
(**5**) **Open embedding and surjectivity.** To prove that every stable bundle E is of the form $\mathbb{E}(\sigma)$, Narasimhan and Seshadri use (what Narasimhan described to me later as) the 'Poincaré continuity method'. They show that the map from the space of conjugacy classes of representations to the moduli of stable bundles is an open embedding and a proper map, and so it is an analytic isomorphism as $S_s(n, m)$ is connected.

In our reformulation of the above material, the overall steps are similar, but there are changes inside each step. These are broadly as follows.
(**1**$*$) **The group $\pi_1(X - x, y)$.** In our reformulation, we have no use for the ramified cover of Poincaré. Instead, we just replace π in (1) by the fundamental group $\pi_1(X - x, y)$ of the punctured Riemann surface, which is a free group on $2g$ generators where g is the genus of X. Moreover, we do not have any need for the integer N of (1), as this new π will give us bundles of all ranks and degrees. The further reformulated steps uniformly work for all values $g \geq 0$ of the genus.
(**2**$*$) **The Deligne construction.** We consider representations $\rho : \pi_1(X - x, y) \to GL(n)$ for which the loop $c \in \pi_1(X - x, y)$ around the puncture x is mapped to $e^{2\pi i m/n} I$ where $m, n \in \mathbb{Z}$ with $n \geq 1$. To such a representation there is naturally associated a holomorphic vector bundle E_ρ on $X - x$ with a holomorphic connection ∇_ρ. The Deligne construction of logarithmic extensions [4] attaches to this data an extension $E(\rho)$ of E_ρ to X, which is a

holomorphic vector bundle of rank n, degree m on X, such that ∇_ρ extends to a logarithmic connection on $E(\rho)$, which has residue $(-m/n)I$ at x. The bundle $E(\rho)$ is naturally isomorphic to the Grothendieck bundle $\mathbb{E}(\sigma)$ where $\sigma : \pi \to GL(n)$ is induced by ρ. This replaces the Grothendieck construction in our reformulation.
(**3**$*$) **Irreducible unitary representations and stability.** By arguments very similar to those in [15], again based on the behaviour of plurisubharmonic functions, we show that if ρ is an irreducible unitary representation with $\rho(c) = e^{2\pi i m/n} I$, then the bundle $E(\rho)$ is stable, and moreover, this gives an injective map from the set of all conjugacy classes of such representations to the set of isomorphism classes of stable bundles.
(**4**$*$) **Moduli space of stable bundles.** We simply appeal to the modern GIT-based construction of the moduli of stable bundles, which allows us to completely bypass the Kodaira-Spencer theory.
(**5**$*$) **Open embedding and surjectivity.** This is almost the same as (5) except that the local deformation theory can now be done much better in the style of Grothendieck-Schlessinger-Artin. Another simplification is to use the Mumford-Seshadri good moduli space $M^{ss}(X, n, m)$ of semistable bundles to compactify the moduli $M^s(X, n, m)$, which makes the properness of the correspondence obvious.

From about 1970, all treatments of the Narasimhan-Seshadri theorem have used algebraic deformation theory and GIT-based moduli of stable and semistable bundles, so (4*) and (5*) are routine for the past five decades or more. Another simplification due to Ramanan (which he discovered in the 1960s, soon after the theorem was published) is to use an argument based on decomposable multivectors in (3), and the same works for (3*).

This note is arranged as follows.

Section 2 sets up the basic definitions and conventions that we will use through out. Section 3 rapidly recalls the well-known material on holomorphic ODEs, logarithmic connections, and Deligne extensions in dimension 1 that we need. Section 4 treats the case of line bundles. Section 5 uses the above machinery to state and prove our reformulation of Weil's theorem on indecomposable bundles. Section 6 states the Main Theorem (Theorem 6.1), which is our reformulation of the Narasimhan-Seshadri Theorem. Section 7 describes the precise relationship between the Grothendieck construction used in [15] and the Deligne construction that we use. Section 8 says how our formulation works for $g = 0$ and $g = 1$, the two cases that were left out of [15]. Section 9 first recalls some well known facts about plurisubharmonic functions, and then proves the statement (1) of the Main Theorem 6.1. Section 10 is devoted to a rapid sketch of the steps (4*) and (5*).
Historical note: I had outlined this reformulation to Professors Narasimhan and Seshadri during the 2015 conference at CMI to mark the 50th anniversary of the theorem. The possibility of such a reformulation was perhaps known to experts for a long time, but the details were not written anywhere. Both of them asked me to write it up as an article. Here it is, ten years later. To my regret, I did not get around to it in their lifetimes.

2 Notations and conventions

Throughout, X will denote a compact Riemann surface of arbitrary genus $g \geq 0$, and $x, y \in X$ two distinct chosen points. Let $r > 1$, $\epsilon > 0$ and let $(U_{r+\epsilon}, z)$ be a holomorphic coordinate chart on X where $z(U_{r+\epsilon})$ is the open disk $|z| < r + \epsilon$ in $\mathbb{C}$ and such that x and y lie in $U_{r+\epsilon}$ and correspond to $z = 0$ and $z = 1$. We denote by $U \subset U_{r+\epsilon}$ the open subset where $|z| < r$. We will denote by $c \in \pi_1(X - x, y)$ the element corresponding to the simple positive loop around x, defined by the map $[0, 1] \to U - x : t \mapsto e^{2\pi i t}$. As usual, a **frame** over y for a local system or a vector bundle on X will mean a basis y^* for its fiber over y.

Following [4], we compose loops (or paths) on the left in defining fundamental groups (or fundamental groupoids). For a pointed universal cover of a space with a base point, the Galois group (deck transformation group) will act on the left on the total space, and it is therefore the opposite of the fundamental group, which will act on the right.

We assume a basic familiarity with the notions of local systems, holomorphic connections, logarithmic connections, monodromy representations etc., as in Deligne [4]. An exposition of these ideas is available in [8]. We recall the following basic definitions for convenience.

2.1 Logarithmic connections on Riemann surfaces and their residues. A logarithmic connection on (X, x) is a pair (E, ∇), which consists of a holomorphic vector bundle E on X equipped with a $\mathbb{C}$-linear homomorphism of sheaves

$$\nabla : E \to \Omega^1_X(\log x) \otimes_{\mathcal{O}_X} E$$

that satisfies the Leibniz rule $\nabla(fv) = df \otimes v + f\nabla(v)$ for any local sections f and v of $\mathcal{O}_X$ and E, respectively. The line bundle $\Omega^1_X(\log x)$ of logarithmic differentials on (X, x) has local free basis dz/z over U. Note that the restriction $(E|_{X-x}, \nabla|_{X-x})$ is a holomorphic connection on $X - x$. The **residue** of (E, ∇) at x is the element $\mathrm{res}_x(\nabla) \in End(E_x)$ defined by the composite homomorphism

$$E \overset{\nabla}{\to} \Omega^1_X(\log x) \otimes_{\mathcal{O}_X} E \overset{P\otimes \mathrm{id}_E}{\to} \mathcal{O}_x \otimes E = E_x$$

(which is $\mathcal{O}_X$-linear, even though ∇ is not so) where $P : \Omega^1_X(\log x) \to \mathcal{O}_x$ denotes the Poincaré residue map, with $P(\mathrm{dz/z}) = 1$. A logarithmic connection on (X, x) is a holomorphic connection on X if and only if its residue is 0.

We introduce various sets that we use later, by means of the following table.

Set	Elements
$Rep(X-x, y, n, \tau)$	representations $\rho : \pi_1(X-x, y) \to GL(n)$ such that $\rho(c) = \tau I$ where $\tau \in \mathbb{C}^\times$.
$Rep(X-x, n, \tau)$	conjugacy classes in $Rep(X-x, y, n, \tau)$.
$URep(X-x, y, n, \tau)$	unitary representations in $Rep(X-x, y, n, \tau)$.
$UR^{irr}(X-x, y, n, \tau)$	irreducible representations in $UR(X-x, y, n, \tau)$.
$UR^{irr}(X-x, n, \tau)$	conjugacy classes in $UR^{irr}(X-x, y, n, \tau)$.
$Con(X-x, y, n, \tau)$	isomorphism classes of triples (F, ∇, y^*) where F is a holomorphic vector bundle on $X-x$ of rank n, $y^* : F_y \overset{\sim}{\to} \mathbb{C}^n$ a frame over y, and $\nabla : F \to \Omega^1_X \otimes F$ a holomorphic connection, with local monodromy $\rho(c) = \tau I$ where $\tau \in \mathbb{C}^\times$ and $\rho : \pi_1(X-x, y) \to GL(n)$ denotes the monodromy representation of (F, ∇, y^*).
$Con(X-x, n, \tau)$	isomorphism classes of pairs (F, ∇) where F is a holomorphic vector bundle on $X-x$ of rank n, and $\nabla : F \to \Omega^1_X \otimes F$ a holomorphic connection, with local monodromy $\rho(c) = \tau I$ where $\tau \in \mathbb{C}^\times$.
$UCon(X-x, y, n, \tau)$	connections in $Con(X-x, y, n, \tau)$ with unitary monodromy.
$UC^{irr}(X-x, y, n, \tau)$	connections in $UCon(X-x, y, n, \tau)$ with irreducible monodromy.
$UC^{irr}(X-x, n, \tau)$	the subset of $Con(X, x, n, \tau)$ where the monodromy (up to conjugacy) is irreducible and unitary.
$Log(X, x, y, n, \lambda)$	isomorphism classes of triples (E, ∇, y^*) where E is a holomorphic vector bundle on X of rank n, y^* is a frame for E over y, and $\nabla : E \to \Omega^1_X(\log x) \otimes E$ a logarithmic connection with residue $\mathrm{res}_x(\nabla) = \lambda I \in End(E_x)$ where $\lambda \in \mathbb{C}$.
$Log(X, x, n, \lambda)$	isomorphism classes of pairs (E, ∇) where E is a holomorphic vector bundle on X of rank n,

Set	Elements
	and $\nabla : E \to \Omega^1_X(\log x) \otimes E$ a logarithmic connection with residue $\mathrm{res}_x(\nabla) = \lambda I \in End(E_x)$ where $\lambda \in \mathbb{C}$.
$ULog(X, x, y, n, \lambda)$	the subset of $Log(X, x, y, n, \lambda)$ where the monodromy is unitary.
$UL^{irr}(X, x, y, n, \lambda)$	the subset of $ULog(X, x, y, n, \lambda)$ where the monodromy is irreducible.
$UL^{irr}(X, x, n, \lambda)$	the subset of $Log(X, x, n, \lambda)$ where the monodromy (up to conjugacy) is irreducible and unitary.

2.2 Remarks on conjugacy of representations. (1) If two unitary representations $\rho_1, \rho_2 : \Gamma \to U(n)$ of a group Γ are conjugate by an element $A \in GL(n)$, then it follows by complete reducibility and Schur's lemma that ρ_1 and ρ_2 are conjugate by an element $B \in U(n)$. (2) Also by Schur's lemma, the inclusion $U(n) \hookrightarrow GL(n)$ induces a bijection from the set of all conjugacy classes (under conjugacy by $U(n)$) of irreducible representation $\rho : \pi_1(X - x, y) \to U(n)$ with local monodromy $\rho(c) = \exp(-m/n)$ to the set of all conjugacy classes (under conjugacy by $GL(n)$) of irreducible representations $\rho : \pi_1(X - x, y) \to GL(n)$ with local monodromy $\rho(c) = \exp(-m/n)$, such that ρ is 'unitarizable', that is, ρ is conjugate under $GL(n)$ to a unitary representation. (3) If $y_0, y_1 \in X - x$, and $\gamma : [0, 1] \to X - x$ is a path with $\gamma(0) = y_0$, $\gamma(1) = y_1$, then there is an induced isomorphism $[\gamma]_* : \pi_1(X - x, y_0) \to \pi_1(X - x, y_1)$ that depends only on the path homotopy class of $[\gamma]$. If $\delta : [0, 1] \to X - x$ is another such path, then $[\delta]_*$ is the conjugate of $[\gamma]_*$ by the element $[\alpha] = [\delta^{-1} \circ \gamma] \in \pi_1(X - x, y_0)$. Hence the set of all conjugacy classes under $GL(n)$ of representations in $Rep(X - x, y_0, n)$ is in a canonical bijection with the set of all conjugacy classes under $GL(n)$ of representations in $Rep(X - x, y_1, n)$, independently of the choice of γ. Hence we drop y from the notation, and denote the set of all conjugacy classes under $GL(n)$ in $Rep(X - x, y, n, \tau)$ simply by $Rep(X - x, n, \tau)$. Similar remark applies to the other sets in the table above, where we have dropped y from the notations.

3 ODEs and the Deligne extension

Let $U \subset \mathbb{C}$ be an open disk with center 0. Let $x \in U$ be the point $z = 0$. Any logarithmic connection $\nabla : \mathcal{O}_U \to \Omega^1_U(\log x) \otimes \mathcal{O}_U$ is uniquely determined by specifying

$$\nabla(e_j) = \frac{dz}{z} \otimes (\textstyle\sum_i A^i_j(z) e_i).$$

where $A(z) = (A^i_j(z))$ is any $n \times n$-matrix of holomorphic functions on U, and e_i is the standard free basis of $\mathcal{O}^n_U$. The residue of ∇ at $z = 0$ is given by the matrix $R = A(0)$. In classical terms, such a logarithmic connection corresponds to the linear system of ODEs

$$\frac{dF(z)}{dz} = -\frac{A(z)}{z} F(z)$$

where $F(z)$ is an $n \times 1$-vector of functions of z.

3.1 Local monodromy for a good residue. With notation as above, suppose that no two eigenvalues of $R = A(0)$ differ by a non-zero integer. Then there exists a unique invertible $n \times n$-matrix of holomorphic functions $P(z)$ on U with $P(0) = I$ such that $P(z)z^{-R}$ is a fundamental solution matrix for the above system, that is,

$$z\frac{d}{dz}(P(z)z^{-R}) = -A(z)P(z).$$

Consequently, the monodromy for the connection around $z = 0$ is $e^{-2\pi i R}$.

Proof This is just Coddington-Levinson [4] Chapter 4, Theorem 4.1. Note that the residue as defined in [4] is -1 times the residue R as per our convention here. Recall that in [4], the matrix valued function $z^{-R} = e^{-R\log z}$ is regarded as a multivalued holomorphic function of z, which becomes single valued over the universal cover of $U - x$ with coordinate $\log z$. Hence the monodromy is the multiplying matrix factor $e^{-2\pi i R}$ on the right by which the fundamental solution Pz^{-R} is modified when $\log z$ changes from 0 to $2\pi i$ as z travels once around $z = 0$ in an anticlockwise sense. □

3.2 Deligne construction of logarithmic extension. Let X be a Riemann surface (not necessarily compact), and let $x, y \in X$, with $x \neq y$. Let (F, ∇, y^*) be a framed holomorphic connection of rank n over $(X - x, y)$. Let $\rho(c) = e^{-2\pi i R} \in GL(n)$ where $R \in M(n)$ is an $n \times n$ matrix over $\mathbb{C}$, such that no two distinct eigenvalues of R differ by a nonzero integer. Then there exists an extension $(E, \phi : E|_{X-x} \overset{\sim}{\to} F)$ of F to a vector bundle E on X, such that ∇ extends as a logarithmic connection on E such that the residue $\mathrm{res}_x(\nabla) \in End(E_x)$ is conjugate to $R \in M(n)$. Moreover, such an extension (E, ϕ) of F is unique up to a unique isomorphism of extensions. To construct (E, ϕ), consider the bundle $\mathcal{O}_U^n$ on the disk $U \subset X$ chosen as in the beginning of Section 2. Let $\nabla_U : \mathcal{O}_U^n \to \Omega_U^1(\log x) \otimes \mathcal{O}_U^n$ be the logarithmic connection defined by $\nabla(e_i) = (dz/z) \otimes \sum_j R_i^j e_j$, where e_i is the standard basis of $\mathcal{O}_U^n$. Note that e_i defines a frame $(e_{i,y})$ over $y \in U$. Then a fundamental solution of ∇_U on $U - x$ is given by the multivalued matrix $z^{-R} = e^{-R\log z}$, and it takes the values I at $\log z = 0$ and $e^{-2\pi i R}$ at $\log z = 2\pi i$. This shows that the monodromy $\rho_U(c) = e^{-2\pi i R}$. Hence there exists a unique isomorphism of framed holomorphic connections

$$\psi : (F, \nabla, y^*)|_{U-x} \to (\mathcal{O}_U^n, \nabla, (e_{i,y}))|_{U-x}$$

over $U - x$. Gluing F and $\mathcal{O}_U^n$ over $U - x$ via ψ defines the extension (E, ϕ). The uniqueness of (E, ϕ) follows from Lemma 3.1 applied to the logarithmic connection $(\underline{Hom}(E_1, E_2), \nabla)$ made from two extensions (E_1, ∇_1) and (E_2, ∇_2), which has the property that no nonzero integer is an eigenvalue of $\mathrm{res}_x(\nabla)$, as by assumption, no two eigenvalues of R differ by a nonzero integer. This has the consequence that the integrable section σ of $\underline{Hom}(E_1, E_2)$ on $X - x$ defined by id_F extends to x to define a flat isomorphism $E_1 \to E_2$. (See [4] or Theorem 4.4 in [8] for more details).

3.3 Note. We will write the extension (E, ϕ) simply as E, and we will say that 'E is the logarithmic extension of (F, ∇) with residue conjugate to R', suppressing ϕ from the notation unless it is explicitly needed for unambiguity.

3.4 Example. It is necessary in the above to assume that no two eigenvalues of R differ by a nonzero integer. For example, let $F = \mathcal{O}_{X-x}^2$, let $\nabla(e_i) = 0$ where e_1, e_2 form the standard basis of $\mathcal{O}_{X-x}^2$, and let y^* be the basis $(e_{i,y})$ over y. Let $R = \mathrm{diag}(1, 0)$. Then F has two possible non-isomorphic extensions $\mathcal{I}_x \oplus \mathcal{O}_X$ and $\mathcal{O}_X \oplus I_x$, on which ∇ extends logarithmically with residues conjugate to R. (Note the isomorphic bundles $\mathcal{I}_x \oplus \mathcal{O}_X$ and $\mathcal{O}_X \oplus I_x$ are not isomorphic as *extensions* of F.)

3.5 Case of scalar local monodromy. Let X be a Riemann surface (not necessarily compact), and let $x, y \in X$, with $x \neq y$. Let $(E, \nabla, y^*) \in Log(X, x, y, n, \lambda)$, that is, E is rank n bundle with a logarithmic connection $\nabla : E \to \Omega_X^1(\log x) \otimes E$ and a frame y^* for E over y, with residue $\mathrm{res}_x(\nabla) = \lambda I \in End(E_x)$ for a given $\lambda \in \mathbb{C}$. Then its restriction $(E|_{X-x}, \nabla|_{X-x}, y^*)$ to $X - x$ lies in $Con(X - x, y, n, e^{-2\pi i\lambda})$, that is, it is a framed holomorphic connection on $(X - x, y)$, whose local monodromy around x is

$$\rho(c) = e^{-2\pi i\lambda} I \in GL(E_y) \overset{y^*}{=} GL(n)$$

Hence restriction defines a map

$$\mathcal{R}_y : Log(X, x, y, n, \lambda) \to Con(X - x, y, n, e^{-2\pi i\lambda})$$

that sends (E, ∇, y^*) to $(E|_{X-x}, \nabla|_{X-x}, y^*)$. The Deligne construction 3.2 gives its inverse map

$$\mathcal{D}_y : Con(X - x, y, n, e^{-2\pi i\lambda}) \to Log(X, x, y, n, \lambda)$$

showing that $\mathcal{R}_y$ and $\mathcal{D}_y$ are bijections. Forgetting the point y and the frame y^*, the above map descends to a bijective map $\mathcal{R} : Log(X, x, n, \lambda) \to Con(X-x, n, e^{-2\pi i\lambda})$ with inverse bijection $\mathcal{D} : Con(X-x, n, e^{-2\pi i\lambda}) \to Log(X, x, n, \lambda)$.

4 The case of line bundles

4.1 The fundamental group of a punctured Riemann surface. As before, let X be a compact Riemann surface of genus $g \geq 0$, and let $x, y \in X$ with $x \neq y$. Let there be given a local holomorphic coordinate chart (U, z) on X such that the image $z(U) \subset \mathbb{C}$ is an open disk of radius > 1, and let $x, y \in U$ with $z(x) = 0$, $z(y) = 1$. The element $c \in \pi_1(X - x, y)$ is then defined by the loop $z(t) = e^{2\pi i t}$ in $U - x$. It is well known that there exist $2g$ distinct elements $a_1, \ldots, a_g, b_1, \ldots, b_g \in \pi_1(X - x, y)$ such that $\pi_1(X - x, y)$ is the free group on these $2g$ elements, and the local positive loop c is given by

$$c = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}.$$

As c is a product of commutators, its homology class is zero, that is, $[c] = 0 \in H_1(X - x) = \pi_1(X - x, y)/(\pi_1(X - x, y), \pi_1(X - x, y))$. Note that the inclusion $X - x \hookrightarrow X$ induces an isomorphism of groups

$$\pi_1(X - x, y)/\langle c\rangle \xrightarrow{\sim} \pi_1(X, y)$$

where $\langle c\rangle$ denotes the smallest normal subgroup of $\pi_1(X - x, y)$ that contains c.

For proving the Main Theorem 6.1, we begin with line bundles on X, that is, we take $n = 1$. Note that every line bundle is a stable vector bundle.

As the group $U(1)$ is abelian and as c lies in the commutator subgroup of $\pi_1(X-x, y)$, for any $\rho : \pi_1(X-x, y) \to U(1)$ we must have $\rho(c) = 1 \in U(1)$. Hence ρ uniquely factors via the quotient $\pi_1(X - x, y) \to \pi_1(X, y)$, to define a representation which we again denote by $\rho : \pi_1(X, y) \to U(1)$. In fact, as $U(1)$ is abelian, the choice of the base point y does not matter, and ρ further factors uniquely via the quotient $\pi_1(X, y) \to H_1(X)$ to define a representation, again denoted by $\rho : H_1(X) \to U(1)$. Also, for any holomorphic connection ∇ on a line bundle L on X, the choice of a base point $y \in X$ and a basis y^* of the fiber L_y over it does not matter for the corresponding monodromy representation $\rho : H_1(X) \to U(1)$, which remains independent of such choices.

4.2 Theorem. *Let X be a compact Riemann surface. Then for any holomorphic line bundle L on X with* $deg(L) = 0$*, there exists a unique representation* $\rho : \pi_1(X) \to U(1)$ *such that L is associated to the representation* ρ*. Equivalently, there exists a unique holomorphic connection* ∇ *on L such that the monodromy of* ∇ *is unitary.*

Proof (following Narasimhan's Trieste lecture notes [11]). We have the following commutative diagram of sheaves on X, with exact rows, where for any of the groups group $A = \mathbb{Z}$, $\mathbb{R}$, $U(1)$, we denote by A_X the sheaf on X of germs of locally constant functions with values in A (which effectively means that we have given the various groups A the discrete topology).

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_X \arrow[r] \arrow[d, equal] & \mathbb{R}_X \arrow[r] \arrow[d] & U(1)_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}_X \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{O}_X^\times \arrow[r] & 0
\end{tikzcd}$$

As the boundary map $\partial : Pic(X) = H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z}_X) = \mathbb{Z}$ sends $[L] \mapsto \deg(L)$, we obtain another commutative diagram with exact rows

$$\begin{tikzcd}
H^1(X, \mathbb{Z}_X) \arrow[r] \arrow[d, equal] & H^1(X, \mathbb{R}_X) \arrow[r] \arrow[d] & H^1(X, U(1)_X) \arrow[r] \arrow[d] & 0 \\
H^1(X, \mathbb{Z}_X) \arrow[r] & H^1(X, \mathcal{O}_X) \arrow[r] & Pic^0(X) \arrow[r] & 0
\end{tikzcd}$$

The map $H^1(X, \mathbb{R}_X) \to H^1(X, \mathcal{O}_X)$ is an isomorphism as X is compact Kähler, hence $H^1(X, U(1)_X) \xrightarrow{\sim} Pic^0(X)$. By the universal coefficient theorem, the natural map $Hom(\pi_1(X), U(1)) = Hom(H_1(X), U(1)) \to H^1(X, U(1)_X)$ is an isomorphism. Hence, as claimed, the map $Hom(\pi_1(X), U(1)) \to Pic^0(X)$ is an isomorphism. ◻

4.3 Example. The unique connection on $L = \mathcal{O}_X$ with unitary monodromy is the de Rham connection (exterior derivative) $d : \mathcal{O}_X \to \Omega^1_X$.

4.4 The natural logarithmic connection on the line bundle $\mathcal{O}_X(mx)$. Note that $\Omega^1_X(\log x)$ is naturally isomorphic to $\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(x)$. The de Rham connection $d : \mathcal{O}_X \to \Omega^1_X$ maps the ideal sheaf $\mathcal{I}_x \subset \mathcal{O}_X$ into $\Omega_X = \Omega_X(\log x) \otimes_{\mathcal{O}_X} \mathcal{I}_x$, which is a logarithmic connection on $\mathcal{I}_x = \mathcal{O}_X(-x)$. Taking duals and tensor powers, this induces a logarithmic connection ∇_m on the line bundle $\mathcal{O}_X(mx)$ for any $m \in \mathbb{Z}$. These logarithmic connections have the following coordinate description in terms of any local coordinate chart (U, z) around x in which x is given by $z = 0$. Note that $\mathcal{O}_X(mx)|_{X-x} = \mathcal{O}_{X-x}$, and $\nabla_m|_{X-x} = d$, the de Rham connection on $\mathcal{O}_{X-x}$. Over U, the sheaf $\mathcal{O}_X(mx)$ has the free basis z^{-m}, and a simple calculation shows that

$$\nabla_m(z^{-m}) = -m\frac{dz}{z} \otimes z^{-m}.$$

In particular, the residue of ∇_m at x is $-m$, and so we have $\deg(\mathcal{O}_X(mx)) = m = -\operatorname{trace}\operatorname{res}_x(\nabla_m)$, as given by Lemma 4.7. As ∇_m restricts to d on $X - x$, the constant functions on $X - x$ are flat sections of $\mathcal{O}_X(mx)$ over $X - x$, and so the monodromy of ∇_m is trivial (that is, ρ takes the constant value 1). In particular, the monodromy of ∇_m is unitary.

4.5 Arbitrary line bundles of degree m. If $\deg(L) = m$, then $\deg(L \otimes \mathcal{O}_X(-mx)) = 0$, so by Theorem 4.2, there exists a unique holomorphic connection $\nabla_{L\otimes\mathcal{O}_X(-mx)}$ on it whose monodromy is unitary. Tensoring the holomorphic connection $(L \otimes \mathcal{O}_X(-mx), \nabla_{L\otimes\mathcal{O}_X(-mx)})$ with $(\mathcal{O}_X(mx), \nabla_m)$, we see that L has a unique logarithmic connection on (X, x) with unitary monodromy, which has trivial local monodromy $\rho(c) = e^{2\pi i m/1} = 1$ and residue $-m = -\mu(L)$.

This concludes the proof of Theorem 6.1 for rank 1 and arbitrary degree m.

4.6 Relation between residue and degree in all ranks. The following lemma, and its appropriate generalization to varieties of higher dimensions, are well known. For easy reference, we include a proof for compact Riemann surfaces.

4.7 Lemma. *Let X be a compact Riemann surface, and let $x \in X$. If (E, ∇) is a logarithmic connection on (X, x), then deg$(E) = -$trace res$_x(\nabla)$.*

Proof The cohomology sequence of the exact sequence $0 \to \Omega^1_X \to \Omega^1_X(\log x) \to \mathcal{O}_x \to 0$, together with the fact that $H^1(X, \Omega^1_X(\log x)) = 0$ by Serre duality, shows that the induced map $H^0(X, \Omega^1_X) \to H^0(X, \Omega^1_X(\log x))$ is an isomorphism. The logarithmic connection $\nabla : E \to \Omega^1_X(\log x) \otimes E$ induces a logarithmic connection $\nabla' : \det(E) \to \Omega^1_X(\log x) \otimes \det(E)$. Note that

$$\mathrm{res}_x(\det(E), \nabla') = \text{trace } \mathrm{res}_x(E, \nabla).$$

If $\deg(E) = m$, then the line bundle $L = \det(E) \otimes \mathcal{O}_X(-mx)$ has a logarithmic connection ∇'' induced by the logarithmic connections on $\det(E)$ and $\mathcal{O}_X(-mx)$. Note that

$$\mathrm{res}_x(L, \nabla'') = \mathrm{res}_x(\det(E), \nabla') + \mathrm{res}_x(\mathcal{O}_X(-mx), \nabla_{-m}) = \text{trace } \mathrm{res}_x(E, \nabla) - m.$$

As L is a line bundle of degree 0, by Lemma 4.2 it admits a holomorphic connection $D : L \to \Omega^1_X \otimes L$. Hence $\nabla'' - D \in H^0(X, \Omega^1_X(\log x)) = H^0(X, \Omega^1_X)$. This shows that $\mathrm{res}_x(L, \nabla'') = \mathrm{res}_x(L, D) = 0$. It follows that trace $\mathrm{res}_x(E, \nabla) = m$ □

5 Logarithmic reformulation of Weil's theorem

The statement (2) of the following theorem is a generalization of Weil's theorem to all degrees, which in another equivalent form already occurs as Proposition 6.2 in [15]. We give a (more transparent) logarithmic reformulation of both the statement and the proof. I had reported the statement (1) in a talk in Strasbourg in 1992, but did not publish it.

5.1 Theorem. *Let X be a compact Riemann surface, and let $x \in X$. Then we have the following.*
(**1**) *Every holomorphic vector bundle E on X admits a logarithmic connection*
$\nabla : E \to \Omega^1_X(\log x) \otimes E$.
(**2**) *For any holomorphic vector bundle E on X, the following three statements are equivalent:*
  (a) E admits a logarithmic connection ∇ with residue $res_x(\nabla) = -\mu(E)I$.
  (b) For every direct summand F of E we have $\mu(F) = \mu(E)$.
  (c) The slope of every indecomposable component of E equals the slope of E.

Proof This is a logarithmic version of Atiyah's proof of Weil's theorem, with some modifications. We assume familiarity with Atiyah's proof (see [1]). The obstruction for the existence of a holomorphic connection on E is the Atiyah class $At(E) \in H^1(X, \Omega^1_X \otimes \underline{End}(E))$. If E is described by an open cover (U_i) and transition functions (g_{ij}), then $At(E)$ is induced by the Cech 1-cocycle $(dg_{ij}g_{ij}^{-1})$ w.r.t. this cover. The analogous argument for logarithmic connections shows that the obstruction for the existence of a logarithmic connection on E over (X, x) is (what we will call as) the *logarithmic* Atiyah class $\alpha_E \in H^1(X, \Omega^1_X(\log x) \otimes \underline{End}(E))$, described by the Cech 1-cocycle $(dg_{ij}g_{ij}^{-1})$.

To prove (1), we will show that $\alpha_E = 0$ for each E. As $\underline{End}(E)$ is self-dual under the trace pairing, the pairing

$$H^0(X, \mathcal{O}_X(-x) \otimes \underline{End}(E)) \times H^1(X, \Omega^1_X(\log x) \otimes \underline{End}(E)) \overset{trace}{\to} H^1(X, \Omega^1_X) = \mathbb{C}$$

is non degenerate by Serre duality. Note that $H^0(X, \mathcal{O}_X(-x) \otimes \underline{End}(E))$ is the subspace of $H^0(X, \underline{End}(E)) = End(E)$ consisting of all endomorphisms $\varphi : E \to E$ such that φ vanishes at the point x. As X is compact, the characteristic polynomial $P(\varphi_u) \in \mathbb{C}[T]$ of $\varphi_u = \varphi|_{E_u}$ on any fiber E_u is independent of $u \in X$, and hence $P(\varphi_u) = P(\varphi_x) = T^n$, where $n = \mathrm{rank}(E)$. Hence $\varphi_u^n = 0$ for all u, so $\varphi^n = 0$. Hence as in [1], there exists a nested sequence (full flag) of vector subbundles $0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$, such that $\varphi(E_i) \subset E_{i-1}$ for all $1 \le i \le n$. We can choose an open cover U_i of X with trivializations of E over U_i, such that the transition functions g_{ij} preserve this full flag. Hence $dg_{ij}g_{ij}$ are upper triangular matrices while φ is given by strictly upper triangular matrices φ_i over U_i. Hence $\varphi_i dg_{ij}g_{ij}^{-1}$ is again strictly upper triangular, so its trace is zero. This shows that $\langle \varphi, \alpha_E \rangle = 0$ for all $\varphi \in H^0(X, \mathcal{F} \otimes \mathcal{I}_x)$, hence $\alpha_E = 0$. Hence (1) is true.

Next we prove (2). A logarithmic connection on $E = E' \oplus E'$ over (X, x) with residue λI induces by inclusions and projections logarithmic connections on E' and E'' over (X, x) with residues λI (here, the same notation I stands for the identity endomorphisms of E_x, E'_x, E''_x for simplicity). Also, by Krull-Remak theorem, E is the direct sum of indecomposable subbundles. So to prove (2), it is enough to show that if E is indecomposable then E admits a logarithmic connection ∇ over (X, x) with $\mathrm{res}_x(\nabla) = \lambda I$ for some $\lambda \in \mathbb{C}$. Then we will necessarily have $\lambda = -\mu(E)$ by Lemma 4.7.

Let $ev_x : \underline{End}(E) \to (\mathcal{O}_X/\mathcal{I}_x) \otimes_{\mathcal{O}_X} \underline{End}(E) = End_{\mathbb{C}}(E_x)$ be the evaluation map, where $\mathcal{I}_x \subset \mathcal{O}_X$ denotes the ideal sheaf of x in X. Let $\mathbb{C}I \subset End_{\mathbb{C}}(E_x)$ be the scalar endomorphisms. Let $\mathcal{F} \subset \underline{End}(E)$ be the coherent sub-$\mathcal{O}_X$-module that is the inverse image of $\mathbb{C}I$ under ev_x. Note that $\mathcal{F}$ is locally free, but it is not a subbundle of $\underline{End}(E)$.

Repeating the arguments of (1) in this context shows that the obstruction to the existence of a logarithmic connection on E over (X, x) with residue of the form λI is defined by the 1-cocycle $\beta_E = (dg_{ij}g_{ij}^{-1})$ in $H^1(X, \Omega^1_X(\log x) \otimes \mathcal{F})$.

The sheaf $\mathcal{F}$ fits in the exact sequence

$$0 \to \underline{End}(E) \otimes \mathcal{I}_x \to \mathcal{F} \to \mathcal{O}_x \to 0$$

where $\mathcal{O}_x = \mathcal{O}_X/\mathcal{I}_x$, and the map $\mathcal{F} \to \mathcal{O}_x$ sends an endomorphism φ to λ where $\varphi_x = \lambda I$. This means the above map $\mathcal{F} \to \mathcal{O}_x$ sends φ to $(1/n)\,\mathrm{trace}(\varphi_x)$. Tensoring with $\mathcal{O}_X(x)$ gives the exact sequence

$$0 \to \underline{End}(E) \to \mathcal{F} \otimes \mathcal{O}_X(x) \to \mathcal{O}_X(x)_x \to 0.$$

As $\underline{End}(E)$ is self-dual under the trace pairing, applying the functor $\underline{Hom}(-, \mathcal{O}_X)$ we get an exact sequence of sheaves

$$0 \to \mathcal{F}^\vee \otimes \mathcal{I}_x \to \underline{End}(E) \to \mathcal{O}_x \to 0$$

where $\underline{Ext}^1(\mathcal{O}_X(x)_x, \mathcal{O}_X)$ is canonically identified with $\mathcal{O}_x$, and the map $\underline{End}(E) \to \mathcal{O}_x$ sends φ to $(1/n)\,\mathrm{trace}(\varphi_x)$. The long exact cohomology sequence of the above short exact sequence of sheaves shows that

$$H^0(X, \mathcal{F}^\vee \otimes \mathcal{I}_x) = \{\varphi \in End(E) \mid \mathrm{trace}(\varphi_x) = 0\}.$$

By Serre duality we have a nondegenerate pairing

$$\langle\text{-}, \text{-}\rangle : H^0(X, \mathcal{F}^\vee \otimes \mathcal{I}_x) \times H^1(X, \Omega^1(\log x) \otimes \mathcal{F}) \to H^1(X, \Omega^1) = \mathbb{C},$$

therefore to show that $\beta_E = 0 \in H^1(X, \Omega^1(\log x) \otimes \mathcal{F})$, we must show that $\langle \varphi, \beta_E \rangle = 0$ for all $\varphi \in End(E)$ such that $\mathrm{trace}(\varphi_x) = 0$. In cocycle terms, we must show that for any such φ, $\mathrm{trace}(\varphi_i \circ dg_{ij}g_{ij}^{-1}) = 0 \in H^1(X, \Omega^1)$.

As by assumption E is indecomposable, each global endomorphism φ of E is of the form $cI + N$ where $c \in \mathbb{C}$ and N is nilpotent. If $\varphi = cI$, then requirement that $\mathrm{trace}(\varphi_x) = 0$ means $cI = 0$. Hence φ is nilpotent. Then as above, we can choose a nested sequence (full flag) of subbundles $0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$, where $n = \mathrm{rank}(E)$, such that $\varphi(E_i) \subset E_{i-1}$ for all $1 \le i \le n$. We can choose an open cover U_i of X with trivializations of E over U_i, such that the transition functions g_{ij} preserve this full flag. Hence $dg_{ij}g_{ij}$ are upper triangular matrices while φ is given by strictly upper triangular matrices φ_i over U_i. Hence $\varphi_i dg_{ij}g_{ij}^{-1}$ is again strictly upper triangular, so its trace is zero. This shows that $\langle \varphi, \beta_E \rangle = 0$ for all $\varphi \in H^0(X, \mathcal{F} \otimes \mathcal{I}_x)$, hence $\beta_E = 0$. □

Combining with the above theorem with the bijection of statement 3.5 gives us the following.

5.2 Theorem. *Let X be a compact Riemann surface, and let $x \in X$. Then we have the following.*
(**1**) *For any indecomposable vector bundle E on X, there exists an indecomposable representation $\rho : \pi_1(X-x) \to GL(n)$ with local monodromy $\rho(c) = e^{-2\pi i \mu(E)} I$ around x, such that E is the underlying vector bundle of the logarithmic connection (E, ∇) on (X, x) associated to ρ by the Deligne construction made by choosing $-\mu(E)I$ as the residue of the logarithmic connection at x.*
(**2**) *Conversely, if $\rho : \pi_1(X - x) \to GL(n)$ is indecomposable, with $\rho(c) = e^{2\pi i m/n} I$ where $m, n \in \mathbb{Z}$ with $n \geq 1$, then the vector bundle E on X which is given by the Deligne construction made by choosing $(-m/n)I$ as the residue of the logarithmic connection at x, is an indecomposable vector bundle on X of rank n and degree m.*

5.3 Existence of indecomposable bundles. If $g(X) = 0$ then $\pi_1(X - x, y)$ is trivial, the only indecomposable representation of $\pi_1(X - x, y)$ is the trivial representation of rank 1, and the only indecomposable vector bundles on X are of rank 1 by the theorem of Grothendieck.

When $g(X) = 1$, $\pi_1(X - x, y)$ is the free group on 2 generators. As proved by Atiyah, when $g(X) = 1$ and $n \geq 1$, there exists an indecomposable vector bundle E on X of rank n and any prescribed degree m. This implies by the above Theorem 5.2 that there exists an indecomposable rank n representation of F_2 with $\rho(c) = e^{2\pi i m/n} I$, for any m, n with $n \geq 1$. Venkataramana provided a direct example of such a representation (see Lemma 8.1 below). This, combined with Theorem 5.2, directly shows that indecomposable bundles exist on elliptic curves for any given rank $n \geq 1$ and degree m, bypassing the results of [2].

When $g(X) \geq 2$, [15] Proposition 9.1 constructs irreducible rank n unitary representations of the free group $\pi_1(X - x, y)$ with $\rho(c) = e^{2\pi i m/n} I$, for any value of m and n with $n \geq 1$. They then deduce the existence of stable bundles of rank n degree m from it by one part of the Narasimhan-Seshadri theorem. This part is then used to prove the remaining part of the theorem. Of course, all stable bundles are indecomposable.

6 Logarithmic reformulation of the Narasimhan-Seshadri theorem

Any stable bundle E on a compact Riemann surface X is indecomposable. So by Theorem 5.2, E arises from some indecomposable representation $\rho : \pi_1(X - x, y) \to GL(n)$ with scalar local monodromy by Deligne construction with scalar residue. However, is there some condition on the monodromy representation, stronger than indecomposability, that guarantees that the bundle E is stable? Does every stable bundle arise this way from a monodromy representation that satisfies this stronger condition? This (and more) was historically answered by the Narasimhan-Seshadri theorem.

The following, which is the main result of this note, is an equivalent logarithmic reformulation of the Narasimhan-Seshadri theorem. We allow any value of g.

6.1 Main Theorem. *Let X be a compact Riemann surface of any genus $g \geq 0$, and let $x, y \in X$ with $x \neq y$. Let $n, m \in \mathbb{Z}$ with $n \geq 1$. Then we have the following.*
(1) If $\rho : \pi_1(X - x, y) \to U(n)$ is an irreducible unitary representation whose local monodromy around the puncture x equals $e^{2\pi i m/n} I$, then the Deligne extension $E(\rho)$ of E_ρ, with residue $(-m/n)I$ at x, is a stable holomorphic vector bundle on X of rank n and degree m. Moreover, two such representations ρ_1 and ρ_2 are conjugate if and only if the bundles $E(\rho_1)$ and $E(\rho_2)$ are isomorphic.
(2) If E is a stable holomorphic vector bundle on X of rank n and degree m, then there exists an irreducible representation $\rho : \pi_1(X - x, y) \to U(n)$, with local monodromy $e^{2\pi i m/n} I$ around x, such that E is isomorphic to $E(\rho)$.
(3) The resulting map $UR^{irr}(X - x, n, e^{2\pi i m/n}) \to M^s(X, n, m) : \rho \mapsto E(\rho)$ is a real analytic isomorphism between

- *(a) the manifold $UR^{irr}(X - x, n, e^{2\pi i m/n})$ of all conjugacy classes of irreducible unitary representations $\rho : \pi_1(X - x) \to U(n)$ with local monodromy $e^{2\pi i m/n} I$ around x, and* *(b) the manifold $M^s(X, n, m)$ of all isomorphism classes of stable holomorphic vector bundles E on X of rank n and degree m.*

6.2 The statement (1) is proved in Section 9. The strategy of proof is copied from [15], but here the logarithmic reformulation leads to somewhat simpler arguments. The statements (2) and (3) are almost identical to the corresponding statements in [15], and we do not repeat the very beautiful proofs given there. In particular, we know by the arguments in [15] that $UR^{irr}(X - x, n, e^{2\pi i m/n})$ and $M^s(X, n, m)$ are real analytic manifolds, and the bijective map

$$UR^{irr}(X - x, n, e^{2\pi i m/n}) \to M^s(X, n, m)$$

of (3) is a tangent-level injection and hence an isomorphism of real analytic manifolds. The original argument in [15] becomes somewhat simplified because of developments between 1965 and 1970, and this partially simplified proof of (2) and (3) (which is standard knowledge since 1970s) is sketched in Section 10 for completeness.

7 Comparison with Grothendieck's construction

7.1 Associated bundles as invariant direct images. Let $p : P \to X$ be a principal bundle under the left action of a group π, and let $\sigma : \pi \to GL(n)$ be a representation. The total space $P(\sigma)$ of the associated vector bundle $P(\sigma) \to X$ is the quotient $\pi\backslash(P \times \mathbb{C}^n)$, where the left action $\pi \times (P \times \mathbb{C}^n) \to P \times \mathbb{C}^n$ is given by $g(y, v) = (gy, \rho(g)v)$. The sheaf of sections $\underline{P(\sigma)}$ of $P(\sigma) \to X$ is then the invariant direct image

$$\underline{P(\sigma)} = p_*^{\pi}(\mathcal{O}_P^n(\sigma))$$

where $\mathcal{O}_P^n(\sigma)$ is the sheaf $\mathcal{O}_P^n$ on P with the obvious left π-action coming from σ, that lifts the left π-action on P.

7.2 Left-right correspondence. Let $G = \pi^{op}$ be the opposite group. Recall that G has the same underlying set that π has, but the multiplication $*$ in G is defined in the reverse order to that in π, that is, $g * h = hg$. Any left π-action $\pi \times Z \to Z : (g, z) \mapsto gz$ on a space Z becomes a right G-action $Z \times G \to Z$ defined by $z * g = gz$. A left principal π-bundle $p : P \to X$ becomes a right principal G-bundle $P \to X$ under the above action. A representation $\sigma : \pi \to GL(n)$ corresponds to a representation $\rho = \sigma^{op} : G \to GL(n)$, defined by $\rho(g) = \sigma(g^{-1})$. The vector bundle $P(\rho)$ is the quotient of $P \times \mathbb{C}^n$ by the right G-action given by $(y, v) * g = (y * g, \rho(g^{-1})v) = (gy, \sigma(g)v) = g(y, v)$. This shows that

$$P(\sigma) = P(\rho).$$

The formulas in 7.1 and 7.2 will be used in our comparison between the Grothendieck construction of invariant direct images and the Deligne construction of logarithmic extensions.

In [5], Grothendieck associates a bundle $\mathbb{E}(\sigma)$ on a compact Riemann surface X to representations of a certain group π. Grothendieck's construction is based on the following lemma, which as he says is essentially a topological fact. It is provable just from the basics of fundamental groups and covering spaces, by attaching open disks to a suitably constructed covering of $X - x$. It is possible that Poincaré proved the lemma via tessallations of the hyperbolic plane, which he had invented.

7.3 Lemma. (Poincaré.) *Let* X *be a compact Riemann surface of genus* $g \geq 1$, *let* $x \in X$, *and let* $N \geq 1$. *Then there exists a Riemann surface* Y *with a holomorphic morphism* $p : Y \to X$ *which satisfies the following properties.* *(1) The space* Y *is simply connected.*

(2) The automorphism group $\pi = Aut(Y/X)$ *acts properly discontinuously on the left on* Y, *and* $p : Y \to X$ *is the corresponding quotient. In particular,* π *acts transitively on all fibers.*
(3) The restriction $p' : Y' = Y - p^{-1}(x) \to X - x = X'$ *of* p *is unramified.*
(4) The inertia subgroup $\pi_{\widehat{x}}$ *at any point* $\widehat{x} \in p^{-1}(x)$ *is cyclic of order* N. *So if* $N = 1$ *then* $p : Y \to X$ *is a universal covering of* X.

Any $p_1 : Y_1 \to X$ *that satisfies (1)-(4) above is isomorphic to* $p : Y \to X$, *where the isomorphism is unique up to composition with an element of the Galois group* π.

7.4 The requirement $g \geq 1$**.** The above lemma is false when $g = 1$ and $N \geq 2$. It is to be noted that our reformulation of Narasimhan-Seshadri theorem via logarithmic connections does not need this lemma, and works uniformly for all values $g \geq 0$ of the genus.

7.5 Relation with $\pi_1(X - x, y)$**.** If $p : Y \to X$ is as above, then its restriction $p' : Y' = Y - p^{-1}(x) \to X - x = X'$ is an unramified Galois covering with Galois group $Aut(Y/X)$. Let $y' \in Y'$ be a point over $y \in X'$. Let $K \subset \pi_1(X', y)$ denote the image of $p'_* : \pi_1(Y', y') \to \pi_1(X', y)$. Then K is the smallest normal subgroup that contains $c^N \in \pi_1(X', y)$. Moreover,

$$\pi^{op} = Aut(Y/X)^{op} = \pi_1(X', y)/K$$

where the opposite group occurs by our convention about composition of loops in fundamental groups. As underlying sets of a group and its opposite group are identical, the image $\overline{c} \in \pi_1(X', y)/K$ of c defines an element $\overline{c}$ of $\pi = Aut(Y/X)$.

7.6 Relationship between the representations of the groups $\pi_1(X - x, y)$ **and** π**.** Any representation $\sigma : \pi \to GL(n)$ defines by composition a representation

$$\rho : \pi_1(X', y) \to \pi_1(X', y)/K = \pi^{op} \overset{\sigma^{op}}{\to} GL(n)$$

where $\sigma^{op} : \pi^{op} \to GL(n)$ is defined by $\sigma^{op}(a) = \sigma(a)^{-1}$. Conversely, any $\rho : \pi_1(X', y) \to GL(n)$ for which $\rho(c^N) = I$ arises from a unique $\sigma : \pi \to GL(n)$.

Next suppose that $n \geq 1$ divides mN where $m \in \mathbb{Z}$. Then any representation $\rho : \pi_1(X - x, y) \to GL(n)$ with $\rho(c) = e^{2\pi i m/n} I$ sends $c^N \mapsto I$, hence factors to define a representation $\overline{\rho} : \pi_1(X - x, y)/K \to GL(n)$. We denote by $\overline{\rho}^{op} : \pi \to GL(n)$ the representation well-defined by the formula

$$\overline{\rho}^{op}(\overline{\alpha}) = \rho(\alpha^{-1}) \in GL(n)$$

where $\alpha \mapsto \overline{\alpha}$ under the quotient map $\pi_1(X - x, y) \to \pi_1(X - x, y)/K$. Thus, every representation $\sigma : \pi \to GL(n)$ for which $\sigma(\overline{c}) = e^{2\pi i m/n} I$ is of the form $\sigma = \overline{\rho}^{op}$ for a unique $\rho : \pi_1(X - x, y) \to GL(n)$ with $\rho(c) = e^{2\pi i m/n} I$.

7.7 Local description. Let $\gamma : [0, 1] \to X$ be an injective path with $\gamma(0) = y$, and $\gamma(1) = x$. Then γ has a unique lift $\widehat{\gamma} : [0, 1] \to Y$ with $\widehat{\gamma}(0) = y'$. Let $x' = \widehat{\gamma}(1)$, which is a point in $p^{-1}(x)$. Then the inertia subgroup $\pi_{x'} \subset \pi$ is the cyclic group $\{e, \overline{c}, \ldots, \overline{c}^{N-1}\}$. The inverse image under p of the open disk U in X (that was chosen at the beginning of Section 2) is a disjoint union of open disks in Y indexed by $p^{-1}(x)$, whose centers are the points of $p^{-1}(x)$. Let W be the component of $p^{-1}(U)$ that contains x'. Then W has a complex coordinate w such that the map $p|_W : W \to U$ is given by $w \mapsto z = w^N$, and the right action of $\overline{c} \in \pi_1(X', y)/K$ on W is given by $w * \overline{c} = \zeta w$ where $\zeta = e^{2\pi i/N}$. Therefore left action of $\overline{c} \in \pi$ on W is given by the formula $\overline{c}w = w * \overline{c}$, and so once again $\overline{c}w = \zeta w$. If $\alpha \in Aut(Y/X) - \pi_{x'}$, then $\alpha(W) \cap W = \emptyset$.

7.8 The Grothendieck construction of the bundle $\mathbb{E}(\sigma)$. With notation as in the above lemma, let $\sigma : \pi \to GL(n)$ be any representation. Then the action of π on Y lifts to an action on the trivial bundle $Y \times \mathbb{C}^n$ over Y, where the action on the fibers is via σ. Let $\mathcal{O}^n_Y(\sigma)$ denote this bundle with the given π-action via σ. Let the sheaf

$$\mathbb{E}(\sigma) = p^{\pi}_*(\mathcal{O}^n_Y(\sigma))$$

denote its **invariant direct image on** X. The sections of $\mathbb{E}(\sigma)$ over an open subset $V \subset X$ are all the sections of $\mathcal{O}^n_Y$ over $p^{-1}(V)$ that are invariant under π. Then $\mathbb{E}(\sigma)$ is a coherent, torsion free $\mathcal{O}_X$-module, so it is a vector bundle on X.

7.9 The natural holomorphic connection ∇ on $\mathbb{E}(\sigma)|_{X-x}$. Let $\rho : \pi_1(X - x, y) \to GL(n)$ be a representation with $\rho(c^N) = I$, so that ρ factors via $\overline{\rho} : \pi_1(X - x, y)/K \to GL(n)$. Let $\sigma = \overline{\rho}^{op} : \pi \to GL(n)$. By statements 7.1 and 7.2, over $X - x$ the bundle $\mathbb{E}(\sigma)$ restricts to the vector bundle associated to the principal $\pi_1(X - x, y)/K$-bundle $Y' \to X'$ (with base point y'), which is the vector bundle E_ρ on $X - x$ with a frame y^* over y and a connection ∇_ρ as in Section 2. If $V \subset X'$ is an open disk, then any flat section of E_ρ over V (means a section of the sheaf $\ker(\nabla)$) has the form (ϕ, ξ) where $\phi : V \to Y'$ is a section, and $\xi \in \mathbb{C}^n$.

7.10 Proposition. *Let $\sigma : \pi \to GL(n)$ be such that $\sigma(\overline{c}) = e^{2\pi i s/N} I$ where $0 \leq s < N$, in particular, $\sigma(\overline{c}^N) = 0$. If such a σ exists then the integer s must be such that N divides sn. Let $m = sn/N$, so that $0 \leq m < n$ and $m/n = s/N$. Let $\rho : \pi_1(X - x, y) \to GL(n)$ be the unique representation such that $\sigma = \overline{\rho}^{op} : \pi \to GL(n)$, in particular, $\rho(c) = \sigma(\overline{c}^{-1}) = e^{-2\pi i m/n} I$. Then the Grothendieck bundle $\mathbb{E}(\sigma)$ is isomorphic as a vector bundle on X with the Deligne extension $E(\rho)$ of E_ρ, with residue $(m/n)I$ at x. It has degree $-sn/N = -m$.*

Proof It is enough to show that natural holomorphic connection ∇ on $\mathbb{E}(\sigma)|_{X-x} = E_\rho$ extends to a logarithmic connection on $\mathbb{E}(\sigma)$ with residue $(s/N)I$ at x. Hence $\deg(\mathbb{E}(\sigma)) = -sn/N$, and as this has to be an integer, it is necessary that N divides sn if σ exists. To see the local behaviour around x of the connection ∇, consider the disk U around x with coordinate z, and the quotient map $W \to U : w \mapsto z = w^N$ under the action of $\pi_{\widehat{x}} = \{\overline{c}, \ldots, \overline{c}^N\}$ with $\overline{c}w = e^{2\pi i/N} w$. If e_i is the standard basis of $\mathbb{C}^n$, then each $h_i(w) = (w, w^s e_i)$ is a $\pi_{\widehat{x}}$-invariant holomorphic section of $\mathcal{O}^n_W(\sigma)$. Hence (h_i) is a free basis of holomorphic sections over U for $\mathbb{E}(\sigma) = p^{\pi}_*(\mathcal{O}^n_Y(\sigma))$. Over a small disk $V \subset U' = U - x$, we have a section $U \to W : z \mapsto z^{1/N}$ of $W \to U$, for a chosen branch $z^{1/N}$, and so $(z^{1/N}, e_i)$ defines a flat local section. Note that $h_i = z^{s/N} e_i$. Hence

$$\nabla(h_i) = \nabla(z^{s/N} e_i) = (s/N)\frac{dz}{z} \otimes z^{s/N} e_i = (s/N)\frac{dz}{z} \otimes h_i$$

hence $\mathrm{res}_x(\nabla) = (s/N)I$ as claimed. □

7.11 Conversely, given any $\rho : \pi_1(X - x, y) \to GL(n)$ with $\rho(c) = e^{-2\pi i m/n}$ for some $0 \leq m < n$, choose any $N \geq 1$ and $0 \leq s < N$ such that $s/N = m/n$. One possible choice is $N = n$ and $s = m$, but there are infinitely many other choices of N, s. Let $p : Y \to X$ be the ramified cover as in Lemma 7.3. Let $\sigma = \overline{\rho}^{op} : \pi \to GL(n)$. Note that $\sigma(\overline{c}) = e^{2\pi i m/n} = e^{2\pi i s/N}$. Then the above shows that $E(\rho)$ (the Deligne extension with residue $(m/n)I$) is naturally isomorphic to the Grothendieck bundle $\mathbb{E}(\sigma)$.

8 The cases of genus 0 and genus 1

The existence of Poincaré's ramified covering, and hence the Grothendieck construction which is based on it, needs $g \geq 1$ in general. That is why it was necessary to assume in [15] that $g \geq 1$. The assumption in [15] that $g \geq 2$, which is actually unnecessary for the validity of the main result there (see the argument below), might have come from the Proposition 9.1 in [15], which indeed needs $g \geq 2$ for it to be true.

In contrast, the Deligne construction, and the Theorem 6.1, work for all $g \geq 0$, so our reformulation does not put any restriction on the genus. Our proof supplements Proposition 9.1 in [15] with the implication (2) ⇒ (1) of Lemma 8.1 and the results of Grothendieck and Atiyah, to cover all $g \geq 0$.

When $g = 0$, the group $\pi_1(X - x, y)$ is trivial, and so there are no irreducible representations of rank $n \geq 2$. On the other side, we know from Grothendieck's decomposition theorem that there are no stable vector bundles on X of rank ≥ 2. This reduces the Theorem 6.1 to the case $n = 1$ when $g = 0$.

When $g = 1$, that is, when X is an elliptic curve, Atiyah's theorem [2] implies that there exists a stable vector bundle on X of rank n and degree m if and only if m and n are coprime. To prove the part (2) of Theorem 6.1 when $g = 1$, we just need to know that there exists an irreducible representation $\rho : \pi_1(X - x, y) \to U(n)$ with $\rho(c) = e^{2\pi i m/n} I$ whenever m and n are coprime. Such an example (due to Venkataramana) is given in the proof of Lemma 8.1.

The part (1) of Theorem 6.1, together with Atiyah's above result, then proves the implication (1) ⇒ (2) of the Lemma 8.1 (we will not need this implication in the proof of the Theorem 6.1). The direct elementary proof of (1) ⇒ (2) that is given below is due to Venkataramana.

8.1 Lemma. *Let n and m be integers with $n \geq 1$, and let $\zeta = e^{2\pi i/n} \in \mathbb{C}$. Let F_2 denote the free group on the two generators a, b. Let $c = aba^{-1}b^{-1}$. Then the following two statements are equivalent.*

(1) There exists an irreducible representation $\rho : F_2 \to U(n)$ such that $\rho(c) = \zeta^m I$.

(2) The integers n and m are coprime.

Moreover, regardless of whether m and n are coprime, there exists an indecomposable representation $\rho : F_2 \to GL(n)$ such that $\rho(c) = \zeta^m I$.

Proof (T.N. Venkataramana). (1) ⇒ (2) : Suppose that n and m are not coprime, in particular, $n \geq 2$. As $\zeta^n = 1$, we can replace m by its residue modulo n, so that $0 \leq m < n$. We now separately consider the two cases (a) $m = 0$ and (ii) $1 < m < n$.

(i) If $m = 0$, then the matrices $\rho(a) = A$ and $\rho(b) = B$ commute, so they are simultaneously diagonalizable. Hence ρ cannot be irreducible as $n \geq 2$.

(ii) Suppose that $1 < m < n$, and n and m are not coprime. Let $h \geq 2$ denote the g.c.d. of n and m, so that $n = ph$ and $m = qh$, where $1 < p < n$ and $1 < q < m$. Note that $mp = nq$.

Suppose that $\rho : F_2 \to U(n)$ is an irreducible representation with $AB = \zeta^m BA$ where $A = \rho(a)$, $B = \rho(b)$. Therefore we have

$$AB^pA^{-1} = (ABA^{-1})^p = \zeta^{mp}B^p = B^p$$

showing that B^p commutes with A. As B^p also commutes with B, by irreducibility of ρ we must have $B^p = \lambda I$ for some $\lambda \in \mathbb{C}^\times$.

Note that A and B do not commute as by assumption $1 < m < n$. Hence A has an eigenvalue α with multiplicity $< n$. Let $0 \neq v \in \mathbb{C}^n$ such that $Av = \alpha v$. Let $V \subset \mathbb{C}^n$ be the subspace spanned by the vectors $v, Bv, \ldots, B^{p-1}v$. As $AB = \zeta^m BA$, for any $k \geq 0$ we have

$$AB^kv = \zeta^{mk}B^kAv = \zeta^{mk}\alpha B^kv.$$

Hence V is stable under A. Also, as $B^p = \lambda I$, V is stable under B. Therefore V is stable under ρ, and $1 \leq \dim(V) \leq p < n$. This contradicts irreducibility of ρ.

$(2) \Rightarrow (1)$: Suppose n and m are coprime. Consider the element $A \in U(n)$ defined in terms of the standard basis (e_r) of $\mathbb{C}^n$ by putting $Ae_r = \zeta^{mr}e_r$ for all $1 \leq r \leq n$, and $Be_1 = e_2, \ldots, Be_{n-1} = e_n, Be_n = e_1$. Then A has n distinct eigenvalues, so all eigenspaces are 1-dimensional, and B cyclically permutes the n eigenspaces of A. It follows that ρ is irreducible. Also, $ABA^{-1}B^{-1} = \zeta^m I$ as required. (This is an example of a Heisenberg representation of F_2 on $U(n)$.) Hence (1) holds.

Finally, suppose that n and m are not coprime. As above, let $h \geq 2$ denote the g.c.d. of n and m, so that $n = ph$ and $m = qh$, where $1 < p < n$ and $1 < q < m$. Note that $q/p = m/n$, and p and q are coprime. As above, we have an irreducible unitary representation $\sigma : F_2 \to U(p)$ with $\sigma(a)u_r = e^{2\pi iq/p}u_r$ for all $1 \leq r \leq p$, and $\sigma(b)u_1 = u_2, \ldots, \sigma(b)u_p = u_1$ where (u_j) is the standard basis of $\mathbb{C}^p$, so that $\sigma(c) = e^{2\pi iq/p}I = e^{2\pi im/n}I$ (this is a Heisenberg representation of F_2 on $U(p)$). Now let the (non-unitary) representation $\tau : F_2 \to GL(h)$ be defined as follows. We put $\tau(a)v_1 = v_1, \tau(a)v_2 = v_2 + v_1, \ldots, \tau(a)v_h = v_h + v_{h-1}$ where (v_j) is the standard basis of $\mathbb{C}^h$, and we put $\tau(b) = I \in GL(p)$. Then for the representation $\rho = \sigma \otimes \tau : F_2 \to GL(\mathbb{C}^p \otimes \mathbb{C}^h) = GL(n)$, we have $\rho(c) = e^{2\pi iq/p}I = e^{2\pi im/n}I$. The only idempotent endomorphisms of $\mathbb{C}^p \otimes \mathbb{C}^h$ that commute with both $\rho(a)$ and $\rho(b)$ are the scalar multiples of I, which shows that ρ is indecomposable. □

9 A unitary ρ is irreducible $\Leftrightarrow$ $E(\rho)$ is stable

All the arguments in this section are analogues of the original arguments in [15], appropriately modified for the Deligne construction in place of the Grothendieck construction, which makes them simpler.

9.1 Plurisubharmonic functions

We recall below some classical facts about harmonic and plurisubharmonic functions on open disks in $\mathbb{C}$.

9.1 Lemma. *Let $U \subset \mathbb{C}$ be an open disk with $0 \in U$. Then the following statements hold.*
(1) If f is a holomorphic function on U, then its real and imaginary parts are real harmonic functions.
(2) If φ is a real harmonic function on the punctured open disk $U - \{0\}$ such that φ is bounded near 0, then φ uniquely extends to a harmonic function on U.
(3) If $\varphi_1, \ldots, \varphi_n$ are real harmonic functions on U, then the function $\varphi_1^2 + \cdots + \varphi_n^2$ is plurisubharmonic on U.
(4) If φ is a real plurisubharmonic function on the open disk U that attains its maximum in U, then f is constant.
(5) If $\varphi_1, \ldots, \varphi_n$ are real harmonic functions on U such that $\varphi_1^2 + \cdots + \varphi_n^2$ is constant, then each φ_i is constant.

9.2 Proof of Theorem 6.1.(1)

Let X be a compact Riemann surface, and let $x, y \in X$ with $x \neq y$. Let n, m be any integers, with $n \geq 1$. Recall from Section 2 that $URep(X - x, y, n, e^{2\pi im/n})$ denotes the set of all unitary representations $\rho : \pi_1(X - x, y) \to U(n)$ with local monodromy $\rho(c) = e^{2\pi im/n}I$ around x, and $ULog(X, x, y, n, -m/n)$ denotes the set of all isomorphism classes of framed logarithmic connections (E, ∇, y^*) whose monodromy representation is unitary, and whose residue at x is $-(m/n)I$. By Section 3 we have a bijection $\mathcal{D}_y : URep(X - x, y, n, e^{2\pi im/n}) \to ULog(X, x, y, n, -m/n)$ defined by taking the Deligne extension of the connection (E_ρ, ∇_ρ) on $X - x$ for the given residue $-(m/n)I$, which has the inverse map $\mathcal{M}_y : ULog(X, x, y, n, -m/n) \to URep(X - x, y, n, e^{2\pi im/n})$ given by taking monodromy representations w.r.t. y^*.

In particular, when $m = 0$, the local monodromy $e^{2\pi im/n}I$ is I, and the residue $-(m/n)I$ is 0. Hence for $m = 0$ we have the equalities $URep(X - x, y, n, e^{2\pi im/n}) = URep(X, y, n)$, the set of all unitary representations of

$\pi_1(X, y)$ of rank n, and $ULog(X, x, y, n, -m/n) = UCon(X, y, n)$ the set of all framed holomorphic connections (E, ∇, y^*) of rank n on (X, y).

9.2 Lemma on global sections: degree 0 case. *With notation as above, let* $(E, \nabla, y^*) \in UCon(X, y, n)$, *in particular,* $\deg(E) = 0$.
(a) The inclusion $\ker(\nabla) \hookrightarrow E$ *induces a natural isomorphism* $\Gamma(X, \ker(\nabla)) \to \Gamma(X, E)$.
(b) Composing the above with the natural isomorphism $(\mathbb{C}^n)^\rho \to \Gamma(X, \ker(\nabla))$, *we have a natural isomorphism* $(\mathbb{C}^n)^\rho \to \Gamma(X, E)$.
(c) The image of the evaluation map $ev_y : \Gamma(X, E) \to E_y = \mathbb{C}^n$ *is the invariant subspace* $(\mathbb{C}^n)^\rho \subset \mathbb{C}^n$, *and the resulting map* $ev_y : \Gamma(X, E) \to (\mathbb{C}^n)^\rho$ *is an isomorphism, inverse to the isomorphism in (b) above.*

Proof As E is associated to a unitary representation, the standard Hermitian metric on $\mathbb{C}^n$ (that is preserved by $U(n)$) defines a Hermitian metric on the bundle E. If $s \in \Gamma(W, E)$ a local section over an open subset $W \subset X$, then the norm square $||s||^2$ is a plurisubharmonic function on W. Hence if $s \in \Gamma(X, E)$ is a global section then $||s||^2$ is a global plurisubharmonic function on X, so it is constant by Lemma 9.1 and the compactness of X. This shows by Lemma 9.1 that s has constant coefficients w.r.t. any local basis of the unitary local system $\ker(\nabla)$, so s is a section of $\ker(\nabla)$. Hence the inclusion $\Gamma(X, \ker(\nabla)) \to \Gamma(X, E)$ is also surjective. This completes the proof of (a). Now (b) and (c) are clear. □

9.3 Lemma on global sections: negative degree case. *Let the unitary representation* $\rho \in URep(X - x, y, n, e^{2\pi i m/n})$ *be the monodromy representation of a framed logarithmic connection* $(E, \nabla, y^*) \in ULog(X, x, y, n, -m/n)$, *in particular,* $\deg(E) = m$. *Then we have the following: If* $m < 0$ *then* $\Gamma(X, E) = 0$.

Proof As $E|_{X-x}$ is associated to the unitary representation ρ, which preserves the standard Hermitian inner product on $\mathbb{C}^n$, it follows that $E|_{X-x}$ has a natural hermitian metric over $X - x$. The integrable sections of $E|_{X-x}$ (means local sections of the sheaf $\ker(\nabla|_{X-x})$) have constant norms w.r.t. this metric. By construction of (E, ∇, y^*) as a Deligne extension of $(E, \nabla, y^*)|_{X-x}$ with residue $-(m/n)I$ at x, we have a trivialization $E|_U \overset{\sim}{\to} \mathcal{O}_U^n$ under which y^* corresponds to the frame $(e_{i,y})$ of $\mathcal{O}_U^n$ over y, where (e_i) is the standard basis of $\mathcal{O}_U^n$, and $\nabla|_{U-x}$ has the logarithmic extension $\nabla_U : \mathcal{O}_U^n \to \Omega_U^1(\log x) \otimes \mathcal{O}_U^n$ for which

$$\nabla_U(e_i) = -(m/n)\frac{dz}{z} \otimes e_i.$$

Hence $s_i(z) = z^{m/n} e_i = e^{m/n \log z} e_i$ are (multivalued) integrable sections of $\mathcal{O}_{U-x}^n$, taking the values $e_{i,y}$ over y for the branch of $\log z$ on which $\log 1 = 0$. Hence at any $z \neq 0$, we have $|z|^{m/n}||e_i(z)|| = ||s_i(z)|| = ||e_i(y)|| = 1$, which implies that for $z \neq 0$, we have

$$||e_i(z)|| = |z|^{-m/n}.$$

Now suppose that $\sigma \in \Gamma(X, E)$. Then the pointwise Hermitian norm $||\sigma||^2$ is a plurisubharmonic function on $X - x$. Over U, have $\sigma|_U = \sum f_j e_j$ for some holomorphic functions f_j on U. Hence on $U - x$, we have

$$||\sigma(z)||^2 = (\textstyle\sum_j |f_j|^2)|z|^{-2m/n}.$$

We have assumed that $m < 0$. Hence $\lim_{z \to 0} |z|^{-2m/n} = 0$, and so

$$\lim_{z \to 0} ||\sigma(z)||^2 = 0.$$

Hence by Lemma 9.1, $||\sigma||^2$ uniquely extends to a plurisubharmonic function on U, which takes the value 0 at $z = 0$. This shows that $||\sigma||^2$ extends from $X - x$ to X as a plurisubharmonic function F on X with $F(x) = 0$. As X is compact, it must attain its maximum. Hence again by Lemma 9.1, F is identically 0. Hence $\sigma = 0$. □

9.4 Corollary. *Let $(E, \nabla, y^*) \in UL(X, x, y, n, -m/n)$ be a logarithmic connection on (X, x) with a frame y^* for E_y over y, such that $res_x(\nabla) = -(m/n)I$ (hence with local monodromy $e^{2\pi im/n}I$ around x) and such that the monodromy representation ρ for $(E, \nabla, y^*)|_{X-x}$ is unitary. Let $L \subset E$ be any holomorphic line subbundle. Then $deg(L) \le m/n = \mu(E)$.*

Proof Let $p = \deg(L)$, so $\deg(L^*) = -p$ for the dual line bundle L^*. By statement 4.5 above, L^* has a logarithmic connection ∇' over (X, x) such that the monodromy of (L^*, ∇') is unitary, the local monodromy around x is 1, and the residue at x is p. Hence the logarithmic connections ∇ and ∇' induce a logarithmic connection ∇'' on $E \otimes L^*$ whose monodromy is again unitary, whose local monodromy around x is $e^{2\pi im/n}I$ and whose residue at x is $(-m/n + p)I$. We have a subbundle $\mathcal{O}_X \subset E \otimes L^*$, so $E \otimes L^*$ has a nowhere vanishing section. Hence by Lemma 9.3, we must have $m/n - p \ge 0$, which proves the lemma. □

9.5 Proposition : Semistability. *Let $(E, \nabla, y^*) \in UL(X, x, y, n, -m/n)$ be a logarithmic connection on (X, x) with a frame y^* for E_y over y, such that* $\mathrm{res}_x(\nabla) = -(m/n)I$ *(hence with local monodromy $e^{2\pi im/n}I$ around x) and such that the monodromy representation ρ for $(E, \nabla, y^*)|_{X-x}$ is unitary. Let $F \subset E$ be any nonzero holomorphic vector subbundle. Then* $\deg(F)/\operatorname{rank}(F) \le \deg(E)/\operatorname{rank}(E)$. *In other words, the underlying bundle E is a semistable vector bundle on X.*

Proof Let $p = \deg(F)$ and $q = \operatorname{rank}(F)$. Then $L = \wedge^q F$ is a line subbundle of $E' = \wedge^q E$ of degree p. The bundle $\wedge^q E$ has rank $r = \binom{n}{q}$ and degree d given by

$$d = \deg(\wedge^q E) = \deg(E)\begin{pmatrix}\operatorname{rank}(E) - 1\\ q-1\end{pmatrix} = m\begin{pmatrix}n-1\\ q-1\end{pmatrix}.$$

The logarithmic connection ∇ on E induces a logarithmic connection ∇' on E', with

$$\mathrm{res}_x(E', \nabla') = (-d/r)I$$

(and therefore with local monodromy $e^{2\pi d/r}I$). The Corollary 9.4 applied to $L \subset E'$ implies that $p \le d/r$, that is,

$$p \le m\begin{pmatrix}n-1\\ q-1\end{pmatrix}/\begin{pmatrix}n\\ q\end{pmatrix} = mq/n.$$

This means $p/q \le m/n$, as desired. □

9.6 The idea of Ramanan. Let V be a vector space and let $W \subset V$ be a finite dimensional vector subspace. Consider the 1-dimensional subspace $\wedge^d W \subset \wedge^d V$ where $d = \dim(W)$. Let $A : V \to V$ be any linear automorphism such that under the induced map $\wedge^d A : \wedge^d V \to \wedge^d V$, we have $(\wedge^d A)(\wedge^d W) = \wedge^d W \subset \wedge^d V$. Then it can be seen that $A(W) = W \subset V$. Soon after the publication of [15], Ramanan used the above observation to significantly simplify the proof given in [15] that the bundles arising from irreducible unitary representations are stable, as it allows the bypassing of the inductive argument in [15] which uses the full theorem for lower ranks. We use Ramanan's idea in the proof of (2) ⇒ (1) below.

9.7 Proposition : Stability. *Let $(E, \nabla, y^*) \in UL(X, x, y, n, -m/n)$ be a logarithmic connection on (X, x) with a frame y^* for E_y over y, such that* $\mathrm{res}_x(\nabla) = -(m/n)I$ *(hence with local monodromy $e^{2\pi im/n}I$ around x) and such that the monodromy representation ρ for $(E, \nabla, y^*)|_{X-x}$ is unitary. Then the following two statements are equivalent.*
(1) The vector bundle E is stable.
(2) The representation ρ is irreducible.

Proof (1) $\Rightarrow$ (2) : This argument is essentially the same as the corresponding argument in [15]. As any unitary representation ρ decomposes into direct sum of irreducible unitary representations ρ_i's, the framed logarithmic connection (E, ∇, y^*) is the direct sum of the corresponding triples (E_i, ∇, y_i^*). Note that $\mathrm{res}_x(\nabla)$ is then the direct sum of $\mathrm{res}_x(\nabla_i)$, so $\mathrm{res}_x(\nabla_i) = (-m/n)I_{n_i}$ where $n_i = \mathrm{rank}(\rho_i) = \mathrm{rank}(E_i)$. A stable vector bundle is not the direct sum of two nonzero bundles. Hence if E_ρ is stable then ρ must be irreducible.

(2) $\Rightarrow$ (1) : We will show that if (1) is false then (2) is false. We already know from Proposition 9.7 that E is semistable. So if it is not stable, then there would exists a subbundle $0 \neq F \neq E$ of E with the same slope $p/q = m/n$ as E where $p = \deg(F)$, $q = \mathrm{rank}(F)$. Then $\wedge^q F \subset \wedge^q E$ is a line subbundle of degree p, hence as proved in Section 4, there exists a unitary representation $\theta : \pi_1(X - x, y) \to U(1)$ with trivial local monodromy $1 = e^{2\pi i p}$ (means a unitary representation $\theta : \pi_1(X, y) \to U(1)$) such that the line bundle $L = \wedge^q F$ admits a logarithmic connection ∇_L on (X, x) with monodromy θ and residue $-p$. The inclusion $L \hookrightarrow \wedge^q E$ defines a nowhere vanishing global section $\sigma \in \Gamma(X, (\wedge^q E) \otimes L^*)$. The value σ_y of σ at y corresponds to the inclusion map $L_y = \wedge^q F_y \hookrightarrow \wedge^q E_y$.

We have a framed logarithmic triple $(\wedge^q E \otimes L^*, \nabla', y^*)$ where ∇' is the logarithmic connection on $E \otimes L^*$ induced by ∇ and ∇_L. A simple calculation shows that $\mathrm{res}_x(\nabla') = 0$, so ∇' is actually a holomorphic connection. By the Lemma 9.2, $\sigma \in \Gamma(X, \ker(\nabla')) = \Gamma(X, (\wedge^q E) \otimes L^*)$. Hence by Lemma 9.2, σ_y is invariant under the monodromy representation $\eta = (\wedge^q \rho) \otimes \theta^*$ of ∇', that is, the inclusion $\wedge^q F_y \hookrightarrow \wedge^q E_y$ is a $\pi_1(X - x, y)$-equivariant map. This means the image of σ_y, which equals $\wedge^q F_y \subset \wedge^q E$, is invariant under ρ. Hence by 9.6 we see that F_y is a ρ-invariant subspace of E_y. By assumption $0 \neq F_y \neq E_y$, hence ρ is not irreducible, a contradiction. □

10 All stable vector bundles arise as $E(\rho)$

This section follows the Poincaré continuity method of [15], with some simplification which becomes possible because we use the Deligne construction for the map $\rho \mapsto E(\rho)$ instead of the Grothendieck construction of the original, and another conceptual simplification (well-known since 1970s) made possible by the Mumford-Seshadri construction of a complex projective moduli variety for semistable vector bundles. We sketch these simplifications. Modulo these, the proof in [15] works verbatim for us, so we do not repeat the details that are common.

10.1 Table comparing notations.

For us	In [15]
$UR(X-x, y, n, e^{2\pi i m/n})$	$U(n, \tau, n)$
$UR^{irr}(X-x, y, n, e^{2\pi i m/n})$	$U_0(n, \tau, n)$
$UR^{irr}(X-x, n, e^{2\pi i m/n})$	$M(n, \tau, n)$
$E(\rho)$	$p_*^\pi(E_\pi(\rho))$
$M^s(X, n, m)$	$S_s(n, m)$
$M^{ss}(X, n, m)$	(not known in 1965)

10.2 Spaces of representations. In particular as shown in [14] and [15], the set of representations $UR(X - x, y, n, e^{2\pi i m/n})$ is a closed real analytic submanifold of $U(n)^{2g}$, and so it is compact. Moreover, it is a Zariski closed real analytic submanifold of the holomorphic manifold $Rep(X - x, y, n, e^{2\pi i m/n})$, which is a Zariski closed holomorphic submanifold of $GL(n)^{2g}$.

10.3 Nonemptyness of the spaces of representations. The Proposition 9.1 of [15] shows that $UR^{irr}(X - x, n, e^{2\pi i m/n}) \neq \emptyset$ for all m, n with $n \geq 1$. We showed in Section 8 that $UR^{irr}(X - x, n, e^{2\pi i m/n}) \neq \emptyset$ for $g = 1$ whenever n and m are coprime. Recall that by [2], stable bundles exist on an elliptic curve only in this case. For $g = 0$, $\pi(X - x) = \{e\}$, and the only case of interest is $n = 1$, where the trivial 1-dimensional representation is irreducible. Therefore our proof of the Theorem 6.1 works uniformly for all $g \geq 0$ and $n \geq 1$, as moreover the

Deligne construction $\rho \mapsto E(\rho)$ works for all $g \geq 0$, unlike the Grothendieck construction in [15] that needs $g \geq 1$ except in special cases.

10.4 Openness of stability. In [15], the openness of the stable locus in the parameter space of any family of vector bundles on a curve of $g \geq 2$ was proved in [15] by induction on rank, using the Narasimhan-Seshadri theorem for lower ranks. A few years later, Narasimhan and Ramanathan [13] used the Quot scheme method for proving the openness of the stable and of the semistable locus for families of vector bundles on projective varieties of any dimension. This new method soon became standard, and it also applies widely to families of decorated sheaves of various kinds.

10.5 Moduli spaces of vector bundles. Following Mumford [10] and Seshadri [18], there exists a good moduli variety $M^{ss}(X, n, m)$ of semistable vector bundles on X of rank n and degree m, and it is a projective variety. Its points correspond to S-equivalence classes of semistable bundles, as defined in [18]. This is the same as the set of isomorphism classes of polystable bundles (means direct sums of stable bundles). As shown by Narasimhan and Ramanan in [12], the moduli variety $M^s(X, n, m)$ of stable bundles is a Zariski open dense subvariety of the smooth locus in $M^{ss}(X, n, m)$ (in fact, Narasimhan and Ramanan [12] show that $M^s(X, n, m)$ is equal to the smooth locus of $M^{ss}(X, n, m)$ in all but a few special low rank, low genus cases). The moduli variety $M^{ss}(X, n, m)$ enjoys the following functorial property. If T is a real analytic manifold, let $\mathcal{A}_{X\times T}$ be the sheaf of germs of complex valued functions f on $X \times T$ such that f is real analytic, and for each $t \in T$, the restriction $f|_{X\times t}$ is a germ of a holomorphic function on $X \times t$. Let E_T be a locally free $\mathcal{A}_{X\times T}$-module of rank n on $X \times T$. Alternatively, we can take E_T to be a vector bundle of rank n on $X \times T$ defined by an open cover of $X \times T$ with transition functions coming from $\mathcal{A}_{X\times T}$. Suppose that each restriction $E_t = (E_T)|_{X\times t}$ is semistable of degree m. Then the induced map of sets $T \to M^{ss}(X, n, m)$ that sends $t \mapsto [E_t]$ (the S-equivalence class of E_t) is a real analytic map. If moreover T is a holomorphic manifold and E is a holomorphic bundle, then the map $T \to M^{ss}(X, n, m) : t \mapsto [E_t]$ is holomorphic. Often in the algebraic geometry literature, this property of the moduli space is stated just for algebraic families, but it holds for even continuous families. This ultimately follows from the following fact: the universal property of a complex Grassmannian also holds for continuous, C^∞, real analytic, holomorphic or algebraic families of subbundles of a trivial bundle. From this elementary fact, it can be concluded by following the construction of Quot schemes and GIT quotients that the classifying map $T \to M^{ss}(X, n, m) : t \mapsto [E_t]$ of a ‘nice’ family of semistable vector bundles on X is a ‘nice’ map of spaces, where the adjective ‘nice’ can mean any one of ‘algebraic’, ‘holomorphic’, ‘real analytic’, ‘C^∞’ or just plain ‘continuous’.

10.6 Holomorphicity of the Deligne construction in families. We have a bijection of sets $\mathcal{D}_y : Con(X - x, y, n, e^{2\pi i m/n}) \to Log(X, x, y, n, -m/n)$ defined the Deligne construction, with inverse bijection given by the restriction map $\mathcal{R}_y : Log(X, x, y, n, -m/n) \to Con(X - x, y, n, e^{2\pi i m/n})$, as seen in Section 3. We claim that by applying the Deligne construction pointwise on the parameter space, a holomorphic family of relative connections (E, ∇) on X parameterized by a holomorphic manifold T gives rise to a holomorphic family of relative logarithmic connections $\mathcal{D}_y(E, \nabla)$ on X parameterized by T (see [16] for the relevant definitions). This being a local statement over T, we can assume that T is a polydisk in $\mathbb{C}^r$. We can choose an open cover (U_i) of X such that each U_i and each $U_{ij} = U_i \cap U_j$ are biholomorphic to open disks. As the relative connection ∇ is flat on the trivial bundle $E|_{U_i\times T}$ over $U_i \times T$, we can choose corresponding trivializations by flat basic sections. Let $U_0 = U$ be the open neighbourhood of $x \in X$ chosen at the beginning of Section 2. Over $U_0 \times t$, the logarithmic extension of ∇_t is described by $\nabla(e_i) = (-m/n)(dz/z) \otimes e_i$ for the standard basis e_i of $\mathbb{C}^n$. This is independent of the point $t \in T$. It follows that the family of Deligne extensions will be described by holomorphic transition functions and holomorphic connection coefficients, so it is a holomorphic family of logarithmic connections on X parameterized by T.

We apply the above to the tautological family of framed holomorphic connections $(\mathbf{E}, \nabla, y^*)$ on $X - x$ parameterized by $T = Con(X - x, y, n, e^{2\pi i m/n})$. We denote by $\rho : \pi_1(X - x, y) \times T \to GL(n)$ the monodromy of this family. Hence the resulting family $\mathbf{E}(\rho)$ of vector bundles on X parameterized by T, which is the family of underlying bundles of the Deligne extension of the family $(\mathbf{E}, \nabla, y^*)$, is a holomorphic family of vector bundles parameterized by T.

10.7 The real analytic proper morphisms $\Phi^{ss} : UR(X-x, y, n, e^{2\pi im/n}) \to M^{ss}(X, n, m)$ and $\Phi^s : UR^{irr}(X-x, y, n, e^{2\pi im/n}) \to M^s(X, n, m)$: By Proposition 9.5, the above holomorphic family of vector bundles on X parameterized by $Con(X-x, y, n, e^{2\pi im/n})$, when restricted to the closed real analytic submanifold $UR(X-x, y, n, e^{2\pi im/n}) \cong UC(X-x, y, n, e^{2\pi im/n})$, is a real analytic family of semistable vector bundles on X of rank n, degree m. Hence its classifying map is a real analytic morphisms $\Phi^{ss} : UR(X-x, y, n, e^{2\pi im/n}) \to M^{ss}(X, n, m)$ to the moduli space $M^{ss}(X, n, m)$ of semistable vector bundles. By Proposition 9.7, the inverse image under Φ^{ss} of the open subset $M^s(X, n, m)$ (moduli of stable bundles) is the open subset $UR^{irr}(X-x, y, n, e^{2\pi im/n})$ of $UR(X-x, y, n, e^{2\pi im/n})$. As $UR(X-x, y, n, e^{2\pi im/n})$ is compact and as $M^{ss}(X, n, m)$ is hausdorff, the map Φ^{ss}, and therefore its base-change $\Phi^s : UR^{irr}(X-x, y, n, e^{2\pi im/n}) \to M^s(X, n, m)$, are proper maps of manifolds (separated and universally closed).

10.8 Connectedness of $M^s(X, n, m)$. This was proved as Lemma 12.3 in [15].

10.9 Φ^s is open. It is proved in [15] that Φ^s is a tangent level surjection hence open. But instead of deducing from this the openness of the induced map $UR^{irr}(X-x, n, e^{2\pi im/n}) \to M^s(X, n, m)$ where $UR^{irr}(X-x, n, e^{2\pi im/n})$ is the quotient manifold of $UR^{irr}(X-x, y, n, e^{2\pi im/n})$ by the conjugate action of $U(n)$, they apply the invariance of domain theorem, leaving out the differential argument. In a conversation with me fifty years later, Narasimhan ascribed it to their youthful enthusiasm for Brouwer's theorem on the invariance of domain. (However, one of the anonymous referees of the present article commented that there is more to this use of invariance of domain, which is related to the actual history of Brouwer's attempt to fix a gap in Poincaré's proof of the uniformization theorem. I am not sufficiently knowledgeable of mathematical history to say anything more on this.)

10.10 Surjectivity of $\Phi^s : UR^{irr}(X-x, y, n, e^{2\pi im/n}) \to M^s(X, n, m)$: The map Φ^s is continuous, proper, open, its domain is nonempty whenever the codomain is nonempty, and the codomain is connected. Hence Φ^s is surjective. The map Φ^s factors via $UR^{irr}(X-x, n, e^{2\pi im/n})$, and the resulting map is bijective and tangent level isomorphism, so a real analytic isomorphism. This completes the sketch of the proofs of (2) and (3) in the Main Theorem.

10.11 Generalizations of the logarithmic reformulation. The Grothendieck construction of bundles as invariant direct images from representations of the Galois groups of Poincaré's ramified covers has also found use by Ramanathan [17] for principal bundles with reductive structure groups, and by Seshadri and by Mehta and Seshadri (see [19] and [9]) for parabolic bundles. It is possible to apply a similar logarithmic reformulation to these cases. This will be given elsewhere.

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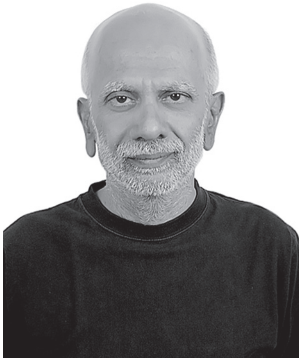

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