



On counting totally imaginary number fields

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Abstract

A number field is said to be a CM-number field if it is a totally imaginary quadratic extension of a totally real number field. We define a totally imaginary number field to be of **CM**-type if it contains a CM-subfield, and of **TR**-type if it does not contain a CM-subfield. For quartic totally imaginary number fields when ordered by discriminant, we show that about 69.95% are of **TR**-type and about 33.05 % are of **CM**-type. For a sextic totally imaginary number field we classify its type in terms of its Galois group and possibly some additional information about the location of complex conjugation in the Galois group.

Mathematics Subject Classification 11R45 · 11R16 · 11R21 · 11R32

1 Introduction

The past few decades have seen a huge amount of work on counting number fields; see, for example, [1–3, 6, 7]; these represent only a tiny sampling of a huge and growing body of related literature which the interested reader can search on MathSciNet. In this article, we introduce a new aspect to the general problem of counting number fields by restricting our attention to totally imaginary number fields and then searching for those which contain a CM subfield. This dichotomy of whether a totally imaginary number field contains a CM subfield or not plays an important role in the arithmetic aspects of automorphic forms. See, for example, [12], where this dichotomy has a subtle influence on reciprocity laws satisfied by the special values of L -functions. One can ask the naïve question: suppose we are given a totally imaginary number field at random, then what is the probability that it contains a CM-subfield? We answer this completely for quartic number fields, and make some modest progress for sextic fields.

A number field, by which we mean a finite extension of \mathbb{Q} , is said to be a CM-number field if it is a totally imaginary quadratic extension of a totally real number field. A totally imaginary number field is said to be of **CM**-type if it contains a CM-subfield, and is said to be of **TR**-type if it does not contain a CM-subfield. One can organize this definition in other ways. Let F be a totally imaginary number field; since the compositum of two totally real fields is again totally real, F contains a maximal totally real subfield, say, F_0 . There is at most one totally imaginary quadratic extension F_1 of F_0 contained inside F ; if such a CM field F_1 exists, then F is of **CM**-type and F_1 is in fact the maximal CM subfield of F ; otherwise, we say F is of **TR**-type, and we put $F_1 = F_0$. Let $n \geq 2$ be an even positive integer. To count totally imaginary number fields F of prescribed degree $n = [F : \mathbb{Q}]$ and bounded discriminant $\delta_F \leq X$, define $N_n^{\mathbf{TI}}(X)$ to be the number of isomorphism classes of totally imaginary number fields F of degree $[F : \mathbb{Q}] = n$, with absolute discriminant $\delta_F \leq X$. To count those totally imaginary fields of **CM**-type, define $N_n^{\mathbf{CM}}(X)$ to be the number of isomorphism classes of totally imaginary number fields

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F of **CM**-type of degree n and $\delta_F \leq X$. We would like to know the asymptotics of the ratio

$$R_n^{\mathbf{CM}}(X) := \frac{N_n^{\mathbf{CM}}(X)}{N_n^{\mathbf{TI}}(X)}, \quad \text{as } X \rightarrow \infty.$$

The question is trivial for $n = 2$ since an imaginary quadratic field is a CM-field; $R_2^{\mathbf{CM}}(X) = 1$. Henceforth, assume that $n \geq 4$.

For quartic extensions ($n = 4$), in Theorem 3.1 we show that $\lim_{X \rightarrow \infty} R_4^{\mathbf{CM}}(X) \approx 0.33052$. The proof uses the results of Bhargava [2] and Cohen–Diaz Y Diaz–Olivier [3].

For sextic extensions ($n = 6$), in Theorem 4.1, we classify the type of a totally imaginary extension in terms of its Galois group and possibly some additional data. If F is a sextic totally imaginary extension, and F_s is its Galois closure, then the Galois group of F_s/\mathbb{Q} is a transitive subgroup of S_6 containing a $(2, 2, 2)$ cycle. There are 11 possible such subgroups of S_6 (up to conjugacy), and for each one of them we can classify the type of F . In particular, we show that if G is S_4 , S_5 , or S_6 then F is of **TR**-type; if G is C_6 , S_3 , D_6 , $S_3 \times C_3$, $A_4 \times C_2$, $S_3 \times S_3$, or $C_3^2 \rtimes D_4$, then F is of **CM**-type, and furthermore in each case one can decide if F itself is a CM field or whether it only contains a proper CM-subfield in terms of the Galois group and its internal structure. The case of D_6 offers some piquant features. The case $G = S_4 \times C_2$ is more complicated than other cases because there are totally imaginary fields of either type with this Galois group; identifying an order 2 element of G that represents complex conjugation serves to distinguish between the two types. Finally, within the framework of currently available results on counting number fields we are unable to predict a precise value for $\lim_{X \rightarrow \infty} R_6^{\mathbf{CM}}(X)$. Even assuming Malle's conjecture (see [Conj. 1.1][10]) on counting number fields it seems difficult to predict a value because of examples like the $S_4 \times C_2$ case, and also because we do not apparently know—even conjecturally—the constants of proportionality denoted $c(k, G)$ for $k = \mathbb{Q}$ in [10].

2 Preliminaries

2.1 Notations

The cyclic group of order n is denoted C_n . The Klein 4-group $C_2 \times C_2$ is denoted V_4 . The dihedral group of order $2n$ is $D_n = \{1, r, r^2, \dots, r^{n-1}\} \rtimes \{1, s\}$, with $r^n = s^2 = 1$ and $sr = r^{-1}s$. The permutation group on n letters, usually taken to be $\{1, 2, \dots, n\}$, is denoted S_n ; and A_n denotes the alternating group—the subgroup of S_n of all even permutation.

2.2 Examples

A number field will always mean a finite extension of \mathbb{Q} , and by its Galois group one means the Galois group of its normal closure. There are several families of examples of totally imaginary number fields of the two types as will become clear throughout the paper. We found it especially helpful to search the LMFDB (the L -functions and modular forms database [9]) for number fields with prescribed constraints, such as degree, signature, Galois group, and whether or not it is a CM field. For example, we can set the degree to be 4, the signature as $[0, 2]$, the Galois group as S_4 or also by its label 4T5 (the fifth group amongst the transitive subgroups of S_4) and then upon hitting search we get a list ordered by discriminant of quartic totally imaginary number fields with Galois group S_4 —the first item in this list is the number field labelled as 4.0.229.1; the parameters in the label stands for degree, number of real embeddings, absolute discriminant, and the last one is item number for all the number fields with the given first three parameters. As another example, if we search for number fields with degree 6, signature $[0, 3]$, and Galois group A_6 (label 6T15), then we get ‘no matches’. Indeed, if F is a number field of signature $[r_1, r_2]$,

i.e., r_1 real embeddings and r_2 pairs of complex embeddings, and suppose r_2 is odd then A_n cannot be the Galois group of F , where $n = r_1 + 2r_2$; see Lem. 4.1.

3 Quartic extensions

In this section, we study the case of quartic number fields; $n = 4$.

A number field is said to be primitive if it admits no intermediate subfields. Clearly, a totally imaginary primitive number field is of **TR**-type. The number field labelled 4.0.229.1 in LMFDB mentioned above is an example of a totally imaginary quartic primitive field and hence of **TR**-type. Given a totally imaginary quartic extension F/\mathbb{Q} , it is either primitive or contains a quadratic subfield. If F contains a real quadratic extension, then F is a **CM**-field; and if F contains an imaginary quadratic extension—which is necessarily a **CM**-field then F is of **CM**-type. Hence, if F is not primitive then it is of **CM**-type; whence:

$$F \text{ is of } \mathbf{TR} \text{ - type} \iff F \text{ is primitive.}$$

Let F be a quartic totally imaginary number field and F_s its Galois closure. Then the Galois group $G = \text{Gal}(F_s/\mathbb{Q})$ is a transitive subgroup of S_4 via its action on the set of 4 roots of an irreducible quartic polynomial of a primitive element generating F . Up to isomorphism, G is one of C_4 , V_4 , D_4 , A_4 , or S_4 . Let us consider each case. If G is C_4 or V_4 then it is not primitive and hence of **CM**-type. If G is D_4 then $H = \text{Gal}(F_s/F)$ is of order 2, and any order 2 subgroup of D_4 is contained in some subgroup, say K , of order 4 of D_4 ; then F_s^K is a quadratic subfield of F ; whence F is of **CM**-type. If G is A_4 (resp., S_4) then $H = \text{Gal}(F_s/F)$ corresponds to a subgroup of order 3 (resp., 6); such a subgroup is maximal; hence by Galois theory F is primitive. To summarize, we have shown

$$F \text{ is primitive} \iff \text{Gal}(F_s/\mathbb{Q}) \text{ is } S_4 \text{ or } A_4.$$

The argument above does not rely on the fact that F is totally imaginary and but only employs Galois theory and elementary group theory.

Thus, to count quartic number fields of **TR**-type (which is equivalent to counting **CM**-type), we need to know the asymptotics for totally imaginary quartic fields with Galois group S_4 and A_4 . Towards this, let us recall some results from the literature on such asymptotics; in the results quoted below, number fields are ordered by discriminant.

- (1) Bhargava [2] showed that approximately 17.111% of quartic fields have Galois group D_4 , and the remaining 82.889% of quartic fields have Galois group S_4 ; in particular, the contribution from the C_4 , V_4 , and A_4 quartic fields are negligible in comparison to the S_4 and D_4 quartic fields.
- (2) Bhargava also proved that asymptotically 30% of the S_4 -quartic fields are totally imaginary; see Thm. 1 of *loc. cit.*
- (3) Cohen–Diaz Y Diaz–Olivier [Prop. 6.2 [3]] proved that asymptotically 71.747% of the D_4 -quartic fields are totally imaginary.

From these results one can deduce the first main theorem of this article:

Theorem 3.1 *When ordered by absolute discriminant, asymptotically, about 66.948% of the totally imaginary quartic fields are of **TR**-type, and the remaining 33.052% of the totally imaginary quartic fields are of **CM**-type, i.e., $\lim_{X \rightarrow \infty} R_4^{CM}(X) \approx 0.33052$.*

Proof Using the results quoted above, the proof is a simple exercise in Bayes’ theorem. Within the sample space of all quartic fields ordered by discriminant (taken to be bounded by X and then letting X go to infinity to get the

required asymptotics), let D stand for the subset of D_4 extensions, S stand for the subset of S_4 extensions, and T stand for the subset of totally imaginary extensions. The results quoted can be rephrased as:

- (1) $P(D) = 0.17111$ and $P(S) = 0.82889$;
- (2) $P(T|S) = 0.3$;
- (3) $P(T|D) = 0.71747$.

We compute then that

$$P(S|T) = \frac{P(T|S) \cdot P(S)}{P(T|S) \cdot P(S) + P(T|D) \cdot P(D)} = \frac{0.30 \times 0.82889}{0.30 \times 0.82889 + 0.71747 \times 0.17111} = 0.66948,$$

i.e., of all totally imaginary quartic fields, about 66.948% have Galois group S_4 and hence of **TR**-type and the remaining 33.052% have Galois group D_4 and hence of **CM**-type. \square

4 Sextic totally imaginary extensions

Before we can state our main theorem for sextic totally imaginary extensions, we need to collect some preliminaries on 6-transitive groups.

4.1 6-transitive groups

Let F be a sextic totally imaginary number field and F_s its Galois closure. Then the Galois group $G = \text{Gal}(F_s/\mathbb{Q})$ is a transitive subgroup of S_6 via its action on the set of 6 roots of an irreducible quartic polynomial of a primitive element generating F . Up to isomorphism, there are 16 possible 6-transitive subgroups of S_6 . However, the six roots being pairwise conjugate, there is an element of G of $(2, 2, 2)$ -cycle decomposition as an element of S_6 , in particular G is not a subgroup of A_6 ; this cuts down the set of possible 6-transitive groups for G to be one of the following eleven:

- (1) (6T1) C_6
- (2) (6T2) S_3
- (3) (6T3) D_6
- (4) (6T5) $S_3 \times C_3$
- (5) (6T6) $A_4 \times C_2$
- (6) (6T8) S_4
- (7) (6T9) $S_3 \times S_3$
- (8) (6T11) $S_4 \times C_2$
- (9) (6T13) $C_3^2 \rtimes D_4$
- (10) (6T14) $S_5 \cong PGL_2(\mathbb{F}_5)$
- (11) (6T16) S_6

The above list can be deduced from the corresponding table in Dummit [App. A [6]]. The label (6Tn) refers to the label of the group as in the database of 6-transitive groups in Magma or GAP.

4.2 A digression on possible Galois groups

That A_6 is not the Galois group of a sextic totally imaginary number field is a special case of the following more general lemma:

Lemma 4.1 *Let F be a number field of signature (r_1, r_2) and degree $n = [F : \mathbb{Q}] = r_1 + 2r_2$. Let $G = \text{Gal}(F_s/\mathbb{Q})$ be the Galois group of a Galois closure F_s of F . If r_2 is odd, then $G \not\cong A_n$.*

Proof The number field F is defined by an irreducible polynomial $f(x) \in \mathbb{Q}[x]$. The polynomial discriminant, $\text{disc}(f)$, is well-defined up to non-zero squares. The discriminant root field of the extension $\mathbb{Q}(\sqrt{\text{disc}(f)})$ is well-defined. The Galois group G is a subgroup of S_n , well-defined up to conjugation. The discriminant root field is the fixed field of $\text{Gal}(F_s/\mathbb{Q}) \cap A_n$. Note that $\text{sign}(\delta_F) = (-1)^{r_2}$ and $\text{sign}(\delta_F) = \text{sign}(\text{disc}(f))$. If r_2 is odd, then $\text{sign}(\text{disc}(f)) = -1$. Thus, $\mathbb{Q}(\sqrt{\text{disc}(f)})$ is an imaginary quadratic extension of \mathbb{Q} . Suppose $\text{Gal}(F_s/\mathbb{Q}) = A_n$, then by Galois theory $\text{Gal}(F_s/\mathbb{Q}(\sqrt{\text{disc}(f)})) \cong \text{Gal}(F_s/\mathbb{Q}) \cap A_n = \text{Gal}(F_s/\mathbb{Q})$, which implies $\mathbb{Q}(\sqrt{\text{disc}(f)}) = \mathbb{Q}$, giving us a contradiction. \square

In particular, if F is a totally imaginary number field of degree n , and $n/2$ is odd, then $\text{Gal}(F_s/\mathbb{Q}) \not\cong A_n$. This applies to our case of $n = 6$. This ends the digression.

4.3 The main result on sextic totally imaginary number fields

Theorem 4.1 *Let F be a sextic totally imaginary number field, F_s its Galois closure, and $G = \text{Gal}(F_s/\mathbb{Q})$ its Galois group.*

- (1) *If G is S_4, S_5 , or S_6 then F is of **TR**-type.*
- (2) *If G is $C_6, S_3, D_6, S_3 \times C_3, A_4 \times C_2, S_3 \times S_3$, or $C_3^2 \rtimes D_4$ then F is of **CM**-type.*
- (3) *If $G = S_4 \times C_2$ then F can be of either type; one can distinguish between the two types by identifying an order 2 element of G that represents complex conjugation.*

4.4 Proof of Theorem 4.1

4.4.1 Preliminary remarks

We still have the basic observation

$$F \text{ is primitive} \implies F \text{ is of } \mathbf{TR} \text{ - type.}$$

Unlike the degree 4 case, examples of F of **TR**-type but not primitive do exist¹. If F is not primitive, then it can only contain quadratic or cubic subextensions. Quadratic subextensions of F can in principle be either real or imaginary; but if a quadratic subextension is real, then F is a cubic totally imaginary extension of a real quadratic field, which by ramification theory is not possible. Thus, the maximal totally real subfield F_0 can either be \mathbb{Q} or a totally real cubic subextension. If F is not primitive and does not contain a cubic subextension, then the maximal totally real subfield F_0 of F is \mathbb{Q} , and F is of **CM**-type. If F is not primitive and does contain a cubic subextension F' , then F' can be totally real or of mixed signature; if F' is totally real, F is of **CM**-type. Otherwise, i.e., if F not primitive and every cubic subfield has mixed signature, then F is of **TR**-type if F does not contain a quadratic subextension. In conclusion, F is of **TR**-type if and only if F contains neither a quadratic extension nor a totally real cubic extension.

4.4.2 $G = C_6$

If $\text{Gal}(F_s/\mathbb{Q}) \cong C_6$, then $F_s = F$ and the subfields of F are Galois. Both quadratic and cubic subfields occur as they correspond to the index 2 and index 3 subgroups of C_6 respectively. By our analysis above, quadratic subfields can only be imaginary. Moreover, a Galois cubic field has to be totally real. Therefore, F is a **CM**-field.

¹ LMFDB number field 6.0.29095.1

4.4.3 $G = S_3$

If $\text{Gal}(F_s/\mathbb{Q}) \cong S_3$, then $F_s = F$. The imaginary quadratic subfield of F corresponds to the subgroup A_3 of S_3 and any cubic subfield of F corresponds to a copy of S_2 in S_3 . Suppose K is a totally real cubic subfield of F , then K/\mathbb{Q} is Galois, because, for $\sigma \in \text{Gal}(F/\mathbb{Q})$, we know that $\sigma(K)$ is also totally real, and if $\sigma(K) \neq K$ then $F = K\sigma(K)$ is totally real which is not the case, hence $\sigma(K) = K$, whence K/\mathbb{Q} is Galois; this implies that $\text{Gal}(F_s/K)$ is an order 2 normal subgroup of S_3 which is not possible. Thus, only mixed signature cubic subfields can appear; hence F is of **CM**-type but not a CM field.

4.4.4 $G = S_5$

If $\text{Gal}(F_s/\mathbb{Q}) \cong S_5$, we will show that F is primitive, hence of **TR**-type. Specifically, F corresponds to a subgroup in S_5 of order 20. Suppose F is not primitive, then there exists a subgroup of $\text{Gal}(F_s/\mathbb{Q})$ of order 40 or 60 containing $\text{Gal}(F_s/F)$ as a maximal subgroup. The maximal subgroups of S_5 are of order 12, 20, 24, or 60^2 . In particular, there is no subgroup of order 40. The only subgroup of S_5 of order 60 is A_5 . But then $\text{Gal}(F_s/F)$ would be a subgroup of order 20 inside $A_5 = \text{Gal}(F_s/F_s^{A_5})$; but A_5 does not have a subgroup of order 20. Thus, F is primitive. Another way to see the maximality of an order 20 subgroup of S_5 , and hence the primitivity of F/\mathbb{Q} , is to note that $S_5 \cong \text{PGL}_2(\mathbb{F}_5)$, which acts transitively on $\mathbb{P}^1(\mathbb{F}_5)$, with the stabilizer of a point therein being the Borel subgroup which is maximal.

4.4.5 $G = S_6$

If $\text{Gal}(F_s/\mathbb{Q})$ is S_6 , then as in the S_5 case, we will see that F is primitive, hence of **TR**-type. If $\text{Gal}(F_s/\mathbb{Q}) \cong S_6$, then $\text{Gal}(F_s/F)$ is a subgroup of S_6 of order 120. Suppose F is not primitive, then there exists a subgroup of $\text{Gal}(F_s/\mathbb{Q})$ of order 240 or 360 containing $\text{Gal}(F_s/F)$ as a maximal subgroup. The maximal subgroups of S_6 are of order 48, 72, 120, or 360^3 . In particular, there is no subgroup of order 240. The only subgroup of S_6 of order 360 is A_6 . But then $\text{Gal}(F_s/F)$ would be a subgroup of order 120 inside $A_6 = \text{Gal}(F_s/F_s^{A_6})$; but A_6 does not have a subgroup of order 120. Thus, F is primitive.

4.4.6 $G = S_4$

If $\text{Gal}(F_s/\mathbb{Q}) \cong S_4$, then $H := \text{Gal}(F_s/F)$ corresponds to an order 4 non-normal subgroup of S_4 . From the lattice of subgroups of S_4 (Appendix A, Fig. 3), we see that H is C_4^j or V_4^j for some $j = 1, 2, 3$; note that H cannot be V_4^n since it is normal in S_4 . Furthermore, H cannot be any of the V_4^j , towards which we need the following easy lemma whose proof is omitted.

Lemma 4.2 *Suppose $E/F/\mathbb{Q}$ is a tower of finite extensions and E/\mathbb{Q} is Galois with Galois group G . Let $H = \text{Gal}(E/F)$. Then $\text{Hom}(F, E)$ is in bijection with the set of cosets G/H . Furthermore, the bijection is G -equivariant.*

If $H = V_4^j$ then it contains two transpositions and their product. If $\mathfrak{c} \in \text{Aut}(\mathbb{C})$ denotes complex conjugation, then, \mathfrak{c} restricted to F_s being an order two element in $G = S_4$ is, up to conjugation, an element of H . Say, $x^{-1}\mathfrak{c}x \in H$, whence, it fixes the coset xH , which is not possible by the above lemma since F is totally imaginary.

Now suppose $H = C_4^j$. The only proper subgroup containing H is the corresponding D_4^j . Hence, F has a unique necessarily cubic subfield—denote this as F_c . We will show that F_c is not totally real, from which we conclude that F is of **TR**-type. Let $\Sigma_{F_c} = \{\sigma_1, \sigma_2, \sigma_3\}$ be the set of embeddings of F_c into \mathbb{C} . Suppose, by the way of seeking a contradiction, that F_c is totally real. Then $\mathfrak{c} \circ \sigma_k = \sigma_k$ for all $k = 1, 2, 3$, which, by the above lemma,

² https://groupprops.subwiki.org/wiki/Subgroup_structure_of_symmetric_group:S5.

³ https://groupprops.subwiki.org/wiki/Subgroup_structure_of_symmetric_group:S6.

is equivalent to saying $\mathfrak{c}(xD_4^j) = xD_4^j$ for all $x \in S_4$. Hence, $\mathfrak{c} \in \bigcap_{x \in S_4} xD_4^jx^{-1} = \bigcap_{j=1,2,3} D_4^j = V_4^n = \{1, (12)(34), (13)(24), (14)(23)\}$, whence,

$$\mathfrak{c} \in \{(12)(34), (13)(24), (14)(23)\}.$$

Given that F is totally imaginary, \mathfrak{c} exchanges pairs of conjugate-embeddings, or equivalently, exchanges pairs of cosets in S_4/H . We look at the action of the possible choices of \mathfrak{c} on different possibilities of H . Assume, without loss of generality that

$$H = \{1, (1234), (13)(24), (1432)\}.$$

- If $\mathfrak{c} = (12)(34)$, we have $\mathfrak{c}((14)H) = (14)H$ since $(14)(13)(24)(14) = (12)(34)$.
- If $\mathfrak{c} = (13)(24)$, we have $\mathfrak{c}H = H$.
- If $\mathfrak{c} = (14)(23)$, we have $\mathfrak{c}((12)H) = (12)H$ since $(12)(13)(24)(12) = (14)(23)$.

In all cases \mathfrak{c} fixes a coset and so the corresponding embedding of F is real—a contradiction since F is totally imaginary. Therefore, F_c has mixed signature, and F is of **TR**-type.

4.4.7 $G = D_6$

If $\text{Gal}(F_s/\mathbb{Q}) \cong D_6$, then $H = \text{Gal}(F_s/F)$ is an order 2 subgroup of D_6 . Since F/\mathbb{Q} is not Galois, we rule out the case of $H = \langle r^3 \rangle$. Any subfield $F' \subset F$ corresponds to either an order 4 or order 6 subgroup of D_6 containing H . By the subgroup lattice of D_6 (Appendix A, Fig. 2), every order 2 subgroup (that is not $\langle r^3 \rangle$) of D_6 is contained in a unique subgroup of order 6 and a unique subgroup of order 4. Thus, F has a unique imaginary quadratic and a unique cubic subfield. Since the order 4 subgroups are not normal, the cubic subfield can be totally real or mixed signature.

We want to determine when the cubic subfield is totally real (resp., mixed signature). Let H_c denote the order 4 subgroup corresponding to the cubic subfield $F_c = F^{H_c}$. Let \mathfrak{c} denote complex conjugation whose restriction to F_s is an element of order 2 in D_6 that we also denote \mathfrak{c} . We know that \mathfrak{c} acts on the set of embeddings Σ_{F_c} of F_c into \mathbb{C} . Lemma 4.2 implies that \mathfrak{c} acts on the set G/H_c . We can determine whether F_c is totally real by analyzing the action of \mathfrak{c} on G/H_c . In particular, F_c is totally real if and only if \mathfrak{c} acts trivially on G/H_c , which is equivalent to $\mathfrak{c}(gH_c) = gH_c$ for all $g \in G$. This gives us the condition $\mathfrak{c} \in \bigcap_{g \in G} gH_c g^{-1}$. By the same argument, if H_q denote the order 6 subgroup corresponding to the imaginary quadratic subfield $F_q = F^{H_q}$, then \mathfrak{c} should exchange pairs of cosets in G/H_q . Similarly, since F is totally imaginary, \mathfrak{c} should exchange pairs of cosets in G/H . From these two cases, we know that $\mathfrak{c} \notin H_q$ and $\mathfrak{c} \notin H$. To summarize, \mathfrak{c} should satisfy:

$$\mathfrak{c} \in \bigcap_{g \in G} gH_c g^{-1}, \quad \mathfrak{c} \notin H_q, \quad \mathfrak{c} \notin H. \quad (*)$$

With these properties, we can now enumerate the possible choices for \mathfrak{c} given different choices of H . From the symmetry in the subgroup lattice of D_6 (Appendix A, Fig. 2), it suffices to consider the cases when $H \in \{\langle s \rangle, \langle sr^2 \rangle, \langle sr^4 \rangle\}$. When $H = \langle s \rangle$, we know $H_c = \langle s, r^3 \rangle$ and $H_q = \langle s, r^2 \rangle$. Since \mathfrak{c} is of order 2 and $\mathfrak{c} \notin H, H_q$, one can only have $\mathfrak{c} \in \{r^3, sr^5, sr^3, sr\}$. It is easy to verify that only $\mathfrak{c} = r^3$ satisfies (*). The other two cases of H are identical to the case $H = \langle s \rangle$. We can therefore conclude that F_c is totally real if and only if \mathfrak{c} is central (since r^3 is the only order 2 central element in D_6). The following two examples show that both cases— F_c being totally real or mixed signature can happen.

- (1) The number field with label 6.0.309123.1 on LMFDB is a totally imaginary sextic field with Galois group D_6 ; it is a CM-field; its unique cubic subfield F_c with label 3.3.321.1 is totally real (c is necessarily central); the unique quadratic subfield F_q is $\mathbb{Q}(\sqrt{-3})$.
- (2) The number field with label 6.0.14283.1 on LMFDB is a totally imaginary sextic field with Galois group D_6 is not a CM-field; its unique cubic subfield F_c with label 3.1.23.1 is of mixed signature; the unique quadratic subfield is $\mathbb{Q}(\sqrt{-3})$. Hence F is of **CM**-type but not a CM-field.

In the spirit of this article, one may formulate the naïve problem:

Problem 4.1 *When ordered by discriminant, determine what percentage of sextic totally imaginary fields with Galois group D_6 are not CM-fields.*

Remark 4.2 We propose two different strategies to tackle this problem, both leveraging on the fact that we have the group isomorphism

$$D_6 \cong S_3 \times C_2.$$

With this isomorphism, one can reduce the study of sextic fields to that of cubic and quadratic extensions. Another common approach used in both is the application of Abelian and Tauberian theorems to convert the problem of counting to that of computing residue of Dirichlet series.

The first proposed strategy is to generalize the computation for D_4 -quartic fields by Cohen–Diaz Y Diaz–Olivier [3]. They consider the Dirichlet series associated to a tower of two quadratic extensions and identify the types of quartic fields that have the same intermediate field structure (D_4 , C_4 , V_4 -fields). Since the distributions of C_4 and V_4 -fields are well-studied, they reduce the problem to counting quadratic extensions with arbitrary base field and specified signatures by subtracting the contribution from C_4 and V_4 . Therefore, they are able to distinguish totally imaginary D_4 -quartic fields that are not CM by varying the signature of the intermediate quadratic field.

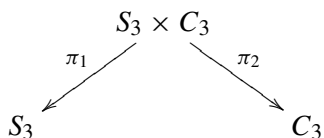
One may hope to mimic this strategy by considering the Dirichlet series associated to towers of extension of S_3 -cubic fields over \mathbb{Q} followed by quadratic extensions over the cubic. By looking at subgroups, we know that S_3 , D_6 , S_4 , and $S_4 \times C_2$ -sextic fields have this intermediate field structure. This Dirichlet series is approachable as we know the counting of S_3 -cubic extensions over \mathbb{Q} with specified signatures from Davenport and Heilbronn [4, 5]. We also know the counting of quadratic extensions from [3]. The difficulty in this strategy lies in the lack of knowledge about S_4 and $S_4 \times C_2$ -fields. Moreover, Malle’s conjecture predicts $S_4 \times C_2$ -fields to dominate D_6 -fields in the asymptotic distribution. Therefore, we might not be able to obtain information about D_6 -fields in this way.

The second potential strategy is to modify the argument by Masri-Thorne-Tsai-Wang [11]. They have studied the distribution of $S_d \times A$ -fields of degree $d|A|$ over \mathbb{Q} without distinguishing signatures. Here A is any abelian group and $d = 3, 4, 5$. Their strategy converts the counting of D_6 -fields into counting pairs (F, K) with F an S_3 -cubic field and K a quadratic extension over \mathbb{Q} using the Product Lemma developed by Wang [13]. The difficulty of their result lies in controlling the discrepancy between the discriminant of the compositum FK and the product of appropriate powers of discriminants of F and K . This discrepancy comes from primes that are ramified in both F and K . The challenge to adapting this strategy to our case lies in extending the description of the effect of ramification to infinite places.

Remark 4.3 Following Altug–Shankar–Varma–Wilson [1] one can ask for variations of the above problem exploiting the structure of D_6 . There exists a unique faithful irreducible two-dimensional representation of D_6 . We can order the totally imaginary sextic fields with Galois group $D_6 = \text{Gal}(F_s/\mathbb{Q})$ using the Artin conductor of this particular two-dimensional representation and then ask the question of determining what percentage of such fields are CM-fields.

4.4.8 $G = S_3 \times C_3$

If $\text{Gal}(F_s/\mathbb{Q}) \cong S_3 \times C_3$, then $H = \text{Gal}(F_s/F)$ corresponds to an order 3 subgroup of $S_3 \times C_3$. Consider the image of H under the canonical projections

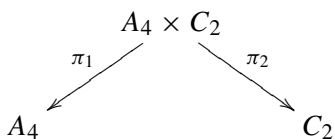


Clearly, $|\pi_2(H)|$ divides 3. Suppose $|\pi_2(H)| = 1$, then we know that $H \subset \ker(\pi_2) = S_3 \times \{1\}$. The unique subgroup of order 3 in S_3 is A_3 , thus, $H \cong A_3 \times \{1\}$, which is normal in $S_3 \times C_3$. This would force F to be Galois over \mathbb{Q} , which is a contradiction. Hence, $|\pi_2(H)| = 3$. Now looking at the order of $\pi_1(H)$ we only have two cases:

- (i) $|\pi_1(H)| = 1$. Then $H = \ker(\pi_1) = \{1\} \times C_3$ which is normal in $S_3 \times C_3$, whence F is Galois over \mathbb{Q} , which is a contradiction.
- (ii) $|\pi_1(H)| = 3$. This implies $H \cong \pi_1(H) \cong A_3$. Furthermore, there is a surjective (hence bijective) map $f : \pi_1(H) = A_3 \rightarrow \pi_2(H) = C_3$ such that $H = \{(x, f(x)) \mid x \in A_3\}$. In fact, f is an isomorphism. Also, there is a unique intermediate subgroup: $H \subset A_3 \times C_3 \subset S_3 \times C_3$. Therefore, $F_s^{A_3 \times C_3}$ is a quadratic subfield of F that is necessarily imaginary. Hence F is of CM-type but not a CM-field.

4.4.9 $G = A_4 \times C_2$

If $\text{Gal}(F_s/\mathbb{Q}) \cong A_4 \times C_2$, then $\text{Gal}(F_s/F)$ corresponds to an order 4 subgroup of $A_4 \times C_2$. To classify such order 4 subgroups, consider the projections



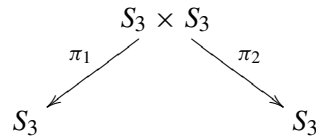
The order of $\pi_1(H)$ divides 4 and the order of $\pi_2(H)$ divides 2. If $\pi_2(H)$ is trivial then $H \subset \ker(\pi_2) = A_4 \times \{1\}$; by the subgroup lattice, the unique subgroup of order 4 in A_4 is V_4 , thus, $H \cong V_4 \times \{1\}$, which is normal in $A_4 \times C_2$; this would force F to be Galois over \mathbb{Q} , which is a contradiction. Hence $\pi_2(H) = C_2$. By order consideration, we can only have two cases:

- (i) $|\pi_1(H)| = 2$. By the exact sequence $1 \rightarrow H \cap \ker(\pi_1) \rightarrow H \xrightarrow{\pi_1} \pi_1(H) \rightarrow 1$, we know that $|H \cap \ker(\pi_1)| = |H|/|\pi_1(H)| = 2$, which implies $\ker(\pi_1) = \{1\} \times C_2 \leq H$. Since $\{1\} \times C_2$ is normal in $A_4 \times C_2$, we would have $F_s^{\{1\} \times C_2}/\mathbb{Q}$ is Galois, which contradicts the fact that F_s is the smallest Galois extension over \mathbb{Q} containing F .
- (ii) $|\pi_1(H)| = 4$. This implies $H \cong \pi_1(H) \cong V_4$. There exists a surjective map $f : \pi_1(H) = V_4 \rightarrow \pi_2(H) = C_2$ such that $H = \{(x, f(x)) \mid x \in V_4\}$. Notice that $H \subset V_4 \times C_2 \triangleleft_3 A_4 \times C_2$. Therefore, $F_s^{V_4 \times C_2}$ is a cubic subfield of F which is Galois. This forces the cubic subfield to be totally real. Hence, F is a CM field.

There is another perhaps conceptually better way to deal with $A_4 \times C_2$ which is to view it as the wreath product $C_2 \wr C_3 \cong C_2^3 \rtimes C_3$. Then the unique order 8 subgroup containing H is the normal C_2^3 , forcing the unique cubic subfield of F to be Galois and totally real; hence F is a CM field. In this case, Klüners [8] sheds some light, however, even if we assume the full statement of Malle’s conjecture, without explicit information about the constants of proportionality, one cannot make progress on the main question of this paper.

4.4.10 $G = S_3 \times S_3$

If $\text{Gal}(F_s/\mathbb{Q}) \cong S_3 \times S_3$, then $H = \text{Gal}(F_s/F)$ is a subgroup of $S_3 \times S_3$ or order 6. To classify all such subgroups. Consider the projections



We must have the orders of $\pi_1(H)$ and $\pi_2(H)$ to be factors of 6.

Suppose $|\pi_2(H)| = 1$. Thus, $H \subset \ker(\pi_2)$, which forces $H \cong S_3 \times \{1\}$ by order consideration; such a subgroup is normal in $S_3 \times S_3$, forcing F to be Galois over \mathbb{Q} , which is a contradiction.

Suppose $|\pi_2(H)| = 2$, then by the exact sequence $1 \rightarrow H \cap \ker(\pi_2) \rightarrow H \xrightarrow{\pi_2} \pi_2(H) \rightarrow 1$, we know that $|H \cap \ker(\pi_2)| = 3$, which implies $C_3 \times \{1\} \leq H$. Since $C_3 \times \{1\}$ is normal in $S_3 \times S_3$, we then have $F_s^{C_3 \times \{1\}}/\mathbb{Q}$ is Galois, which contradicts F_s being the smallest Galois extension over \mathbb{Q} containing F .

Suppose $|\pi_2(H)| = 3$, then by the same exact sequence $|H \cap \ker(\pi_2)| = 2$, which implies $C_2 \times \{1\} \subset H$. This implies that $H = C_2 \times C_3$ or $H = \{(x, f(x)) \mid x \in S_3\}$ for f a surjective map $f : \pi_1(H) = S_3 \rightarrow \pi_2(H) = C_3$; the latter is ruled out as there is no nontrivial morphism from S_3 to C_3 , and the former is ruled out because it contains $C_3 \times \{1\}$ arguing then as in the previous paragraph.

Suppose $|\pi_2(H)| = 6$. We will rule out $\{1\} \times S_3$ as above. The only remaining possible H is of the form

$$H = \{(x, f(x)) \mid x \in S_3\},$$

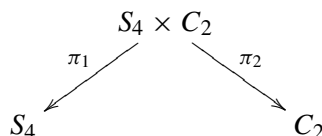
for f an automorphism of S_3 . Consider the following homomorphism

$$\phi : S_3 \times S_3 \rightarrow \{\pm 1\}, \quad (x, y) \mapsto \text{sgn}(x) \text{sgn}(y)$$

and let $K = \ker(\phi)$. Since ϕ is surjective, we note that K is a subgroup of order 18 containing H for all f . Thus, F_s^K is a quadratic subfield of F . By order consideration, H is maximal in K . Thus, any other subgroup L of $S_3 \times S_3$ containing H has to contain an element of the form (x, y) , where $\text{sgn}(x) \neq \text{sgn}(y)$. Since the H 's are isomorphic for different choices of f , we will work with $f = 1_{S_3}$ for simplicity, in which case H is the usual diagonal S_3 in the product $S_3 \times S_3$. We claim that $|L| \neq 12$. Suppose otherwise, then for all $l \in L$, we should have $l^2 \in H$. By our observation, L needs to contain a $(1 - \text{cycle}, 2 - \text{cycle})$ or $(2 - \text{cycle}, 3 - \text{cycle})$ (or exchanging the first and second copy). It is clear that $(2 - \text{cycle}, 3 - \text{cycle})^2 \notin H$. Without loss of generality we can take the $(1 - \text{cycle}, 2 - \text{cycle})$ to be $(1, (12))$. Multiplying by $((23), (23))$ on the right, we get $((23), (231))$, whose square does not lie in the diagonal. Thus, we arrive at a contradiction and $|L| \neq 12$. We conclude that K is the only subgroup containing H . Hence, F contains a unique quadratic subfield, which is necessarily imaginary, whence, F is of **CM**-type but not a CM-field.

4.4.11 $G = S_4 \times C_2$

If $\text{Gal}(F_s/\mathbb{Q}) \cong S_4 \times C_2$, then $H := \text{Gal}(F_s/F)$ corresponds to an order 8 subgroup. To classify all such subgroups. Consider the projections



The order of $\pi_1(H)$ must divide 8.

Suppose $|\pi_1(H)| = 1$, then $H \subset \ker(\pi_1) = \{1\} \times C_2$, which is impossible.

Suppose $|\pi_1(H)| = 2$, then by the exact sequence $1 \rightarrow H \cap \ker(\pi_1) \rightarrow H \xrightarrow{\pi_1} \pi_1(H) \rightarrow 1$, we get $|H \cap \ker(\pi_1)| = |H|/|\pi_1(H)| = 4$, which is impossible.

Suppose $|\pi_1(H)| = 4$, then $|H \cap \ker(\pi_1)| = 2$, or equivalently $\ker(\pi_1) = \{1\} \times C_2 \leq H$. Since $\{1\} \times C_2$ is normal in $S_4 \times C_2$, we would have $F_s^{\{1\} \times C_2} / \mathbb{Q}$ is Galois contradicting F_s being the smallest Galois extension over \mathbb{Q} containing F .

Suppose $|\pi_1(H)| = 8$. Consider two sub-cases depending on the order of $\pi_2(H)$:

Suppose that $|\pi_1(H)| = 8$ and $|\pi_2(H)| = 1$, then $H \cong D_4 \times \{1\}$ since D_4 is the unique order 8 subgroup of S_4 . By the group lattice of S_4 (Appendix A, Fig. 3) we see that D_4 contains the unique Klein-4 subgroup that is normal in S_4 denoted by V_4^n . But then $F_s^{V_4^n \times \{1\}} / \mathbb{Q}$ is Galois, contradicting F_s being the smallest Galois extension over \mathbb{Q} containing F .

Suppose now that $|\pi_1(H)| = 8$ and $|\pi_2(H)| = 2$. Then $\pi_1|_H$ is injective, hence $H \cong \pi_1(H) \cong D_4$. From the group lattice of S_4 (Appendix A, Fig. 3) we see that S_4 contains three copies of D_4 denoted D_4^j for $j = 1, 2, 3$. Each D_4^j contains V_4^n , a copy of C_4 denoted by C_4^j , and a non-normal copy of V_4 denoted by V_4^j . Assume without loss of generality that $\pi_1(H) \cong D_4^j$ for a particular j . Since $|\pi_2(H)| = 2$, there exists a surjective map $f : \pi_1(H) \cong D_4^j \rightarrow \pi_2(H) = C_2$ such that

$$H = \{(x, f(x)) \mid x \in D_4^j\}.$$

Observe that since π_1 is injective, f needs to be a function. Since $(1, 1) \in H$, we necessarily have $f(1) = 1$. Moreover, $(x, f(x))(y, f(y)) = (xy, f(x)f(y))$ combined with the fact that f is a function shows that $f(xy) = f(x)f(y)$. Therefore, f is a group homomorphism. Thus, H is completely determined by the datum (D_4^j, f) . Since $\ker(f)$ is normal in D_4^j and of order 4, it could be V_4^n, V_4^j , or C_4^j . If $\ker(f) = V_4^n$, then $V_4^n \times \{1\} \leq H$ is normal in $S_4 \times C_2$ and we have a contradiction as before. Thus, we can only have $\ker(f) = V_4^j$ or $\ker(f) = C_4^j$. In each case, the kernel determines f completely. Observe that

$$H = (D_4^j, f) \subset_2 D_4^j \times C_2 \subset_3 S_4 \times C_2,$$

where the subscript indicates the index. Therefore, $D_4^j \times C_2$ corresponds to a cubic subfield of F . Since $D_4^j \times C_2$ is not normal, the cubic subfield can be totally real or mixed signature. In fact, this is the unique cubic subfield in F , or equivalently, $D_4^j \times C_2$ is the unique index 3 subgroup of $S_4 \times C_2$ containing H . To see this, consider a subgroup K such that

$$H \subset_2 K \subset_3 S_4 \times C_2.$$

Since $|K| = 16$, we necessarily have $\pi_1(K) \cong D_4^j$ and $\pi_2(K) = C_2$. Let $(x, y) \in K \setminus H$, then $(x, -y) \in H$, which implies $(x^{-1}, -y) \in H$. Thus, $(x, y)(x^{-1}, -y) = (1, -1) \in K$ and we conclude that $\{1\} \times C_2 \leq K$. Consequently, for any $(x, y) \in K$, we have $(1, y) \in K$ and $(x, y)(1, y) = (x, 1) \in K$. Thus, $D_4^j \times \{1\} \leq K$, which forces $K \cong D_4^j \times C_2$.

Suppose there exists H' such that $H \subset_3 H'$. Then we must have $\pi_1(H') = S_4$ and $\pi_2(H') = C_2$. In other words, H' is also of the form (S_4, f') , where $f'|_{D_4^j} = f$. Since $|\ker(f')| = 12$, we know $\ker(f') = A_4$. Since $\ker(f) \subset \ker(f')$ but also since neither C_4^j nor V_4^j is contained in A_4 , we get a contradiction. Thus, F has no quadratic subfields.

We want to determine when the unique cubic subfield is totally real or of mixed signature. Following the case of D_6 in §4.4.7, we need to examine the action of complex conjugation c on the set of embeddings of the cubic subfield into \mathbb{C} . Note that c should be of the form

$$(e_1, e_2), \text{ where } e_1 \in \{C_2^{11}, C_2^{21}, C_2^{31}, C_2^1, \dots, C_2^6\} \text{ and } e_2 \in C_2; \quad (e_1, e_2) \neq (1, 1).$$

Since F is totally imaginary, the action of \mathfrak{c} on the set of cosets gH is nontrivial (and in particular exchanges pairs of cosets in G/H). This implies

$$\mathfrak{c} \notin H \quad (*).$$

Let $H_{\mathfrak{c}} = D_4^j \times C_2$ be the subgroup corresponding to the cubic subfield, then the analysis in 4.4.7 tells us that

$$F_{\mathfrak{c}} \text{ is totally real} \iff \mathfrak{c} \in \bigcap_{g \in G} gH_{\mathfrak{c}}g^{-1} \iff \mathfrak{c} \in V_4^n \times C_2 \quad (**).$$

First, consider the case when $\ker(f) = C_4^j$. Without loss of generality, take $j = 1$. Among all the pairs (e_1, e_2) , we know from the information of $\ker(f)$ combined with the group lattice that those in the following subsets (excluding the element $(1, 1)$) satisfy (*):

- (i) $C_2^{11} \times \{-1\}$;
- (ii) $C_2^{j'1} \times \{1\}$, $j' = 2, 3$;
- (iii) $C_2^i \times \{1\}$, $i = 1, 2$;
- (iv) $C_2^i \times \{\pm 1\}$, $i = 3, \dots, 6$.

Among these, case (i) and case (ii) satisfy (**). Thus, when \mathfrak{c} is in case (i) or (ii), $F_{\mathfrak{c}}$ is totally real and F is a CM -field. If \mathfrak{c} is in case (iii) or (iv), $F_{\mathfrak{c}}$ is of mixed signature and F is of \mathbf{TR} -type.

Now consider the case when $\ker(f) = V_4^j$. Without loss of generality we can take $j = 1$. Among all the pairs (e_1, e_2) , we know from the information of $\ker(f)$ combined with the group lattice that those in the following subsets (excluding the element $(1, 1)$) satisfies (*):

- (i) $C_2^{11} \times \{-1\}$;
- (ii) $C_2^{j'1} \times \{1\}$, $j' = 2, 3$;
- (iii) $C_2^i \times \{-1\}$, $i = 1, 2$;
- (iv) $C_2^i \times \{\pm 1\}$, $i = 3, \dots, 6$.

Among these, case (i) and case (ii) satisfy (**). Thus, when \mathfrak{c} is in case (i) or (ii), $F_{\mathfrak{c}}$ is totally real and F is CM . If \mathfrak{c} is in case (iii) or (iv), $F_{\mathfrak{c}}$ is of mixed signature and F is of \mathbf{TR} -type.

To summarize in general for $j = 1, 2, 3$, if $\mathfrak{c} = (e_1, -1)$ with $e_1 \in C_2^{j1}$ or $\mathfrak{c} = (e_1, 1)$ with $e_1 \in C_2^{j'1}$ ($j' \neq j$), then F is CM . Otherwise, F is of \mathbf{TR} -type.

As in the $A_4 \times C_2$ case, there is another way to deal with $S_4 \times C_2$ which is to view it as the wreath product $C_2 \wr S_3 \cong C_2^3 \rtimes S_3$ which might afford a partial simplification of the above discussion; once again Klüners [8] sheds some light in this situation, but does not answer the main question of this paper. More interestingly, one may ask for the analogue of Prob. 4.1 for $S_4 \times C_2$ -extensions:

Problem 4.2 *When ordered by discriminant, determine what percentage of sextic totally imaginary fields with Galois group $S_4 \times C_2$ are CM -fields.*

4.4.12 $G = C_3^2 \rtimes D_4$

If $\text{Gal}(F_S/\mathbb{Q}) \cong C_3^2 \rtimes D_4$, then $H = \text{Gal}(F_S/F)$ corresponds to an order 12 subgroup of $C_3^2 \rtimes D_4$. We first exhibit a particular presentation of $C_3^2 \rtimes D_4$ in S_6 . Denote the 6 letters to permute as $\{1, 2, 3, a, b, c\}$. Let

$$C_3^2 = \langle (123) \rangle \times \langle (abc) \rangle \quad D_4 = \langle (1a2b)(3, c) \rangle \times \langle (ab) \rangle$$

It is easily seen that $r = (1a2b)(3, c)$ is the rotation element and $s = (ab)$ is the reflection element. The action of D_4 on C_3^2 can be explicated as:

$$r(123)r^{-1} = (1a2b)(3c)(123)(1b2a)(3c) = (abc), \quad s(123)s = (123),$$

$$r(abc)r^{-1} = (1a2b)(3c)(abc)(1b2a)(3c) = (132), \quad s(abc)s = (acb).$$

In order to classify all order 12 subgroups of G , we need the following seemingly well-known result⁴ classifying groups of order 12:

Proposition 4.1 *Every group of order 12 is isomorphic to one of the following:*

- (i) $\mathbb{Z}/12 \cong \mathbb{Z}/3 \times \mathbb{Z}/4$,
- (ii) $(\mathbb{Z}/2)^2 \times \mathbb{Z}/3$,
- (iii) $(\mathbb{Z}/2)^2 \rtimes \mathbb{Z}/3$ (one isomorphism class),
- (iv) $\mathbb{Z}/3 \rtimes (\mathbb{Z}/2)^2$ (one isomorphism class),
- (v) $\mathbb{Z}/3 \rtimes \mathbb{Z}/4$ (one isomorphism class).

Since D_4 has no element of order 3, an order 3 element of G can only come from the C_3^2 component. Thus, type (ii) and (iii) cannot exist in the ambient group. Moreover, since an element of order 2 or 4 can only come from the D_4 component and the action of D_4 on C_3^2 is not trivial, H can only be of type (iv) or (v).

Thus, we know that H is of the form $A \rtimes K$ with $A \leq C_3^2$, $|A| = 3$ and $K \leq D_4$, $|K| = 4$. The choices for A are: $C_3 \times \{1\}$, $\{1\} \times C_3$, or $\Delta_{C_3, f} = \{(x, f(x)) \mid x \in C_3\}$ for some isomorphism $f : C_3 \rightarrow C_3$. There are two distinct choices of f . Suppose $f : C_3 = \langle x \rangle \rightarrow C_3 = \langle y \rangle$. Then there exist f_1 and f_2 characterised by $f_1(x) = y$ and $f_2(x) = y^2$. Therefore, in all, there are four choices for A :

$$C_3 \times \{1\}, \quad \{1\} \times C_3, \quad \Delta_{C_3, f_1}, \quad \Delta_{C_3, f_2}.$$

It is easy to that possible choices for K are:

$$\langle r \rangle, \quad V_1 = \langle r^2 \rangle \times \langle s \rangle, \quad V_2 = \langle r^2 \rangle \times \langle sr \rangle.$$

We want to see if various K acts on the different choices of A by conjugation. By explicit computation using the action of D_4 explicated above, we have obtain the following list of feasible K for a given A :

A	K
$C_3 \times \{1\}$	V_1
$\{1\} \times C_3$	V_1
Δ_{C_3, f_1}	V_2
Δ_{C_3, f_2}	V_2

(4.4)

Let $H = A \rtimes K$ with one of the pair (A, K) in the table. We want to find the possible subgroups L such that $H \subsetneq L \subsetneq G = C_3^2 \rtimes D_4$.

First note that we can have $L = C_3^2 \rtimes K$, which cuts out a quadratic subfield of F since $[L : H] = 3$; this quadratic subfield is necessarily imaginary and so F is of **CM**-type.

Next, we claim that G does not have a proper subgroup L with $[L : H] = 2$, i.e., F does not have any cubic subfields. Suppose otherwise, then we must have the following commutative diagram. There exist sections in the top and bottom rows because G and H are semidirect products.

⁴ <https://kconrad.math.uconn.edu/blurbs/grouptheory/group12.pdf>.

$$\begin{array}{ccccccc}
 1 & \longrightarrow & C_3^2 & \longrightarrow & G & \begin{array}{c} \xrightarrow{\pi_2} \\ \xleftarrow{s} \end{array} & D_4 \longrightarrow 1 \\
 & & \uparrow & & \uparrow & & \parallel \\
 1 & \longrightarrow & C_3^2 \cap L & \longrightarrow & L & \xrightarrow{\pi_2} & D_4 \longrightarrow 1 \\
 & & \parallel & & \uparrow & & \uparrow \\
 1 & \longrightarrow & A & \longrightarrow & H & \begin{array}{c} \xrightarrow{\pi_2} \\ \xleftarrow{s} \end{array} & K \longrightarrow 1
 \end{array}$$

We want to show that L is also a semidirect product, i.e., there exists a section $s' : D_4 \hookrightarrow L$ compatible with $s : D_4 \rightarrow G$. Suppose, by the way of seeking a contradiction, that such an s' does not exist. We will separate the analysis into two cases based on the choice of K .

First consider $K = V_1 = \langle s, r^2 \rangle$, then $A = C_3 \times \{1\}$ or $\{1\} \times C_3$ based on the feasible pairs of (A, K) as in (4.4). To say that s' does not exist is to say that $s(D_4) \not\subset L$. Since from the commutative diagram we know that $s(V_1) \subset L$, we must have $(1, sr) \notin L$. Since $L \twoheadrightarrow D_4$, there exists some element $(x, sr) \in L$ with $x \neq 1$ and $x \in A$. Moreover, we must have $(x, sr)^2 \in H$ by the assumption on index. We check which choices of x satisfy these criteria. For simplicity let $\alpha = (123)$ and $\beta = (abc)$.

choice of (x, sr)	$(x, sr)^2$	$(x, sr)^2 \in H?$
(α, sr)	$(\alpha, sr)^2 = (\alpha sr \alpha r^{-1} s^{-1}, 1) = (\alpha \beta^2, 1)$	no
(α^2, sr)	$(\alpha^2, sr)^2 = (\alpha^2 sr \alpha^2 r^{-1} s^{-1}, 1) = (\alpha^2 \beta, 1)$	no
(β, sr)	$(\beta, sr)^2 = (\beta sr \beta r^{-1} s^{-1}, 1) = (\beta \alpha^2, 1)$	no
(β^2, sr)	$(\beta^2, sr)^2 = (\beta^2 sr \beta^2 r^{-1} s^{-1}, 1) = (\beta^2 \alpha, 1)$	no

Thus, the element $(x, sr) \in L$ as required cannot exist. Therefore, s' exists and $L = A \rtimes D_4$.

Next consider $K = V_2 = \langle sr, r^2 \rangle$, then $A = \Delta_{C_3, f_1} = \langle \alpha \beta \rangle$ or $\Delta_{C_3, f_2} = \langle \alpha \beta^2 \rangle$ based on the feasible pairs of (A, K) as in (4.4). To say that s' does not exist is to say that $s(D_4) \not\subset L$. Since from the commutative diagram we know that $s(V_2) \subset L$, we must have $(1, s) \notin L$. Since $L \twoheadrightarrow D_4$, there exists some element $(x, s) \in L$ with $x \neq 1$ and $x \in A$. Moreover, we must have $(x, s)^2 \in H$ by the assumption on index. We check which choices of x satisfy these criteria.

choice of (x, s)	$(x, s)^2$	$(x, s)^2 \in H?$
$(\alpha \beta, s)$	$(\alpha \beta, s)^2 = (\alpha \beta s \alpha \beta s^{-1}, 1) = (\alpha^2, 1)$	no
$(\alpha^2 \beta^2, s)$	$(\alpha^2 \beta^2, s)^2 = (\alpha^2 \beta^2 s \alpha^2 \beta^2 s^{-1}, 1) = (\alpha, 1)$	no
$(\alpha \beta^2, s)$	$(\alpha \beta^2, s)^2 = (\alpha \beta^2 s \alpha \beta^2 s^{-1}, 1) = (\alpha^2, 1)$	no
$(\alpha^2 \beta, s)$	$(\alpha^2 \beta, s)^2 = (\alpha^2 \beta s \alpha^2 \beta s^{-1}, 1) = (\alpha, 1)$	no

Thus, the element $(x, s) \in L$ as required cannot exist. Therefore, s' exists and $L = A \rtimes D_4$.

From our analysis above, any subgroup L with $[L : H] = 2$ must be of the form $L = A \rtimes D_4$. However, from the table we see that D_4 cannot stabilize any choice of A since $\langle r \rangle$ does not act on any A by conjugation. Thus, F does not have any cubic subfields.

In conclusion, any totally imaginary sextic field F with $\text{Gal}(F_s/\mathbb{Q}) = C_3^2 \rtimes D_4$ has a unique subfield which is imaginary quadratic and hence F is of CM-type but not a CM-field.

4.5 To summarise:

The following table is a summary of the sextic totally imaginary fields:

Galois group	Subfield	CM	CM-type but not CM	TR-type
C_6	imaginary quadratic, totally real cubic	•		
S_3	imaginary quadratic, mixed signature cubic		•	
D_6 , Case (i)	imaginary quadratic, totally real cubic	•		
D_6 , Case (ii)	imaginary quadratic, mixed signature cubic		•	
$A_4 \times C_2$	totally real cubic	•		
$S_4 \times C_2$, Case (i)	totally real cubic	•		
$S_4 \times C_2$, Case (ii)	mixed signature cubic			•
$S_3 \times C_3$	imaginary quadratic		•	
$C_3^2 \rtimes D_4$	imaginary quadratic		•	
$S_3 \times S_3$	imaginary quadratic		•	
S_4	mixed signature cubic			•
$S_5 \cong PGL_2(\mathbb{F}_5)$	None			•
S_6	None			•

Appendix A Lattice of subgroups for D_4 , D_6 , and S_4

Figures 1, 2 and 3.

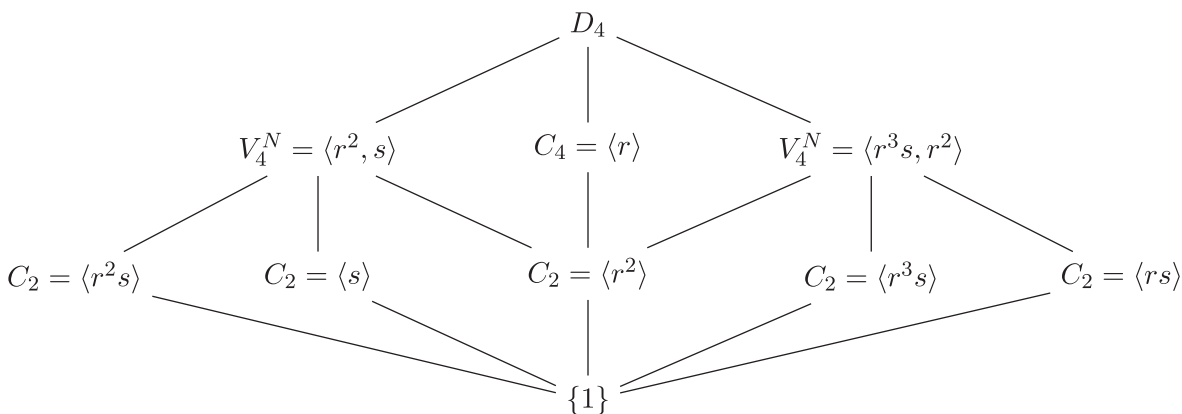


Fig. 1 Subgroup Lattice of D_4

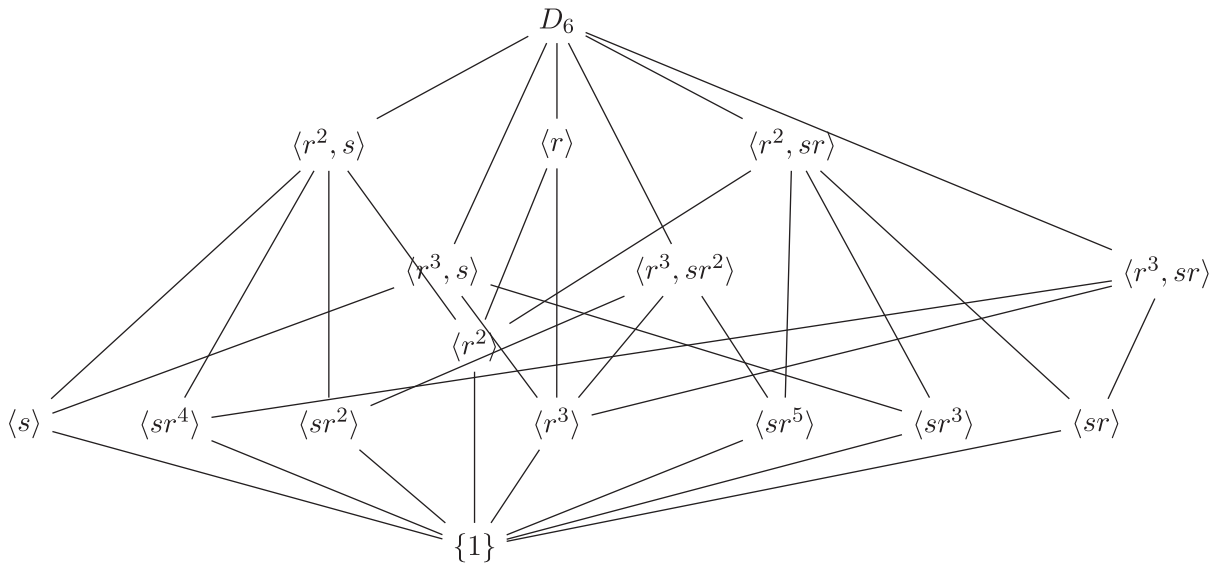


Fig. 2 Subgroup Lattice of D_6

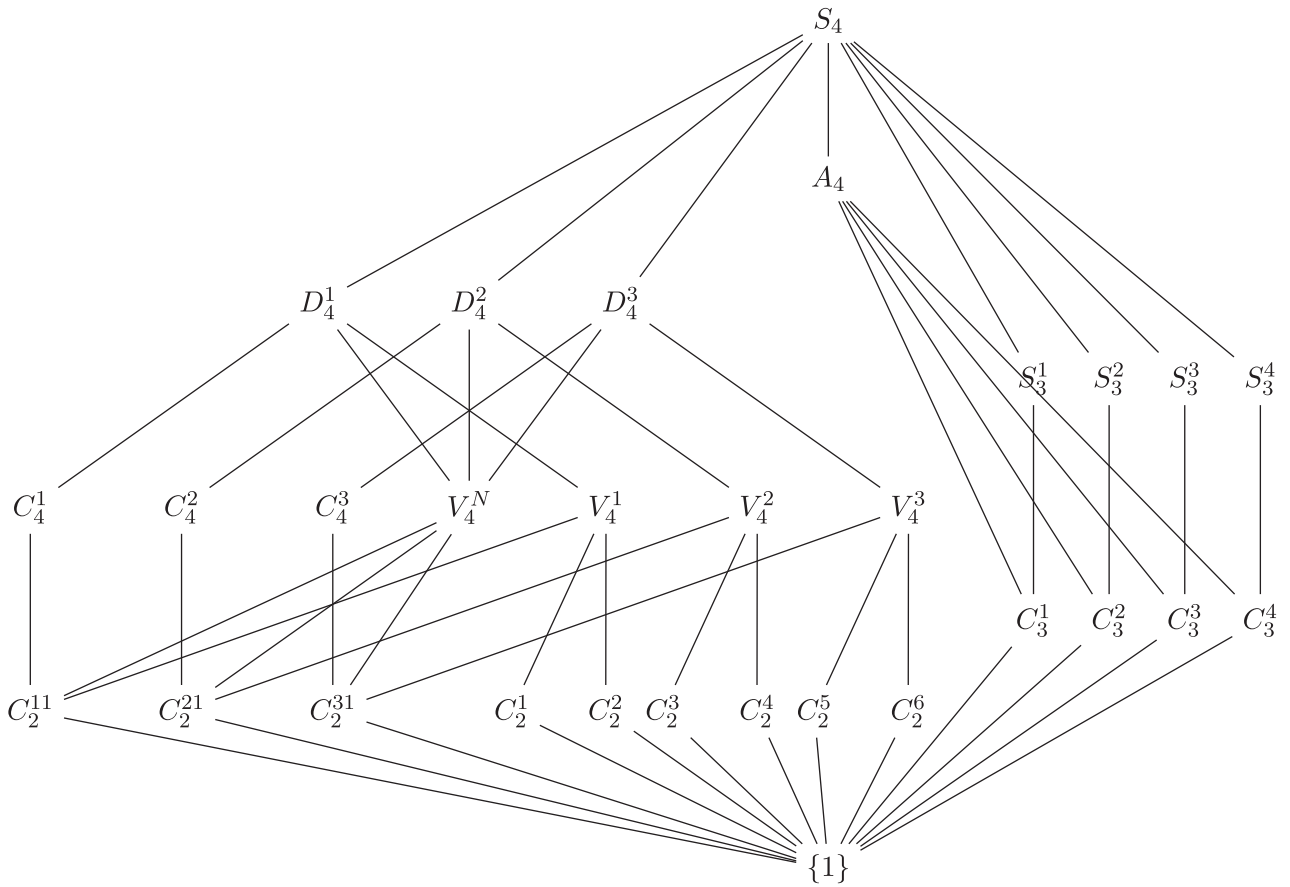


Fig. 3 Subgroup Lattice of S_4 (subgroups of S_3 omitted)

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