ORIGINAL RESEARCH

Elephant random walk with varying memory

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Abstract

Schütz and Trimper (*Phys. Rev. E* **70,** 045101 (2004)) introduced the 'elephant random walk (ERW)' which, like the simple random walk studied over the last century has increments $+1$ and -1, however the ERW keeps a track of its past steps. Harbola *et al.* (*Phys. Rev. E*, **90**, 022136 (2014)) studied a unidirectional ERW where the increments are 0 and $+1$. Gut and Stadmüller (*J. Appl. Probab.* **58,** 805–829 (2021), *Stat. Probab. Lett.* **189,** 109598 (2022)) extended the study of the ERW 'when the elephant has only a restricted memory' and Laulin (*Electron. Commun. Probab.*, **27**, Paper No. 54 (2022)) studied an ERW model with a 'smooth amnesia'. Here we discuss these models of restricted memory. We also provide a new proof of a result of Miyazaki and Takei (*J. Stat. Phys.*, **181**, 587–602 (2020)). Finally we end with some problems connected with these models.

Keywords Elephant random walk · Martingale · Phase transitions

AMS Classification 60K35

1 Introduction

The standard Elephant random walk (ERW) is a time-inhomogeneous Markov chain described as follows: let $p \in [0, 1]$, $s \in [0, 1]$ and $\{U_n : n \geq 1\}$ be a sequence of independent random variables with U_n having a uniform distribution over $\{1, \ldots, n\}$. Consider a sequence $X_1, X_2, \ldots$ of random variables taking values in $\{+1, -1\}$ given by

$$X_1 = \begin{cases} +1 & \text{with probability } s \\ -1 & \text{with probability } 1 - s, \end{cases} \tag{1.1}$$

and, for $n \geq 1$,

$$X_{n+1} = \begin{cases} X_{U_n} & \text{with probability } p \\ -X_{U_n} & \text{with probability } 1 - p; \end{cases} \tag{1.2}$$

we assume here that X_1 is independent of the sequence $\{U_n : n \geq 1\}$.

The ERW is given by $\{S_n : n \geq 1\}$ where

$$S_n = X_1 + \cdots + X_n. \tag{1.3}$$

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It may be easily seen that, taking $\mathcal{F}_n$ to be the σ-algebra generated by $X_1, \ldots, X_n$, for $n \geq 1$, we have

$$\begin{aligned} P(S_{n+1} = S_n \pm 1 \mid \mathcal{F}_n) &= P(X_{n+1} = \pm 1 \mid \mathcal{F}_n) \\ &= \frac{n \pm S_n}{2n} \cdot p + \frac{n \mp S_n}{2n} \cdot (1-p) = \frac{1}{2}\left(1 \pm (2p-1) \cdot \frac{S_n}{n}\right), \end{aligned} \tag{1.4}$$

thus the ERW is a time-inhomogeneous Markov chain.

This model was initially studied by physicists Schütz and Trimper [29]; the references in Laulin [21] is indicative of the attention this model received from both physicists and mathematicians in recent years.

Observe that for the memory parameter p,

- when $p = 1/2$, except for the first step, it is the standard simple symmetric random walk,
- when $p > 1/2$ the walk has a propensity to be in the same direction as in the past,
- when $p < 1/2$, it has a propensity of walking in the opposite direction.

For S_n as defined in (1.3), we have (see [3, 4, 10, 11, 13, 19, 24]).

(1) for $p \in (0, 1)$,

$$\lim_{n\to\infty} \frac{S_n}{n} = 0 \quad \text{a.s. and in } L^m \text{ for any } m \geq 1, \tag{1.5}$$

(2) for $p \in (0, 3/4)$,

$$\frac{S_n}{\sqrt{n}} \xrightarrow{\text{d}} N\left(0, \frac{1}{3-4p}\right) \quad \text{as } n \to \infty, \tag{1.6}$$

$$\limsup_{n\to\infty} \pm \frac{S_n}{\sqrt{2n \log\log n}} = \frac{1}{\sqrt{3-4p}} \quad \text{a.s.}, \tag{1.7}$$

(3) for $p = 3/4$,

$$\frac{S_n}{\sqrt{n \log n}} \xrightarrow{\text{d}} N(0, 1) \quad \text{as } n \to \infty, \tag{1.8}$$

$$\limsup_{n\to\infty} \pm \frac{S_n}{\sqrt{2n \log n \log\log\log n}} = 1 \quad \text{a.s.} \tag{1.9}$$

(4) for $p \in (3/4, 1)$, there exists a random variable M s.t.

$$\lim_{n\to\infty} \frac{S_n}{n^{2p-1}} = M \quad \text{a.s. and in } L^2, \tag{1.10}$$

where

$$\mathbb{E}[M] = \frac{2s-1}{\Gamma(2p)}, \quad \mathbb{E}[M^2] > 0, \quad \mathbb{P}(M \neq 0) = 1,$$

and

$$\frac{S_n - Mn^{2p-1}}{\sqrt{n}} \xrightarrow{\text{d}} N\left(0, \frac{1}{4p-3}\right) \quad \text{as } n \to \infty. \tag{1.11}$$

The strong law of large numbers (1.5) is proved in [4, 10], each in a different way. Also the central limit theorem ((1.6) and (1.8)) and the law of the iterated logarithm ((1.7) and (1.9)) can be proved by various approaches, see [3, 4, 10, 11]. The existence of the supercritical limit (1.10) is established in [3, 4, 10], and the almost-sure positivity of M is proved in [13] (see [24] for related results). The Gaussian fluctuation (1.11) in the supercritical phase is proved in [19].

Gut and Stadmüller [15, 16] extended the study of the ERW "when the elephant has only a restricted memory, for example remembering only the most remote step(s) and/or the most recent one(s)".

Laulin [22] studied an ERW model with a 'smooth amnesia' in the sense that, for $\beta > 0$ and $\{\beta_{n+1} : n \in \mathbb{N}\}$ independent random variables given by

$$P(\beta_{n+1} = k) = \begin{cases} \dfrac{\beta+1}{n} \cdot \dfrac{\mu_k}{\mu_{n+1}} & \text{for } 1 \le k \le n \\ 0 & \text{otherwise} \end{cases} \tag{1.12}$$

where

$$\mu_n = \frac{\Gamma(n+\beta)}{\Gamma(n)\Gamma(\beta+1)} \sim \frac{n^\beta}{\Gamma(\beta+1)} \quad \text{as } n \to \infty. \tag{1.13}$$

Here $x_n \sim y_n$ means $x_n/y_n \to 1$ as $n \to \infty$. With (1.1) and, for $n \ge 1$,

$$X_{n+1} = \begin{cases} X_{\beta_{n+1}} & \text{with probability } p \\ -X_{\beta_{n+1}} & \text{with probability } 1-p, \end{cases} \tag{1.14}$$

the ERW $S_n = X_1 + \cdots + X_n$ exhibits a phase transition (Laulin [22]).

- it is diffusive for $p < (4\beta+3)/(4\beta+4)$,
- it is superdiffusive for $p > (4\beta+3)/(4\beta+4)$.

Harbola *et al.* [17] studied a unidirectional ERW given as follows: $s \in [0, 1]$, $p \in (0, 1)$ and $q \in [0, 1]$,

$$S_1 = X_1 := \begin{cases} +1 & \text{with probability } s \\ 0 & \text{with probability } 1-s, \end{cases} \tag{1.15}$$

and, with $\{U_k : k \in \mathbb{N}\}$ as earlier,

$$\begin{aligned} &\text{if } X_{U_n} = +1 \text{ then } X_{n+1} := \begin{cases} +1 & \text{with probability } p \\ 0 & \text{with probability } 1-p, \end{cases} \\ &\text{if } X_{U_n} = 0 \text{ then } X_{n+1} := \begin{cases} +1 & \text{with probability } q \\ 0 & \text{with probability } 1-q. \end{cases} \end{aligned} \tag{1.16}$$

For the unidirectional random walk $S_n := \sum_{k=1}^n X_k$, with $\{X_k : k \in \mathbb{N}\}$ as in (1.15) and (1.16), Harbola *et al.* [17] showed that

$$E[S_n] \sim \begin{cases} \dfrac{qn}{1-q} & \text{if } q > 0 \\ \dfrac{sn^p}{\Gamma(1+p)} & \text{if } q = 0. \end{cases}$$

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When $q = 0$ and $s = 1$ the walk is the 'laziest elephant random walk (LERW)':

$$S_0 = 0,\ S_n := \sum_{k=1}^{n} X_k \text{ with } X_1 \equiv 1,\ X_{n+1} = \begin{cases} X_{U_n} & \text{with probability } p \\ 0 & \text{with probability } 1 - p. \end{cases} \tag{1.17}$$

In the next section we study the Gut and Stadmüller [16] model. In Section 3, we define the unidirectional ERW with a power law memory and present a complete understanding of the model. In Section 4, we present a different proof of the results of Miyazaki and Takei [23] regarding the law of large numbers for the LERW given in (1.17).

2 ERW recalling the remote past

We begin this section with a formal construction of the model studied by Gut and Stadmüller [16]. Let $X_1, X_2, \ldots$ be the increments of an ERW as given in (1.1) and (1.2). Consider a sequence $\{m_n : n \in \mathbb{N}\}$ of positive integers such that

$$\gamma_n := \frac{m_n}{n} \leq 1,\ \lim_{n\to\infty} m_n = +\infty \text{ and } \gamma := \lim_{n\to\infty} \gamma_n \in [0, 1]. \tag{2.1}$$

Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be an independent collection with $\mathcal{U}_n := \{U_{k,n} : m_n < k \leq n\}$ being a collection of i.i.d. uniform random variables over $\{1, \ldots, m_n\}$. Define

$$Y_k^{(n)} := \begin{cases} X_k & \text{for } 1 \leq k \leq m_n \\ X_k^{(n)} & \text{for } m_n < k \leq n, \end{cases} \tag{2.2}$$

where, for $m_n < k \leq n$,

$$X_k^{(n)} = \begin{cases} X_{U_{k,n}} & \text{with probability } p \\ -X_{U_{k,n}} & \text{with probability } 1 - p. \end{cases} \tag{2.3}$$

This provides us a triangular array of random variables $\{\{S_k^{(n)} : 1 \leq k \leq n\} : n \in \mathbb{N}\}$ given by

$$S_k^{(n)} := \sum_{i=1}^{k} Y_i^{(n)}; \tag{2.4}$$

$\{S_k^{(n)} : 1 \leq k \leq n\}$ is a random walk with step size $+1$ or -1. Let

$$T_n := S_n^{(n)}. \tag{2.5}$$

The sequence $\{T_n : n \in \mathbb{N}\}$ does not have any random walk interpretation and it is this process which Gut and Stadtmüller [16] called the process an *ERW with gradually increasing memory*, in contrast to the model with $m_n \equiv m$ (i.e. the elephant remembers only a finite part of the first steps), studied in [15].

The theorem below presents the strong law of large numbers and central limit theorem for the process $\{T_n : n \geq 1\}$:

Theorem 1 (Roy, Takei and Tanemura [28]) *Suppose* (2.1) *holds for a sequence* $\{m_n : n \in \mathbb{N}\}$*. We have, as* $n \to \infty$*,* *(i)* (LLN) *for* $p \in (0,1)$ *and* $c \in (\max\{(2p-1), 1/2\}, 1)$,

$$\lim_{n\to\infty} \frac{\gamma_n T_n}{(m_n)^c} = 0 \quad \textit{almost surely}; \tag{2.6}$$

(ii) (CLT) *for* $a_n = \gamma_n + (2p-1)(1-\gamma_n)$, $a = \gamma + (2p-1)(1-\gamma)$ *we have*

$$\frac{\gamma_n T_n}{\sqrt{m_n}} \xrightarrow{d} N\left(0, \frac{a^2}{3-4p} + \gamma(1-\gamma)\right) \textit{ for } p \in (0, 3/4), \tag{2.7}$$

$$\frac{\gamma_n T_n}{\sqrt{m_n \log m_n}} \xrightarrow{d} N\left(0, \frac{(1+\gamma)^2}{4}\right) \textit{ for } p = 3/4, \tag{2.8}$$

$$\lim_{n\to\infty} \frac{\gamma_n T_n}{(m_n)^{2p-1}} \overset{a.s.}{=} aM \textit{ for } p \in (3/4, 1), \tag{2.9}$$

where M *is a random variable as given in* (1.10).

Moreover, for $p \in (3/4, 1)$,

$$\frac{\gamma_n T_n - M \cdot a_n (m_n)^{2p-1}}{\sqrt{m_n}} \xrightarrow{d} N\left(0, \frac{a^2}{4p-3} + \gamma(1-\gamma)\right) \quad \textit{as } n \to \infty. \tag{2.10}$$

Proof The notation S_n used in this proof will always denote the standard ERW given in (1.3).
Fix $n \in \mathbb{N}$. Along with $\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$, defined in (1.3) let

$$\mathcal{G}^{(n)}_{m_n} = \mathcal{F}_\infty := \sigma(\{X_i : i \in \mathbb{N}\}) = \sigma(\{X_1\} \cup \{U_i : i \in \mathbb{N}\})$$

and, for $k \in (m_n, n] \cap \mathbb{N}$

$$\begin{aligned} \mathcal{G}^{(n)}_k &:= \sigma(\{X_i : i \in \mathbb{N}\} \cup \{X^{(n)}_i : m_n < i \le k\}) \\ &= \sigma(\{X_1\} \cup \{U_i : i \in \mathbb{N}\} \cup \{U_{i,n} : m_n < i \le k\}). \end{aligned}$$

Equations (1.4) and (2.3) yield

$$\mathbb{P}(X^{(n)}_k = \pm 1 \mid \mathcal{G}^{(n)}_{k-1}) = \frac{1}{2}\left(1 \pm (2p-1) \cdot \frac{S_{m_n}}{m_n}\right) \quad \text{for } k \in (m_n, n] \cap \mathbb{N}; \tag{2.11}$$

so that,

$$\begin{aligned} \mathbb{E}[Y^{(n)}_k \mid \mathcal{F}_\infty] &= (2p-1) \cdot \frac{S_{m_n}}{m_n} \quad \text{for } k \in (m_n, n] \cap \mathbb{N}, \\ \mathbb{V}[Y^{(n)}_k \mid \mathcal{F}_\infty] &= \mathbb{E}[(Y^{(n)}_k)^2 \mid \mathcal{F}_\infty] - (\mathbb{E}[Y^{(n)}_k \mid \mathcal{F}_\infty])^2 \\ &= \begin{cases} 0 & \text{for } k \in [1, m_n] \cap \mathbb{N} \\ 1 - (2p-1)^2 \cdot \left(\dfrac{S_{m_n}}{m_n}\right)^2 & \text{for } k \in (m_n, n] \cap \mathbb{N}. \end{cases} \end{aligned} \tag{2.12}$$

Also

$$\mathbb{E}[T_n - S_{m_n} \mid \mathcal{F}_\infty] = \sum_{k=m_n+1}^{n} \mathbb{E}[X_k^{(n)} \mid \mathcal{F}_\infty] = (2p-1)(n-m_n) \cdot \frac{S_{m_n}}{m_n}, \tag{2.13}$$

thus,

$$A_n := \mathbb{E}[T_n \mid \mathcal{F}_\infty] = S_{m_n} + \mathbb{E}[T_n - S_{m_n} \mid \mathcal{F}_\infty] = \frac{a_n S_{m_n}}{\gamma_n} \tag{2.14}$$

and, for $B_n := T_n - A_n$, we have

$$\gamma_n T_n = \gamma_n (A_n + B_n) = a_n S_{m_n} + \gamma_n B_n. \tag{2.15}$$

First we obtain the CLT i.e. Theorem 1 (ii). Assume that $p \in (0, 3/4)$. By (2.14) and (1.6), we have that

$$\frac{\gamma_n A_n}{\sqrt{m_n}} = \frac{a_n S_{m_n}}{\sqrt{m_n}} \xrightarrow{\mathrm{d}} a \cdot N\left(0, \frac{1}{3-4p}\right) \quad \text{as } n \to \infty. \tag{2.16}$$

Also, under $\mathbb{P}(\cdot \mid \mathcal{F}_\infty)$, $\{X_k^{(n)} : k \in (m_n, n] \cap \mathbb{N}\}$ being a collection of i.i.d. random variables,

$$B_n = \sum_{k=m_n+1}^{n} \{X_k^{(n)} - \mathbb{E}[X_k^{(n)} \mid \mathcal{F}_\infty]\} \tag{2.17}$$

is a sum of centred i.i.d. random variables with

$$\begin{aligned} \mathbb{V}\left[\frac{\gamma_n B_n}{\sqrt{m_n}} \,\middle|\, \mathcal{F}_\infty\right] &= \frac{(\gamma_n)^2}{m_n} \cdot (n-m_n) \cdot \left\{1 - (2p-1)^2 \cdot \left(\frac{S_{m_n}}{m_n}\right)^2\right\} \\ &= \gamma_n(1-\gamma_n) \cdot \left\{1 - (2p-1)^2 \cdot \left(\frac{S_{m_n}}{m_n}\right)^2\right\} \qquad (2.18) \\ &\to \gamma(1-\gamma) \text{ a.s., from (1.5), as } n \to \infty. \qquad (2.19) \end{aligned}$$

Thus, for $\xi \in \mathbb{R}$, the characteristic function of $\gamma_n B_n / \sqrt{m_n}$ is

$$\begin{aligned} &\mathbb{E}\left[\exp\left(i \frac{\xi \gamma_n B_n}{\sqrt{m_n}}\right) \,\middle|\, \mathcal{F}_\infty\right] \\ &= \left[1 - \frac{\xi^2 \gamma_n}{2n} \cdot \left\{1 - (2p-1)^2 \cdot \left(\frac{S_{m_n}}{m_n}\right)^2\right\} + o\left(\frac{\gamma_n}{n}\right)\right]^{n-m_n} \\ &\to \exp\left(-\frac{\xi^2}{2} \cdot \gamma(1-\gamma)\right) \quad \text{as } n \to \infty \text{ a.s.,} \qquad (2.20) \end{aligned}$$

and, from (2.16), the characteristic function of $\gamma_n A_n / \sqrt{m_n}$ is

$$\mathbb{E}\left[\exp\left(i \frac{\xi \gamma_n A_n}{\sqrt{m_n}}\right)\right] \to \exp\left(-\frac{(a\xi)^2}{2(3-4p)}\right) \quad \text{as } n \to \infty. \tag{2.21}$$

Combining the above two expressions and using the bounded convergence theorem, from (2.15), we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(i\frac{\xi\gamma_n T_n}{\sqrt{m_n}}\right)\right] \\
&= \mathbb{E}\left[\exp\left(i\frac{\xi\gamma_n A_n}{\sqrt{m_n}}\right)\cdot\mathbb{E}\left[\exp\left(i\frac{\xi\gamma_n B_n}{\sqrt{m_n}}\right)\,\middle|\,\mathcal{F}_\infty\right]\right] \\
&\to \exp\left(-\frac{(a\xi)^2}{2(3-4p)}\right)\cdot\exp\left(-\frac{\xi^2}{2}\cdot\gamma(1-\gamma)\right) \text{ as } n\to\infty
\end{aligned}$$

which completes the proof of (2.7).

To prove (2.8) note that, for $p = 3/4$, $a_n = (1+\gamma_n)/2$ and,

(a) from (2.19) and (1.5), we have

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{\gamma_n B_n}{\sqrt{m_n}}\right)^2\right] &= \gamma_n(1-\gamma_n)\left\{1-(2p-1)^2\cdot\mathbb{E}\left[\left(\frac{S_{m_n}}{m_n}\right)^2\right]\right\} \\
&\to \gamma(1-\gamma) \quad \text{as } n\to\infty,
\end{aligned} \tag{2.22}$$

i.e. $\gamma_n B_n/\sqrt{m_n\log m_n} \to 0$ as $n\to\infty$ in L^2;

(b) from (2.14) and (1.8), we have

$$\frac{\gamma_n A_n}{\sqrt{m_n\log m_n}} = \frac{a_n S_{m_n}}{\sqrt{m_n\log m_n}} \xrightarrow{\text{d}} \frac{1+\gamma}{2}\cdot N(0,1) \quad \text{as } n\to\infty; \tag{2.23}$$

an application of Slutsky's lemma completes the proof of (2.8).

Before we prove (2.9) and (2.10) we first obtain (2.6). Fix $p\in(0,1)$ and $\gamma\in[0,1]$. Since $|X_k^{(n)} - \mathbb{E}[X_k^{(n)} \mid \mathcal{F}_\infty]| \le 1$ from Azuma's inequality (Lemma 1 in [2]) we have that, for $\lambda\in\mathbb{R}$,

$$\begin{aligned}
\mathbb{E}[\exp(\lambda\gamma_n B_n)\mid\mathcal{F}_\infty] &= \mathbb{E}\left[\exp\left(\lambda\gamma_n\sum_{k=m_n+1}^{n}\{X_k^{(n)}-\mathbb{E}[X_k^{(n)}\mid\mathcal{F}_\infty]\}\right)\,\middle|\,\mathcal{F}_\infty\right] \\
&\le \exp((\lambda\gamma_n)^2(n-m_n)/2),
\end{aligned}$$

and, for $x>0$

$$\mathbb{P}(|\gamma_n B_n|\ge x) \le 2\exp\left(-\frac{x^2}{2\gamma_n(1-\gamma_n)m_n}\right).$$

Thus, for any $\varepsilon>0$, we have

$$\sum_{n=1}^{\infty}\mathbb{P}(|\gamma_n B_n|\ge\sqrt{2(1+\varepsilon)\gamma_n(1-\gamma_n)m_n\log n}) \le \sum_{n=1}^{\infty}\frac{2}{n^{1+\varepsilon}},$$

which on an application of the Borel–Cantelli lemma implies

$$\limsup_{n\to\infty}\frac{\gamma_n B_n}{\sqrt{2\gamma_n(1-\gamma_n)m_n\log n}} \le 1 \quad \text{a.s.} \tag{2.24}$$

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Also, for $c \in (1/2, 1)$,

$$\frac{2\gamma_n(1-\gamma_n)m_n \log n}{(m_n)^{2c}} = \frac{2(1-\gamma_n)\log n}{n(m_n)^{2c-2}} \leq \frac{2(1-\gamma_n)\log n}{n^{2c-1}} \to 0 \quad \text{as } n \to \infty,$$

thus,

$$\lim_{n\to\infty} \frac{\gamma_n B_n}{(m_n)^c} = 0 \quad \text{a.s. for } c \in (1/2, 1) \tag{2.25}$$

To prove (2.6) we note that

(a) for $p \in (0, 3/4)$, from (1.7) and (2.14) we have that

$$\limsup_{n\to\infty} \frac{\gamma_n A_n}{\sqrt{2m_n \log\log m_n}} \leq \frac{a}{\sqrt{3-4p}} \quad \text{a.s.,}$$

so, by (2.25), (2.6) holds for any $c \in (1/2, 1)$;

(b) for $p = 3/4$, from (1.9) and (2.14) we have that

$$\limsup_{n\to\infty} \frac{\gamma_n A_n}{\sqrt{2m_n \log m_n \log\log\log m_n}} \leq \frac{1+\gamma}{2}, \quad \text{a.s.}$$

again (2.25), implies that (2.6) holds for any $c \in (1/2, 1)$;

(c) for $p \in (3/4, 1)$, almost sure convergence in (2.9) follows from (2.23) and (2.25).

Thus (2.6) holds for any $c \in (2p-1, 1)$.

Finally we prove (2.9) and (2.10).

Assume that $p \in (3/4, 1)$. First, from (2.22), $\gamma_n B_n/(m_n)^{2p-1} \to 0$ as $n \to \infty$ in L^2. Next, from (2.14) and (1.10), $\gamma_n A_n/(m_n)^{2p-1} \to aM$ as $n \to \infty$ a.s. and in L^2. The almost sure convergence in (2.9) follows from (2.25).

From (2.15), we have

$$\gamma_n T_n - a_n \cdot M \cdot (m_n)^{2p-1} = a_n\{S_{m_n} - M \cdot (m_n)^{2p-1}\} + \gamma_n B_n,$$

and, using (1.11), (2.20) and an argument as in the proof of (2.7), yields (2.10). □

3 A unidirectional elephant random walk with a power law memory

In this section we study a model similar to that of Laulin [22], however for the LERW given by (1.17) with a memory distribution (1.12) for $\beta > -1$. In particular, our model is given by $S_0 = 0$, and with $\{\beta_{n+1} : n \in \mathbb{N}\}$ as in (1.12) for $\beta > -1$,

$$S_n := \sum_{k=1}^{n} X_k \text{ with } X_1 \equiv 1,\ X_{n+1} = \begin{cases} X_{\beta_{n+1}} & \text{with probability } p \\ 0 & \text{with probability } 1-p. \end{cases} \tag{3.1}$$

For this model we obtain three distinct rates of growth of S_n depending on the parameter β. We employ a martingale method.

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Let $\mathcal{F}_0$ be the trivial σ-field, and $\mathcal{F}_n := \sigma(X_1, \dots, X_n)$. From (1.12), we have

$$\begin{aligned} E[X_{n+1} \mid \mathcal{F}_n] &= p \cdot E[X_{\beta_{n+1}} \mid \mathcal{F}_n] = p \cdot \sum_{k=1}^{n} X_k P(\beta_{n+1} = k) \\ &= \frac{p(\beta+1)}{n\mu_{n+1}} \sum_{k=1}^{n} X_k \mu_k = \frac{p(\beta+1)}{n\mu_{n+1}} \cdot \Sigma_n, \end{aligned} \tag{3.2}$$

where $\Sigma_n := \sum_{k=1}^{n} X_k \mu_k$ for $n \in \mathbb{N}$. Noting that $\Sigma_1 = X_1\mu_1 = 1$ a.s. we have

$$E[\Sigma_{n+1} \mid \mathcal{F}_n] = \Sigma_n + E[X_{n+1}\mu_{n+1} \mid \mathcal{F}_n] = \left(1 + \frac{p(\beta+1)}{n}\right) \Sigma_n.$$

For $\gamma > -1$, let

$$c_n(\gamma) := \frac{\Gamma(n+\gamma)}{\Gamma(n)\Gamma(\gamma+1)} \sim \frac{n^{\gamma}}{\Gamma(\gamma+1)} \quad \text{as } n \to \infty. \tag{3.3}$$

Note that $\mu_n = c_n(\beta)$. Put

$$M_n := \frac{\Sigma_n}{c_n(p(\beta+1))}. \tag{3.4}$$

Since $\{M_n\}$ is a non-negative martingale, there exists a non-negative random variable M_∞ such that $\lim_{n\to\infty} M_n = M_\infty$ a.s. As a consequence, as $n \to \infty$

$$E[\Sigma_n] = c_n(p(\beta+1)) \cdot E[\Sigma_1] = c_n(p(\beta+1)) \sim \frac{n^{p(\beta+1)}}{\Gamma(p(\beta+1))}. \tag{3.5}$$

From (3.2) and (3.5) we have for $\beta > -1$,

$$E[S_n] = \begin{cases} \dfrac{p(\beta+1)}{p(\beta+1)-\beta} \cdot \dfrac{c_n(p(\beta+1))}{c_n(\beta)} + \dfrac{\beta}{\beta - p(\beta+1)} & \text{if } \beta \neq \dfrac{p}{1-p} \\ \displaystyle\sum_{k=0}^{n-1} \frac{\beta}{k+\beta} & \text{if } \beta = \dfrac{p}{1-p}. \end{cases}$$

Hence

- If $-1 < \beta < p/(1-p)$ then $\lim_{n\to\infty} \dfrac{E[S_n]}{n^{p(\beta+1)-\beta}} = C(p, \beta)$, where $C(p, \beta) := \dfrac{1}{p(\beta+1)-\beta} \cdot \dfrac{\Gamma(\beta+1)}{\Gamma(p(\beta+1))}$.
- If $\beta = p/(1-p)$ then $\lim_{n\to\infty} \dfrac{E[S_n]}{\log n} = \beta$.
- If $\beta > p/(1-p)$ then $\lim_{n\to\infty} E[S_n] = \dfrac{\beta}{\beta - p(\beta+1)}$, so $S_\infty := \lim_{n\to\infty} S_n < +\infty$ a.s.

From the above we have $E[S_\infty] = +\infty$ for $\beta = p/(1-p)$, however

Proposition 2 (Roy, Takei and Tanemura [26]) *If* $\beta = p/(1-p)$ *then* $P(S_\infty < +\infty) = 1$.

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We provide an outline of the proof of this proposition which uses coupling with an auxiliary multi-type branching process.

Outline of the proof of Proposition 2. With μ_n as defined in (1.13), let $\{q(x, y) : x, y \in \mathbb{N}^2 \cap \{x < y\}\}$, be defined as

$$q(x, y) = p \cdot P(\beta_y = x) = \frac{p(\beta + 1)}{y - 1} \cdot \frac{\mu_x}{\mu_y}. \tag{3.6}$$

We remark that for $\beta > 0$ and $k \in \mathbb{N}$,

$$\sum_{x=k+1}^{\infty} q(k, x) = \frac{p(\beta + 1)}{\beta}. \tag{3.7}$$

Now we construct the branching process. Let $\{Z_1, Z_2, \ldots\}$ be a process where Z_k denotes particles of the k-th generation and $\mathbb{N}$ be the space of types of particles.

(1) The first generation Z_1 consists of only one particle of type $y^{(1)} \equiv 1$.
(2) In case there is no particle in the n-th generation, we stop. Otherwise, a particle of type $y^{(n)}$ in the n-th generation gives birth independently to a $(n + 1)$-th generation particle of type $k > y^{(n)}$ with probability $q(y^{(n)}, k)$. We assume that
    (i) the number and types of children of two distinct particles are independent of each other,
    (ii) the events that a particle has one child of type k and another of type ℓ ($k \neq \ell$) are independent of each other. We note here that, for $n \geq 3$, there may be two particles in Z_n of the same type born of two distinct parents.

From (3.7) we see that the expected number of children of a particle of any type is $p(\beta + 1)/\beta$, which is smaller or equal to one if and only if $\beta \geq p/(1 - p)$. We are able to prove that the branching process dies out in this case, by the same argument as the Galton-Watson branching process, although the progeny distribution is different and depends on the type. By using a comparison argument, we can derive Proposition 2 from the extinction of the branching process at the critical point. □

In Theorem 2.3 in [26] it is shown that if $-1 < \beta < p/(1 - p)$ then $P(M_\infty > 0) > 0$, and

$$S_n \sim C(p, \beta) M_\infty n^{p(\beta+1)-\beta} \quad \text{a.s. on } \{M_\infty > 0\}.$$

Taking

$$\Omega_\infty(p, \beta) := \{M_\infty > 0,\ S_n \sim C(p, \beta) M_\infty n^{p(\beta+1)-\beta} \text{ as } n \to \infty\}, \tag{3.8}$$

we have the following proposition.

Proposition 3 (Roy, Takei and Tanemura [27]) *Assume that* $p \in (0, 1)$.

(i) If $\beta \in (-1, 0]$ *then* $P(\Omega_\infty(p, \beta)) = 1$.
(ii) If $\beta \in (0, p/(1 - p))$ *then* $0 < P(S_\infty = \infty) < 1$ *and* $P(\Omega_\infty(p, \beta) \mid S_\infty = \infty) = 1$.

A sketch of the proof of the above propositions using a graphical representation of process is given later. This allows us to have a definitive understanding of the asymptotic behaviour of S_n in the different regimes as presented in Table 1.

Table 1 Behaviour of S_n in different regimes

Regime	Asymptotic behaviour
$-1 < \beta \le 0$	$P(\Omega_\infty(p, \beta)) = 1.$
$0 < \beta < \dfrac{p}{1-p}$	$0 < P(S_\infty = +\infty) < 1, P(\Omega_\infty(p, \beta) \mid S_\infty = +\infty) = 1.$
$\beta = \dfrac{p}{1-p}$	$E[S_n] \sim \beta \log n$, but $P(S_\infty < +\infty) = 1.$
$\beta > \dfrac{p}{1-p}$	$E[S_\infty] < +\infty$, so $P(S_\infty < +\infty) = 1.$

Moreover, taking

$$W_n := S_n - C(p, \beta) M_\infty n^{p(\beta+1)-\beta}, \tag{3.9}$$

we obtain the central limit theorem for $\{W_n\}$ in different regimes of β. Let η be a non-negative random variable defined by

$$\eta = \sqrt{\frac{p^2(\beta+1)^2 + \beta^2}{(p(\beta+1)-\beta)^2} \cdot C(p, \beta) \cdot M_\infty}. \tag{3.10}$$

The following theorem is derived by applying the following martingale CLT, which is Exercise 6.2 based on Theorem 6.1 in Häusler and Luschgy [18], p.86, with $\mathcal{G}_{n,k} = \mathcal{F}_k$.

Theorem 4 (Roy, Takei and Tanemura [27]) *Assume that* $p \in (0, 1)$ *and let* $N \overset{d}{=} N(0, 1)$.

(i) If $\beta \in (-1, p/(1-p))$ *then* $\dfrac{W_n}{\sqrt{n^{p(\beta+1)-\beta}}} \xrightarrow{d} \eta \cdot N$ *as* $n \to \infty$, *where* N *is independent of* η.

(ii) If $\beta \in (-1, 0]$ *then* $P(\eta > 0) = 1$ *and* $\dfrac{W_n}{\eta\sqrt{n^{p(\beta+1)-\beta}}} \xrightarrow{d} N$ *as* $n \to \infty$. *If* $\beta \in (0, p/(1-p))$ *then* $\{\eta > 0\} = \{S_\infty = \infty\}$ *a.s., and* $\dfrac{W_n}{\eta\sqrt{n^{p(\beta+1)-\beta}}} \xrightarrow{d} N$ *as* $n \to \infty$ *under* $P(\cdot \mid S_\infty = \infty)$.

We prepare a graphical representation of the model to prove Proposition 3. Let α_k be independent random variables such that $\alpha_k = 1$ with probability p and $= 0$ with probability $1-p$. For $j, k \in \mathbb{N}$ with $j < k$ we write $j \Leftarrow k$ if $\alpha_k = 1$ and $\beta_k = j$, and $j \leftarrow k$ if there is an increasing sequence $\{\ell_i\}_{i=0}^p$ of $\mathbb{N}$ with $\ell_0 = j, \ell_p = k$ such that $\ell_i \Leftarrow \ell_{i+1}, i = 0, 1, \ldots, p-1$. Let

$$\begin{aligned}
&\eta^{(0)} = \{1\}, \quad \eta^{(1)} = \{i \in \mathbb{N} : \beta_i = 1\} =: \{Y_j^{(1)}\}_{j=1}^{\#\eta^{(1)}}, \\
&\eta^{(m)} = \{i \in \mathbb{N} : k \Leftarrow i \text{ for some } k \in \eta^{(m-1)}\} =: \{Y_j^{(m)}\}_{j=1}^{\#\eta^{(m)}}, m \ge 2,
\end{aligned}$$

where $Y_j^{(m)} < Y_{j+1}^{(m)}$, $j \in \mathbb{N}$. We set $\eta_n^{(m)} = \eta^{(m)} \cap \{1, 2, \ldots, n\}$. We introduce another process defined as

$$\begin{aligned}
&\zeta^{(m,j)} = \{Y_j^{(m)}\} \cup \{i : Y_j^{(m)} \leftarrow i\}, \quad j = 1, 2, \ldots, \#\eta^{(m)}, \\
&\zeta_n^{(m,j)} = \zeta^{(m,j)} \cap \{1, 2, \ldots, n\}.
\end{aligned}$$

We put $\xi = \{k \in \mathbb{N} : X_k = 1\}$ and $\xi_n := \xi \cap \{1, 2, \ldots, n\}$. Then we have

$$\xi = \eta^{(0)} \cup \eta^{(1)} \cup \left\{ \bigcup_{j=1}^{\infty} \zeta^{(1,j)} \right\}. \tag{3.11}$$

We note that $S_n = \#\xi_n$, and $\#\eta^{(1)} \begin{cases} = \infty & \text{a.s. if } \beta \in (-1, 0] \\ < \infty & \text{a.s. if } \beta \in (0, \infty). \end{cases}$

Outline of the proof of Proposition 3. We introduce a modified version of the process.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of 0's and 1's, and let

$$\mathbb{S} := \{k \in \mathbb{N} : x_k = 1\}. \tag{3.12}$$

Let $\{\tilde{\beta}_{n+1} : n \in \mathbb{N}\}$ be a collection of independent random variables on the same probability space but with a probability measure $P^{\mathbb{S}}$ given by

$$P^{\mathbb{S}}(\tilde{\beta}_{n+1} = k) = \begin{cases} w(n,k) := \dfrac{x_k \mu_k}{\sum_{\ell=1}^{n} \mu_\ell} & \text{for } 1 \le k \le n \\ 1 - \sum_{\ell=1}^{n} w(n,\ell) & \text{for } k = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

For $s_1 = k \in \mathbb{N}$, let

$$\tilde{X}_\ell = 0 \text{ for } 0 \le \ell \le k-1,\ \tilde{X}_k = 1, \text{ and}$$
$$\text{for } n \ge k,\ \tilde{X}_{n+1} = \begin{cases} x_{n+1} \tilde{X}_{\tilde{\beta}_{n+1}} & \text{with probability } p \\ 0 & \text{with probability } 1-p. \end{cases}$$

The modified models for S_n and Σ_n are given by $\tilde{S}_0 = \Xi_0 = 0$,

$$\tilde{S}_n := \sum_{k=1}^{n} \tilde{X}_k \quad \text{and} \quad \tilde{\Sigma}_n := \sum_{k=1}^{n} \mu_k \tilde{X}_k,\ n \in \mathbb{N}, \tag{3.14}$$

respectively. Let $1 \le s_1 < s_2 < \cdots$ be the ordering of all elements of $\mathbb{S}$ and

$$m_n = m_n(\mathbb{S}) := \#\{k \in \mathbb{N} : s_1 < k \le n, x_k = 0\}. \tag{3.15}$$

We assume that $\{x_n\}_{n\in\mathbb{N}}$ satisfies the following: there exists $N_0 = N_0(\mathbb{S})$ such that

$$m_n \le n^{p(\beta+1)-\beta} \quad \text{for all } n \ge N_0. \tag{3.16}$$

Note that $p(\beta+1) - \beta \in (0, 1)$ for $\beta \in (-1, p/(1-p))$. Then we have the following estimates. See Lemma 2.4 in [27] for the proof.

Lemma 5 *Let* $p \in (0, 1)$. *For all* $\mathbb{S}$ *as in* (3.12) *satisfying* (3.16), *we have*

(i) For $\beta \in (-1, 0)$, *there is a positive constant* $K = K(p, \beta)$ *not depending on* $\mathbb{S}$ *such that* $P^{\mathbb{S}}(\tilde{S}_n \asymp n^{p(\beta+1)-\beta}) \ge 1/K$. *Here* $a_n \asymp b_n$ *means that as* $n \to \infty$, $ca_n \le b_n \le Ca_n$ *for some* $0 < c \le C < \infty$, *which may depend*

on $\beta > -1$, $p \in (0,1)$, *and* $\mathbb{S}$.

(ii) Suppose $m \in \mathbb{N}$ *with* $s_m = N_0$, *where* N_0 *is defined in* (3.16). *Put*

$$\hat{\mathbb{S}} = \{\hat{s}_i\}_{i\in\mathbb{N}} := \{s_{m-1+i}\}_{i\in\mathbb{N}}. \tag{3.17}$$

For $\beta \in [0, p/(1-p))$, *there is a positive constant* $K = K(p,\beta)$ *not depending on* $\mathbb{S}$ *such that* $P^{\hat{\mathbb{S}}}(\tilde{S}_n \asymp n^{p(\beta+1)-\beta}) \geq 1/K$.

We set

$$\Lambda_j^{(m)} = \begin{cases} \left\{ \lim_{n\to\infty} \dfrac{\#\zeta_n^{(m,j)}}{n^{p(\beta+1)-\beta}} = 0 \right\}, & \text{if } j \leq \#\eta^{(m)}, \\ \Omega, & \text{otherwise.} \end{cases} \tag{3.18}$$

We consider the case $\beta \in (-1, 0)$. The proof of the other case can be carried out by repeating similar techniques. For details, refer to [26]. Because $\lim_{n\to\infty} \dfrac{S_n}{n^{p(\beta+1)-\beta}}$ exists,

$$\begin{aligned} P\left(\lim_{n\to\infty} \frac{S_n}{n^{p(\beta+1)-\beta}} = 0 \right) &\leq P\left(\bigcap_{j=1}^{\infty} \Lambda_j^{(1)} \right) \\ &= P\left(\Lambda_1^{(1)}\right) \prod_{j=1}^{\infty} P\left(\Lambda_{j+1}^{(1)} \,\middle|\, \bigcap_{\ell=1}^{j} \Lambda_\ell^{(1)} \right). \end{aligned} \tag{3.19}$$

Note that $k \notin \bigcup_{\ell=1}^{j} \zeta^{(1,\ell)}$ means $\beta_k \notin \bigcup_{\ell=1}^{j} \zeta^{(1,\ell)}$. Then, for any $\mathbb{S} \subset \mathbb{N}$, β_k under the conditional probability $P\left(\cdot \,\middle|\, \mathbb{N} \setminus \bigcup_{\ell=1}^{j} \zeta^{(1,\ell)} = \mathbb{S} \right)$ stochastically dominates $\tilde{\beta}_k$ in (3.13) for any $k \in \mathbb{S}$. Under the conditional probability, β_k, $k \in \mathbb{S}$ are independent, so, for any $A_i \subset \mathbb{S}$ and $\{k_i\}_{i=1}^m \in \mathbb{S}$, $m \in \mathbb{N}$,

$$P\left(\beta_{k_i} \in A_i,\ 1 \leq i \leq m \,\middle|\, \mathbb{N} \setminus \bigcup_{\ell=1}^{j} \zeta^{(1,\ell)} = \mathbb{S} \right) \geq P^{\mathbb{S}}(\tilde{\beta}_{k_i} \in A_i,\ 1 \leq i \leq m). \tag{3.20}$$

On the event $\bigcap_{\ell=1}^{j} \Lambda^{(1,\ell)}$, $\mathbb{S} = \mathbb{N} \setminus \bigcup_{\ell=1}^{j} \zeta^{(1,\ell)}$ satisfies the condition (3.16). Then from Lemma 5 (i) and (3.20),

$$P\left(\Lambda_{j+1}^{(1)} \,\middle|\, \bigcap_{\ell=1}^{j} \Lambda_\ell^{(1)} \right) \leq 1 - \frac{1}{K},$$

and by (3.19) we have $P\left(\lim_{n\to\infty} \dfrac{S_n}{n^{p(\beta+1)-\beta}} = 0 \right) = 0$. □

4 The laziest elephant random walk

In this section, we prove the law of large numbers (Theorem 6 below) for the laziest elephant random walk (LERW) $\{S_n : n \geq 0\}$ given in (1.17). The proof found in Miyazaki and Takei [23] is based on the computation of the factorial moments. Here we give another proof adapted from Bercu [5], which is much more transparent and elegant.

For any $a \in \mathbb{R}$, the Pochhammer symbol of a is defined by

$$a^{(0)} := 1, \quad \text{and} \quad a^{(m)} := a(a+1)\cdots(a+m-1) \text{ for } m \in \mathbb{N}.$$

Note that for any $m \in \mathbb{Z}_+$ and any $a, b \in \mathbb{R}$,

$$(a+b)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} a^{(k)} b^{(m-k)}. \tag{4.1}$$

For the sake of completeness, we give a short proof of (4.1). The generalized binomial theorem says that

$$(1+x)^{\alpha} = \sum_{m=0}^{\infty} \binom{\alpha}{m} x^m \quad \text{for } \alpha \in \mathbb{R} \text{ and } |x| < 1.$$

Since $\binom{-a}{0} = 1 = a^{(0)}$ and

$$\binom{-a}{m} = \frac{(-a)(-a-1)\cdots(-a-m+1)}{m!} = \frac{(-1)^m a^{(m)}}{m!} \quad \text{for } m \in \mathbb{N},$$

we have

$$\sum_{m=0}^{\infty} \frac{(-1)^m a^{(m)}}{m!} x^m = (1+x)^{-a} \quad \text{for } a \in \mathbb{R} \text{ and } |x| < 1.$$

From the obvious identity $(1+x)^{-(a+b)} = (1+x)^{-a} \cdot (1+x)^{-b}$, we can see that

$$\frac{(-1)^m (a+b)^{(m)}}{m!} = \sum_{k=0}^{m} \frac{(-1)^k a^{(k)}}{k!} \cdot \frac{(-1)^{m-k} b^{(m-k)}}{(m-k)!},$$

which is equivalent to (4.1).

We compute $E[S_n^{(m)}]$, the Pochhammer moments (a.k.a. the rising factorial moments) of S_n. For any $m \in \mathbb{N}$,

$$S_{n+1}^{(m)} = (S_n + X_{n+1})^{(m)} = \sum_{k=0}^{m} \binom{m}{k} S_n^{(k)} X_{n+1}^{(m-k)}, \tag{4.2}$$

which implies that

$$E[S_{n+1}^{(m)} \mid \mathcal{F}_n] = \sum_{k=0}^{m} \binom{m}{k} S_n^{(k)} E[X_{n+1}^{(m-k)} \mid \mathcal{F}_n]. \tag{4.3}$$

Using $(X_{n+1})^2 = X_{n+1}$, it is straightforward to see that $X_{n+1}^{(k)} = k! \cdot X_{n+1}$ for any $k \in \mathbb{N}$. This together with $E[X_{n+1} \mid \mathcal{F}_n] = (pS_n)/n$ implies that

$$E[S_{n+1}^{(m)} \mid \mathcal{F}_n] = S_n^{(m)} + \sum_{k=0}^{m-1} \binom{m}{k} S_n^{(k)} \cdot (m-k)! \cdot \frac{pS_n}{n}$$

$$= S_n^{(m)} + \frac{m! \cdot p}{n} \sum_{k=0}^{m-1} \frac{1}{k!} S_n^{(k)} \cdot S_n,$$

and hence

$$E[S_{n+1}^{(m)}] = E[S_n^{(m)}] + \frac{m! \cdot p}{n} \sum_{k=0}^{m-1} \frac{1}{k!} E[S_n^{(k)} \cdot S_n]. \tag{4.4}$$

Noting that $E[S_n^{(k)} \cdot S_n] = E[S_n^{(k+1)}] - k \cdot E[S_n^{(k)}]$,

$$\begin{aligned} E[S_{n+1}^{(m)}] &= E[S_n^{(m)}] + \frac{m! \cdot p}{n} \left\{ \frac{E[S_n^{(1)}]}{0!} + \sum_{k=1}^{m-1} \left(\frac{E[S_n^{(k+1)}]}{k!} - \frac{E[S_n^{(k)}]}{(k-1)!} \right) \right\} \\ &= E[S_n^{(m)}] + \frac{m! \cdot p}{n} \cdot \frac{E[S_n^{(m)}]}{(m-1)!} = \left(1 + \frac{mp}{n}\right) E[S_n^{(m)}]. \end{aligned} \tag{4.5}$$

Since $S_1^{(m)} = X_1^{(m)} = 1^{(m)} = m!$, we have $E[S_1^{(m)}] = m!$ for $m \in \mathbb{N}$. From (4.5),

$$E[S_n^{(m)}] = E[S_1^{(m)}] \cdot \prod_{k=1}^{n-1} \left(1 + \frac{mp}{k}\right) = \frac{m! \cdot \Gamma(n+mp)}{\Gamma(n)\Gamma(1+mp)}. \tag{4.6}$$

The LERW $\{S_n : n \geq 0\}$ is the unidirectional elephant random walk in Section 3, with $\beta = 0$. Noting that $\{\Sigma_n : n \geq 0\} = \{S_n : n \geq 0\}$, $M_n := S_n/c_n(p)$ is a non-negative martingale, and there exists a non-negative random variable M_∞ such that $\lim_{n\to\infty} M_n = M_\infty$ almost surely.

The *Mittag–Leffler function* is defined by

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)} \qquad (\alpha, z \in \mathbb{C}).$$

Note that $E_1(z) = e^z$ (See e.g. [8], p. 315). The random variable X is *Mittag–Leffler distributed* with parameter $p \in [0, 1]$ if

$$E[e^{\lambda X}] = E_p(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(1+kp)} \qquad \text{for } |\lambda| < 1.$$

Thus the k-th moment of X is $\dfrac{k!}{\Gamma(1+kp)}$, and this distribution is determined by moments (see [8], p. 329 and p. 391). If $p = 0$ (resp. $p = 1$), then X has the exponential distribution with mean one (resp. X concentrates on $\{1\}$). For $p \in (0, 1)$, the probability density function $f_p(x)$ of X is

$$f_p(x) = \frac{\rho_p(x^{-1/p})}{p x^{1+1/p}} \qquad \text{for } x > 0,$$

where $\rho_p(x)$ is the density function of the one-sided stable(p) distribution. In particular, $f_{1/2}$ is the density function of the standard half-normal distribution:

$$f_{1/2}(x) = \sqrt{\frac{2}{\pi}} e^{-x^2/2} \qquad \text{for } x > 0.$$

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Theorem 6 (Miyazaki and Takei [23]) *Assume that $p \in (0,1)$. For any $m \in \mathbb{N}$,*

$$\lim_{n\to\infty} E\left[\left(\frac{S_n}{n^p}\right)^m\right] = \frac{m!}{\Gamma(1+mp)}. \tag{4.7}$$

Thus the martingale $\{M_n : n \geq 1\}$ is L^m-bounded for any m, and the almost sure limit

$$\mathcal{ML} := \lim_{n\to\infty} \frac{S_n}{n^p} = \frac{M_\infty}{\Gamma(1+p)} \tag{4.8}$$

has a Mittag–Leffler distribution with parameter p. In particular, $P(\mathcal{ML} > 0) = 1$.

Proof of Theorem 6: We prove (4.7) by induction. For $m = 1$, we have

$$E[S_n] = E[S_n^{(1)}] = \frac{\Gamma(n+p)}{\Gamma(n)\Gamma(1+p)} \sim \frac{n^p}{\Gamma(1+p)} \quad \text{as } n \to \infty.$$

Assume that $\ell \in \mathbb{N}$ and

$$E[S_n^m] \sim \frac{m! \cdot n^{mp}}{\Gamma(1+mp)} \quad \text{for } m = 1, \ldots, \ell \tag{4.9}$$

hold true. Using the Maclaurin expansion $x^{(\ell+1)} = x^{\ell+1} + \sum_{k=0}^{\ell} c_{\ell+1,k}\, x^k$,

$$E[S_n^{\ell+1}] + \sum_{k=0}^{\ell} c_{\ell+1,k} \cdot E[S_n^k] = E[S_n^{(\ell+1)}] = \frac{(\ell+1)! \cdot \Gamma(n+(\ell+1)p)}{\Gamma(n)\Gamma(1+(\ell+1)p)}.$$

By (4.9), we obtain

$$\lim_{n\to\infty} E\left[\left(\frac{S_n}{n^p}\right)^{\ell+1}\right] = \frac{(\ell+1)!}{\Gamma(1+(\ell+1)p)},$$

which is (4.7) with $m = \ell + 1$. This completes the proof. □

Remark 1 Eq. (4.1) is a consequence of the fact that $\{x^{(m)}\}$ is a Sheffer sequence of binomial type. See Chapter 19 of [25]. The coefficients $\{c_{m,k}\}$ in $x^{(m)} = \sum_{k=0}^{m} c_{m,k}\, x^k$ are called unsigned Stirling numbers of the first kind. See e.g. p.224 of [30].

5 Some questions

In this section we list some questions connected to the previous three sections.

First, in the context of the triangular array set-up of Section 2 we may have an alternate construction of an ERW which recalls its remote past. Consider an ERW, $W_{n+1} := W_n + Z_{n+1}$ given by its increments

$$W_1 = Z_1 = \begin{cases} +1 & \text{with probability } s \\ -1 & \text{with probability } 1-s, \end{cases} \tag{5.1}$$

and

$$Z_{n+1} = \begin{cases} Z_{V_n} & \text{with probability } p \\ -Z_{V_n} & \text{with probability } 1-p, \end{cases} \tag{5.2}$$

where V_n is a uniform random variable over $\{1, \ldots, m_n\}$ (m_n being as in Section 2) and $\{V_n : n \in \mathbb{N}\}$ is an independent collection.

This is the linear setting of Roy, Takei and Tanemura [28]. It is apparent that the methods used in obtaining Theorem 1 will not work in this setting. A challenging question is to obtain suitable law of large numbers (LLN) and central limit theorem (CLT) for this linear setting.

In the context of the unidirectional ERW with power law memory, described in Sections 3 and 4 a natural question is to extend Theorem 6 for $-1 < \beta < 0$ when $\Omega_\infty(p, \beta)$ (defined in (3.6)) occurs almost surely.

Also for the unidirectional ERW with power law memory law of iterated logarithm and functional central limit theorems are not yet known.

In one dimension, several properties of the limit in the supercritical regime are studied by Guérin, Laulin and Raschel [13] and Guérin, Laulin, Raschel, and Simon [14]. What can we say about the supercritical limit for the ERW with stops, studied in Bercu [5]? In particular, the positivity problem is still open.

The higher dimensional analogue of the standard ERW is studied in Bercu and Laulin [7], and the recurrence problem is solved by Qin [24] (see also Curien and Laulin [12]). How about the recurrence problem for the higher dimensional analogue of the ERW with power law memory introduced in Chen and Laulin [9]?

Conflict of Interest There is no competeing interest.

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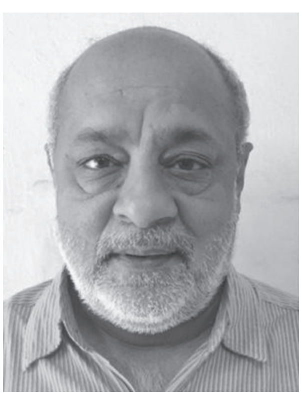

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