ORIGINAL RESEARCH

Partial differential equations in the hyperbolic space : A survey

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Abstract
In this article, we survey several results on partial differential equations in the hyperbolic space. The topics covered include various Sobolev and functional inequalities and related problems, as well as semilinear elliptic, heat, Schrödinger, and wave equations in this setting.

Keywords Hyperbolic space · Sobolev Inequalities · Heat · Schrödinger · Wave

Mathematics Subject Classification 35-02 · 35R01 · 35J60 · 35K58

1 Introduction

Partial Differential Equations (PDEs) on manifolds form an important area of mathematics, connecting various topics such as geometry, analysis, and mathematical physics. Among non-Euclidean geometries, the hyperbolic space occupies a special place due to its constant negative curvature and rich symmetric structure. The study of PDEs in hyperbolic space not only deepens our understanding of geometric analysis but also reveals phenomena absent in the Euclidean setting, such as the influence of curvature on decay, regularity, and uniqueness properties of solutions.

The study of PDEs in hyperbolic space also appears in various other contexts. For example, it is known that PDEs related to the so-called Hardy–Sobolev–Maz'ya type equations can be lifted to the hyperbolic space using certain transformations after establishing partial symmetry (see [29, 30, 60]). Under the assumption of partial symmetry of functions, the PDEs related to the Grushin operators can also be lifted to the hyperbolic space, as shown in [12]. Therefore, many aspects of these problems can be inferred from the corresponding results in the hyperbolic space. Two of the most important aspects that make the hyperbolic setting quite different from the Euclidean case are the volume growth and the spectrum. While in Euclidean space the volume of an open ball $B(x, R)$ is of the order R^n, in the hyperbolic space it grows exponentially, namely as $e^{(n-1)R}$ as $R \to \infty$. This exponential growth of volume affects the behaviour of solutions of many nonlinear problems at infinity, for example, their decay. We know that the L^2- spectrum of $-\Delta$ in $\mathbb{R}^n$ is $[0, \infty)$, while it is $[\frac{(n-1)^2}{4}, \infty)$ for the Laplace–Beltrami operator in the hyperbolic space. This spectral gap also makes the study of elliptic-type PDEs in the hyperbolic space different from that in Euclidean space in many respects. Another important aspect of the hyperbolic space is the presence of an ideal boundary, which in the case of the ball model can be identified with the boundary of the ball. Unlike Euclidean space, one may even consider solutions with prescribed values at this ideal boundary in the case of hyperbolic space.

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In recent years, there have been many works dealing with partial differential equations in the hyperbolic space, often motivated by the mathematical challenges it presents or by their necessity in other contexts such as those mentioned earlier. The study of PDEs in this setting requires optimal inequalities. There have been many works dealing with Sobolev inequalities, Moser–Trudinger–Adams inequalities, Hardy's inequalities, etc., in the hyperbolic space. In recent years, the stability of inequalities has also become a major topic of study, and substantial progress has been made in this direction. The rich symmetric structure of hyperbolic space also makes it an ideal setting to investigate the symmetry properties of solutions to partial differential equations. Many results have been obtained in this direction. The existence and multiplicity of solutions to various PDEs, as well as concentration phenomena for critical semilinear equations, have also been investigated in this context. In this article, we will survey some of these results.

By a hyperbolic space we mean a complete, simply connected Riemannian manifold of constant sectional curvature -1. There are many models of hyperbolic space, the most prominent being the ball model, the upper half-space model, and the hyperboloid model. For the study of partial differential equations, when explicit calculations are needed, one may choose any of these convenient representations. For more details on hyperbolic space, we refer to [73].

Notation We denote a general hyperbolic space of dimension n by $\mathbb{H}^n$, its Riemannian volume element by $dv_{\mathbb{H}^n}$, and the corresponding gradient and Laplace–Beltrami operator by $\nabla_{\mathbb{H}^n}$ and $\Delta_{\mathbb{H}^n}$, respectively. The L^p and Sobolev spaces in this context are denoted by $L^p(\mathbb{H}^n)$ and $W^{1,p}(\mathbb{H}^n)$, respectively.

2 Sobolev embedding and other functional inequalities

Sobolev inequalities are fundamental tools in the analysis of partial differential equations. In particular, for studying PDEs on hyperbolic space, it is essential to establish sharp versions of Sobolev and related functional inequalities.

Sobolev inequalities on Riemannian manifolds have been a topic of intense study for many decades. Their existence, best constants, and extremals have been of interest to many mathematicians; see [47] for a detailed discussion on this topic. In the case of hyperbolic space, one of the earliest known inequality is the isoperimetric inequality established in [75] which in modern terminology states:

Isoperimetric Inequality: Let E be a measurable subset of $\mathbb{H}^n$ then $P(E) \geq P(B)$, for any geodesic ball B satisfying $\mu(E) = \mu(B)$ where μ is the Riemmannian measure and $P(E)$ is the perimeter of E in $\mathbb{H}^n$.

It also follows from the general results in [64] and [78] that the Poincaré inequality

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^p dv_{\mathbb{H}^n} \geq \left(\frac{n-1}{p}\right)^p \int_{\mathbb{H}^n} |u|^p dv_{\mathbb{H}^n} \tag{2.1}$$

holds for functions $u \in W^{1,p}(\mathbb{H}^n)$. Moreover, the constant appearing in the above inequality is optimal. We also have the Euclidean-type Sobolev inequality

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^p dv_{\mathbb{H}^n} \geq C_{n,p} \left(\int_{\mathbb{H}^n} |u|^{p^*} dv_{\mathbb{H}^n}\right)^{\frac{p}{p^*}}, \quad p^* = \frac{np}{n-p} \tag{2.2}$$

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which holds for all $u \in W^{1,p}(\mathbb{H}^n)$, $1 \le p < n$; see Chapter 8 of [47] for a proof of this inequality in more general settings.

Combining (2.1) and (2.2), we obtain the Poincaré–Sobolev inequality, which states:

Poincaré–Sobolev Inequality: Let $1 \le p < n$. Then, for any $q \in (p, p^*]$ and $\lambda < \left(\frac{n-1}{p}\right)^p$, there exists an optimal constant $S_{\lambda,q} (= S_{\lambda,n,p,q}) > 0$ such that

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^p dv_{\mathbb{H}^n} - \lambda \int_{\mathbb{H}^n} |u|^p dv_{\mathbb{H}^n} \ge S_{\lambda,q} \left(\int_{\mathbb{H}^n} |u|^q dv_{\mathbb{H}^n} \right)^{\frac{p}{q}} \tag{2.3}$$

holds for all $u \in W^{1,p}(\mathbb{H}^n)$.

One can easily show that the above inequality is not true for $\lambda > \left(\frac{n-1}{p}\right)^p$. We also expect this inequality to hold for $\lambda = \left(\frac{n-1}{p}\right)^p$, which is known when $p = 2$ (see [60]), $p = n$ (see [62]), and for $\frac{2n}{n-1} \le p < n$ when $n \ge 4$ (see [66]).

Hardy Inequality: Another inequality that plays an important role in the analysis of PDEs is the so-called Hardy inequality. In the Euclidean case, it reads as

$$\int_{\mathbb{R}^n} |\nabla u(x)|^2 dx \ge \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2} dx, \quad \forall u \in C_c^\infty(\mathbb{R}^n),\ n \ge 3.$$

In the last three decades, many improvements of this inequality have been obtained, and various issues related to it have been a topic of intense study. We now review the developments of this type of inequality in the hyperbolic space.

Among many other results, Carron (see [27]) established the Hardy inequality

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u(x)|^2 dv_{\mathbb{H}^n} \ge \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{H}^n} \frac{|u(x)|^2}{d^2(x, x_0)} dv_{\mathbb{H}^n}, \quad \forall u \in C_c^\infty(\mathbb{H}^n),\ n \ge 3,$$

where x_0 is any fixed point in $\mathbb{H}^n$, and $d(x, x_0)$ denotes the hyperbolic distance between x and x_0.

An improvement of this inequality was obtained in [17], where it was shown that

$$\begin{aligned} \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u(x)|^2 dv_{\mathbb{H}^n} - \left(\frac{n-1}{2}\right)^2 \int_{\mathbb{H}^n} u^2 dv_{\mathbb{H}^n} \\ \ge \frac{1}{4} \int_{\mathbb{H}^n} \frac{|u(x)|^2}{d^2(x, x_0)} dv_{\mathbb{H}^n} + C_n \int_{\mathbb{H}^n} \frac{|u(x)|^2}{\sinh^2(d(x, x_0))} dv_{\mathbb{H}^n}, \end{aligned}$$

where $C_n = \frac{(n-1)(n-3)}{4}$. The above inequality is also sharp in a certain sense, as explained in [17]. An optimal version of the Hardy inequality in the hyperbolic space was established in [19]. The above inequality can be viewed as a Hardy inequality with one pole; improvements of this type, dealing with multiple poles, were obtained in [18].

The above Hardy inequality also has a p-version for $1 < p < n$, which states

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u(x)|^p dv_{\mathbb{H}^n} \geq \left(\frac{n-p}{p}\right)^p \int_{\mathbb{H}^n} \frac{|u(x)|^p}{(d(x,x_0))^p} dv_{\mathbb{H}^n}, \quad \forall u \in C_c^\infty(\mathbb{H}^n).$$

See [33], Theorem 6.5, for a proof and more general weighted versions of these inequalities. A vesion of the Caffarelli-Kohn-Nirenberg inequality was established in [74] and as a consequence the embedding of $W^{1,2}(\mathbb{H}^n)$ in to the Lorentz space was also derived.

Moser–Trudinger and Adams Inequalities: It is not very difficult to see that the best constant in (2.3) when $q = p^*$, say S_{λ,p^*}, tends to zero as $p \to n$. The Sobolev embedding corresponding to this limiting case $p = n$ is known as the Moser–Trudinger inequality, which states that

$$\sup_{u \in W^{1,n}(\mathbb{H}^n), \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^n dv_{\mathbb{H}^n} \leq 1} \int_{\mathbb{H}^n} E_{n-1}(\alpha |u|^{\frac{n}{n-1}}) dv_{\mathbb{H}^n} < \infty \tag{2.4}$$

if and only if $\alpha \leq \alpha_n := n\omega_{n-1}^{\frac{1}{n-1}}$, where $E_k(x) = \sum_{i=k}^{\infty} \frac{x^i}{i!}$. Moreover, when $\alpha > \alpha_n$, the above supremum is infinite. This inequality was established in the two-dimensional case in [59] (see also [2] for an alternative proof) and extended to higher dimensions in the Master's theses of Gabriele Mancini and Luca Battaglia. It was further developed in [55], covering various other extensions. There have been many other improvements of this inequality incorporating singularities and weakening the norm. We mention a recent result from [67], which states that

$$\sup_{u \in W^{1,n}(\mathbb{H}^n), \int_{\mathbb{H}^n} [|\nabla_{\mathbb{H}^n} u|^n - (\frac{n-1}{n})^n |u|^n] dv_{\mathbb{H}^n} \leq 1} \int_{\mathbb{H}^n} \frac{E_{n-1}(\alpha(1-\beta/n)|u|^{\frac{n}{n-1}})}{(\tanh(d(x,x_0)/2))^\beta} dv_{\mathbb{H}^n} < \infty \tag{2.5}$$

for $0 \leq \beta < n$ and $x_0 \in \mathbb{H}^n$. References for many previous developments beginning with [62] and [55], and subsequent advancements, can be found in the references of [67].

The Adams inequalities, which concern the embedding of the Sobolev space $W^{k,p}(\mathbb{H}^n)$ when $kp = n$, were established in [35] and [51]. The inequality in [51] was motivated by the problem of prescribing the Q-curvature, which amounts to studying the Paneitz operator with exponential nonlinearity. On the other hand, the results of [35] established that

$$\sup_{u \in W^{k,p}(\mathbb{H}^n), \int_{\mathbb{H}^n} |\nabla^k_{\mathbb{H}^n} u|^p dv_{\mathbb{H}^n} \leq 1} \int_{\mathbb{H}^n} E_{[p-1]}(\alpha |u|^{\frac{p}{p-1}}) dv_{\mathbb{H}^n} < \infty \tag{2.6}$$

if and only if $\alpha \leq \alpha_0(k,n)$, where $\alpha_0(k,n)$ is explicitly given, $[p-1]$ denotes the largest integer less than or equal to $p-1$, and $\nabla^k_{\mathbb{H}^n}$ denotes the k-th order gradient.

As in the case of the Moser–Trudinger inequality, there have been many extensions of these inequalities to various settings, such as fractional spaces, exact growth versions, and singular versions. For further developments, we refer to [50, 56, 57, 68] and the references therein.

3 Extremals and stability

In this section, we provide a detailed discussion of the optimal Sobolev inequality (2.3), focusing initially on the case $p = 2$:

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dv_{\mathbb{H}^n} - \lambda \int_{\mathbb{H}^n} |u|^2 dv_{\mathbb{H}^n} \geq S_{\lambda,q} \left(\int_{\mathbb{H}^n} |u|^q dv_{\mathbb{H}^n} \right)^{\frac{2}{q}}, \quad u \in W^{1,2}(\mathbb{H}^n), \tag{3.1}$$

where $n \geq 2$, $0 \leq \lambda \leq \frac{(n-1)^2}{4}$, and $2 < q \leq 2^*$.

In [60], it was shown that the best constant $S_{\lambda,q}$ in (3.1) is attained if and only if $n \geq 2$ and $2 < q < 2^*$, or when $q = 2^*$, $n \geq 4$, and $\frac{n(n-2)}{4} < \lambda \leq \frac{(n-1)^2}{4}$. Moreover, the uniqueness of extremals (up to isometry and multiplication by a constant) follows from the uniqueness result for (5.1) (see below) established in [60].

Explicit forms of extremals are known in certain special cases, allowing for the computation of $S_{\lambda,q}$ in these instances (see [60]). When $q = 2^*$, the results of [60] show that $S_{\lambda,2^*} \leq S$, where S denotes the best constant in the Euclidean Sobolev inequality:

$$S = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^2 dx : \int_{\mathbb{R}^n} |u|^{2^*} dx = 1, \ u \in C_c^1(\mathbb{R}^n) \right\}. \tag{3.2}$$

Furthermore, $S_{\lambda,2^*} = S$ when $n = 3$ or $n = 4$ and $0 \leq \lambda \leq \frac{n(n-2)}{4}$ (see [60], [13]), and strict inequality in the remaining cases.

In the general case $1 < p < n$, $S_{\lambda,q}$ is achieved when $p < q < p^*$ and $S_{\lambda,p^*} \leq S_p$ where S_p is the best constant in the Euclidean Sobolev inequality defined like (3.2) in this context, see [34] for details.

From a variational perspective, two crucial aspects of (3.1) are: (i) the non-degeneracy properties of the extremals, and (ii) the compactness properties of the associated energy functional

$$J(u) := \frac{1}{2} \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dv_{\mathbb{H}^n} - \frac{\lambda}{2} \int_{\mathbb{H}^n} |u|^2 dv_{\mathbb{H}^n} - \frac{1}{q} \int_{\mathbb{H}^n} |u|^q dv_{\mathbb{H}^n}, \quad u \in W^{1,2}(\mathbb{H}^n), \tag{3.3}$$

or, more precisely, the convergence behavior of its Palais–Smale sequences. The first aspect was fully analyzed in [40], where it was shown that the kernel of the linearized operator has dimension n. In [23], general Palais–Smale sequences of J were studied and shown, in general, to consist of a solution of the corresponding PDE together with a finite number of bubbles of two types: hyperbolic bubbles, generated by translating a solution to infinity, and point-concentrating bubbles, analogous to standard Aubin–Talenti bubbles.

Quantitative stability addresses the question: if a function nearly attains the optimal constant in an inequality, how close is it to an actual minimizer? Let $U \in W^{1,2}(\mathbb{H}^n)$ be an extremal, i.e., equality holds in (3.1) for $u = U$ with $U \neq 0$. Then the set of all extremals of (3.1) is

$$\mathcal{E} := \{cU \circ T : c \in I\!R \setminus \{0\}, \ T \text{ is an isometry}\}.$$

For $u \in W^{1,2}(\mathbb{H}^n)$, denote its distance to $\mathcal{E}$ by

$$\mathrm{dist}(u,\mathcal{E}) := \inf_{v\in\mathcal{E}} \left(\int_{\mathbb{H}^n} (|\nabla_{\mathbb{H}^n}(u-v)|^2 - \lambda(u-v)^2) dv_{\mathbb{H}^n} \right)^{\frac{1}{2}},$$

and define the deficit $\delta(u)$ in the inequality as

$$[\delta(u)]^2 := \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dv_{\mathbb{H}^n} - \lambda \int_{\mathbb{H}^n} |u|^2 dv_{\mathbb{H}^n} - S_{\lambda,q} \left(\int_{\mathbb{H}^n} |u|^q dv_{\mathbb{H}^n} \right)^{\frac{2}{q}}.$$

It was shown in [21] that there exists a constant $C(n,\lambda,q) > 0$ such that

$$C(n,\lambda,q)\,\delta(u) \leq \mathrm{dist}(u,\mathcal{E}), \quad \forall u \in W^{1,2}(\mathbb{H}^n).$$

The same work establishes quantitative stability for Palais–Smale sequences at the lowest ”bad” energy level: if u_k is a Palais–Smale sequence of J at level $\frac{q-2}{2q}(S_{\lambda,q})^{\frac{q}{q-2}}$, then

$$C(n,\lambda,q)\,\mathrm{dist}(u_k,\mathcal{E}) \leq \|DJ(u_k)\|_{H^{-1}(\mathbb{H}^n)}, \quad \forall k.$$

Further results on general Palais–Smale sequences were obtained in [22], where sharp quantitative stability estimates for the Poincaré–Sobolev inequality were proved in dimensions $3 \leq n \leq 5$ for $p > 2$. Furthermore the conditions on n, p are optimal. These results rely on refined estimates of the interactions of hyperbolic bubbles and their derivatives.

A quantitative version of the isoperimetric inequality in the hyperbolic space was established in [25]. They showed that for any $R > 0$, there exists a constant $C_{n,R} > 0$ such that the inequality

$$P(E) - P(B) \geq C_{n,R}\beta^2(E)$$

holds for all geodesic balls B of radius $r \in (0, R]$ and any subset E of finite perimeter with $\mu(E) = \mu(B)$, where $\beta(E)$ is the L^2 oscillation index of E as defined in (1.5) of [25].

The existence of extremals for the Moser–Trudinger inequality (2.4) remains an open problem. A maximizing sequence may fail to converge due to either concentration of mass at a point or loss of mass at infinity. Following Euclidean analogues, one can estimate the maximum attained if the sequence concentrates; similarly, if the sequence vanishes at infinity, the corresponding maximum can be calculated. Interestingly, the maximum attained via vanishing exceeds that of the value attained via concentration. Hence, understanding whether a maximizing sequence loses mass at infinity is central to the problem. Work in this direction remains inconclusive (see [61, 62]). However, the existence of extremals has been established for certain modified energy functionals (see, e.g., [62]).

4 Symmetry and symmetrization

In this section, we discuss the symmetry properties of solutions to semilinear and quasilinear elliptic problems in the hyperbolic space. We begin by recalling symmetrization and related properties in this setting.

Let $f : \mathbb{H}^n \to \mathbb{R}$ be a function satisfying $\mu(\{x \in \mathbb{H}^n : |f(x)| > t\}) < \infty$ for all $t > 0$, where μ denotes the volume measure. Its distribution function μ_f is defined by

$$\mu_f(t) := \mu(\{x \in \mathbb{H}^n : |f(x)| > t\}), \quad \forall t > 0,$$

and its decreasing rearrangement is

$$f^*(t) = \sup\{s > 0 : \mu_f(s) > t\}, \quad \forall t > 0.$$

For a fixed point $o \in \mathbb{H}^n$, the Schwarz symmetrization $f^\#$ is defined as the radial function

$$f^\#(x) := f^*(\mu(B(o, d(o, x)))), \quad x \in \mathbb{H}^n.$$

The Pólya–Szegö inequality (see [7], Chapter 7) then states that, for any $p \in [1, \infty)$,

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u^\#|^p dv \le \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^p dv, \quad \forall u \in W^{1,p}(\mathbb{H}^n).$$

This symmetrization tool, combined with the principle of symmetric criticality (see [69]), can be used to produce radial solutions to many semilinear and quasilinear PDEs in hyperbolic space, and to reduce minimization problems in $W^{1,p}(\mathbb{H}^n)$ to their radial subspaces, thereby simplifying estimates considerably (see [34, 59]).

While the above approach yields radially symmetric solutions, an important and challenging question is whether all positive solutions of semilinear and quasilinear PDEs in hyperbolic space are radially symmetric.
A fundamental tool in establishing the symmetry of positive solutions for elliptic PDEs in Euclidean space is the *method of moving planes*, introduced by Serrin in the study of over-determined problems, and later adapted by Gidas, Ni, and Nirenberg to prove symmetry for semilinear elliptic PDEs (see [42]).

In bounded domains in the hyperbolic space, the method of moving planes was employed by Prajapat and Kumaresan to establish results analogous to those of Serrin and Gidas–Ni–Nirenberg (see [52, 53]). The ”planes” in this context are totally geodesic hyperplanes, so that reflection across a hyperplane is an isometry and commutes with the Laplace–Beltrami operator. The method was further developed for certain semilinear elliptic equations in the whole hyperbolic space in [3]. A consequence of these results is that any solution of

$$-\Delta_{\mathbb{H}^n} u - \lambda u = u^{q-1}, \quad u \ge 0 \text{ in } \mathbb{H}^n, \quad \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dv_{\mathbb{H}^n} < \infty$$

is radially symmetric with respect to some point in $\mathbb{H}^n$ when $\lambda \le \frac{n(n-2)}{4}$ and $2 < q \le \frac{2n}{n-2}$. Later, this was refined in [60] to hold for $\lambda \le \frac{(n-1)^2}{4}$. In particular, this implies that extremals of (2.3), when they exist, are radially symmetric for $p = 2$. One of the main challenges in this approach is initiating the moving plane from infinity, which is managed using the Sobolev inequality (2.3) and small-measure-type arguments.

The Euler–Lagrange equation corresponding to an extremal of (2.3) is

$$-\Delta_p^{\mathbb{H}^n} u - \lambda u^{p-1} = u^{q-1}, \quad u \ge 0 \text{ in } \mathbb{H}^n, \quad \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^p dv_{\mathbb{H}^n} < \infty, \tag{4.1}$$

where $\Delta_p^{\mathbb{H}^n}$ denotes the p-Laplace operator, $\Delta_p^{\mathbb{H}^n} u := \mathrm{div}_{\mathbb{H}^n}(|\nabla_{\mathbb{H}^n} u|^{p-2} \nabla_{\mathbb{H}^n} u)$. Symmetry questions for (4.1) are particularly challenging due to the lack of a strong comparison principle and the difficulty of moving planes from infinity.

The method of moving planes for the p-Laplace equation in Euclidean space has been studied since the mid-1990s. The symmetry of solutions to the Euler–Lagrange equation associated with the Sobolev inequality,

$$-\Delta_p u = u^{q-1}, \quad u \geq 0 \text{ in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} |\nabla u|^p dx < \infty,$$

was established in a series of works (see [76] and references therein). Analogous developments in Euclidean space suggest that, in order to apply the moving plane method to solutions of (4.1), one requires regularity estimates for u as well as sharp upper and lower bounds for u and $|\nabla_{\mathbb{H}^n} u|$ at infinity.

In [34], it was established that any solution of (4.1) is radially symmetric with respect to some point $O \in \mathbb{H}^n$. That is, there exists a strictly positive, decreasing function $\Phi : [0, \infty) \to \mathbb{R}_+$ such that

$$u(x) = \Phi(\mathrm{dist}_{\mathbb{H}^n}(O, x)), \quad \Phi \in C^1 \text{ at } t = 0, \quad \Phi'(0) = 0, \quad \Phi \text{ smooth for } t > 0.$$

Moreover, Φ satisfies the pointwise bounds $ce^{-\alpha t} \leq \Phi(t) \leq Ce^{-\alpha t}$ for all $t \geq 0$, and $ce^{-\alpha t} \leq -\Phi'(t) \leq Ce^{-\alpha t}$ for all $t \geq 1$, where $c, C > 0$ depend on Φ. Furthermore, $\lim_{t\to\infty} \Phi'(t)/\Phi(t) = -\alpha$, where α is the unique positive root of $|\alpha|^{p-2}\alpha(n-1-(p-1)\alpha) = \lambda$ satisfying $\alpha \in \left(\frac{n-1}{p}, \frac{n-1}{p-1}\right]$.

A crucial part of the proof involves deriving sharp upper and lower bounds for u and its gradient. The upper and lower bounds for u are obtained via suitable barriers, while the gradient estimate from above follows from the Harnack inequality. Establishing the lower bound is more delicate and is achieved using blow-up analysis and a classification result for eigenfunctions with a prescribed singularity at infinity.

Symmetry of nonnegative solutions for a higher-order Brezis–Nirenberg problem associated with the Paneitz operator was established in [54] using an integral representation, which circumvents difficulties arising from the lack of a strong comparison principle.

One-dimensional symmetry questions for

$$-\Delta_{\mathbb{H}^n} u = f(u), \quad f = F',$$

with F a double-well potential, were investigated in [24]. It was shown that any bounded global solution reduces to a function of one variable if it has prescribed asymptotic boundary values at infinity and if it is invariant under a cohomogeneity-one subgroup of the isometry group of $\mathbb{H}^n$. One-dimensional symmetry results for the fractional version of this equation have been studied in [43].

5 Semilinear and quasilinear elliptic problems

In this section, we review some of the PDEs studied in the hyperbolic space. Unlike Euclidean space, it is known that bounded, non-constant harmonic functions exist in hyperbolic space. The solutions of $\Delta_{\mathbb{H}^n} u = 0$ with prescribed values at the boundary at infinity have been extensively studied in hyperbolic space and, more generally, in negatively curved complete Riemannian manifolds (see [4, 79]).

For semilinear elliptic equations, the Brezis–Nirenberg problem in bounded domains and geodesic balls in hyperbolic space was investigated in [77]. When considered on the whole space $\mathbb{H}^n$ with a finite energy condition, this problem becomes the Euler–Lagrange equation for (3.1), namely

$$-\Delta_{\mathbb{H}^n} u - \lambda u = u^{q-1}, \quad \text{in } \mathbb{H}^n, \quad \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dv_{\mathbb{H}^n} < \infty. \tag{5.1}$$

Positive solutions of (5.1) have been analyzed in detail in [60], where necessary and sufficient conditions for existence were obtained. When $\lambda > \frac{(n-1)^2}{4}$, the operator on the left-hand side is no longer positive definite, so positive solutions require $\lambda \leq \frac{(n-1)^2}{4}$. In the subcritical regime $2 < q < 2^*$, the problem admits a solution if and only if $\lambda < \frac{(n-1)^2}{4}$. In the borderline case $\lambda = \frac{(n-1)^2}{4}$, solutions exist in a larger energy space. In the critical case $q = 2^*$, solutions exist if and only if $\frac{n(n-2)}{4} < \lambda \leq \frac{(n-1)^2}{4}$ and $n \geq 4$; in particular, there are no solutions for $n = 3$. The uniqueness of solutions to (5.1) holds up to isometries, except in the case $n = 2$ and $\lambda \geq \frac{2q}{(q+2)^2}$. Infinite-energy and singular solutions were studied in [8, 16, 26].

Existence and multiplicity of sign-changing solutions were studied in [23, 39]. In the subcritical case, there are infinitely many sign-changing solutions (see [23]). In the critical case, there are no sign-changing solutions when $0 \leq \lambda \leq \frac{n(n-2)}{4}$, and infinitely many sign-changing solutions when $n \geq 7$ and $\lambda > \frac{n(n-2)}{4}$ (see [39]).

The existence and non-existence of positive solutions for the Brezis–Nirenberg problem with a logarithmic perturbation in hyperbolic space was studied in [41]. Several related problems addressing existence, nonexistence, and multiplicity have also been investigated (see [20, 32, 36]). Another class of semilinear problems considered are Henon-type equations:

$$-\Delta_{\mathbb{H}^n} u = (\sinh(\text{dist}(x, x_0)))^{\alpha} |u|^{p-1} u,$$

see [46] and references therein.

The equation (4.1), which is the Euler–Lagrange equation for (2.3), represents the quasilinear version of the Brezis–Nirenberg problem. It was shown in [34] that, for $q < p^*$, (4.1) admits a solution. Some partial results are also available for the critical case (see [34]).

The higher-order Brezis–Nirenberg problem in hyperbolic space, involving the Paneitz operator, was studied using harmonic analysis techniques in [54].

Semilinear noncompact problems in dimension two take the form $-\Delta_{\mathbb{H}^n} u = f(x, u)$, where f is of exponential type. These problems were considered in [37], where existence results were established after careful analysis of Palais–Smale sequences. Stability of solutions for $f(u) = e^u$ was investigated in [15].

Various existence and multiplicity results were proved for solutions with and without one-dimensional symmetry for problems of the form $-\Delta_{\mathbb{H}^n} u = f(u) = F'(u)$, where F is a double-well potential (see [24]). Multiple-layer solutions were established in [63], and existence results for corresponding fractional equations are found in [43].

Phase transition problems in hyperbolic space were investigated in [72]. Specifically, local minimizers of the energy functional

$$J(u, \Omega) = \frac{1}{2} \int_{\Omega} |\nabla_{\mathbb{H}^n} u|^2 dv_{\mathbb{H}^n} + \frac{1}{4\epsilon^2} \int_{\Omega} (1 - u^2)^2 dv_{\mathbb{H}^n}$$

taking values 1 and -1 at the boundary at infinity were studied, and the limiting behavior of these solutions as $\epsilon \to 0$ was analyzed.

In the fully nonlinear context, some works, such as [1], explore the relationship between absolutely minimizing Lipschitz extensions of functions defined on subsets of hyperbolic space and the viscosity solutions of PDEs arising from the associated variational problems.

6 Evolution problems

Various evolution equations, including the heat equation, Schrödinger equation, Navier–Stokes type equations, and their semilinear versions, have been studied in the hyperbolic space. We review some key results in these areas.

The linear heat equation

$$u_t - \Delta_{\mathbb{H}^n} u = 0 \quad \text{in } \mathbb{H}^n \times (0, \infty), \quad u(x, 0) = u_0$$

has been extensively studied. Under reasonable assumptions, the solution can be explicitly written as

$$u(x, t) = \int_{\mathbb{H}^n} P(x, y, t) u_0(y) dv_{\mathbb{H}^n}(y),$$

where $P(x, y, t)$ is the heat kernel on $\mathbb{H}^n$; see [44] for an explicit formula. In analogy with the Euclidean case, the asymptotic behavior of solutions with initial data in L^1 has been analyzed. In [81], it was shown that for radially symmetric solutions with finite mass, the solution converges to a multiple of the fundamental solution.

A well-studied problem in the Euclidean setting is the blow-up of solutions to the semilinear heat equation with power-type nonlinearity, characterized by the Fujita exponent : finite-time blow-up for all solutions occurs if the exponent is below a critical value (Fujita exponent), while global solutions exist only for powers above this exponent. This phenomenon has been investigated in hyperbolic space, where the large-time behavior of the heat kernel differs from the Euclidean case. In [9], it was shown that the hyperbolic situation resembles that of bounded Euclidean domains. It was also shown that in order to observe the Fujita phenomenon in hyperbolic space, the nonlinearity must include a weight depending exponentially on time. These results were improved in [82], and an analogous problem by fixing the weight function as a polynomial in t and searching for the growth in u to exhibit the Fujita Phenomenon was studied in [38]. Similar developments can be seen in [45].

Another evolution problem is the porous-medium equation

$$u_t - \Delta_{\mathbb{H}^n} u^m = 0, \quad m > 1.$$

In [80], the fundamental solution was shown to be radial, compactly supported, and non-increasing in the radial variable. Its short-time asymptotics are close to the Euclidean case, while long-time behavior includes a logarithmic term due to curvature effects. A fractional version of this equation,

$$u_t + (-\Delta_{\mathbb{H}^n})^s u^m = 0, \quad 0 < s < 1, m > 1,$$

was studied in [14].

The Schrödinger equation in $\mathbb{H}^n$ has been studied extensively due to its connection with harmonic analysis. The solution of the initial value problem

$$i u_t + \Delta_{\mathbb{H}^n} u = 0, \quad u(x, 0) = u_0,$$

can be expressed as $u(x, t) = u_0 * S_t(x)$, where the kernel S_t has an explicit formula; see Section 3 of [5] for details and estimates.

The semilinear Schrödinger Cauchy problem

$$i u_t + \Delta_{\mathbb{H}^n} u = f(u), \quad u(x, 0) = u_0$$

relies fundamentally on estimates on the solutions of the linear problem $iu_t + \Delta_{\mathbb{H}^n} u = f(x,t), \quad u(x,0) = u_0$, known as Strichartz estimates :

$$\|u\|_{L_t^p L_x^q} \leq C\|u_0\|_{L_x^2} + C\|f\|_{L_t^{p'} L_x^{q'}}.$$

Initial progress for establishing such inequalities for radial functions was made in [10, 70], further developed in [11]. Strichartz estimates for a wider class of admissible pairs (p, q) were obtained by Anker [5]. Using these estimates, well-posedness and scattering results for semilinear equations with power-type nonlinearities have been established. In particular, global well-posedness and scattering in $W^{1,2}(\mathbb{H}^n)$ for defocusing power-type nonlinearities were proved in [49], and for energy-critical defocusing problems in [48]. Certain ranges of pairs for which Strichartz estimates fail were identified in [28].

For the wave equation in $\mathbb{H}^n$, Strichartz estimates for radial data were obtained in [71]. The three-dimensional hyperbolic case was analyzed in [65], establishing dispersive and Strichartz estimates, with applications to small-data global well-posedness for semilinear wave equations. These results were generalized to higher dimensions in [6]. There have been many developments in these directions, for example the recent work [58] on global well-posedness and scattering for the defocusing conformal nonlinear wave equation.

Other evolution equations, such as the Navier–Stokes equation, have also been studied in hyperbolic space. Novel phenomena distinct from Euclidean settings have been observed; see [31].

7 Concluding remarks

As we have discussed in this article, a considerable amount of work in various directions has been carried out in the study of PDEs in hyperbolic space. There are many more results and directions that we have not touched upon in order to keep the length of this article within a reasonable limit. Even among the topics covered, several fundamental questions remain unanswered. We conclude this article by mentioning a few of them.

Question 1 Does there exist an extremal for the Moser–Trudinger inequality (2.4)?
The above question will be answered affirmatively if one can show that the supremum in (2.4) is strictly greater than β_n, where

$$\beta_n := \frac{n^{n-1}\omega_{n-1}}{(n-1)!}\left(\frac{n}{n-1}\right)^n.$$

This would also imply that, in the case of nonexistence of an extremal, the supremum in (2.4) is equal to β_n , see [61, 62] for details.

Question 2 Existence of an extremal for (2.3) when $q = p^*$?
The range of λ for which the extremal exists has been completely characterized for $p = 2$ in [60]. For general p, only partial results are available (see [34]). The uniqueness of extremals of (2.3) also remains open, except in the case $p = 2$.

Question 3 Can one compute the best constants $S_{\lambda,q}$ in (2.3)?
These constants are known only in a few cases, as mentioned in Section 3.

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