ORIGINAL RESEARCH

Multiplicities and density functions in commutative algebra

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Abstract
In this expository article we discuss the numerical invariant *Hilbert-Kunz multiplicity* and provide numerical characterizations of integral dependence of ideals, where the main focus is on the graded set up.
To emphasize the subtlety of the characteristic p techniques to non-expert readers, we give some self contained proofs and sketches of some proofs. In the later part we give an introduction of the theory of density functions and their applications to Hilbert-Kunz multiplicities and briefly describe how these functions along with basic analysis provide numerical characterizations of integral dependence of ideals.

1 Introduction

Multiplicities are numerical invariants which were introduced to study 'singularities' in commutative algebra and geometry. One of the earliest examples is the *Hilbert-Samuel multiplicity*, which was introduced by D. Rees [44]. A positive characteristic variant of the Hilbert-Samuel multiplicity is *Hilbert-Kunz multiplicity* e_{HK}, which was introduced in a formal algebraic way by P. Monsky.

Other than singularities, Hilbert-Samuel multiplicity also characterizes the *integral dependence* of ideals which are of finite colengths. The other multiplicities such as ε-multiplicities, j-multiplicities were introduced to characterize the integral dependence of ideals where the ideals under consideration are not necessarily of finite colengths. Similarly the Hilbert-Kunz multiplicity characterizes the *tight closure dependence* of the ideals. Over the time Hilbert-Kunz multiplicity has turned out to be an important characteristic p-invariant in other ways; here we will give some examples to demonstrate that.

In this expository article we discuss these multiplicities. However the material is far from being exhaustive and mostly focuses on the graded set up, which means the rings are standard graded rings and the ideals are graded ideals.

To emphasize the subtlety of the characteristic p techniques to non-expert readers, we give a self contained proof of the existence of Hilbert-Kunz multiplicity and show that it is an additive function. For this we have drawn on material from the paper [40] of P. Monsky and also from the expository article of C. Huneke [27]. We also pose some open questions. Here we make use of standard notations and terminology as well as basic results from commutative algebra (see [35]).

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2 Hilbert-Kunz multiplicity e_{HK}

Throughout this section $(R, \mathbf{m}, k)$ will denote a Noetherian local ring of prime characteristic $p > 0$ with maximal ideal $\mathbf{m}$ and residue field k. Also R is of dimension $d \geq 1$.

Since the characteristic of R is $p > 0$ the endomorphism map

$$F : R \longrightarrow R \quad \text{given by} \quad a \rightarrow a^p$$

is a ring homomorphism.

Further for any integer $e \geq 1$ we get the e^{th}-iterated Frobenius endomorphism $F^e : R \longrightarrow R$ where $a \rightarrow a^{p^e}$. This gives the ring R a different R-module structure, *i.e.*, via the map F^e. To emphasize this module structure on R, we would write the map $F^e : R \longrightarrow R$ as $F^e : R \longrightarrow F_*^e R$ and would denote the elements of $F_*^e R$ as $F_*^e r$ if $r \in R$. With this notation $F_*^e R$ is a module over R with the module structure given as follows:

(1) If $F_*^e r_1, F_*^e r_2 \in F_*^e R$ then $F_*^e r_1 + F_*^e r_2 = F_*^e(r_1 + r_2)$ and
(2) if $r_1 \in R$ and $r_2 \in F_*^e R$ then $r_1 \cdot F_*^e r_2 = F_*^e(r_1^{p^e} r_2)$.

Similarly any R-module M has another R-module structure, namely via the map F^e. Again we denote this module structure on M by $F_*^e M$ and the element of $F_*^e M$ as $F_*^e m$, if $m \in M$.

With this notation M and $F_*^e M$ both are the same as an additive group but the scalar multiplication on $F_*^e M$ is given as follows:

$$\text{if} \quad r \in R \quad \text{and} \quad m \in F_*^e M \quad \text{then} \quad r \cdot F_*^e m = F_*^e(r^{p^e} m).$$

Remark 2.1 We note that $F_*^e M$ as $F_*^e R$-module is the same as M as R-module, *i.e.*, for $F_*^e r \in F_*^e R$ and $F_*^e m \in F_*^e M$ we have $F_*^e r \cdot F_*^e m = F_*^e(rm)$. Hence we have a commutative diagram

$$\begin{array}{ccc} F_*^e R \times F_*^e M & \longrightarrow & F_*^e M \\ || & & || \\ R \times M & \longrightarrow & M, \end{array}$$

where we have identified the ring $F_*^e R$ with R via the map $F_*^e r \rightarrow r$, similarly we have identified the $F_*^e M$ with M via the map $F_*^e m \rightarrow m$.

Now if I is an ideal of R with generators $\{x_1, x_1, \ldots, x_n\}$ and $q = p^e$ then

$$I \cdot F_*^e R = \sum_i F_*^e(x_i^q R) \simeq \sum_i x_i^q R =: I^{[q]},$$

where the second identification is as mentioned in Remark 2.1.

E. Kunz has been one of the early ones to realize the importance of positive characteristic methods in commutative algebra. We proceed by recalling two fundamental results of his (in [32]) in characteristic p.

Theorem 1 (E. Kunz). *If R is a Noetherian ring of prime characteristic $p > 0$ then it is regular if and only if the map $F^e : R \longrightarrow F_*^e R$ is flat for some $e > 0$ (equivalently for all $e > 0$).*

In particular if R is F-finite, that is, if $F_* R$ is a finite R-module, then the theorem of Kunz implies that the notion of R being regular ring is same as $F_*^e R$ being a locally free R-module, for some e (equivalently for all $e > 0$).

The following theorem of Kunz shows that the colengths of the Frobenius powers of the maximal ideal decide the regular property of the ring.

Theorem 2 (E. Kunz). *Let $(R, \mathbf{m}, k)$ be a Noetherian local ring of dimension d and characteristic $p > 0$. Then for $e \geq 1$ and $q = p^e$, we have*

$$\ell_R(R/\mathbf{m}^{[q]}) \geq q^d.$$

Moreover if the equality holds for some $e \geq 1$ then R is regular.

Remark 2.2 To study these length functions we can assume without loss of generality that $(R, \mathbf{m})$ is complete and the residue field is algebraically closed.

For this we take the $\mathbf{m}$-adic completion $\widehat{R}$ of R. Now $\widehat{R}$ contains a coefficient field k. Then we take an algebraic closure $\bar{k}$ of the field k. Let $S = \widehat{R}\widehat{\otimes}_k\bar{k}$ then

(1) S is a complete local ring with maximal ideal $\mathbf{m}S$ and
(2) S is a faithfully flat extension of R and
(3) the residue field of S is $S/\mathbf{m}S = \bar{k}$, an algebraically closed field.

Now, if I is an $\mathbf{m}$-primary ideal and M is a finitely generated R-module then

$$\ell_R\Big(\frac{M}{I^{[q]}M}\Big) = \ell_S\Big(\frac{M}{I^{[q]}M} \otimes_R S\Big) = \ell_S\Big(\frac{M \otimes_R S}{I^{[q]}M \otimes_R S}\Big) = \ell_S\Big(\frac{M \otimes_R S}{I^{[q]}(M \otimes_R S)}\Big),$$

where the first and second equality follow from property (1) and (2) ively.

Remark 2.3 Let I and M be as in Remark 2.2. Then to study the length function $\ell_R M/I^{[q]}M)$ we can assume that $(R, \mathbf{m})$ is a complete local ring where $R/\mathbf{m}$ is an algebraically closed field. In particular R is F-finite and moreover, for any $e \geq 1$,

$$\ell_R\Big(\frac{F_*^e M}{I \cdot F_*^e M}\Big) = \ell_{F_*^e R}\Big(\frac{F_*^e M}{I \cdot F_*^e M}\Big) \cdot \ell_k\Big(\frac{F_*^e k}{k}\Big) = \ell_{F_*^e R}\Big(\frac{F_*^e M}{F_*^e I^{[q]}M}\Big) = \ell_R\Big(\frac{M}{I^{[q]}M}\Big). \tag{2.1}$$

Notation 2.4 (1) M is a finitely generated R-module and in that case $\mu_R(M)$ denotes the minimal number of generators of M.
(2) I denotes an $\mathbf{m}$-primary ideal of R.
(3) $e(I, M) = \lim_{n\to\infty} \ell_R(M/I^n M)d!/n^d$ denotes the (Hilbert-Samuel) multiplicity of M with respect to I.
(4)

$$\Lambda(R) = \{P \in \operatorname{Spec} R \mid \dim\ R/P = \dim\ R\}.$$

(5) If $f, g : \mathbb{N} \longrightarrow \mathbb{R}$ are two functions then we denote $f(n) = O(g(n))$ if there exists a positive constant C such that $|f(n)| \leq C \cdot g(n)$, for all n.

Now we state and prove the following lemma which is the first assertion of Theorem 2 due to Kunz.

Lemma 2.5 *Let $(R, \mathbf{m}, k)$ be a Noetherian local ring of dimension d and characteristic $p > 0$. Then for $e \geq 1$ and $q = p^e$*

$$\ell_R(R/\mathbf{m}^{[q]}) \geq q^d.$$

Proof We can assume that R is complete and has algebraically closed residue field k. Further we can assume R is a domain by replacing R by R/P, where $P \in \Lambda(R)$.

We choose (see Theorem 14.14 in [35]) a reduction ideal $(x_1, \ldots, x_d)$ of $\mathbf{m}$. Now consider the ring homomorphism $A = k[[X_1, \ldots, X_d]] \longrightarrow R$ mapping $X_i \to x_i$. Then, by Theorem 8.4 in [Ma], R is a finitely generated A-module, and therefore the map $A \longrightarrow R$ is an injective map.

Consider the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & R \\ \downarrow & & \downarrow \\ F_*^e A & \longrightarrow & F_*^e R \end{array}$$

It is easy to check that $F_*^e A$ is a free A-module of rank $p^{ed} = q^d$, generated by $\{F_*^e(x_1^{i_1}), \cdots, F_*^e(x_d^{i_d}) \mid 0 \le i_j < q = p^e\}$. We have

$$\operatorname{rank}_A F_*^e R = \big(\operatorname{rank}_R F_*^e R\big)\big(\operatorname{rank}_A R\big) = \big(\operatorname{rank}_{F_*^e A}(F_*^e R)\big)\big(\operatorname{rank}_A F_*^e A\big).$$

But $\operatorname{rank}_A R = \operatorname{rank}_{F_*^e A}(F_*^e R)$ which implies $\operatorname{rank}_R F_*^e R = \operatorname{rank}_A F_*^e A = q^d$.

If $T = R \setminus \{0\}$ then

$$\operatorname{rank}_R F_*^e R = \mu_{T^{-1}R}(T^{-1}R \otimes_R F_*^e R) \le \mu_R(F_*^e R) = \ell_R\Big(\frac{F_*^e R}{\mathbf{m} \cdot F_*^e R}\Big) = \ell_R\left(R/\mathbf{m}^{[q]}\right).$$

□

Lemma 2.6 *For all $e \ge 0$ and $q = p^e$ we have*

$$\ell_R(M/I^{[q]}M) = O(q^{\dim M}).$$

Proof Let $t = \mu(I)$ then $I^{tq} \subset I^{[q]}$, for all q. Then $\ell_R(M/I^{[q]}M) \le \ell_R(M/I^{tq}M)$. If $P(x) = c_0x^{d_1} + c_1x^{d_1-1} + \cdots + c_{d_1}$ is the Hilbert-Samuel polynomial of M with respect to I, where $c_0 \ne 0$, then $d_1 = \dim M$ and for $q \gg 0$, $\ell_R(M/I^{tq}M) = P(tq)$. Now for

$$C = (d_1+1)t^{d_1} \cdot \max\{|c_i|\} \quad \text{we have} \quad \ell_R(M/I^{[q]}M) \le C(q^{\dim M}).$$

□

Lemma 2.7 *Let M and N be two finitely generated R-modules. Let $d = \dim\ R$. If $M_P \simeq N_P$, for each $P \in \Lambda(R)$ then for all $q = p^n$*

$$|\ell_R(M/I^{[q]}M) - \ell_R(N/I^{[q]}N)| = O(q^{d-1}).$$

Proof Let $S = R \setminus \bigcup_{P \in \Lambda(R)} P$. Then $S^{-1}R$ is an Artinian ring and $S^{-1}R \simeq \prod_{P \in \Lambda(R)} R_P$. This gives an isomorphism of $S^{-1}R$-modules

$$S^{-1}M = M \otimes_R S^{-1}R = \prod_{P \in \Lambda(R)} M_P \simeq \prod_{P \in \Lambda(R)} N_P = S^{-1}N, \tag{2.2}$$

well defined up to multiplication by a unit in $S^{-1}R$. Now the isomorphism $\operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N) \simeq S^{-1}\operatorname{Hom}_R(M, N)$ implies that there is a map $\phi : M \longrightarrow N$ such that $S^{-1}\phi$ is a map as given in (2.2). In particular if $K = \operatorname{coker}\ \phi$ then $\dim\ K \le d - 1$.

The exact sequence

$$M/I^{[q]}M \longrightarrow N/I^{[q]}N \longrightarrow K/I^{[q]}K \longrightarrow 0$$

gives

$$\ell_R(N/I^{[q]}N) \leq \ell_R(M/I^{[q]}M) + \ell_R(K/I^{[q]}K) \leq \ell_R(M/I^{[q]}M) + O(q^{d-1}).$$

Now reversing the role of M and N we complete the proof of the lemma. □

Lemma 2.8 *Let* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *be a short exact sequence of finitely generated* R*-modules and let* $\dim\ A = d \geq 1$*. Then*

$$\ell_R(M/I^{[q]}M) = \ell_R(M'/I^{[q]}M') + \ell_R(M''/I^{[q]}M'') + O(q^{d-1}).$$

Proof Again we can assume R is complete and has an algebraically closed residue field.
Case (a) R is a reduced ring.

Let $\{P_1, \ldots, P_n\}$ be the set of minimal primes of R and let $S = R \setminus (P_1 \cup \cdots \cup P_n)$. Then R_{P_i} is a field and therefore $(M' \oplus M'')_{P_i} \simeq M_{P_i}$ for each i. By Lemma 2.7

$$\ell_R\Big(\frac{M}{I^{[q]}M}\Big) = \ell_R\Big(\frac{M' \oplus M''}{I^{[q]}(M' \oplus M'')}\Big) + O(q^{d-1}) = \ell_R\Big(\frac{M'}{I^{[q]}M'}\Big) + \ell_R\Big(\frac{M''}{I^{[q]}M''}\Big) + O(q^{d-1}).$$

Case (b) R is not a reduced ring. Let nilrad(R) denote the nil radical of R. We choose $q_0 = p^{s_0}$ such that $(\text{nilrad}(R))^{q_0} = 0$. Hence $F_*^{s_0}M$ is annihilated by nilrad(R). Moreover

$$0 \longrightarrow N' = F_*^{s_0}M' \longrightarrow N = F_*^{s_0}M \longrightarrow N'' = F_*^{s_0}M'' \longrightarrow 0$$

is an exact sequence of $F_*^{s_0}R$-modules. Then it is an exact sequence of R-modules via the map $F_*^{s_0}$ and therefore of $R/\text{nilrad}(R)$-modules. Now by Case (a)

$$\ell_R(N/I^{[q]}N) = \ell_R(N'/I^{[q]}N') + \ell_R(N''/I^{[q]}N'') + O(q^{d-1}).$$

By (2.1), for an R-module $N_0 = F_*^{s_0}M_0$

$$\ell_R\Big(\frac{N_0}{I^{[q]}N_0}\Big) = \ell_R\Big(\frac{F_*^sN_0}{I \cdot F_*^sN_0}\Big) = \ell_R\Big(\frac{F_*^{s+s_0}M_0}{I \cdot F_*^{s+s_0}M_0}\Big) = \ell_R\Big(\frac{M_0}{I^{[qq_0]}M_0}\Big).$$

This gives (taking $q > q_0$)

$$\ell_R(M/I^{[q]}M) = \ell_R(M'/I^{[q]}M') + \ell_R(M''/I^{[q]}M'') + O((q/q_0)^{d-1}).$$

But $O(q/q_0)^{d-1}) = O(q^{d-1})$ as q_0 is fixed. □

Definition 2.9 We recall the following standard notations. If $\{x_n\}_n$ is a bounded sequence of elements in $\mathbb{R}$ then

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} y_n \quad \text{where} \quad y_n = \inf\{x_m \mid m \geq n\}$$

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and similarly

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} z_n \quad \text{where} \quad z_n = \sup\{x_m \mid m \geq n\}.$$

Since $\{y_n\}_n$ and $\{z_n\}_n$ are both monotonic sequences, both limits do exist.

Exercise 2 (see the last section) proves that

$$e(I)/d! \leq \liminf_{q\to\infty} \ell_R(R/I^{[q]})/q^d \leq \limsup_{q\to\infty} \ell_R(R/I^{[q]})/q^d \leq e(I).$$

(These limits are well defined by Lemma 2.6).

Lemma 2.10 *Let $(R, \mathfrak{m}, k)$ be a complete local domain of dimension $d \geq 1$ such that k is an algebraically closed field. Then there exists a constant $e_{HK}(R, I) > 0$ such that*

$$\ell_R(R/I^{[q]}) = e_{HK}(R, I)q^d + O(q^{d-1}).$$

Proof Let $r = \operatorname{rank}_R F_*R$, then for $S = R \setminus \{0\}$ we have an $S^{-1}R$-linear isomorphism

$$\widetilde{\phi} : \oplus^r S^{-1}R \longrightarrow S^{-1}R \otimes_R F_*R.$$

Since F_*R is a finite R-module, the map $\widetilde{\phi}$ lifts to a map $\phi : \oplus^r R \longrightarrow F_*R$, after multiplying $\widetilde{\phi}$ by a unit in $S^{-1}R$. This gives us a short exact sequence of R-modules

$$0 \longrightarrow \oplus^r R \longrightarrow F_*R \longrightarrow \operatorname{coker} \phi \longrightarrow 0,$$

where $\dim \operatorname{coker} \phi \leq d - 1$.

Therefore

$$O(q^{d-1}) = \left| r\ell_R\left(\frac{R}{I^{[q]}}\right) - \ell_R\left(\frac{F_*R}{I^{[q]} \cdot F_*R}\right)\right| = \left| r\ell_R(R/I^{[q]}) - \ell_R(R/I^{[qp]})\right|.$$

This gives

$$\left|(r/(qp)^d)\ell_R(R/I^{[q]}) - (1/(qp)^d)\ell_R(R/I^{[qp]})\right| = O(1/q).$$

Let $\{C_n\}_n$ be the sequence given by $C_n = (1/q^d)\ell_R\left(R/I^{[q]}\right)$, where $q = p^n$. Then we can write the above equality as

$$\left|(r/p^d)C_n - C_{n+1}\right| = O(1/q).$$

Therefore

$$\limsup_{n\to\infty} C_n = \limsup_{n\to\infty} C_{n+1} = \limsup_{n\to\infty} (r/p^d)C_n,$$
$$\liminf_{n\to\infty} C_n = \liminf_{n\to\infty} C_{n+1} = \liminf_{n\to\infty} (r/p^d)C_n.$$

Suppose $r/p^d > 1$ then

$$\limsup_{n\to\infty} (r/p^d)C_n = (r/p^d) \limsup_{n\to\infty} C_n > \limsup_{n\to\infty} C_n.$$

Suppose $r/p^d < 1$ then

$$\liminf_{n\to\infty} (r/p^d)C_n = (r/p^d) \liminf_{n\to\infty} C_n < \liminf_{n\to\infty} C_n.$$

Therefore $r/p^d = 1$ and $\{C_n\}_n$ is a Cauchy sequence converging to a constant $e_{HK}(R, I)$.

Now the assertion that $e_{HK}(R, I) > 0$ follows from Exercise (a). □

Theorem 2.11 *Let $(R, \mathbf{m}, k)$ be a Noetherian local ring of dimension d. Let M be a finitely generated R-module. Then there exists constant $e_{HK}(M, I) \in \mathbb{R}_{\geq 0}$ such that for all q*

$$\ell_R(M/I^{[q]}M) = e_{HK}(M, I)q^d + O(q^{d-1}). \tag{2.3}$$

Further, the associativity formula holds:

$$e_{HK}(M, I) = \sum_{P\in\Lambda(R)} e_{HK}(R/P, I)\ell_{R_P}(M_P).$$

Proof To prove the assertion (2.3), without loss of generality we can assume (from Remark 2.2) that R is a complete local ring with algebraically closed residue field. Applying Lemma 2.8 to a prime filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M,$$

where $M_{i+1}/M_i = R/P_i$, for $0 \leq i < n$ and where $P_i \in \operatorname{Spec} R$, we get

$$\begin{aligned}\ell_R(M/I^{[q]}M) &= \sum_{0\leq i<n} \ell_R\Big(\frac{M_{i+1}/M_i}{I^{[q]}(M_{i+1}/M_i)}\Big) + O(q^{d-1})\\ &= \sum_{\dim M_{i+1}/M_i=d} e_{HK}(M_{i+1}/M_i, I)q^d + O(q^{d-1}),\end{aligned}$$

where the second equality follows from Lemma 2.10. Hence follows the first assertion of the theorem, that is, (2.3) holds.

Now consider a short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

of finitely generated R-modules. Then, by Lemma 2.8 it follows that for all q,

$$e_{HK}(M, I)q^d = e_{HK}(N, I)q^d + e_{HK}(M/N, I)q^d + O(q^{d-1})$$

and therefore $e_{HK}(M, I) = e_{HK}(N, I) + e_{HK}(M/N, I)$.

Now applying this additivity property of e_{HK} to a prime filtration (over R, which may not be related to the earlier one over $\widehat{R}$)

$$0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_m = M,$$

of M where $N_{i+1}/N_i = R/Q_i$, for $0 \leq i < m$ and some $Q_i \in \operatorname{Spec} R$, we get

$$\ell_R(M/I^{[q]}M) = \sum_{\dim N_{i+1}/N_i = d} e_{HK}(N_{i+1}/N_i, I)q^d + O(q^{d-1}).$$

Since for $Q \in \Lambda(R)$, the number of times R/Q occurs as N_{i+1}/N_i is equal to $\ell_{R_Q}(M_Q)$, the second assertion of the theorem follows. □

Definition 2.12 We define the *Hilbert-Kunz (HK) multiplicity* of M with respect to I as the constant given by

$$e_{HK}(M, I) = \lim_{q\to\infty} \ell_R(M/I^{[q]}M)/q^d.$$

Corollary 2.13 *If* $(R, \mathbf{m}, k)$ *is a domain then*

$$e_{HK}(M, I) = e_{HK}(R, I)(\text{rank } M).$$

Remark 2.14 To study $e_{HK}(M, I)$ it is enough to know $e_{HK}(R/P, I)$, where $\dim R/P = \dim R$. Also $\dim M < \dim R$ if and only if $e_{HK}(M, I) = 0$.

Corollary 2.15 *If* $(R, \mathbf{m}, k)$ *is a ring of dimension 1 then* $e_{HK}(R, I) = e(R, I)$.

3 Some properties of e_{HK}

Remark 3.1 We have

$$e_{HK}(R, I) = e_{HK}(\widehat{R}, I\widehat{R}) = \sum_{P\in\Lambda(\widehat{R})} e_{HK}(\widehat{R}/P, I\widehat{R}).$$

In particular the lower dimensional components of R and $\widehat{R}$ do not contribute to $e_{HK}(R, I)$. Hence to study the implication of e_{HK} on any property of R, we need to assume that R is formally unmixed, that is, for every associated prime P of $\widehat{R}$, we have $\dim \widehat{R}/P = \dim R$.

For example if $R = k[[x, y, z]]/(xy - xz)$ then by Theorem 2.11 $e_{HK}(R) = 1$ but R is not regular.

Definition 3.2 Let R^o denote the complement of the union of all minimal primes of a ring R. An element $x \in R$ is the *tight closure* of an ideal J of R, if there is $c \in R^o$ such that $cx^q \in I^{[q]}$, for $q = p^e \gg 0$.

We denote $I^* = \{x \in R \mid x \in \text{ tight closure of } I\}$. It is easy to check that I^* is an ideal in R.

We have seen that analogous to multiplicity the HK multiplicity satisfies associativity formula. The following result shows that the HK multiplicity and the tight closures of the ideals have a relation very similar to the relation between multiplicity and the integral closures of the ideals.

Theorem 3.3 *Let* $(R, \mathbf{m}, k)$ *be a Noetherian local ring of dimension d. Let I be an* $\mathbf{m}$*-primary ideal and suppose* $J \subseteq I \subseteq J^*$. *Then* $e_{HK}(R, I) = e_{HK}(R, J)$.

Conversely, if R is formally unmixed and $J \subseteq I$, *then*

$$e_{HK}(R, I) = e_{HK}(R, J) \implies I \subseteq J^*$$

where for an ideal I, the ideal I^* *denotes the tight closure of I.*

Proof We prove the first assertion which is the easier part. If $J = \langle x_1, \ldots, x_n \rangle$ then $J^{[q]} = \langle x_1^q, \ldots, x_n^q \rangle$. We can find $c \in R^o$ such that $cJ^{[q]} \subset I^{[q]}$ for all q. Hence $J^{[q]}/I^{[q]}$ is a $R/(I^{[q]}, c)$-module. So there exists m and a surjective map

$$\oplus^m R/(I^{[q]}, c) \longrightarrow J^{[q]}/I^{[q]}$$

of $R/(I^{[q]}, c)$-modules. Therefore Theorem 2.11 for R/cR-modules gives

$$\ell_R\left(J^{[q]}/I^{[q]}\right) \leq m\ell_R\left(R/(I^{[q]}, c)\right) = m\ell_{R/cR}\left(R/(I^{[q]}, c)\right) = O(q^{d-1})$$

and for the sequence of R-modules

$$0 \longrightarrow J^{[q]}/I^{[q]} \longrightarrow R/I^{[q]} \longrightarrow R/J^{[q]} \longrightarrow 0$$

Lemma 2.8 gives $e_{HK}(I) = e_{HK}(J)$.

The converse follows easily if we assume the following result due to Aberbach [1]: If $x \in R$ such that $x \notin I^*$ then there exists $k \geq 1$ such that for all q, we have $(I^{[q]} : x^q) \subset \mathbf{m}^{\lfloor q/k \rfloor}$. For this we consider the canonical exact sequence

$$0 \longrightarrow R/(I^{[q]} : x^q) \longrightarrow R/I^{[q]} \longrightarrow R/(I^{[q]}, x^q) \longrightarrow 0$$

which for $q \gg 0$ gives

$$\ell_R(R/I^{[q]}) - \ell_R(R/(I^{[q]}, x^q)) = \ell_R(R/(I^{[q]} : x^q)) \geq \ell_R(R/\mathbf{m}^{\lfloor q/k \rfloor}).$$

Note that the Hilbert-Samuel polynomial of R with respect to $\mathbf{m}$ is

$$\ell_R(R/\mathbf{m}^n) = c_0 n^d + c_1 n^{d-1} + \cdots + c_d \quad \text{for} \quad n \gg 0.$$

Therefore for a choice of $q \gg 0$ such that

$$\lfloor q/k \rfloor \geq 2d \max\{|c_1|, \ldots, |c_d|\} \quad \text{and} \quad q \geq 2k^2$$

we have $\ell_R(R/\mathbf{m}^{\lfloor q/k \rfloor}) \geq (c_0 - 1/2)(q/2k)^d$. This implies $e_{HK}(R, I) > e_{HK}(R, I + xR)$. □

3.1 Estimates on HK multiplicity

Bound constraints on e_{HK} reflect on the characteristic p-singularities. A detailed survey on these results can be found in [27]. Here we will discuss some more recent work on bound estimates. By the result (see Theorem 2) of Kunz we know that e_{HK} is always bounded below by 1. Here we state a result due to Watanabe-Yoshida [61] which characterizes rings achieving the minimum $e_{HK}(R, \mathbf{m})$. Again the statement is analogous to the fundamental result due to Nagata characterizing the rings with minimal multiplicity.

Theorem 3.4 *Let $(R, \mathbf{m}, k)$ be a Noetherian local ring of dimension d which is formally unmixed. Then*

$$e_{HK}(R) = 1 \iff R \text{ is regular}.$$

Examples (see Example 3.6) show that e_{HK} need not be an integer and may depend on the characteristic of the ring. So the natural attempt is to give a characteristic-free lower bound on the HK multiplicity away from 1. The first such bound was proved by [2] which was improved later by [8] as follows.

If $(R, \mathbf{m})$ is a non regular ring of dimension $d \geq 2$ then $e_{HK}(R) \geq 1 + (1/d!d^d)$.

We recall the following conjecture of [63] along with a result of Gessel-Monsky [21] which indicates that still this is not the best possible: First we recall a few notations.

Let $R_{p,d} = K[x_0, \ldots, x_d]/(x_0^2 + \cdots + x_d^2)$ and let m_d denote the constants occurring as the coefficients of the following expression

$$\sec(x) + \tan(x) = 1 + \sum_{d=1}^{\infty} m_d x^d, \quad \text{where} \quad |x| < \pi/2,$$

where

$$m_3 = \frac{1}{3!}, \quad m_4 = \frac{5}{4!}, \quad m_5 = \frac{16}{5!}, \quad m_6 = \frac{61}{6!} \quad \text{etc.}$$

Theorem ([21]) $\lim_{p \to \infty} e_{HK}(R_{p,d}, \mathbf{m}) = 1 + m_d$.

Conjecture ([63]) *Let $p > 2$ be prime and $K = \bar{\mathbb{F}}_p$.*

(a) *If $(A, \mathbf{m}_A, K)$ is a formally unmixed non regular local ring of dimension d. Then $e_{HK}(A, \mathbf{m}_A) \geq e_{HK}(R_{p,d}, \mathbf{m}) \geq 1 + m_d$.*
(b) *If $e_{HK}(A, \mathbf{m}_A) = e_{HK}(R_{p,d}, \mathbf{m})$ then the $\mathbf{m}$-adic completion $\widehat{A}$ of A is isomorphic to $R_{p,d}$ as local rings.*

This conjecture has been proved for $d \leq 6$, see [3, 63]. Moreover the second inequality of the assertion (a) is proved recently in Theorem 4.3 of [54] for general d provided $p \geq d - 3$. Also the first inequality of the assertion (a) is proved in Theorem 4.6 of [17] provided $(A, \mathbf{m}_A, K)$ is a complete intersection rings of dimension $d \geq 2$. In particular combining these two results we have the following.

Theorem 3.5 *If $d \geq 4$ and $p \geq d - 3$ and $(A, \mathbf{m}_A, K)$ is a complete intersection but non regular local ring of d. Then $e_{HK}(A, \mathbf{m}) \geq e_{HK}(R_{p,d}, \mathbf{m}) \geq 1 + m_d$.*

More recently other alternate proofs covering the smaller primes, have been put out by [36] and subsequently by [42].

Also [30] have revisited the conjecture and have conjectured a lower bound for the class of rings which are non regular and are not quadric hypersurfaces.

The following computation due to Han-Monsky [24] of the Fermat quartic is very significant in the theory of HK multiplicity.

Example 3.6 Let $R = k[[x, y, z]]/(x^4 + y^4 + z^4)$, where char $k = p$. Then

$$\begin{aligned} e_{HK}(R, (x, y, z)) &= 3 + (1/p^2), \text{ if } \quad p \equiv \pm 3 \pmod 8) \\ &= 3, \qquad\qquad\quad \text{ if } \quad p \equiv \pm 1 \pmod 8). \end{aligned}$$

This example implies that

(1) e_{HK} depends on the characteristic of the ring. Moreover it depends on the congruence class of p.
(2) Under any hyperplane section e_{HK} can fail to remain same: Because if there is $x \in \mathbf{m}$ such that $e_{HK}(R/xR) = e_{HK}(R)$ then $e_{HK}(R) = e_{HK}(R/xR) = e(R/xR) \in \mathbb{N}$, whereas the above example implies otherwise.

(3) This example gives rise to an example of a vector bundle (see the next section) asserting that *reduction mod p* property is not open for the 'Frobenius semistability property' of V.

Failure of invariance under hyperplane section is one of the reasons which makes computations of e_{HK} difficult. However, we will see some examples in the next section exhibiting that the HK multiplicity is a more refined invariant compared to the multiplicity.

3.2 Behaviour of e_{HK} in families of ideals

(1) Let ϕ : Spec $R \longrightarrow \mathbb{R}_{\geq 0}$ be the map given by $P \to e_{HK}(R_P, I_P)$. It was shown in [46] that it is upper semi-continuous under some mild conditions on R.
(2) The growth of $e_{HK}(R, I^k)$ as $k \to \infty$ has been studied by [23] and [62]. In the graded situation in [50, 51] it was shown that

$$\lim_{k\to\infty} \frac{e_{HK}(M, I^k) - e_0(M, I^k)/d!}{k^{d-1}} = \frac{e_0(M, I)}{2(d-2)!} - \frac{E_1(M, I)}{(d-1)!} \geq \frac{e_0(M, I)}{2(d-1)\ p^{e_0(M,I)}},$$

where $e_0(M, I)$ and $e_1(M, I)$ are the coefficients of the Hilbert-Samuel polynomial of M with respect to I and $E_1(M, I) = \lim_{q\to\infty} e_1(M, I^{[q]})/q^d$ does exist. This formula was later generalized to local case in [47].
(3) An affirmative answer to the following question will give a well defined notion of e_{HK} in characteristic 0. **Open question.** Let $(R, \mathbf{m}, k)$ be a Noetherian local ring where k is a field of characteristic 0. Let $(R_p, \mathbf{m}_p, k_p)$ denote *reduction mod p* of $(R, \mathbf{m}, k)$. Then

$$\text{does} \quad \lim_{p\to\infty} e_{HK}(R_p, \mathbf{m}_p) \quad \text{exist?}$$

An affirmative answer is obvious for toric rings, monomial rings (see [10, 18, 60]) where the e_{HK} is independent of the characteristic. In the case of two dimensional standard graded rings [49] or when R is a diagonal hypersurface [21] of any dimension the answer is affirmative though $e_{HK}(R_p, \mathbf{m}_p)$ depends on the p .

4 Some applications

4.1 e_{HK} and semistability of a vector bundle

In the case R is a graded ring the invariant e_{HK} and the char p behaviour of the associated *syzygy bundle* on Proj R are closely related. Here we elaborate when R is two dimensional, equivalently when Proj R is a curve.

First we recall some relevant facts about vector bundles on a nonsingular projective curve. For more details on material related to semistable bundles reader can refer to [29]. Given a vector bundle (locally free sheaf of $\mathcal{O}_X$-modules) V on a projective curve X is *semistable* if for any subbundle W (locally free subsheaf) of V we have $\mu(W) \leq \mu(V)$, where $\mu(E) = \deg(E)/\text{rank}\,(E)$ denotes the μ-slope of E.

A semistable bundle has cohomologically nice properties. We recall a fundamental result on vector bundles due to Harder-Narasimhan. Every vector bundle V has a (unique) filtration

$$0 \subset E_1 \subset \cdots \subset E_n \subset V$$

of subbundles such that

(1) each E_{i+1}/E_i is semistable and

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(2) $\mu(E_1) > \mu(E_2/E_1) > \cdots > \mu(V/E_n)$.

This filtration is now called the Harder-Narasimhan (HN) filtration of V.

It is known that semistability property is an open property for *reduction mod* p: Suppose V is a vector bundle on a nonsingular curve X which is defined over a field of characteristic 0 and if V_p denotes the *reduction mod* p of V, then V semistable implies that V_p is semistable for $p \gg 0$.

It is known that if V is a semistable vector bundle on X and $\pi : Y \longrightarrow X$ is a a finite map of nonsingular projective curves then π^*V is a semistable vector bundle on Y where by a finite map means that if Spec A is an affine open set of X then $\pi^{-1}(\text{Spec } A) = \text{Spec } B$ is affine open set of Y and the induced homomorphism $A \longrightarrow B$ is finite. If X_p is *reduction mod* p of X then the Frobenius $F : X_p \longrightarrow X_p$ is a finite map, It is natural to ask that if V is semistable then is F^*V_p is semistable for $p \gg 0$?

The following relation between HK multiplicity and the semistability behaviour of the relevant syzygy bundle gives a negative answer.

Let R be a standard graded domain of dimension two over a perfect field k. Then $C = \text{Proj } R$ is an irreducible projective curve. Let X be the normalization of C and let $h_1, \ldots, h_s$ be a set of homogeneous generators of $\mathbf{m}$. Then we have a canonical short exact sequence of sheaves of $\mathcal{O}_X$-modules:

$$0 \longrightarrow V \longrightarrow \oplus^s \mathcal{O}_X \longrightarrow \mathcal{O}_X(1) \longrightarrow 0.$$

where V is a vector bundle on X of rank $s-1$.

Note that for a given vector bundle V *syzygy bundle* (associated to the above data) on a nonsingular projective curve we can associate a *strong HN data* $\big((a_1(V), a_2(V), \ldots,), \quad (r_1(V), r_2(V), \ldots,)\big)$ as follows. Given a vector bundle V there is $s_0 \gg 0$, see Theorem 2.7 in [L] due to Langer, such that $F^{*s_0}(V)$ has a *strong HN filtration*, that is, if

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_m \subset V_{m+1} = F^{*s_0}V$$

is the HN filtration of $F^{*s_0}V$ then for $s \geq s_0$ the filtration

$$0 \subset F^{*s-s_0}V_1 \subset F^{*s-s_0}V_2 \subset \cdots \subset F^{*s-s_0}V_m \subset F^{*s-s_0}V_{m+1} = F^{*s}V$$

is the HN filtration of $F^{*s}V$. We define $a_i(V) = \mu(V_{i+1}/V_i)/p^{s_0}$ and $r_i(V) = \text{rank } (V_{i+1}/V_i)$.

Theorem 4.1 ([7, 48]). *With the notations as above we have*

$$e_{HK}(R) = \frac{\deg(X)}{2}\Big(\sum_{i=0}^{m}(r_i(V)a_i(V)^2 - s)\Big).$$

Theorem 4.2 ([48]). *Let $R = k[X, Y, Z]/(h)$ be a graded domain, where h is a homogeneous polynomial of degree $d \geq 3$. Then*

$$e_{HK}(R) = (3d/4) + (l^2/4dp^{2s}),$$

where $l \leq d(d-3)$ and s are nonnegative integers with the following description: For $s_1 < s$ the bundle $F^{s_1*}V$ is semistable and therefore $0 \subset F^{s_1*}V$ is its HN filtration. Further $F^{s*}V$ has strong HN filtration and is given by

$$[(1)]0 \subset \mathcal{L} \subset F^{s*}V \quad \textit{such that} \quad \mu(\mathcal{L}) = (-dp^s + l)/2 \quad \textit{and} \quad \mu(F^{*s}V/\mathcal{L}) = (-dp^s - l)/2.$$

In particular this bound on l gives a dictionary between e_{HK} and the integers s and l provided $p > d(d-3)$.

Applying this to the Example 3.6, for char $p \geq 5$, we get

(1) If $p \equiv \pm 1 \pmod 8$ then $F^{s*}V$ is semistable for all $s \geq 0$.
(2) If $p \equiv \pm 3 \pmod 8$ then F^*V has the HN filtration $\mathcal{L} \subset F^*V$ where $\mu(\mathcal{L}) = \mu(F^*V) + 2$. In particular F^*V is not semistable.

Building on the idea of 'taxicab distance' due to Han-Monsky, one can compute (see [50, 51]) $e_{HK}(R)$ for any explicit trinomial plane curve $R = k[X, Y, Z]/(h)$ and therefore exhibit interesting and varied examples of the Frobenius semistability behaviour of the syzygy bundle V. In fact under *reduction mod p*, the Frobenius semistability behaviour is a function of the congruence class of p modulo an integer determined explicitly in terms of the exponents of the monomials appearing in h.

4.2 Tiling of a convex polytope and asymptotic growth of $e_{HK}(R, \mathbf{m}^k)$

We know that (see [18] and [60]) that e_{HK} of a toric ring does not depend on the characteristic of the ring. However e_{HK} reflects on some combinatorial aspect of the toric rings.

Definition 4.3 A rational convex polytope P in $\mathbb{R}^{d-1}$ *tiles* the space $\mathbb{R}^{d-1}$ with respect to the lattice $\mathbb{Z}^{d-1}$ if there exists $\lambda \in \mathbb{R}_{>0}$ such that

(1) $\bigcup_{z \in \mathbb{Z}^{d-1}}(\lambda P + z) = \mathbb{R}^{d-1}$, and
(2) $\dim\big((\lambda P + z) \cap (\lambda P + z')\big) < d - 1$ if $z \neq z'$.

For the basic facts which are used here about toric varieties one can refer to [20].

Let (X, D) be a toric pair of dimension $d - 1 \geq 1$ and where X is a projective toric variety and D is a torus invariant (very ample, Cartier) divisor. Let $R = \oplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}(mD))$ (see Section 3.4 of [20]) denote the associated homogeneous coordinate ring with the graded maximal ideal $\mathbf{m}$. It is well known that a toric pair (X, D) corresponds to a *very ample* convex integral polytope $P_{X,D}$ in $\mathbb{R}^{d-1}$.

Theorem 4.4 ([38]). *For a given toric pair (X, D) of dimension $d - 1$, where $d \geq 2$ and for associated ring $(R, \mathbf{m})$ as above we have*

$$\lim_{k \to \infty} \frac{e_{HK}(\mathbf{m}^k) - e(\mathbf{m}^k)/d!}{k^{d-1}} \geq \left[\frac{d-1}{d}\right]\left[\frac{e(\mathbf{m})}{(d-1)!}\right]^{\frac{d-2}{d-1}}$$

and the equality holds if and only if the associated polytope $P_{X,D}$ tiles the space $\mathbb{R}^{d-1}$ with respect to the lattice $\mathbb{Z}^{d-1}$.

In other words the normalized asymptotic growth of $e_{HK}(R, \mathbf{m}^k)$ is the slowest iff $P_{X,D}$ tiles $\mathbb{R}^{d-1}$ with respect to $\mathbb{Z}^{d-1}$.

Further we note that such a tiling property of any rational polytope P in $\mathbb{R}^{d-1}$ can be formulated in terms of the Hilbert-Kunz multiplicity as follows: We choose $m_1 \geq 0$ such that $m_1 P$ is an integral polytope. Then by Corollary 2.2.18 in [11] the polytope mP, with $m = (d-2)m_1$, is a very ample integral convex polytope. In particular, there is a toric pair (X, D) such that the associated polytope $P_{X,D} = mP$.

5 density functions

In this section we discuss the notion of *density functions* and briefly describe their applications to multiplicities. Here we work in the graded setup.

Roughly speaking, a density function for a given algebraic invariant, say ϵ, is an integrable function $f_\epsilon : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ which gives a *new measure* μ_{f_ϵ} on $\mathbb{R}$ such that the integration $\int_E f_\epsilon$ on a subset $E \subset \mathbb{R}$ is the measure of the invariant ϵ on E. The notion of density function, though seemingly complicated, keeps track of the behaviour of the 'original' invariant in more than one way, and also carries information about other related invariants.

5.1 HK density function

This function was introduced to study Hilbert-Kunz multiplicity, which we know is a hard numerical invariant to compute. However it turns out that this function carries the information about another char p invariant, namely the F-threshold $c^I(\mathbf{m})$ of I at $\mathbf{m}$, which is the support of the Hilbert-Kunz density function. See also [39, 56], and for some recent works see [37] on these kind of density functions.

Definition 5.1 A *graded pair* (R, I) means R is a standard graded ring over a perfect field of characteristic $p > 0$ and I is a homogeneous ideal of finite colength. We also assume that $\dim\ R = d \geq 2$.

For a graded pair (R, I) consider a sequence $\{f_n(R, I) : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}\}_{n\in\mathbb{N}}$ of functions given by

$$f_n(R, I)(x) = \frac{\ell_R(R/I^{[q]})_{\lfloor xq \rfloor}}{q^{d-1}}, \quad \text{where} \quad q = p^n.$$

Then we define the *HK density (HKd) function* $f_{R,I} : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ as $f_{R,I}(x) = \lim_{n\to\infty} f_n(R, I)(x)$. The following theorem asserts that this is indeed a well defined map.

Theorem 5.2 ([52]) *For a graded pair (R, I) the sequence $\{f_n(R, I)\}_{n\in\mathbb{N}}$ converges uniformly to the continuous compactly supported function $f_{R,I}$ and*

$$\int_0^\infty f_{R,I}(x)dx = e_{HK}(R, I).$$

Properties

(1) (Associativity) The associativity formula holds for the HK density function.
(2) (Tight closure) If R is an equidimensional ring and (R, I) and (R, J) two graded pairs with $I \subseteq J$ are two graded ideals of R then

$$f_{R,I} \cong f_{R,J} \ \text{ if and only if } \ J \subseteq I^*,$$

where I^* denotes the *tight closure* of I in R.
(3) (Multiplicative property): If (R, I) and (S, J) are two graded pairs and $H_R(x) = e(R)x^{d-1}(d-1)!$, where $e(R)$ denotes the multiplicity of R with respect to its irrelevant maximal ideal $\mathbf{m}$ and $d = \dim\ R$, then the Segre product $(R\#S, I\#J)$ satisfies

$$H_{R\#S}(x) - f_{R\#S,I\#J}(x) = \big(H_R(x) - f_{R,I}(x)\big)\big(H_S(x) - f_{S,J}(x)\big).$$

Recall that the set of compactly supported continuous functions are in bijective correspondence with the set of their Fourier transforms which are holomorphic functions. Further if $\widehat{f_{R,I}}$ denotes the Fourier transform of $f_{R,I}$ then $\widehat{f_{R,I}}(0) = e_{HK}(R, I)$. This suggests possible applications of harmonic analysis in the study of HK multiplicities.

Two applications of HK density function we have already stated:

(1) To give a lower bound on the HK multiplicity of quadratic hypersurfaces as in the earlier mentioned conjecture of [63] (see Theorem 4.3 of [54]) and

(2) relating tiling property a convex polytope with the growth of $e_{HK}(\mathbf{m}^k)$ (see Theorem 6.3 of [38]).

Now we give a third application of HK density function.

Since $f_{R,I}$ is compactly supported function we can define the notion of maximal support $\alpha(R, I) = \sup\{x \mid f_{R,I}(x) \neq 0\}$. This relates to another char p invariant, namely F-threshold $c^I(\mathbf{m})$ of $\mathbf{m}$ with respect to I, which is defined (see [DsNbP]) as

$$c^I(\mathbf{m}) = \lim_{q\to\infty} \frac{\min\{r \mid \mathbf{m}^{r+1} \subseteq I^{[q]}\}}{q}.$$

Theorem 5.3 ([55]) *Let (R, I) be a graded pair, where R is a domain such that its normalization S is strongly F-regular on its punctured spectrum. Then $\alpha(R, I) = c^I(\mathbf{m})$.*

In particular the equality $\alpha(R, I) = c^I(\mathbf{m})$ holds for any two dimensional standard graded pair (R, I), where R is a domain.

Note if dim $R = 2$ then we can describe the function $f_{R,I}$ and relate $c^I(\mathbf{m})$ to the strong HN data of the syzygy bundle as described in Section 4. Using this relation and an example due to Gieseker [22] of a family of vector bundles with certain Frobenius semistability property on a nonsingular projective curve, one gives an affirmative answer to a question posed by Mustaţă-Takagi-Watanabe ([41] as follows.

Theorem 5.4 ([53]) *Given a prime p and an integer $g > 1$, there exists a two dimensional standard graded normal $\mathbb{Q}$-Gorenstein domain R with the graded maximal ideal $\mathbf{m}$ such that the set of F-thresholds of $\mathbf{m}$ has accumulation points, where Proj $R = X$ is a nonsingular projective curve of genus g over a field of char p.*

Open question. Let (R, I) be a standard graded pair and $\mathbf{m}$ be the graded maximal ideal of R. Then, is $\alpha(R, I) = c^I(\mathbf{m})$?

5.2 Density functions for integral dependence criteria

Definition 5.5 Let R be commutative ring with ideals I and J.

(1) An element $r \in R$ is *integral* over I it satisfies a monic polynomial

$$x^n + r_1x^{n-1} + r_2x^{n-2} + \cdots + r_n = 0, \quad \text{where } n \geq 1 \text{ and } r_i \in I^i.$$

The set of integral elements over I is an ideal, and is called the integral closure $\overline{I}$ of I. The ideals I and J are *integrally dependent* if one is contained in another and $\overline{I} = \overline{J}$.

(2) If $A \to B$ is a homomorphism of commutative rings then $A \to B$ is an *integral extension* if every element x of B is integral over A, *i.e.*, it satisfies a monic polynomial

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0, \quad \text{where } n \geq 1 \text{ and } a_i \in A.$$

One of the nice things about integral closure is that this extension behaves well with respect to localizations and integral extensions (see [28] for details):

(1) (Localization) If I is an ideal in a commutative ring R and S is a multiplicatively closed set in R then $S^{-1}(\overline{I}) = \overline{S^{-1}I}$.

(2) (Integral extension) If $A \subseteq B$ is an integral extension of commutative rings, and I is an ideal in R. Then

$$\overline{IB} \cap A = \overline{I}.$$

In particular if X is a scheme and $\mathcal{O}_X$ its structure sheaf and $\mathcal{I}$ a sheaf of ideals in $\mathcal{O}_X$, then the localization property of the integral closure of an ideal gives a well defined notion of the integral closure $\overline{\mathcal{I}}$ of $\mathcal{I}$ in $\mathcal{O}_X$, which is given as follows. On an open affine subset U of X we define $H^0(U, \overline{\mathcal{I}}) = \overline{H^0(U, \mathcal{I})}$, where $\overline{H^0(U, \mathcal{I})}$ is the integral closure of $H^0(U, \mathcal{I})$ in $H^0(U, \mathcal{O}_X)$. Therefore, if $X = \text{Spec } R$ is affine then $H^0(X, \overline{\mathcal{I}}) = \overline{I}$ is the integral closure of I in R.

On the other hand if $R = \oplus_{m\geq 0} R_m$ is a standard graded Noetherian algebra over a field R_0, I a homogeneous ideal of R such that $R_1 \not\subseteq I$, and $\mathcal{I}$ be the ideal sheaf associated to I on $V = \text{Proj } R$. Then the ideal sheaf associated to the integral closure $\overline{I}$ is the ideal sheaf $\overline{\mathcal{I}}$ in $\mathcal{O}_V$.

In this setup we have another way of reinterpreting the integral closure (see Definition 9.6.2 and Remark 9.6.4 in [34]).

(1) Let X be a normal variety and $\mathcal{I} \subseteq \mathcal{O}_X$ be a nonzero ideal sheaf. Let $\nu\colon X^+ \to X$ be the normalization of the blow up of X along $\mathcal{I}$. Let E be the exceptional divisor of ν so that $\mathcal{I}\mathcal{O}_{X^+} = \mathcal{O}_{X^+}(-E)$. Then $\nu_*\mathcal{O}_{X^+}(-E) = \overline{\mathcal{I}}$.
(2) Further, if $f : Y \longrightarrow X$ is a proper birational map surjective map between normal varieties with the property that $\mathcal{I}\mathcal{O}_Y = \mathcal{O}_Y(-E)$ for some effective Cartier divisor E on Y. Then f factors through ν and consequently $f_*\mathcal{O}_Y(-E) = \overline{\mathcal{I}}$.

In other words, two nonzero ideals $\mathcal{I} \subseteq \mathcal{J} \subset \mathcal{O}_X$, where X denotes an affine normal variety, are *integrally dependent* (or have the same *integral closure*) if and only if $\mathcal{I}\mathcal{O}_{X^+} = \mathcal{J}\mathcal{O}_{X^+}$, where $X^+ \longrightarrow X$ is the normalization of the blow up along $\mathcal{J}$. This notion is used in singularity theory. In dealing with the questions which depend only on the integral closure of an ideal, it is often convenient to replace $\mathcal{J}$ by a smaller ideal $\mathcal{I}$ which has the same integral closure. The notion of integral closure makes sense in any Noetherian commutative ring.

The integral dependence of ideals of finite colength in a local ring has a well known numerical characterization, namely equality of Hilbert-Samuel multiplicities, due to D. Rees: Let $(R, \mathbf{m})$ a Noetherian analytically unramified local ring of dimension d and let $I \subseteq J$ be two ideals of finite colengths in R then

$$e(R, I) = e(R, J) \iff \overline{I} = \overline{J},$$

where $e(R, I) = \lim_{n\to\infty} \ell(R/I^n)/n^d$ denotes the Hilbert-Samuel multiplicity (also known as multiplicity) of R with respect to I.

Attempts to give a numerical characterization for ideals which might not necessarily be of finite colength led to numerical invariants like j-multiplicity, ε-multiplicity (see [4, 19, 31, 58, 59]) which require computing the invariant at several localizations, hence not readily amenable to computations. Also there exists a notion of multiplicity sequence (see [5, 9, 43]) which gives a numerical characterization of integral dependence.

Here we give a list of numerical characterizations of the integral dependence of I and J in graded setup. In particular any of the multiplicities, namely, the polar multiplicities, the ε-multiplicities or the j-multiplicities of the truncated ideals $I[Y]_{\geq c}$ and $J[Y]_{\geq c}$ in $R[Y]$ characterizes the integral dependence of I and J.

A novelty of this approach is that it does not involve localization and only requires checking computable and well-studied invariants like Hilbert-Samuel multiplicities. Existence of appropriate density functions play a central role in the proofs.

We recall the definitions and properties of the three density functions which were used in the proof, namely, (1) adic density function, (2) saturation density function and (3) epsilon density function.

Notation 5.6 Let k be a field and $R = \oplus_{m\geq 0} R_m$ be a standard graded finitely generated equidimensional algebra over $R_0 = k$ of dimension $d \geq 2$. Let $\mathbf{m} = \oplus_{m\geq 1} R_m$ be the unique homogeneous maximal ideal of R. Let $I \subseteq J$ be two nonzero homogeneous ideals in R. Let $d(I)$ and $d(J)$ be the maximum generating degrees of I and J respectively, and set $\mathbf{d} = \max\{d(I), d(J)\}$. For the sake of simplifying the statement we further assume that R is a domain. For a given choice of homogeneous generators of I, let the set of degrees of its generators be $d_1, \ldots, d_l$.

By reindexing we may assume that $d_1 < \cdots < d_l$ and denote $d(I) = d_l$. The *saturation* of I is defined to be the graded ideal $\widetilde{I} = I :_R \mathbf{m}^\infty = \{f \in R \mid f \cdot \mathbf{m}^c \subset I \text{ for some } c \in \mathbb{N}\}$.

Definition 5.7

(1) Let $f_n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be the function given by

$$f_n(x) = \frac{\ell_k\big((I^n)_{\lfloor xn \rfloor}\big)}{n^{d-1}/d!}.$$

The *adic density function* of I is the function $f_{\{I^n\}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by

$$f_{\{I^n\}}(x) = \limsup_{n \to \infty} f_n(x).$$

(2) Let $g_n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be the function given by

$$g_n(x) = \frac{\ell_k\big((I^n :_R \mathbf{m}^\infty)_{\lfloor xn \rfloor}\big)}{n^{d-1}/d!}.$$

The *saturation density function* of I is the function $f_{\{\widetilde{I^n}\}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by

$$f_{\{\widetilde{I^n}\}}(x) = \limsup_{n \to \infty} g_n(x).$$

(3) Let $f_n(\varepsilon) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be the function given by

$$f_n(\varepsilon)(x) = \frac{\ell_k\Big(\big(H^0_{\mathbf{m}}(R/I^n)\big)_{\lfloor xn \rfloor}\Big)}{n^{d-1}/d!}.$$

The *epsilon density* (ε-density) *function* of I is the function $f_{\varepsilon(I)} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by

$$f_{\varepsilon(I)}(x) = \limsup_{n \to \infty} f_n(\varepsilon)(x).$$

We shall now list the properties of these density functions from [14].

Theorem 5.8 *Let R and I be as in Notations 5.6. Then the following statements are true.*

(i) The sequence $\{f_n\}_{n \in \mathbb{N}}$ converges (locally uniformly) to $f_{\{I^n\}}$ on the set $\mathbb{R}_{\geq 0} \setminus \{d_1\}$.
(ii) Further,

$$f_{\{I^n\}}(x) = \begin{cases} 0 & \forall x \in [0, d_1), \\ \mathbf{p}_1(x) & \forall x \in (d_1, d_2], \\ \mathbf{p}_j(x) & \forall x \in [d_j, d_{j+1}] \text{ and } j = 2, \ldots, l, \end{cases}$$

where $d_{l+1} = \infty$, and for each $j = 1, \ldots, l$, $\mathbf{p}_j(x)$ is a nonzero polynomial of degree $\leq d-1$ with rational coefficients. Moreover, $\mathbf{p}_l(x)$ has degree $d-1$.

(iii) *For any real number $c > 0$, we have*

$$\int_0^c f_{\{I^n\}}(x)dx = \lim_{n\to\infty} \frac{\sum_{m=0}^{\lfloor cn\rfloor} \ell_k\left((I^n)_m\right)}{n^d/d!}.$$

Next we recall the relevant notations to list the properties of the saturation density functions (see [25] for definitions). Consider the projective variety $V = \text{Proj } R$ with a very ample invertible sheaf $\mathcal{O}_V(1)$. Let $\mathcal{I}$ be the ideal sheaf associated to I on V. Let

$$\pi : X = \mathbf{Proj}\left(\oplus_{n\geq 0}\mathcal{I}^n\right) \longrightarrow V$$

be the blow up of V along $\mathcal{I}$. Then $\mathcal{I}$ becomes locally principal on X, i.e., there is an effective Cartier divisor E on X (namely the exceptional divisor of π) such that $\mathcal{I}\mathcal{O}_X = \mathcal{O}_X(-E)$. Let H be the pullback of a hyperplane section on V. Define the constants

$$\alpha_I = \min\{x \in \mathbb{R}_{\geq 0} \mid xH - E \quad \text{is pseudo effective}\} \quad \text{and} \quad \beta_I = \min\{x \in \mathbb{R}_{\geq 0} \mid xH - E \quad \text{is nef}\},$$

where we have $0 \leq \alpha_I \leq d_1$ and $\beta_I \leq d_l = d(I)$. If I is not an ideal of finite colength then E is an effective divisor of positive degree which implies that $\alpha_I > 0$.

We recall the following result from [14].

Theorem 5.9 *Let R and I be as in Notations 5.6. Then the following statements are true:*

(i) *The sequence $\{g_n\}_{n\in\mathbb{N}}$ converges (locally uniformly) to $f_{\{\tilde{I^n}\}}$ on the set $\mathbb{R}_{\geq 0}$. Furthermore,*

$$f_{\{\tilde{I^n}\}}(x) = d \cdot \text{vol}_X(xH - E).$$

(ii) *The function $f_{\{\tilde{I^n}\}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous.*

(iii) *We have $f_{\{\tilde{I^n}\}}(x) = 0$ for all $x \leq \alpha_I$, and $f_{\{\tilde{I^n}\}}$ is a strictly increasing continuously differentiable function on the interval (α_I, ∞).*

(iv) *If $x \geq \beta_I$, then*

$$f_{\{\tilde{I^n}\}}(x) = d \cdot (xH - E)^{d-1} = \sum_{i=0}^{d-1} (-1)^{d-1-i} \frac{d!}{(d-i-1)!i!}(H^i \cdot E^{d-1-i})x^i,$$

where $H^{d-1-i} \cdot E^i$ denotes the intersection number of the Cartier divisors H^{d-1-i} and E^i.

(v) *For any real number $c > 0$,*

$$\int_0^c f_{\{\tilde{I^n}\}}(x)dx = \lim_{n\to\infty} \frac{\sum_{m=0}^{\lfloor cn\rfloor} \ell_k\left(\left(\tilde{I^n}\right)_m\right)}{n^d/d!}.$$

(vi) *We have $f_{\{\tilde{I^n}\}}(x) = f_{\{I^n\}}(x)$ for all $x > d(I)$.*

(vii) *If I is of finite colength then $f_{\{\tilde{I^n}\}}(x) = d \cdot e(R)x^{d-1}$, where $e(R)$ denotes the Hilbert-Samuel multiplicity of R. Here, $\alpha_I = \beta_I = 0$.*

Using the above two density functions one can prove the properties of the ε-density function leading to the invariant ε-multiplicity, where we recall that the notion of epsilon multiplicity, which was introduced in [UV08], and defined as follows: if $(R, \mathbf{m})$ is a d-dimensional Noetherian local ring and I is an ideal in R, then the *epsilon multiplicity* $\varepsilon(I)$ of I is defined as

$$\varepsilon(I) = \limsup_{n\to\infty} \frac{\ell_R\left(H^0_{\mathbf{m}}(R/I^n)\right)}{n^d/d!},$$

where $\ell_R(-)$ denotes the length as an R-module. In particular, if I is $\mathbf{m}$-primary then it coincides with the usual multiplicity of I. Note that we can also write

$$\varepsilon(I) = \limsup_{n\to\infty} \frac{\ell_R\left(H^0_{\mathbf{m}}(R/I^n)\right)}{n^d/d!} = \limsup_{n\to\infty} \frac{\ell_R\left(\widetilde{I^n}/I^n\right)}{n^d/d!},$$

as by definition $H^0_{\mathbf{m}}(R/I^n) = \widetilde{I^n}/I^n$. It was shown by [12] that the 'lim sup' in the definition can be replaced by 'lim' under mild conditions on the ring R.

Theorem 5.10 *Let R and I be as in Notations 5.6. Then the following statements are true:*

(i) The sequence $\{f_n(\varepsilon)\}_{n\in\mathbb{N}}$ converges (locally uniformly) to $f_{\varepsilon(I)}$ on the set $\mathbb{R}_{\geq 0}\setminus\{d_1\}$. Moreover,

$$f_{\varepsilon(I)}(x) = f_{\{\widetilde{I^n}\}}(x) - f_{\{I^n\}}(x), \quad \textit{for all} \ \ x \in \mathbb{R}_{\geq 0}.$$

(ii) The function $f_{\varepsilon(I)}$ is continuous everywhere possibly except at $x = d_1$. Moreover, it is continuously differentiable outside the finite set $\{\alpha_I, d_1, \ldots, d_l\}$.
(iii) The support of the function $f_{\varepsilon(I)}$ is contained in the closed interval $[\alpha_I, d(I)]$.
(iv) The ε-multiplicity of I is given by

$$\varepsilon(I) = \int_0^\infty f_{\varepsilon(I)}(x)dx.$$

All the above density functions remain invariant up to integral closure (see Theorem 6.2 in [14]).

Theorem 5.11 *Let R and I be as in Notations 5.6. Let $J = \overline{I}$ be the integral closure of the ideal I in R. Then the following statements are true:*

(i) $f_{\{I^n\}}(x) = f_{\{J^n\}}(x)$ for all $x \in \mathbb{R}_{\geq 0}$.
(ii) $f_{\{\widetilde{I^n}\}}(x) = f_{\{\widetilde{J^n}\}}(x)$ for all $x \in \mathbb{R}_{\geq 0}$.
(iii) $f_{\varepsilon(I)}(x) = f_{\varepsilon(J)}(x)$ for all $x \in \mathbb{R}_{\geq 0}$.

Conversely the equality of the adic density functions implies the integral dependence ([15]). As a consequence we get numerical characterizations (in terms of various multiplicities) for the integral dependence of ideals as follows.

Notation 5.12 (1) For any integer $c \geq \mathbf{d}$, let $I_{\geq c} = \oplus_{m\geq c} I_m$ and $J_{\geq c} = \oplus_{m\geq c} J_m$ be the corresponding truncated ideals in R. Notice that $I_{\geq c} = I \cap \mathbf{m}^c$ and $J_{\geq c} = J \cap \mathbf{m}^c$.
(2) Let

$$R[It] = \oplus_{(m,n)\in\mathbb{N}^2}(I^n)_m t^n \quad \text{and} \quad R[Jt] = \oplus_{(m,n)\in\mathbb{N}^2}(J^n)_m t^n$$

be the bigraded Rees algebras of I and J respectively.

(3) Further consider the $(c,1)$-diagonal sub algebras of $R[It]$ and $R[Jt]$ respectively, i.e.,

$$R[It]_{\Delta_{(c,1)}} = \oplus_{n\geq 0}(I^n)_{cn}t^n \quad \text{and} \quad R[Jt]_{\Delta_{(c,1)}} = \oplus_{n\geq 0}(J^n)_{cn}t^n.$$

(4) Define $S = R[Y]$, where Y is an indeterminate with $\deg Y = 1$, and $\mathbf{n} = \mathbf{m} + (Y)$ be the unique homogeneous maximal ideal of S. Let $\mathsf{I} = IS$ and $\mathsf{J} = JS$ be the extensions of the ideals I and J in S respectively.
(5) For $c \geq \mathbf{d}$, similarly define the truncated ideals $\mathsf{I}_{\geq c} = \oplus_{m\geq c}\mathsf{I}_m$ and $\mathsf{J}_{\geq c} = \oplus_{m\geq c}\mathsf{J}_m$ in S.
(6) We also consider the $(c,1)$-diagonal sub algebras

$$S[\mathsf{I}t]_{\Delta_{(c,1)}} = \oplus_{n\geq 0}\left(\mathsf{I}^n\right)_{cn}t^n \quad \text{and} \quad S[\mathsf{J}t]_{\Delta_{(c,1)}} = \oplus_{n\geq 0}\left(\mathsf{J}^n\right)_{cn}t^n$$

of the Rees algebras $S[\mathsf{I}t]$ and $S[\mathsf{J}t]$ respectively.
(7) Further, recall that the Hilbert-Samuel multiplicity of a $d(\geq 1)$-dimensional finitely generated graded k-algebra $A = \oplus_{m\geq 0}A_m$, is given by

$$e(A) := \lim_{m\to\infty} \frac{\ell_k(A_m)}{m^{d-1}/(d-1)!}.$$

Theorem 5.13 (Theorem 1.1 in [15]) *With Notations 5.12 we have*

$$\overline{I} = \overline{J} \quad \textit{if and only if} \quad e\left(S[\mathsf{I}t]_{\Delta_{(c,1)}}\right) = e\left(S[\mathsf{J}t]_{\Delta_{(c,1)}}\right) \quad \textit{for some (every) integer} \quad c > \mathbf{d}.$$

Further, if I is of finite colength then

$$e\left(S[\mathsf{I}t]_{\Delta_{(c,1)}}\right) = c^d e(R) - e(I,R) \quad \textit{and} \quad e\left(S[\mathsf{J}t]_{\Delta_{(c,1)}}\right) = c^d e(R) - e(J,R).$$

Notation 5.14 Further assume that R is a domain. From [45] or Theorem 4.2 in [HT03], we know that there exist constants $m_0 \geq 0$ and $n_0 \geq 0$ such that for all integers $m \geq d(I)n + m_0$ and $n \geq n_0$, the length function $\ell_k\left((I^n)_m\right)$ agrees with a polynomial in m and n, i.e.,

$$\ell_k\left((I^n)_m\right) = \sum_{i=0}^{d-1} \frac{e_i(R[It])}{i!(d-1-i)!} m^i n^{d-1-i} + \text{lower degree terms},$$

where the coefficients $e_i(R[It])$ are integers for all $i = 0, \ldots, d-1$. We shall refer to $e_i(R[It])$ as the i^{th} *RA-multiplicity of* $R[It]$, where 'RA' stands for the Rees algebra.

Remark 5.15 The i^{th} RA-multiplicity $e_i(R[It])$ is the intersection number $H^i \cdot E^{d-1-i}$, where H and E are divisors as in Theorem 5.9. Therefore the saturation density function for I can also be written as

$$f_{\{\widetilde{I^n}\}}(x) = d \cdot \sum_{i=0}^{d-1} \binom{d-1}{i} e_i(R[It])x^i \quad \text{for all real numbers} \quad x \geq d(I). \tag{5.1}$$

Further, $e_{d-1}(R[It]) = H^{d-1} = e(R)$ and therefore is independent of the ideal I.

We give another numerical characterization which involves ε-multiplicity.

Theorem 5.16 (Theorem 5.6 in [15]) *Adopt Notations 5.14 and further assume that R is a domain. Then the following statements are true:*

(1) $\ell_R\left(\overline{J}/\overline{I}\right) < \infty$ *if and only if* $e_i(R[It]) = e_i(R[Jt])$ *for all* $0 \leq i < \dim(R/I)$.
(2) $\overline{J} = \overline{I}$ *if and only if* $\varepsilon(I) = \varepsilon(J)$ *and* $e_i(R[It]) = e_i(R[Jt])$ *for all* $0 \leq i < \dim(R/I)$.

We note that both the above theorems are a natural generalization of the classical theorem of Rees [44] in the graded situation.

These RA multiplicities and the mixed multiplicities I can be retrieved from each other, where recall that the mixed multiplicities $e_i(J|I)$ for two ideals J and I are integers introduced by Bhattacharya [Bha57] in the following way.

Let $(R, \mathbf{m})$ be a Noetherian local ring of dimension d. Let J be an $\mathbf{m}$-primary ideal and I be an arbitrary ideal with positive height. Then by [Bha57] for all $u \gg 0$ and $v \gg 0$, the bivariate Hilbert function $\ell_R\left(J^u I^v / J^{u+1} I^v\right)$ agrees with a bivariate numerical polynomial $Q(u, v)$ of total degree $d-1$. Moreover, we can write

$$Q(u, v) = \sum_{i=0}^{d-1} \frac{e_i(J|I)}{i!(d-1-i)!} u^{d-1-i} v^i + \text{lower degree terms}, \tag{5.2}$$

Notation 5.17 Let R, $\mathbf{m}$, and k be as in Notations 5.6. Further assume that R is a domain. Let I be a nonzero homogeneous ideal in R. Let $\beta > 0$ be an integer such that I is generated in degrees $\leq \beta$ and $I_{\geq \beta}$ denotes the truncated ideal $\oplus_{m \geq \beta} I_m$.

Then (Lemma 5.3 and Proposition 5.4 in [13]) one has

$$e_i\big(R[It]\big) = \sum_{j=0}^{d-1-i} (-1)^j \binom{d-1-i}{j} \beta^j e_{d-1-i-j}\big(\mathbf{m}|I_{\geq \beta}\big) \qquad \text{for all } i = 0, \ldots, d-1, \text{ and} \tag{5.3}$$

$$e_i\big(\mathbf{m}|I_{\geq \beta}\big) = \sum_{j=0}^{i} \binom{i}{j} \beta^j e_{d-1-i+j}(R[It]) \qquad \text{for all } i = 0, \ldots, d-1. \tag{5.4}$$

Moreover, if dim $(R/I) = i_0$ then by Corollary 4.5 in [57]

$$e_j\big(\mathbf{m}|I_{\geq \beta}\big) = \beta^j e(R) \quad \text{for all } j = 0, \ldots, d - i_0 - 1.$$

Further we can relate all these multiplicities with j-multiplicity of I (Theorem 1.3 in [15]).

Definition 5.18 Let $(R, \mathbf{m})$ be a d-dimensional Noetherian local ring and $I \subseteq R$ be an ideal. The *j-multiplicity* of I is defined to be the integer

$$j(I) = \lim_{n \to \infty} \frac{\ell_R\left(H^0_{\mathbf{m}}\left(I^n/I^{n+1}\right)\right)}{n^{d-1}/(d-1)!}.$$

Theorem 5.19 *Following Notations 5.6 we further assume that R is a domain. Then the following statements are equivalent:*

(i) $\overline{I} = \overline{J}$.
(ii) $e_i(S[\mathsf{I}t]) = e_i(S[\mathsf{J}t])$ *for all i, where* $0 \leq i \leq \dim R/I$.
(iii) $e_i\big(\mathbf{n}|\mathsf{I}_{\geq \mathbf{d}}\big) = e_i\big(\mathbf{n}|\mathsf{J}_{\geq \mathbf{d}}\big)$ *for all i, where* $0 \leq d - i \leq \dim R/I$.
(iv) $\varepsilon\big(\mathsf{I}_{\geq c}\big) = \varepsilon\big(\mathsf{J}_{\geq c}\big)$ *for some integer (all integers)* $c > \mathbf{d}$.
(v) $j\big(\mathsf{I}_{\geq c}\big) = j\big(\mathsf{J}_{\geq c}\big)$ *for some integer (all integers)* $c > \mathbf{d}$.

We now give a brief outline of the methods used in the proofs.

As stated above, given a homogeneous ideal I in R we have three real-valued nonnegative density functions, namely the *adic density function* $f_{\{I^n\}}$, the *saturation density function* $f_{\{\widetilde{I^n}\}}$, and the *ε-density function* $f_{\varepsilon(I)}$, which respectively measure the 'growth' of $\{I^n\}_n$, $\{\widetilde{I^n}\}_n$ and $\varepsilon(I)$ on a real scale.

In order to give a numerical characterization for $\overline{I} = \overline{J}$, we first provide a numerical characterization for the weaker assertion, namely the finiteness of $\ell_R(\overline{J}/\overline{I})$.

Main point is the observation that the saturation density functions for I and J agree at some (every) integer $c > \mathbf{d}$ if and only if $\ell_R(\overline{J}/\overline{I}) < \infty$, which is same as the assertion that the equality $e\big(R[It]_{\Delta_{(c,1)}}\big) = e\big(R[Jt]_{\Delta_{(c,1)}}\big)$ holds. Hence Theorem 5.16 (2). Note that both $R[It]_{\Delta_{(c,1)}}$ and $R[Jt]_{\Delta_{(c,1)}}$ are d-dimensional standard graded Noetherian domains over R_0. Thus using saturation density functions, we give the following numerical characterization of finiteness of $\ell_R(\overline{J}/\overline{I})$ in terms of Hilbert-Samuel multiplicities as follow.

(1) $\ell_R(\overline{J}/\overline{I}) < \infty$.
(2) $f_{\{\widetilde{I^n}\}}(x) = f_{\{\widetilde{J^n}\}}(x)$ for all $x \geq 0$.
(3) $e\big(R[It]_{\Delta_{(c,1)}}\big) = e\big(R[Jt]_{\Delta_{(c,1)}}\big)$ for some integer $c > \mathbf{d}$.

We note that the equality $\overline{I} = \overline{J}$ holds if and only if $\ell_S(\bar{\mathrm{J}}/\bar{\mathrm{I}}) < \infty$. But as discussed above, this is equivalent to the assertion that the equality $e\big(S[\mathrm{I}t]_{\Delta_{(c,1)}}\big) = e\big(S[\mathrm{J}t]_{\Delta_{(c,1)}}\big)$ holds. Hence Theorem 5.13.

Now, by construction, for a given graded ideal I, the adic density function $f_{\{I^n\}}$ is a point wise limit of integrable step functions, and therefore using the Lebesgue's dominated convergence theorem, we can express the saturated density function $f_{\{\widetilde{\mathrm{I}^n}\}}$ as integrals of the adic density function $f_{\{I^n\}}$. On the other hand for ideals $I \subseteq J$ the function $f_{\{J^n\}} - f_{\{I^n\}}$ is a nonnegative function which is continuous at almost all points. From this we deduce that the equality of the adic density functions is same as the equality of the saturation density functions for the respective extended ideals. Thus we establish the equivalence of the following statements:

(1) $\overline{I} = \overline{J}$.
(2) $f_{\{I^n\}}(x) = f_{\{J^n\}}(x)$ for all $x \geq 0$.
(3) $f_{\{\widetilde{\mathrm{I}^n}\}}(x) = f_{\{\widetilde{\mathrm{J}^n}\}}(x)$ for all $x \geq 0$.
(4) $e\big(S[\mathrm{I}t]_{\Delta_{(c,1)}}\big) = e\big(S[\mathrm{J}t]_{\Delta_{(c,1)}}\big)$ for some (every) integer $c > \mathbf{d}$.

Using the above equivalences one easily proves Theorem 5.19 (i), (ii), (iii) and Theorem 5.16 (2).

Since the numerical characterizations are in terms of well-studied invariants, it is expected that these techniques and results will have applications from both theoretical and computational points of view.

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