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ORIGINAL RESEARCH

On spectral spread and trace norm of Sombor matrix

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Abstract For a simple graph G with vertex set $\{v_1, \ldots, v_n\}$ and degree sequence $\{d_1, \ldots, d_n\}$, the Sombor matrix $S(G)$ of G is an $n \times n$ matrix, whose (i, j)-th entry is $\sqrt{d_i^2 + d_j^2}$, if v_i and v_j are adjacent and 0, otherwise. The multi-set of the eigenvalues of $S(G)$ is known as the Sombor spectrum of G, denoted by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, where μ_1 is the Sombor spectral radius of G. The absolute sum of the Sombor eigenvalues is known as the trace norm (Sombor energy) of G. The spectral spread of $S(G)$ is defined by $s(S(G)) = \mu_1 - \mu_n$. In this article, we establish various sharp bounds for $s(S(G))$ in terms of various graph parameters like order, Schatten norm, Frobenius norm, trace norm, Forgotten topological index, Sombor index and many other invariants. We give complete characterization of the extremal graphs attaining these bounds. As a consequence of $s(S(G))$, we present the bounds on the trace norm of $S(G)$ along with the graphs attaining them.

Keywords Adjacency matrix · trace norm (energy) · Sombor index · Sombor matrix · spectral spread

Mathematics Subject Classification 05C50 · 05C12 · 15A18

1 Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the set of all square matrices of order n with entries from complex field $\mathbb{C}$. For $M \in \mathbb{M}_n(\mathbb{C})$, the square roots of the eigenvalues of MM^* or M^*M are known as the *singular values*, where M^* is the complex conjugate of M. As MM^* is positive semi-definite, so the singular values of M are non negative real numbers, denoted by $\sigma_i(M)$ and are indexed from largest to smallest as: $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_n(M)$. The *Schatten p-norm* of $M \in \mathbb{M}_n(\mathbb{C})$, denoted by $\|M\|_{\mathcal{P}}$, is the p-th root of the sum of the p-th powers of $\sigma_i(M)$'s, that is

$$\|M\|_{\mathcal{P}} = (\sigma_1^p(M) + \sigma_2^p(M) + \cdots + \sigma_n^p(M))^{\frac{1}{p}},$$

where $1 \leq p \leq n$. For $p = 2$, The Schatten 2-norm is the *Frobenius norm*, denoted by $\|M\|_F$, defined by

$$\|M\|_F^2 = \sigma_1^2(M) + \sigma_2^2(M) + \cdots + \sigma_n^2(M).$$

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The Schatten 1-norm is the sum of all singular values and is known as the *trace norm* of M, that is, $\|M\| = \sigma_1(M) + \sigma_2(M) + \cdots + \sigma_n(M)$. The sum of first k singular values is the *Ky Fan k-norm* or *nuclear norm*, that is for $p = 1$ and $1 \leq k \leq n$, we have

$$\|M\|_k = \sigma_1(M) + \sigma_2(M) + \cdots + \sigma_k(M).$$

The largest singular value $\|M\|_1 = \sigma_1(M)$ of M and is the *spectral norm*. It is well known that for a *normal* matrix M, $\sigma_i(M) = |\lambda_i(M)|$, where $\lambda_i(M)$, $i = 1, 2, \ldots, n$, are the eigenvalues of M. More about the Schatten p-norm and the Ky Fan k-norm about can be seen in [13,15,29,30,36].

For $M \in \mathbb{M}_{n\times n}(\mathbb{C})$, the diameter of its spectrum is called the spread (spectral spread) $s(M)$, that is $s(M) = \max\limits_{ij} |\lambda_i(M) - \lambda_j(M)|$, where maximum is taken over all pair of eigenvalues of M. For *Hermitian* matrix M, $s(M) = \lambda_1(M) - \lambda_n(M)$. If an eigenvalue λ of M occurs with multiplicity k, we represent it as $\lambda^{[k]}$. There is considerably vast literature on the spread of matrix M [27,28,39].

Consider a simple graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, edge set $E(G)$ and degree sequence $\{d_1, d_2, \ldots, d_n\}$. The number of elements in $V(G)$ is the *order* n and that of $E(G)$ is *size* m of G. If vertices u and v are adjacent, we write $u \sim v$, otherwise $u \nsim v$. By e we denote the edge uv of G. If degree of each vertex is same, then G is said to be *regular* graph. The complete graph is denoted by K_n, complete bipartite graph by $K_{a,b}$, complete split graph with clique number ω by $CS_{\omega,n-\omega}$, empty graph (totally disconnected graph) by $\overline{K}_n$. More graph theoretic notations can be found in [3].

The *adjacency* matrix $A(G) = (a_{ij})$ of G is a square matrix of order n, where a_{ij} is 1, if $v_i \sim v_j$ and 0, if $v_i \nsim v_j$. The matrix $A(G)$ is real and symmetric. We list its eigenvalues in non-increasing order as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The largest eigenvalue λ_1 is known as the *spectral radius* of G. By Perron-Frobenius theorem, λ_1 is unique, $|\lambda_i| \leq \lambda_1$ and its associated eigenvector say X has positive entries. In theoretical chemistry, the trace norm $|\lambda_1| + |\lambda_2| + \cdots + |\lambda_n|$ of $A(G)$ is studied under the name energy (see, Gutman [9]), denoted by $\mathcal{E}(A(G))$, and has its origin in theoretical chemistry, where it helps in approximating the π-electron energy of unsaturated hydrocarbons. There is a wealthy literature about the trace norm, and the other spectral invariants of $A(G)$, see [7,30,34,40]. The spread of $A(G)$ is defined as $s(A(G)) = \lambda_1 - \lambda_n$ and its properties were investigated in [1,2,8,25].

In mathematical and chemical literature, there are several well known degree-based graph invariants (also known as topological indices) that have several uses in chemical studies. They play a very crucial role in theoretical chemistry especially in quantitative structure-activity relationship (in short QSAR) and quantitative structure-property relationship (QSPR) related studies [14]. The first Zagreb index Z_1 and the forgotten topological index F of a graph G are defined as

$$Z_1(G) = \sum_{i=1}^{n} d_i^2 = \sum_{v_i v_j \in E(G)} (d_i + d_j) \quad \text{and} \quad F(G) = \sum_{i=1}^{n} d_i^3 = \sum_{v_i v_j \in E(G)} (d_i^2 + d_j^2).$$

Gutman (2021) [10] introduced a new topological index known as the *Sombor index*, denoted by $SO(G)$, defined as

$$SO(G) = \sum_{v_i v_j \in E(G)} \sqrt{d_i^2 + d_j^2}.$$

Das et al. [5] presented novel bounds for $SO(G)$ and deduced its relations with the Zagreb indices of G. In [38], extremal properties of Sombor index were discussed for several families of graphs and it was proved that the cycle C_n has the minimum Sombor index among all unicyclic graphs. Other chemical applicability and molecular properties of $SO(G)$ can be found in [4,6,20,23,24,35,37].

The *Sombor matrix* $S(G)$ of a graph G is a square matrix of order n, defined by

$$S(G) = (s_{ij}) = \begin{cases} \sqrt{d_i^2 + d_j^2} & \text{if } v_i \sim v_j \\ 0 & \text{if } v_i \nsim v_j. \end{cases}$$

It is clear that $S(G) = r\sqrt{2}A(G)$, if G is regular graph of degree r. Evidently, $S(G)$ is a real real symmetric matrix. We denote its eigenvalues by μ_i's and order them as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. The set of all eigenvalues (including multiplicities) of $S(G)$ is known as the *Sombor spectrum* of $S(G)$. The largest eigenvalue of $S(G)$

is called as the *Sombor spectral radius* of G, which for real symmetric matrices is same as the *spectral norm* of G. By Perron-Frobenius theorem, μ_1 is unique, $|\mu_i| \leq \mu_1$ and its corresponding eigenvector Y have positive entries. The trace norm (Sombor energy) of $S(G)$, is defined by

$$\mathcal{E}(S(G)) = \sum_{i=1}^{n} |\mu_i|.$$

Various paper on spectral properties of $S(G)$, like the Sombor eigenvalues, the Sombor spectral norm, the trace norm, the Sombor Estrada index, relation of trace norm of $A(G)$ with $S(G)$, $SO(G)$ and others spectral invariants can be seen in [11,12,16,18,21,22,31–33,41]. The spread (Sombor spectral spread) of $S(G)$ is defined as $s(S(G)) = \mu_1 - \mu_n$. For r regular graphs, $S(G) = r\sqrt{2}A(G)$, so $s(S(G)) = 2\sqrt{2}s(A(G))$, $\mathcal{E}(S(G)) = r\sqrt{2}\mathcal{E}(A(G))$. The aim of this article is to explore the spectral graph invariant $s(S(G))$ and relate with the the other spectral invariants like, the trace norm (Sombor energy), the spectral norm, the Schatten p-norm and the other spectral invariants of $S(G)$. And the more important is identifying the extremal graphs for $s(S(G))$.

In the present article, we explore the properties and establish the bounds (Section 2) of $s(S(G))$ and in Section 3, we relate $s(S(G))$ with the trace norm of $S(G)$ and present sharp bounds for $\mathcal{E}(S(SG))$. We end up article with some comments for future work (Section 4).

2 Bounds for the spread of $S(G)$

The real sequence $b_1' \geq b_2' \geq \cdots \geq b_m'$ is said to interlace the real sequence $b_1 \geq b_2 \geq \cdots \geq b_n$ $(m < n)$ if $b_i \geq b_i' \geq b_{n-m+i}$, for $i = 1, 2, \ldots, m$ and the interlacing is said to be tight if there exists a positive integer $k \in [0, m]$ such that

$$b_i = b_i' \quad \text{for } i = 1, 2, \ldots, k \text{ and } b_{n-m+i} = b_i' \quad \text{for } k+1 \leq i \leq m.$$

The following classical result establishes the relation between the eigenvalues of a real symmetric matrix and the eigenvalues of its principal submatrix.

Theorem 2.1 **(Interlacing Theorem,** *[13]) Let $M \in \mathbb{M}_n(\mathbb{R})$ be a real symmetric matrix and A be its principal submatrix of order m, $(n \geq m)$, respectively. Then the eigenvalues of M and A satisfy the following interlacing relation*

$$\lambda_i(M) \geq \lambda_i(A) \geq \lambda_{i+n-m}(M), \qquad \textit{for } 1 \leq i \leq m.$$

Consider a square matrix M of order n in block form

$$M = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,s-1} & A_{1,s} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,s-1} & A_{2,s} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{s-1,1} & A_{s-1,2} & \cdots & A_{s-1,s-1} & A_{s-1,s} \\ A_{s,1} & A_{s,2} & \cdots & A_{s,s-1} & A_{s,s} \end{pmatrix},$$

whose rows and columns are partitioned according to a partition $\pi = \{P_1, P_2, \ldots, P_s\}$ of the index set $I = \{1, 2, \ldots, n\}$. The quotient matrix $Q = (q_{ij})_{s\times s}$ (see [3]) is a square matrix of order s, such that (i, j)-th entry of Q is the average row sum of the block A_{ij} of M. The partition P is said to be *equitable* (regular), if each block $A_{i,j}$ of M has constant row sum and in this case Q is called an *equitable quotient* matrix.

The following result gives the relation between the eigenvalues of M and the eigenvalues of Q.

Theorem 2.2 *([3]) Let M be a real symmetric matrix of order n and Q be its quotient matrix of order m, $(n > m)$. Then the following hold.*

(i) *If the partition π of I of matrix M is not equitable, then the eigenvalues of Q interlace the eigenvalues of M, that is*

$$\lambda_i(M) \geq \lambda_i(Q) \geq \lambda_{i+n-m}(M), \qquad \textit{for } i = 1, 2, \ldots, m.$$

(ii) *If the partition π of I of matrix M is equitable, then the spectrum of Q is contained in the spectrum of M.*

For matrix $S(G)$, the Schatten t-norm is $\|S(G)\|_{\mathcal{P}}^t = |\mu_1|^t + |\mu_2|^t + \cdots + |\mu_n|^t$, the Frobenius norm is $\mu_1^2 + \mu_2^2 + \cdots + \mu_n^2 = \|S(G)\|_F^2$, and the trace norm (see introduction) is $\mathcal{E}(S(G)) = |\mu_1| + |\mu_2| + \cdots + |\mu_n|$. We recall that $\|S(G)\|_F^2 = 2F$ (see, Lin [18]), where $F = \sum_{v_i v_j \in E(G)} (d_i^2 + d_j^2)$ is the forgotten topological index of G.

Next, we state the following known result, which will be used later.

Lemma 2.3 *([32]) Let G be a connected graph with order $n \geq 3$. Then G has two distinct Sombor eigenvalue if and only if G is a complete graph.*

Lemma 2.4 *([32]) A connected bipartite graph G with order $n \geq 3$ has three distinct Sombor eigenvalues if and only if G is isomorphic to $K_{a,n-a}$.*

Lemma 2.5 *([32]) Let G be a connected graph of order n and $\{v_1, v_2, \ldots, v_\alpha\}$ be the independent set of G satisfying the condition $N(v_i) = N(v_j)$ for each $1 \leq i, j \leq \alpha$. Then 0 is the Sombor eigenvalue of G with multiplicity at least $\alpha - 1$.*

Lemma 2.6 *([32]) Let G be a connected graph with n vertices and $\omega = \{v_1, v_2, \ldots, v_\omega\}$ induces a clique in G satisfying $N(v_i) \setminus \omega = N(v_j) \setminus \omega,\ 1 \leq i, j \leq \omega$. Then $-(w - 1 + \beta)\sqrt{2}$ is the Sombor eigenvalue of G with multiplicity at least $\omega - 1$, where β is the number of vertices in $V(G) \setminus \omega$ that are adjacent to each vertex of ω.*

The very first results gives the upper bound for $s(S(G))$ in terms the spectral norm μ_1 and the Schatten t-norm.

Theorem 2.7 *Let G be a connected graph of order $n \geq 2$, and $t \geq 2$ be a positive even integer. Then*

$$s(S(G)) \leq \mu_1 + \sqrt[t]{\|S(G)\|_{\mathcal{P}}^t - \mu_1^t} \leq 2\sqrt[t]{\frac{\|S(G)\|_{\mathcal{P}}^t}{2}},$$

with equality holding if and only if $G \cong K_{a,n-a}$.

Proof For $t \geq 2n_1$, n_1 is a positive integer, it is known that $\sum_{i=1}^{n} \mu_i^t = \|S(G)\|_{\mathcal{P}}^t$. Now, we have

$$\mu_1^t + |\mu_n|^t = |\mu_1|^t + |\mu_n|^t \leq \sum_{i=1}^{n} |\mu_i|^t = \sum_{i=1}^{n} \mu_i^t,$$

with equality if and only if the Sombor spectrum of G is $\{\mu_1, 0^{[n-1]}, \mu_n\}$. So, from this we obtain

$$-\sqrt[t]{\|S(G)\|_{\mathcal{P}}^t - \mu_1^t} \leq \mu_n \leq \sqrt[t]{\|S(G)\|_{\mathcal{P}}^t - \mu_1^t}.$$

Therefore, by the definition of Sombor spread, we have

$$s(S(G)) = \mu_1 - \mu_n \leq \mu_1 + \sqrt[t]{\|S(G)\|_{\mathcal{P}}^t - \mu_1^t}.$$

with equality if and only if $\mu_2 = \mu_3 = \cdots = \mu_{n-1} = 0$ and from $\sum_{i=1}^{n} \mu_i = 0$, we get $\mu_1 = -\mu_n$. This implies that G has three distinct Sombor eigenvalue μ_1, 0 with multiplicity $n - 1$ and μ_n. So, by Lemma 2.4, $G \cong K_{a,n-a}$. Consider the function

$$f(x) = x + \sqrt[t]{\|S(G)\|_{\mathcal{P}}^t - x^t}.$$

Now, $f'(x) > 0$ implies that $f(x)$ is increasing for $x \leq \sqrt[t]{\frac{\|S(G)\|_{\mathcal{P}}^t}{2}}$ and decreasing for $x \geq \sqrt[t]{\frac{\|S(G)\|_{\mathcal{P}}^t}{2}}$. Thus, with $f(x) \leq f\left(\sqrt[t]{\frac{\|S(G)\|_{\mathcal{P}}^t}{2}}\right)$ we have

$$s(S(G)) \leq \mu_1 + \sqrt[t]{\|S(G)\|_{\mathcal{P}}^t - \mu_1^t} \leq \sqrt[t]{\frac{\|S(G)\|_{\mathcal{P}}^t}{2}} + \sqrt[t]{\|S(G)\|_{\mathcal{P}}^t - \frac{\|S(G)\|_{\mathcal{P}}^t}{2}} = 2\sqrt[t]{\frac{\|S(G)\|_{\mathcal{P}}^t}{2}}. \tag{2.1}$$

If equality holds in (2.1), then $\mu_1^t = (-\mu_n)^t$ and $\mu_2 = \cdots = \mu_{n-1} = 0$. So, it implies that $\mu_n^t = \frac{\|S(G)\|_{\mathcal{P}}^t}{2}$ and $s(S(G)) = 2\mu_1$ and by Lemma 2.4 equality holds for $G \cong K_{a,n-a}$. Conversely, it is easy to verify that equality holds if G is the complete bipartite. □

Noting that $\|S(G)\|_F^2 = \sum_{i=1}^{n} \mu_i^2 = 2F$, we have the following immediate consequence of the above theorem.

Corollary 2.8 *Let G be a connected graph of order $n \geq 2$, $t = 2$ and let F be its forgotten topological index. Then*

$$s(S(G)) \leq \mu_1 + \sqrt{\|S(G)\|_F^2 - \mu_1^2} = \mu_1 + \sqrt{2F - \mu_1^2} \leq 2\sqrt{F} = \frac{2\|S(G)\|_F}{\sqrt{2}},$$

with equalities holding on both sides if and only if G is the complete bipartite graph.

Theorem 2.7 can be generalized in the following result.

Theorem 2.9 *Let G be a connected graph of order $n \geq 2$, $t \geq 2$ be an even integer and $k \geq 2$ be the multiplicity of μ_n. Then*

$$s(S(G)) \leq \mu_1 + \sqrt[t]{\frac{\|S(G)\|_{\mathcal{P}}^t - \mu_1^t}{k}},$$

with equality if and only if G is the regular complete $(k+1)$-multipartite graph.

Proof Let G be a connected graph such that the multiplicity of μ_n is k, so as in proof of Theorem 2.7, we have

$$\mu_1^t + k|\mu_n|^t = |\mu_1|^t + k|\mu_n|^t \leq \sum_{i=1}^{n} |\mu_i|^t = \sum_{i=1}^{n} \mu_i^t = \|S(G)\|_{\mathcal{P}}^t,$$

with equality if and only if $S(G)$ has the eigenvalue 0 with multiplicity $n-k-1$ and the eigenvalue μ_n with multiplicity k. Thus, we obtain

$$-\sqrt[t]{\frac{\|S(G)\|_{\mathcal{P}}^t - \mu_1^t}{k}} \leq \mu_n \leq \sqrt[t]{\frac{\|S(G)\|_{\mathcal{P}}^t - \mu_1^t}{k}}.$$

Therefore, by the definition of the Sombor spread, we have

$$s(S(G)) = \mu_1 - \mu_n \leq \mu_1 + \sqrt[t]{\frac{\|S(G)\|_{\mathcal{P}}^t - \mu_1^t}{k}}. \tag{2.2}$$

with equality if and only if $\mu_2 = \mu_3 = \cdots = \mu_{n-k-1} = 0$, $\mu_{n-k} = \mu_{n-k+1} = \cdots = \mu_{n-1} = \mu_n$ and from $\sum_{i=1}^{n} \mu_i = 0$, we get $\mu_1 = -k\mu_n$. Next, we show that $G \cong K_{\underbrace{p,\, p,\, p,\, \ldots,\, p}_{k+1}}$. Under some suitable labelling, the Sombor matrix of G is

$$S(G) = \begin{pmatrix} \mathbf{0}_p & r\sqrt{2}J_p & \ldots & r\sqrt{2}J_p & r\sqrt{2}J_p \\ r\sqrt{2}J_p & \mathbf{0}_p & \ldots & r\sqrt{2}J_p & r\sqrt{2}J_p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r\sqrt{2}J_p & r\sqrt{2}J_p & \ldots & \mathbf{0}_p & r\sqrt{2}J_p \\ r\sqrt{2}J_p & r\sqrt{2}J_p & \ldots & r\sqrt{2}J_p & \mathbf{0}_p \end{pmatrix},$$

where $r = kp$ and J is the matrix of all ones. By Lemma 2.5, 0 is the eigenvalue of $S(G)$ with multiplicity $n-(k+1)$ and the remaining $k+1$ eigenvalue of $S(G)$ are the eigenvalues of the following equitable quotient matrix

$$Q = \begin{pmatrix} 0 & rp\sqrt{2} & \dots & rp\sqrt{2} & rp\sqrt{2} \\ rp\sqrt{2} & 0 & \dots & rp\sqrt{2} & rp\sqrt{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ rp\sqrt{2} & rp\sqrt{2} & \dots & 0 & rp\sqrt{2} \\ rp\sqrt{2} & rp\sqrt{2} & \dots & rp\sqrt{2} & 0 \end{pmatrix}_{k+1}.$$

The matrix Q can be written as $Q = rp\sqrt{2}\big(J_{k+1} - I_{k+1}\big)$ and it is know that the spectrum (see, [13] P. 45) of J_{k+1} is $\{k+1, 0^{[k]}\}$. Thus, the spectrum of Q consists of the simple eigenvalue $rpk\sqrt{2} = (kp)^2\sqrt{2}$ and the eigenvalue $-rp\sqrt{2} = -kp^2\sqrt{2}$ with multiplicity k. By (2.2), we have $\mu_1 + \sqrt[t]{\frac{\|S(G)\|_{\mathcal{P}}^t - \mu_1^t}{k}} = (kp)^2\sqrt{2} + \sqrt[t]{\frac{((kp)^2\sqrt{2})^t + k(kp^2\sqrt{2})^t - ((kp)^2\sqrt{2})^t}{k}} = (kp)^2\sqrt{2} + kp^2\sqrt{2} = kp^2\sqrt{2}(k+1)$, which is same as the spread of $s(S(K_{\underbrace{p, p, p, \dots, p}_{k+1}})) = k^2p^2\sqrt{2} - kp^2\sqrt{2}$. Thus, equality holds if and only if G is the regular complete $k+1$-multipartite graph. □

We illustrate Theorem 2.9 for the Peterson graph with the help of computer calculations (Wolframe Mathematica). The Sombor spectrum of the Peterson graph G is

$$\Big\{12.7279, (4.24264)^{[5]}, (-8.48528)^{[4]}\Big\}.$$

Also, we note that $k = 4$, $\mu_1 = 12.7279$, $\mu_n = -8.48528$ and the spread of G is $s(S(G)) = 12.7279 + 8.48528 = 21.2132$

- For $t = 2$, $\|S(G)\|_{\mathcal{P}}^2 = 539.999$ and by Theorem 2.9, we have $s(S(G)) \le 22.449$.
- For $t = 4$, $\|S(G)\|_{\mathcal{P}}^4 = 48599.8$ and by Theorem 2.9, we have $s(S(G)) \le 21.3743$.
- For $t = 6$, $\|S(G)\|_{\mathcal{P}}^6 = 5.77363 \times 10^6$ and by Theorem 2.9, we have $s(S(G)) \le 21.2406$.
- For $t = 8$, $\|S(G)\|_{\mathcal{P}}^8 = 7.96758 \times 10^8$ and by Theorem 2.9, we have $s(S(G)) \le 21.2183$.
- For $t = 10$, $\|S(G)\|_{\mathcal{P}}^{10} = 1.19324 \times 10^{11}$ and by Theorem 2.9, we have $s(S(G)) \le 21.2142$.
- For $t = 12$, $\|S(G)\|_{\mathcal{P}}^{12} = 1.86325 \times 10^{13}$ and by Theorem 2.9, we have $s(S(G)) \le 21.2133$.
- For $t = 14$, $\|S(G)\|_{\mathcal{P}}^{14} = 2.96828 \times 10^{15}$ and by Theorem 2.9, we have $s(S(G)) \le 21.2132$.

So, with this calculation for $t = 14$ in the Schatten norm, the approximated value of $s(S(G))$ of Theorem 2.9 is same as 21.2132, the four decimal place value of the spread of the Peterson graph. Next result gives the upper bound for $s(S(G))$.

Theorem 2.10 *Let G be a connected graph of order $n \ge 2$ and let $t \ge 2$ be an even integer. Then*

$$s(S(G)) \le \mu_1 + \sqrt[t]{\frac{\|S(G)\|_{\mathcal{P}}^t - \mu_1^t}{n-1}},$$

with equality holding if and only if G is the complete graph.

Proof Similar to the proof of Theorems 2.7 and 2.9. □

Depending on the number of positive Sombor eigenvalues and the number of negative Sombor eigenvalues, we have the following result for the Sombor spread of G.

Theorem 2.11 *The following hold for a connected graph G of order $n \ge 2$ and an even positive integer $t \ge 2$.*

(i) *If G has k $(1 \le k \le n-1)$ positive Sombor eigenvalues and $\|S(G)\|_{\mathcal{P}}^t \ge k\mu_1^t$, then*

$$s(S(G)) \ge \mu_1 + \sqrt{\frac{\|S(G)\|_{\mathcal{P}}^t - k\mu_1^t}{n-k}}.$$

Equality is attained if and only if $G \cong K_n$.

(ii) *If G has k $(1 \leq k \leq n-1)$ negative Sombor eigenvalues and $\|S(G)\|_{\mathcal{P}}^t \geq k\mu_n^t$, then*

$$s(S(G)) \geq \sqrt{\frac{\|S(G)\|_{\mathcal{P}}^t - k\mu_n^t}{n-k}} - \mu_n,$$

with equality holding if and only if $G \cong K_n$.

Proof With the given hypothesis, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0 \geq \mu_{k+1} \geq \cdots \geq \mu_n$, it follows $\mu_i^t \leq \mu_1^t$, for $1 \leq i \leq k$ and $\mu_i^t \leq \mu_n^t$, for $k+1 \leq i \leq n$. So, $\|S(G)\|_{\mathcal{P}}^t \leq k\mu_1^t + (n-k)\mu_n^t$ and we get

$$\mu_n \leq -\sqrt{\frac{\|S(G)\|_{\mathcal{P}}^t - k\mu_1^t}{n-k}}.$$

Therefore, the Sombor spread of G is

$$s(S(G)) = \mu_1 - \mu_n \geq \mu_1 + \sqrt{\frac{\|S(G)\|_{\mathcal{P}}^t - k\mu_1^t}{n-k}}.$$

Assume that equality occurs. Then $\mu_i = \mu_1$, for $1 \leq i \leq k$ and $\mu_i = \mu_n$, for $k+1 \leq i \leq n$. Thus G has two distinct Sombor eigenvalues μ_1 and μ_n with multiplicities k and $n-k$, respectively. So by Lemma 2.3, G is isomorphic to K_n. For other way round, we have

$$\mu_1 + \sqrt{\frac{\|S\|_t^t - k\mu_1^t}{n-k}} = n-1+\sqrt{\frac{(n-1)^t + (n-1) - (n-1)^t}{n-1}} = n-1+1 = n = s(S(G))(K_n).$$

This proves the equality case of (i). Now, proof of (ii) is similar to that of part (i). □

The following inequality can be found in [17].

Lemma 2.12 *([17]) Let $\nu_1, \nu_2, \ldots, \nu_n$ and $a_1, a_2, \ldots, a_n$ be real numbers such that $\nu \leq \nu_i \leq M$ and $a \leq a_i \leq m$, for $1 \leq i \leq n$. Then*

$$\left|\frac{1}{n}\sum_{i=1}^{n}\nu_i a_i - \frac{1}{n}\sum_{i=1}^{n}\nu_i \frac{1}{n}\sum_{i=1}^{n}a_i\right| \leq \frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\right)(M-\nu)(m-a),$$

where $\lfloor . \rfloor$ denotes the floor function.

Next results is an application of Lemma 2.12 and established a lower bound for $s(S(G))$ in terms of n and the Frobenius norm of $S(G)$.

Theorem 2.13 *For a graph G of order n*

$$s(S(G)) \geq 2\|S(G)\|_F \sqrt{\frac{2n}{2n^2 - 1 + (-1)^{n+2}}}$$

with equality holds if and only if $G \cong \overline{K}_n$.

Proof Let $\nu_i = a_i = \mu_i$ in Lemma 2.12. We have

$$\left|\frac{1}{n}\sum_{i=1}^{n}\mu_i^2 - \frac{1}{n}\sum_{i=1}^{n}\mu_i \frac{1}{n}\sum_{i=1}^{n}\mu_i\right| \leq \frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\right)(\mu_1-\mu_n)(\mu_1-\mu_n), \tag{2.3}$$

which implies that

$$\frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\right)s(S(G))^2 \geq \left|\frac{1}{n}\sum_{i=1}^{n}\mu_i^2\right| = \frac{\|S(G)\|_F^2}{n}.$$

Therefore

$$s(S(G))^2 \geq \frac{\|S(G)\|_F^2}{n} \cdot \frac{1}{\frac{1}{n}\left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n}\left\lfloor \frac{n}{2} \right\rfloor\right)}.$$

It is easy to verify that (see, [19]) $\frac{1}{\frac{1}{n}\lfloor \frac{n}{2} \rfloor\left(1-\frac{1}{n}\lfloor \frac{n}{2} \rfloor\right)} = \frac{1}{\frac{1}{8n^2}(2n^2-1+(-1)^{n+2})}$ and with this, we obtain

$$s(S(G)) \geq \|S(G)\|_F \sqrt{\frac{8n}{2n^2 - 1 + (-1)^{n+2}}} = 2\|S(G)\|_F \sqrt{\frac{2n}{2n^2 - 1 + (-1)^{n+2}}}.$$

Equality occurs if and only if equality holds in (2.3), which is possible if and only if $\mu_1 = \mu_2 = \cdots = \mu_n$. □

With $\sum_{i=1}^{n} \mu_i^2 = 2F$, the following corollary follows from Theorem 2.13.

Corollary 2.14 *Let G be a graph of order $n \geq 2$. Then*

$$s(S(G)) \geq 4\sqrt{\frac{nF}{2n^2 - 1 + (-1)^{n+2}}}.$$

The equality is attained either for an empty graph or $G \cong \overline{K}_n$.

In the next result, we present the upper bounds for $s(S(G))$ in terms of the clique number of G.

Theorem 2.15 *Let G be a connected graph consisting of a clique of size $\omega \geq 3$ and let $t \geq 2$ be an even integer. Then the following holds.*

(i)

$$s(S(G)) \leq \mu_1 + \sqrt[t]{\|S(G)\|_{\mathcal{P}}^t - \mu_1^t - (\omega - 1)\Big((\omega - 1)\sqrt{2}\Big)^t},$$

with equality holding if and only if $G \cong K_n$.

(ii) *If each vertex of the clique share the common neighbourhood of size η, then*

$$s(S(G)) \leq \mu_1 + \sqrt[t]{\|S(G)\|_{\mathcal{P}}^t - \mu_1^t - (w - 1)\Big((\omega - 1 + \eta)\sqrt{2}\Big)^t},$$

with equality holding if and only if G is the complete split graph.

Proof Since G contains a clique of size $\omega \geq 3$, so by Lemmas 2.6, 2.2 and by interlacing property of Theorem 2.1, G has at least ω Sombor eigenvalues (including μ_n) less than or equal to $-(\omega-1)\sqrt{2}$, that is $\mu_i \leq -(\omega-1)\sqrt{2}$ for $i = 1, 2, \ldots, \omega - 1$. Thus,

$$\mu_1^t + (\omega-1)\Big((\omega-1)\sqrt{2}\Big)^t + |\mu_n|^t = |\mu_1|^t + (\omega-1)\Big((\omega-1)\sqrt{2}\Big)^t + |\mu_n|^t \leq \|S(G)\|_{\mathcal{P}}^t, \tag{2.4}$$

which further implies that

$$-\sqrt[t]{\|S(G)\|_{\mathcal{P}}^t - (\omega-1)\Big((\omega-1)\sqrt{2}\Big)^t} \leq \mu_n \leq \sqrt[t]{\|S(G)\|_{\mathcal{P}}^t - \mu_1^t - (\omega-1)\Big((\omega-1)\sqrt{2}\Big)^t}.$$

Equality holds if equality occurs in (2.4). Thus there may be two possibilities: first possibility is $\mu_i = -(\omega-1)\sqrt{2}$, for $i = 1, 2, 3, \ldots, \omega - 1$ and $\mu_n = -(\omega-1)\sqrt{2}$. In this case G has two distinct Sombor eigenvalues and by Lemma 2.3, G is the complete graph. The second possibility is $\mu_i = -(\omega-1)\sqrt{2}$, for $i = 1, 2, \ldots, \omega - 1$ and $\mu_n < -(\omega-1)\sqrt{2}$. Since $\mu_i = -(\omega-1)\sqrt{2}$, for $i = 1, 2, \ldots, \omega - 1$, so by Lemma 2.6, it follows that each vertex of the clique share no external neighbourhood (other than the vertices of clique itself). Therefore, ω must be equal to n and $\mu_n = -(\omega-1)\sqrt{2}$. Conversely, if $G \cong K_n$, then it is easy to see that equality holds in (2.4). Now, by definition of the Sombor spread, we have

$$s(S(G)) = \mu_1 - \mu_n \le \mu_1 + \sqrt[t]{\|S(G)\|_{\mathcal{P}}^t - (\omega-1)\Big((\omega-1)\sqrt{2}\Big)^t},$$

with equality holding if and only if $G \cong K_n$.
(ii). We partition the vertex set of G into three disjoint sets $V = V_1 \cup V_2 \cup V_3$, where V_1 is the vertex set of clique of cardinality ω, V_2 is the vertex set shared by each vertex of V_1, say it is of cardinality η and $V_3 = V \setminus (V_1 \cup V_2)$. With this hypothesis and Lemma 2.6 $-(\omega-1+\eta)\sqrt{2}$ is the Sombor eigenvalue of G with multiplicity $\omega-1$. We assume that $\eta \ne 0$, otherwise it reduces to previous case. So, with this information, we have

$$\mu_1^t + (\omega-1)((\omega-1+\eta)\sqrt{2})^t + \mu_n^t \le \|S(G)\|_{\mathcal{P}}^t. \tag{2.5}$$

Thereby, we obtain $\mu_n^t \le \|S(G)\|_{\mathcal{P}}^t - (w-1) - ((\omega-1+\eta)\sqrt{2})^t$ and it follows that

$$-\sqrt[t]{\|S(G)\|_{\mathcal{P}}^t - \mu_1^t - (w-1)\Big((\omega-1+\eta)\sqrt{2}\Big)^t} \le \mu_n \le \sqrt[t]{\|S(G)\|_{\mathcal{P}}^t - \mu_1^t - (w-1)\Big((\omega-1+\eta)\sqrt{2}\Big)^t}.$$

Therefore, the spread of G satisfies

$$s(S(G)) = \mu_1 - \mu_n \le \mu_1 + \sqrt[t]{\|S(G)\|_{\mathcal{P}}^t - \mu_1^t - (w-1)\Big((\omega-1+\eta)\sqrt{2}\Big)^t}.$$

Equality holds if all inequalities hold. From (2.5), it follows that G has exactly three distinct non zero Sombor eigenvalues. This implies that V_2 must be an independent set and by Lemma 2.5, it contributes 0 in equation 2.5. This further implies that V_3 is an empty set and it is clear that μ_1 and μ_n are the remaining two non zero Sombor eigenvalues of its associated equitable quotient matrix. Thus, G is the complete split graph with independent set V_2. Conversely, assume that G is the complete split graph with clique size ω and independent set of size η, where $\omega + \eta = n$. By [32], the Sombor spectrum of G is

$$\left\{0^{[n-\omega-1]}, \big(-(n-1)\sqrt{2}\big)^{[\omega-1]}, \frac{1}{2}\Big(\sqrt{2}(n\omega - n - \omega + 1) \pm \sqrt{\triangle}\Big), \right\}$$

where $\triangle = 4n^3\omega - n^2(2\omega^2 + 12\omega - 2) + n(4\omega^3 + 4\omega^2 + 12\omega - 4) - 4\omega^4 - 2\omega^2 - 4\omega + 2$. Now, it is easy to see that $s(S(G)) = 2\sqrt{\triangle} = \mu_1 + \sqrt[t]{\|S\|_t^t - \mu_1^t - (w-1)\big((\omega-1+\eta)\sqrt{2}\big)^t}$ and equality holds. □

The following results gives the lower bound for $s(S(G))$ in terms of the independence number associated with graph G.

Theorem 2.16 *Let G be a connected graph of order n and let I be an independence set of cardinality $\alpha \ge 2$. Then*

$$s(S(G)) \ge \frac{\sqrt{\alpha^2\,(2SO(G) - B_0 - B_1)^2 + 4\alpha B_0 B_1\,(n-\alpha)}}{\alpha(n-\alpha)},$$

where $SO(G)$ is the Sombor index of G, B_0 and B_1 are defined in (2.6). Equality holds if and only if degree of each vertex in I and in $V \setminus I$ are some constants and each vertex in $V \setminus I$ have same neighbourhood in I.

Proof We first index the the vertices of I, so that the Sombor matrix of G can be written as

$$S(G) = \begin{pmatrix} \mathbf{0}_\alpha & M_{\alpha\times n-\alpha} \\ M^T & B_{n-\alpha} \end{pmatrix},$$

where $\mathbf{0}$ is the zero matrix, M^T is the transpose of M and B is the sub matrix corresponding to the non independent vertices. The quotient matrix of above matrix is

$$Q = \begin{pmatrix} 0 & \frac{B_0}{\alpha} \\ \frac{B_1}{n-\alpha} & \frac{2SO(G)-B_0-B_1}{n-\alpha} \end{pmatrix},$$

where

$$B_0 = \sum_{i=1}^{\alpha} r_i(M), \qquad B_1 = \sum_{i=1}^{n-\alpha} r_i(M^T), \tag{2.6}$$

and r_i denotes the i-th row sum of M. The eigenvalues of Q are

$$\frac{\alpha(2SO(G) - B_0 - B_1) \pm \sqrt{\alpha^2 \left(2SO(G) - B_0 - B_1\right)^2 + 4\alpha B_0 B_1 (n-\alpha)}}{2\alpha(n-\alpha)}.$$

Therefore, by Theorems 2.1 and 2.2, $\mu_1(S(G)) \geq \mu_1(Q) \geq \mu_2(S(G)) \geq \mu_2(Q) \geq \mu_n(S(G))$, which implies that

$$s(S(G)) = \mu_1 - \mu_n \geq \frac{\sqrt{\alpha^2 \left(2SO(G) - B_0 - B_1\right)^2 + 4\alpha B_0 B_1 (n-\alpha)}}{\alpha(n-\alpha)}.$$

Equality holds if and only if Q is an equitable quotient matrix, that is, each non zero block M, M^T and B of $S(G)$ have constant row sum. Thus, $B_0 = \sum_{i=1}^{\alpha} r_0(M) = \alpha r_0$, $B_1 = \sum_{i=1}^{n-\alpha} r'_0(M^T) = (n-\alpha)r'_0$ and $2SO(G) - B_0 - B_1 = (n-\alpha)r''_0$. So, the eigenvalues of Q are $\frac{1}{2}\left(r''_0 \pm \sqrt{(r''_0)^2 + 4r_0r'_0}\right)$ and spread is $s(S(G)) = \sqrt{(r''_0)^2 + 4r_0r'_0}$. This can happen if and only if vertex degrees on I and on $V \setminus I$ are some constants, equivalently each vertex of I is adjacent to same number of vertices in $V \setminus I$ or vice versa. □

Clearly, equality holds in Theorem 2.16 for graph like the complete graph minus edge $K_n - e$, the star graph $K_{1,n-1}$, the complete split graph and many other graphs. Ans it remains open to completely characterize graphs attaining equality in Theorem 2.16.

The join of two vertex disjoint graphs G_1, G_2 denoted by $G_1 \vee G_2$ is obtained from $G_1 \cup G_2$ by including all the edges between the vertices of G_1 and the vertices in G_2. To avoid triviality, we assume that at least one G_i is not complete.

The next result gives the lower bound for $s(S(G))$ of the join of two graphs.

Theorem 2.17 *Let $G = G_1 \vee G_2$ be the join of two graphs G_1 and G_2 of order n_1 and n_2, respectively. Then*

$$s(S(G)) \geq \frac{2\sqrt{(SO(G_1)n_2 - SO(G_2)n_1)^2 + \eta^2 n_1 n_2}}{n_1 n_2},$$

where $SO(G_i), i = 1, 2$ is the Sombor index of G_i in G, $\eta = \sum_{v_iu_j \in e(G_1,G_2)} \sqrt{l_i^2 + t_j^2}$, $e(G_1, G_2)$ is the set of edges between each vertex of G_1 to every vertex of G_2 and $l_i,\ i = 1, 2, \ldots, n_1$ and $t_i,\ i = 1, 2, \ldots, n_2$ is the degree sequence of G. Equality holds if and only if both G_1 and G_2 are regular graphs.

Proof Let $\{d_1, d_2, \ldots, d_{n_1}\}$ and $\{d'_1, d'_2, \ldots, d'_{n_2}\}$ be the degree sequence of G_1 and G_2, respectively. Clearly, the degree sequence of G is $d_i + n_2 = l_i,\ i = 1, 2, \ldots, n_1$ and $d'_i + n_1 = t_i$ for $i = 1, 2, \ldots, n_2$. We first label the vertices $\{v_1, v_2, \ldots, v_{n_1}\}$ of G_1 and then the vertices $\{u_1, u_2, \ldots, u_{n_2}\}$ of G_2. Under this labelling, the Sombor matrix of G is

$$S(G) = \begin{pmatrix} S(G_1) & B_{n_1 \times n_2} \\ B^T & S(G_2) \end{pmatrix},$$

where $S(G_i),\ i = 1, 2$ are the Sombor matrices of G_1 and G_2 in G. The matrix B is given by

$$\begin{pmatrix} \sqrt{l_1^2 + t_1^2} & \sqrt{l_1^2 + t_2^2} & \cdots & \sqrt{l_1^2 + t_{n_2}^2} \\ \sqrt{l_2^2 + t_1^2} & \sqrt{l_2^2 + t_2^2} & \cdots & \sqrt{l_2^2 + t_{n_2}^2} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{l_{n_1}^2 + t_1^2} & \sqrt{l_{n_1}^2 + t_2^2} & \cdots & \sqrt{l_{n_1}^2 + t_{n_2}^2} \end{pmatrix}.$$

The quotient matrix of $S(G)$ is

$$\begin{pmatrix} \frac{2\sum\limits_{v_iv_j\in E(G_1)}\sqrt{l_i^2+l_j^2}}{n_1} & \frac{\sum\limits_{v_iu_j\in e(G_1,G_2)}\sqrt{l_i^2+t_j^2}}{n_1} \\ \frac{\sum\limits_{v_iu_j\in e(G_1,G_2)}\sqrt{l_i^2+t_j^2}}{n_2} & \frac{2\sum\limits_{u_iu_j\in E(G_2)}\sqrt{t_i^2+t_j^2}}{n_2} \end{pmatrix} = \begin{pmatrix} \frac{2SO(G_1)}{n_1} & \frac{\eta}{n_1} \\ \frac{\eta}{n_2} & \frac{2SO(G_2)}{n_2} \end{pmatrix}, \tag{2.7}$$

where $e(G_1, G_2)$ is the set of edges between each vertex of G_1 to every vertex of G_2, $SO(G_i),\ i = 1, 2$ is the Sombor index of G_i in G and $\eta = \sum\limits_{v_iu_j\in e(G_1,G_2)} \sqrt{l_i^2 + t_j^2}$. The eigenvalues of (2.7) are

$$\frac{SO(G_1)n_2 + SO(G_2)n_1 \pm \sqrt{(SO(G_1)n_2 - SO(G_2)n_1)^2 + \eta^2 n_1n_2}}{n_1n_2}.$$

By the interlacing property of Theorems 2.1 and 2.2, the Sombor spread of G satisfies

$$s(S(G)) \geq \frac{\sqrt{(SO(G_1)n_2 - SO(G_2)n_1)^2 + \eta^2 n_1n_2}}{n_1n_2}.$$

If both G_1 and G_2 are regular, then partition of $G_1 \vee G_2$ is regular. So, row sum of the quotient matrix given in (2.7) is constant and each eigenvalue of the regular quotient matrix is the eigenvalue of $S(G)$, that is, the spectrum of matrix given in (2.7) is contained in the spectrum of $S(G)$. Furthermore, it is known that the smallest and the largest eigenvalues of the quotient matrix are the smallest and the largest eigenvalues of a non negative matrix. Thus, the equality holds if and only if both G_1 and G_2 are regular graphs. □

3 The trace norm of $S(G)$

The following class of real polynomials $P(a_1, a_2)$ was considered by Lupaş in [26]. The elements of $P(a_1, a_2)$ are of the form $p_n(y) = y^n + a_1y^{n-1} + a_2y^{n-2} + b_3y^{n-3} + \cdots + b_{n-2}y^2 + b_1y + b_0$, where a_1 and a_2 are given real numbers. Let $y_1 \geq y_2 \geq \cdots \geq y_n$ be the zeros of polynomial $p_n(y) \in P(a_1, a_2)$ and let $\overline{y} = \frac{1}{n}\sum\limits_{i=1}^{n} y_i$ and $z = n\sum\limits_{i=1}^{n} y_i^2 - \left(\sum\limits_{i=1}^{n} y_i\right)^2$. Then, followings holds

$$\begin{aligned} \overline{y} + \frac{1}{n}\sqrt{\frac{z}{n-1}} &\leq y_1 \leq \overline{y} + \frac{1}{n}\sqrt{(n-1)z}, \\ \overline{y} - \frac{1}{n}\sqrt{\frac{z(i-1)}{n-i+1}} &\leq y_i \leq \overline{y} + \frac{1}{n}\sqrt{\frac{(n-i)}{i}z}, \qquad \text{for } i = 2, 3, \ldots, n-1, \\ \overline{y} - \frac{1}{n}\sqrt{z(n-1)} &\leq y_n \leq \overline{y} - \frac{1}{n}\sqrt{\frac{z}{n-1}}. \end{aligned} \tag{3.8}$$

The following result immediately follows from the above fact about the zeros $\mu_1 \geq \mu_2 \geq \cdots \geq$ of the characteristic polynomial of the Sombor matrix of G. As $\overline{y} = \frac{1}{n}\sum\limits_{i=1}^{n} y_i = 0$ and $z = n\sum\limits_{i=1}^{n} y_i^2 - \left(\sum\limits_{i=1}^{n} y_i\right)^2 = n\|S(G)\|_F^2$. So, we have the following result.

Proposition 3.1 *Let G be a connected graph of order $n \geq 2$. Then we have the following inequalities for the eigenvalues of $S(G)$*

(i)

$$\frac{\|S(G)\|_F}{\sqrt{n(n-1)}} \leq \mu_1 \leq \sqrt{\frac{n-1}{n}}\|S(G)\|_F.$$

(ii)

$$-\sqrt{\frac{i-1}{n(n-i+1)}}\|S(G)\|_F \leq \mu_i \leq \sqrt{\frac{n-i}{ni}}\|S(G)\|_F, \quad for\ i = 2, 3, \ldots, n-1.$$

(iii)

$$-\sqrt{\frac{n-1}{n}}\|S(G)\|_F \leq \mu_n \leq -\frac{1}{\sqrt{n-1}}\|S(G)\|_F.$$

Theorem 3.2 *Let G be a graph of order $n \geq 2$. Then*

$$\mathcal{E}(S(G)) \leq s(S(G)) + \sqrt{(n-2)(\|S(G)\|_F^2 - s^2(S(G)) - 2\mu_1\mu_n)},$$

with equality holding if and only if $|\mu_2| = |\mu_3| = \cdots = |\mu_{n-1}|$, that is, equality hold for graphs like K_n and $K_{a,n-a}$.

Proof By applying Cauchy-Schwarz Inequality to vectors $(|\mu_2|, |\mu_3|, \ldots, |\mu_{n-1}|)$, we have

$$\begin{aligned}
\mathcal{E}(S(G)) = \sum_{i=1}^{n} |\mu_i| &= |\mu_1| + |\mu_n| + \sum_{i=2}^{n-1} |\mu_i| \\
&\leq \mu_1 - \mu_n + \sqrt{(n-2)\sum_{i=2}^{n-1} \mu_i^2} \\
&= \mu_1 - \mu_n + \sqrt{(n-2)(\|S(G)\|_F^2 - \mu_1^2 - \mu_n^2)} \\
&= \mu_1 - \mu_n + \sqrt{(n-2)(\|S(G)\|_F^2 - (\mu_1 - \mu_n)^2 - 2\mu_1\mu_n)} \\
&= s(S(G)) + \sqrt{(n-2)(\|S(G)\|_F^2 - s^2(S(G)) - 2\mu_1\mu_n)}. \qquad (3.9)
\end{aligned}$$

Equality holds if and only if equality holds in (3.9), which is possible if and only if $|\mu_2| = |\mu_3| = \cdots = |\mu_{n-1}|$. By Lemmas 2.3 and 2.4, it is clear that K_n and $K_{a,n-a}$ satisfy the required condition of equality. □

It remains open to identify all the graphs satisfying the equality conditions, that is, $|\mu_2| = |\mu_3| = \cdots = |\mu_{n-1}|$.

By using the spectral spread bounds of $S(G)$ and Proposition 3.1, the following are the immediate consequences of Theorem 3.2.

Corollary 3.3 *Let G be a graph of order $n \geq 2$ with Frobenius norm $\|S(G)\|_F^2$ and forgotten topological index F. Then the following holds.*

(i)

$$\mathcal{E}(S(G)) \leq \|S(G)\|_F \left(\sqrt{2} + \sqrt{(n-2)\Big(1 - 4\Big(\frac{2n}{2n^2 - 1 + (-1)^{n+2}}\Big)^2 + \frac{2(n-1)}{n}\Big)}\right)$$

(i)

$$\mathcal{E}(S(G)) \leq \sqrt{2F} \left(\sqrt{2} + \sqrt{(n-2)\Big(1 - 4\Big(\frac{2n}{2n^2 - 1 + (-1)^{n+2}}\Big)^2 + \frac{2(n-1)}{n}\Big)}\right)$$

Proof The proof follows from Theorem 3.2, Corollaries and 2.8 and 2.14. □

4 Comments

From [32], the Sombor spectrum of the complete graph is $\{(n-1)^2\sqrt{2}, (-\sqrt{2}(n-1))^{[n-1]}\}$ and that of $K_n \setminus e$ is

$$\left\{0, (-(n-1)\sqrt{2})^{[n-3]}, \frac{1}{2}\left(\sqrt{2}(n^2-4n+3) \pm \sqrt{2n^4-36n^2+88n-62}\right)\right\}.$$

The Sombor spread of K_n is $s(S(K_n)) = \sqrt{2}(n-1)^2 + \sqrt{2}(n-1) = \sqrt{2}n(n-1)$ and that of $K_n \setminus e$ is $s(S(K_n \setminus e)) = \sqrt{2n^4-36n^2+88n-62}$.

Now, assume that

$$s(S(K_n)) \geq s(S(K_n - e)).$$

After simplification, we obtain

$$-2(2n^3 - 19n^2 + 44n - 31) \geq 0,$$

which is not true for $n \geq 7$. Also, with computations, we have the following observation:

$$s(S(K_3)) = 8.48528 > 6.3256 = s(S(K_3 - e)), s(S(K_4)) = 16.9706 > 16.0333 = s(S(K_4 - e))$$
$$s(S(K_5)) = 28.2843 > 26.9815 = s(S(K_5 - e)), s(S(K_6)) = 42.4264 > 41.9762 = s(S(K_6 - e)).$$

Therefore, the Sombor spread of $K_n, n \geq 7$ increases upon edge deletion. Similarly, it is easy to see that the Sombor spread of the complete bipartite graph decreases upon edge deletion. Thus, Sombor spread may decrease or may increase for an arbitrary subgraph H of G. In case H is an induced subgraph of G, does Sombor spread of H decrease/increase? This Sombor spread problem of edge deletion can be studied for arbitrary graphs. More precisely, we have the following problem.

Problem 1 *Let G be a connected graph of order n and e be an edge of G. Then Characterize the graphs for which $s(S(G)) > s(S(G-e))$, $s(S(G)) < s(S(G-e))$ and $s(S(G)) = s(S(G-e))$.*

Since, the spread of $S(G)$ is related to the Schatten p-norm, the Frobenius norm and the trace norm. So, it it natural to study the Schatten p-norm of $S(G)$ and establish the bounds for it along with the characterization of extremal graphs.

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