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ORIGINAL RESEARCH

Computing the 2-nilpotent multiplier of 2-generator p-groups of class 2

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Abstract The complete classification of 2-generator p-groups of class 2 has been achieved by Ahmad et al. (2012). The various homological functors of these groups, such as the nonabelian tensor square, the nonabelian exterior square and the Schur multiplier, are computed in recent past by Bacon and Kappe (1993), Kappe et al. (1999), and Magidin and Morse (2010). It is also specified which of these groups are capable and which are unicentral. As a generalized topic, the study of nilpotent multiplier of groups is motivated by its application in the isologism theory of P. Hall (1940). Another important reason to study them is that, determining the structure of nilpotent multiplier of groups is essential in studying generalized capability and covering groups, and would be generally useful in developing such a framework. In this paper, we give a complete description of the 2-nilpotent multiplier of 2-generator p-groups of class 2 as well as their triple tensor product and triple exterior product. In addition, we determine which of these groups are 2-capable.

Keywords Nilpotent group · Nonabelian tensor product · Nilpotent multiplier · Capable group

Mathematics Subject Classification Primary 19C09; Secondary 20D15

1 Introduction and Motivation

The concept of nonabelian tensor product of groups was introduced by Brown and Loday in [10]. This construction has its origin in algebraic K-theory and in homotopy theory. Following the work of Miller [25], Dennis [13] introduced the nonabelian tensor square in a search for "homology functors" which are intimately related to K-theory. Subsequently extending certain classical ideas of Whitehead [27], Brown and Loday discovered a topological significance for this tensor square.

When two groups M and N act on each other satisfying certain compatibility conditions [9, pp. 178-179], and act on themselves by conjugation, the nonabelian tensor product $M \otimes N$ can be defined as a group generated by the symbols $m \otimes n$, where $m \in M$ and $n \in N$, with defining relations

$$mm' \otimes n = ({}^m m' \otimes {}^m n)(m \otimes n) \quad \text{and} \quad m \otimes nn' = (m \otimes n)({}^n m \otimes {}^{n'} n),$$

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for all $m, m' \in M$ and $n, n' \in N$, where ${}^m m' = mm'm^{-1}$. It was shown in [14] with homological methods, involving the original approach in [9,10], that $M \otimes N$ is a finite group (or a p-group), when M and N are finite groups (or p-groups).

Since the compatible actions are satisfied when a group acts on itself by conjugation, it leads to define the nonabelian tensor square $M \otimes M$. The nonabelian exterior square $M \wedge M$ is the quotient of $M \otimes M$ modulo the central subgroup $\nabla(M)$ generated by the elements $m \otimes m$ for all $m \in M$. The image of the generator $m \otimes m'$ in $M \wedge M$ is written $m \wedge m'$. The commutator map induces a homomorphism $[\ ,\ ] : M \wedge M \longrightarrow M$, and the results of [25] show that there is an exact sequence

$$0 \longrightarrow H_2(M) \longrightarrow M \wedge M \longrightarrow M,$$

where $H_2(M)$ is the second homology group $H_2(M, \mathbb{Z})$ and is called the Schur multiplier of M.

Group theoretical aspects of nonabelian tensor squares have been studied extensively by several authors, starting with the paper of Brown, Johnson and Robertson [9]. An important perspective of the research includes computing the nonabelian tensor square of groups explicitly and describing them in more standard terms. For larger finite groups, crossed pairing have been used as a computational technique. Let M and N be as above, and let L be a group. Following the notation and terminology of [10], a function $\Phi : M \times N \longrightarrow L$ is called a crossed pairing if

$$\Phi(mm', n) = \Phi({}^m m', {}^m n)\Phi(m, n) \quad and \quad \Phi(m, nn') = \Phi(m, n)\Phi({}^n m, {}^n n'),$$

for all $m, m' \in M$ and $n, n' \in N$. Clearly any crossed pairing from $M \times N$ to L determines a unique homomorphism $\Phi^* : M \otimes N \longrightarrow L$ such that $\Phi^*(m \otimes n) = \Phi(m, n)$.

In order to achieve more related structural results of a group M, we need a generalization of $M \wedge M$. By a crossed module we mean a group homomorphism $\mu : M \longrightarrow P$ together with an action of P on M which satisfies the following conditions ($p \in P, m, m' \in M$):

$$\mu({}^p m) = {}^p \mu(m) \quad \text{and} \quad {}^{\mu(m)} m' = {}^m m'.$$

Let $\upsilon : N \longrightarrow P$ be another crossed module. Then clearly M and N act on each other via P, and the first crossed module rule implies that these actions are compatible. The second crossed module rule says that the actions of M and N on themselves by conjugation are given by action via P.

Definition 1.1 ([10, *Definition 2.11*]) The nonabelian exterior product $M \wedge^P N$ is obtained from $M \otimes N$ by imposing the additional relations $m \otimes n = 1$ for all $(m, n) \in M \times N$ such that $\mu(m) = \upsilon(n)$. The image of a general element $m \otimes n$ in $M \wedge^P N$ is written $m \wedge n$.

The kernel of the natural homomorphism $M \otimes N \longrightarrow M \wedge^P N$ includes all the elements $m \otimes n$ satisfying $\mu(m) = \upsilon(n)$, and is central in $M \otimes N$. Also recall from [10, Proposition 2.15] that, the group $M \otimes N$ arose in a certain universal crossed square which is constructed with crossed modules μ and υ.

Now we consider the crossed module $[\ ,\ ] : M \wedge M \longrightarrow M$ in which the action of M on $M \wedge M$ is defined by ${}^n(m \wedge m') = {}^n m \wedge {}^n m'$, for all $m, m', n \in M$. On the other hand the identity homomorphism $M \longrightarrow M$ is also a crossed module with M acting by conjugation. It follows that the action of $M \wedge M$ on M is given by ${}^{(m \wedge m')} n = {}^{[m,m']} n$, where $[m, m'] = mm'm^{-1}m'^{-1}$. So by Definition 1.1 we can form the triple exterior product $\wedge^3 M = (M \wedge M) \wedge M$ which is obtained from $(M \wedge M) \otimes M$ by imposing the additional relations $(m \wedge m') \otimes [m, m'] = 1$. Note that the triple tensor product $\otimes^3 M = (M \otimes M) \otimes M$ may be defined analogously by using the same crossed modules $[\ ,\ ] : M \otimes M \longrightarrow M$ and $id : M \longrightarrow M$. Furthermore the c-fold exterior product $\wedge^c M$ (or tensor product $\otimes^c M$) is defined recursively (see [11, p. 410]).

The first study in this direction was done by Ellis and McDermott [17], who obtained an upper bound for the order of c-fold tensor product of prime power groups, $c \geqslant 2$, and then has been studied in a number of papers [3,12,16,22]. In particular, there have been some attempts to establish the explicit structure of 3-fold tensor (exterior) product of certain groups [18,21]. This paper is devoted to develop such a framework by computing the triple tensor (exterior) product of 2-generator p-groups of class 2. The classification of this family of groups was first carried out in [5] for $p > 2$, and in [23] for $p = 2$, and then was completed by Ahmad et al. [1] (see Theorem 1.2 in Section 2).

Theorem 1.2 *([1]) Let p be a prime, $n > 2$ a positive integer. Every 2-generated p-group of class exactly 2 and order p^n corresponds to an ordered 5-tuple of integers, $(\alpha, \beta, \gamma; \rho, \sigma)$, such that:*

(i) $\alpha \geqslant \beta \geqslant \gamma \geqslant 1$;
(ii) $\alpha + \beta + \gamma = n$;
(iii) $0 \leqslant \rho, \sigma \leqslant \gamma$;

where $(\alpha, \beta, \gamma; \rho, \sigma)$, *corresponds to the group presented by*

$$G = \langle a, b \mid [a,b]^{p^\gamma} = [a,b,a] = [a,b,b] = 1, a^{p^\alpha} = [a,b]^{p^\rho}, b^{p^\beta} = [a,b]^{p^\sigma} \rangle.$$

Moreover:
FAMILY 1. *If* $\alpha > \beta$, *then G is isomorphic to:*

(a) $(\alpha, \beta, \gamma; \rho, \gamma)$ *when* $\rho \leqslant \sigma$.
(b) $(\alpha, \beta, \gamma; \gamma, \sigma)$ *when* $0 \leqslant \sigma < \sigma + \alpha - \beta \leqslant \rho$ *or* $\sigma < \rho = \gamma$.
(c) $(\alpha, \beta, \gamma; \rho, \sigma)$ *when* $0 \leqslant \sigma < \rho < min(\gamma, \sigma + \alpha - \beta)$.

FAMILY 2. *If* $\alpha = \beta > \gamma$ *or* $\alpha = \beta = \gamma$ *and* $p > 2$, *then G is isomorphic to* $(\alpha, \beta, \gamma; min(\rho, \sigma), \gamma)$.
FAMILY 3. *If* $\alpha = \beta = \gamma$ *and* $p = 2$, *then G is isomorphic to:*

(a) $(\alpha, \beta, \gamma; min(\rho, \sigma), \gamma)$ *when* $0 \leqslant min(\rho, \sigma) < \gamma - 1$.
(b) $(\alpha, \beta, \gamma; \gamma - 1, \gamma - 1)$ *when* $\rho = \sigma = \gamma - 1$.
(c) $(\alpha, \beta, \gamma; \gamma, \gamma)$ *when* $min(\rho, \sigma) \geqslant \gamma - 1$ *and* $max(\rho, \sigma) = \gamma$.

The groups listed in Family 1(a) through Family 3(c) are pairwise non-isomorphic.

Several remarkable results about the 2-generator p-groups of class 2 have been proven [24], including the description of their various homological functors, such as the nonabelian tensor (exterior) square. This, however allows us to obtain a number of substantial results among them the invariant factors of the triple tensor (exterior) product, for each 2-generator p-group of class 2 in terms of its canonical parameters α, β, γ, ρ, and σ. We use C_{p^m} to denote the cyclic group of order p^m.

Theorem 1.3 *Let* $G = G_p(\alpha, \beta, \gamma; \rho, \sigma)$ *be any* 2*-generator p-group of class* 2 *with presentation as in Theorem 1.2 with* $p > 2$. *Then*

$$\otimes^3 G \cong \begin{cases} C_{p^\alpha} \times (C_{p^\beta})^7 \times (C_{p^\rho})^4 & if \quad \rho \leqslant \sigma \\ C_{p^\alpha} \times C_{p^{\beta+\tau}} \times (C_{p^\beta})^6 \times (C_{p^\sigma})^4 & if \quad \rho > \sigma \end{cases}$$

$$\wedge^3 G \cong \begin{cases} (C_{p^\beta})^2 \times (C_{p^\rho})^3 & if \quad \rho \leqslant \sigma \\ C_{p^{\beta+\tau}} \times C_{p^\beta} \times (C_{p^\sigma})^3 & if \quad \rho > \sigma \end{cases}$$

where $\tau = min(\rho - \sigma, \alpha - \beta)$.

Theorem 1.4 *Let* $G = G_2(\alpha, \beta, \gamma; \rho, \sigma)$ *be any* 2*-generator* 2*-group of class* 2 *with presentation as in Theorem 1.2. Then*

$$\otimes^3 G \cong \begin{cases} C_{2^\alpha} \times (C_{2^\beta})^7 \times (C_{2^\rho})^4 & if \quad \rho < \sigma \\ C_{2^\alpha} \times C_{2^{min(\alpha,\beta+1)}} \times C_{2^{\beta+\rho-\sigma}} \times (C_{2^\beta})^5 \times (C_{2^\sigma})^4 & if \quad \sigma \leqslant \rho < \gamma \\ C_{2^\alpha} \times C_{2^{min(\alpha,\beta+1)}} \times C_{2^{\beta+\tau}} \times (C_{2^\beta})^5 \times (C_{2^\sigma})^4 & if \quad \sigma < \rho = \gamma \\ C_{2^\alpha} \times (C_{2^{\beta+\delta_{\sigma\beta}}})^2 \times (C_{2^\beta})^5 \times (C_{2^{\gamma-\delta_{\alpha\gamma}}})^2 \times (C_{2^{\gamma-\delta_{\beta\gamma}}})^2 & if \quad \sigma = \rho = \gamma \end{cases}$$

$$\wedge^3 G \cong \begin{cases} (C_{2^\beta})^2 \times (C_{2^\rho})^3 & if \quad \rho < \sigma \\ (C_{2^{\beta+\rho-\sigma}})^2 \times (C_{2^\sigma})^3 & if \quad \sigma \leqslant \rho < \gamma \\ C_{2^{\beta+\tau}} \times C_{2^\beta} \times (C_{2^\sigma})^3 & if \quad \sigma < \rho = \gamma \\ (C_{2^{\beta+\delta_{\sigma\beta}}})^2 \times C_{2^\gamma} \times (C_{2^{\gamma-\delta_{\beta\gamma}}})^2 & if \quad \sigma = \rho = \gamma \neq \alpha \\ C_{2^{\beta+1}} \times C_{2^\beta} \times (C_{2^{\beta-1}})^3 & if \quad \sigma = \rho = \gamma = \beta = \alpha \end{cases}$$

where $\tau = min(\rho - \sigma, \alpha - \beta)$, *and* δ_{ij} *is Kronecker's delta.*

Let a group G be presented as the quotient of a free group F by a normal subgroup R. Given a positive integer c, the c-nilpotent multiplier of the group G is the abelian group

$$\mathcal{M}^{(c)}(G) = (R \cap \gamma_{c+1}(F))/\gamma_{c+1}(R, F),$$

where $\gamma_1(R, F) = R$, $\gamma_{c+1}(R, F) = [\gamma_c(R, F), F]$ and $\gamma_{c+1}(F) = \gamma_{c+1}(F, F)$. It was shown by Baer [7] that these invariants are, up to group isomorphism, independent of the choice of free presentation of G. In particular, $\mathcal{M}^{(1)}(G)$ is the Schur multiplier of G. It is well-known that if G is finite, then $\mathcal{M}^{(c)}(G)$ is finite too. The crucial aspect of the research into the c-nilpotent multipliers of groups includes either establishing their structures, or estimating their sizes and exponents. One reason for studying the c-nilpotent multiplier is its relevance to the isologism theory of P. Hall [19,20].

This paper focuses on the $\mathcal{M}^{(2)}(G)$. Burns and Ellis proposed in [11, p. 427] that by applying [11, Theorem 2.9], it should be possible to determine the second nilpotent multiplier of groups for which the tensor square has been computed. This motivates us further to carry out such computations for achieving the triple tensor (exterior) product of 2-generator p-groups of class 2. Let $\gamma_3^\sharp(G)$ be the quotient group $\wedge^3 G/\tau(G)$, where $\tau(G)$ is the normal subgroup of $\wedge^3 G$ generated by the elements

$$\langle x, y, z\rangle = ((x \wedge y) \wedge {}^y z)((y \wedge z) \wedge {}^z x)((z \wedge x) \wedge {}^x y)),$$

for all $x, y, z \in G$. The following commutative diagram in which both rows are exact is the consequence of Theorem 2.9 in [11]:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\gamma_3^\sharp(G) \longrightarrow G) & \longrightarrow & \gamma_3^\sharp(G) & \longrightarrow & \gamma_3(G) & \longrightarrow & 1 \\ & & \downarrow & & \cong\downarrow & & \cong\downarrow & & \\ 0 & \longrightarrow & (R \cap \gamma_3(F))/\gamma_3(R, F) & \longrightarrow & \gamma_3(F)/\gamma_3(R, F) & \longrightarrow & \gamma_3(F/R) & \longrightarrow & 1 \end{array} \tag{1}$$

where the homomorphism $\gamma_3^\sharp(G) \longrightarrow G$ is induced from the homomorphism $[\ ,\ ,\ ] : \wedge^3 G \longrightarrow G$. It gives us immediately the natural isomorphism $\mathcal{M}^{(2)}(G) \cong \ker(\gamma_3^\sharp(G) \longrightarrow G)$. In Section 3 we will show that the above two homomorphisms coincide when G is a 2-generator nilpotent group of class 2, and as a result we obtain $\mathcal{M}^{(2)}(G) \cong \wedge^3 G$.

A group G is said to be capable if it is isomorphic to the central factor group $M/Z(M)$ of some group M. Capability plays a significant role in P. Hall's classification scheme for p-groups up to isoclinism [19]. This notion has been widely studied for years as an important source of groups with interesting properties. For instance, the characterization of 2-generator capable p-groups of class 2 was initiated by Bacon and Kappe in 2003 [6], and subsequently completed by Magidin and Morse in 2010 [24].

Extending the above terminology by Burns and Ellis [11] led to the definition of c-capability, $c \geqslant 1$, in which a group G of the form $G \cong M/Z_c(M)$ is called c-capable where $Z_c(M)$ is the c-th term of the upper central series of M. In particular, 1-capable groups are just the capable groups, and 2-capable groups can be expressed as the central factor group of a capable group. Another aim of this paper is to characterize 2-capable groups among the family of 2-generator p-groups of class 2. It is well-known that there is a fruitful relation between the concepts of 2-capability and 2-nilpotent multiplier as well as triple exterior product of groups (see [11, Lemma 2.1(vii)] and [21, Proposition 2.7]). Moreover, characterizing of 2-capable groups enables us to study more efficiently the class of nilpotent groups of class $\leqslant 2$, for they fall into the same isologism class, called 2-isoclinism class [11, pp. 406-407]. In preparation for our next main result we state the following.

Theorem 1.5 *A 2-generator p-group of class two is 2-capable if and only if it is capable.*

In Section 3, we describe our approach to determining the 2-nilpotent multiplier and 2-capability. It is based on computing the triple (exterior) tensor products, which needs some computational evidence for a typical 2-generator p-groups of class two. For this reason, in Section 2, we summarize some of the notations and well-known results that we use throughout. The GAP programs used in this context can be found in the last section, in which we provide some details on the GAP output for explicit computations of specific groups in our class, which allowed us to directly conjecture the results of the paper.

2 Preliminaries

Let us start with stating the following presentation which has an essential role in order to describe the nonabelian tensor (exterior) squares of 2-generator p-groups of nilpotnecy class 2.

Theorem 2.1 *([24]) Let $G = G_p(\alpha, \beta, \gamma; \rho, \sigma)$ be a 2-generator p-group of class 2 with presentation as in Theorem 1.2. Then $G \otimes G$ is a homomorphic image of the abelian group generated by the six elements*

$$a \otimes a, \quad b \otimes b, \quad (a \otimes b)(b \otimes a), \quad a \otimes b, \quad a \otimes [a, b], \quad b \otimes [a, b],$$

subject to the relations:

$$\begin{aligned}
&(a \otimes a)^{2^{\alpha+1}} = (b \otimes b)^{\beta+1} = ((a \otimes b)(b \otimes a))^{2^{\beta}} = 1,\\
&(a \otimes b)^{2^{\alpha+1}} = (a \otimes [a, b])^{2^{min(\rho+1,\gamma)}} = (b \otimes [a, b])^{2^{min(\sigma+1,\rho+1,\gamma)}} = 1,\\
&(a \otimes a)^{2^{\alpha}} = (a \otimes [a, b])^{2^{\rho}}, \qquad (b \otimes b)^{2^{\beta}} = (b \otimes [a, b])^{2^{\sigma}},\\
&(a \otimes b)^{2^{\beta}} = (a \otimes [a, b])^{2^{\sigma}} (b \otimes [a, b])^{2^{\beta-1}}
\end{aligned}$$

if $p = 2$; and if $p > 2$ then subject to the relations:

$$\begin{aligned}
&(a \otimes a)^{p^{\alpha}} = (b \otimes b)^{p^{\beta}} = ((a \otimes b)(b \otimes a))^{p^{\beta}} = 1,\\
&(a \otimes b)^{p^{\alpha}} = (a \otimes [a, b])^{p^{\rho}} = (b \otimes [a, b])^{p^{min(\rho,\sigma)}} = 1,\\
&(a \otimes b)^{p^{\beta}} = (a \otimes [a, b])^{p^{\sigma}}.
\end{aligned}$$

In particular, the exterior squares $G \wedge G$ is generated by the images of the three generators $a \otimes b$, $a \otimes [a, b]$, and $b \otimes [a, b]$ under the natural homomorphism $G \otimes G \longrightarrow G \wedge G$. We need to use the invariant factors of exterior squares in the proof of our main results.

Proposition 2.2 *([24, Proposition 34]) Let $G = G_p(\alpha, \beta, \gamma; \rho, \sigma)$ be any 2-generator p-group of class 2 with presentation as in Theorem 1.2 and $p > 2$. Then*

$$G \wedge G \cong \begin{cases} C_{P^{\beta}} \times (C_{p^{\rho}})^2 & if \quad \rho \leqslant \sigma \\ C_{p^{\beta+\tau}} \times (C_{p^{\sigma}})^2 & if \quad \rho > \sigma \end{cases}$$

where $\tau = min(\alpha - \beta, \rho - \sigma)$. The cyclic factors correspond to the subgroup generated by $a \otimes b$, $a \otimes [a, b]$, and $b \otimes [a, b]$, respectively, when $\rho \leqslant \sigma$; and to the subgroups generated by $a \otimes b$, $(a \otimes b)^{-p^{\beta-\sigma}} a \otimes [a, b]$, and $b \otimes [a, b]$, respectively, when $\rho > \sigma$.

Proposition 2.3 *([24, Proposition 45]) Let $G = G_2(\alpha, \beta, \gamma; \rho, \sigma)$ be any 2-generator 2-group of class 2 with presentation as in Theorem 1.2. Then*

$$G \wedge G \cong \begin{cases} C_{2^{\beta}} \times (C_{2^{\rho}})^2 & if \quad \rho < \sigma \\ C_{2^{\beta+\rho-\sigma}} \times (C_{2^{\sigma}})^2 & if \quad \sigma \leqslant \rho < \gamma \\ C_{2^{\beta+\tau}} \times (C_{2^{\sigma}})^2 & if \quad \sigma < \rho = \gamma \\ C_{2^{\beta+\delta_{\sigma\beta}}} \times C_{2^{\gamma-\delta_{\alpha\gamma}}} \times C_{2^{\gamma-\delta_{\beta\gamma}}} & if \quad \sigma = \rho = \gamma \end{cases}$$

where $\tau = min(\rho - \sigma, \alpha - \beta)$, and δ_{ij} is Kronecker's delta.

Let $\otimes^3 G$ be the triple tensor product of a group G and $x, y, z \in G$. To simplify the notation we shall identify $x \otimes y \otimes z$ with $(x \otimes y) \otimes z$. If $a \otimes b \in G \otimes G$ and $c \in G$, the defining action of $G \otimes G$ on G gives ${}^{(a\otimes b)}cc^{-1} = {}^{[a,b]}cc^{-1} = [a, b, c]$. So the relation (8) of [9, Proposition 3] implies that

$${}^{(a\otimes b\otimes c)}(x \otimes y \otimes z) = {}^{[a,b,c]}(x \otimes y \otimes z).$$

It follows when G is a nilpotent group of class at most 3, then $\otimes^3 G$ is an abelian group. When one tries to figure out the structure of $\otimes^3 G$, then definitely some applicable identities are needed. For this reason, we note the following two useful relations.

Lemma 2.4 *([3]) Let G be a nilpotent group of class two. Then for any $x, y, z, t \in G$ and any integers m, n, and p,*

(i) $x^n \otimes y^m \otimes z^p = (x \otimes y \otimes z)^{nmp}(z \otimes [x, y] \otimes z)^{nm\binom{p}{2}}(x \otimes [x, y] \otimes z)^{mp\binom{n}{2}}([y, x] \otimes y \otimes z)^{np\binom{m}{2}}$,
(ii) $(x \otimes y \otimes [z, t])(z \otimes [x, y] \otimes t)(t \otimes [y, x] \otimes z) = 1$.

Remark 2.5 Let G be a nilpotent group of class 2. Evidently for any $x, y, z \in G$ and any integer n, $(x \otimes y)^n \otimes z = (x \otimes y \otimes z)^n$. So the exponent of $\otimes^3 G$ divides the exponent of $G \otimes G$.

Next we express a helpful expansion formula, which can be verified directly using tensor calculus and Lemma 2.4. This result will be applied to construct the required crossed pairings in Section 3.

Lemma 2.6 *Let G be a 2-generator p-group of class 2 with presentation as in Theorem 1.2. Then for any $u = (a \otimes b)^r (a \otimes [a, b])^s (b \otimes [a, b])^t \in G \wedge G$ and any $g = a^m b^n [a, b]^q \in G$:*

$$u \otimes g = (a \otimes b \otimes a)^{rm}(a \otimes b \otimes b)^{rn}(a \otimes [a, b] \otimes a)^{r\binom{m}{2}+sm}(a \otimes [a, b] \otimes b)^{sn-rq}$$
$$(b \otimes [a, b] \otimes a)^{rnm+tm+rq}(b \otimes [a, b] \otimes b)^{r\binom{n}{2}+tn}.$$

3 Proof of The Main Theorems

Now we are ready to prove the main theorems. Let start with the odd prime case.

3.1 The p odd case

Throughout this section $G = G_p(\alpha, \beta, \gamma; \rho, \sigma)$ is assumed to be as in Theorem 1.2 with $p > 2$. It is confirmed in [24, Section 3.1] that $G \otimes G \cong \nabla(G) \times (G \wedge G)$, in which as G^{ab} is isomorphic to $C_{p^\alpha} \times C_{p^\beta}$, then $\nabla(G) \cong C_{p^\alpha} \times (C_{p^\beta})^2$, where the invariant cyclic factors can be taken to the subgroups generated by $a \otimes a$, $b \otimes b$ and $(a \otimes b)(b \otimes a)$, respectively. One could easily verify that the conditions of Proposition 10 of [9] allows to decompose $\otimes^3 G$ so that

$$\otimes^3 G \cong (\nabla(G) \otimes G) \times ((G \wedge G) \otimes G).$$

In addition, since $\nabla(G)$ and G act on each other trivially, it follows from [10, Proposition 2.4] that $\nabla(G) \otimes G \cong \nabla(G) \otimes G^{ab} \cong C_{p^\alpha} \times (C_{p^\beta})^5$. The cyclic factors correspond to the subgroup generated by $a \otimes a \otimes a$, $a \otimes a \otimes b$, $b \otimes b \otimes a$, $b \otimes b \otimes b$, $(a \otimes b)(b \otimes a) \otimes a$, and $(a \otimes b)(b \otimes a) \otimes b$, respectively. Therefore it remains to be shown what the abelian invariants of $(G \wedge G) \otimes G$ are.

By performing the relevant computations using the HAP package of GAP for a number of these groups, it quickly becomes apparent that $(G \wedge G) \otimes G$ will have six invariants of different orders regarding to disjoint cases $\rho \leqslant \sigma$ and $\rho > \sigma$. We now establish these observation more precisely. Since $G \wedge G \cong \langle a \otimes b, a \otimes [a, b], b \otimes [a, b] \rangle$ by Proposition 2.2, we expect from Lemma 2.6 that

$$(G \wedge G) \otimes G \cong \langle a \otimes b \otimes a, a \otimes b \otimes b, a \otimes [a, b] \otimes a, a \otimes [a, b] \otimes b, b \otimes [a, b] \otimes a, b \otimes [a, b] \otimes b \rangle. \quad (2)$$

The following result gives the order of these six generators.

Proposition 3.1 *Let $G = G_p(\alpha, \beta, \gamma; \rho, \sigma)$ be a 2-generator p-group of class 2 with presentation as in Theorem 1.2 with $p > 2$. Then the order of $a \otimes b \otimes a$ is p^β if $\rho \leqslant \sigma$, and $p^{\beta+\tau}$ if $\rho > \sigma$. The order of $a \otimes b \otimes b$ is p^β. The order of $a \otimes [a, b] \otimes a$ is p^ρ if $\rho \leqslant \sigma$, and $p^{\sigma+\tau}$ if $\rho > \sigma$. In addition $|a \otimes [a, b] \otimes b| = |b \otimes [a, b] \otimes a| = |b \otimes [a, b] \otimes b| = p^{min(\sigma,\rho)}$.*

Proof Keeping Remark 2.5 together with Proposition 2.2 in mind, it is enough to show that the order of generators can be divided by the values given in the assertion. Of course there are two exceptional cases for which we should first change the order values correspond to the cyclic factors in Proposition 2.2.

Assume that $\rho \leqslant \sigma$. It follows from the natural homomorphism $\otimes^3 G \longrightarrow \otimes^3 G^{ab}$ that p^β divides both $|a \otimes b \otimes a|$ and $|a \otimes b \otimes b|$ as required. We also conclude that each of $a \otimes b \otimes a$ and $a \otimes b \otimes b$ are independent

from the other generators of $\otimes^3 G$ when mapping to $\otimes^3 G^{ab}$. The proof of other cases can be made by constructing a suitable crossed pairing $(G \otimes G) \times G \longrightarrow C_{p^\rho}$. For instance, Lemma 2.6 implies that the map

$$(G \otimes G) \times G \longrightarrow C_{p^\rho}, \quad (u, g) \longmapsto r\binom{m}{2} + sm$$

is a crossed pairing which induces a homomorphism $\otimes^3 G \longrightarrow C_{p^\rho}$, maps $a \otimes [a, b] \otimes a$ to the class of 1 modulo p^ρ and every other generator to 0, establishing the order and the independence for $a \otimes [a, b] \otimes a$.

Suppose $\rho > \sigma$. Obviously the relation $(a \otimes b)^{p^\beta} = (a \otimes [a, b])^{p^\sigma}$ in Theorem 2.1 implies the relation

$$(a \otimes b \otimes g)^{p^\beta} = (a \otimes [a, b] \otimes g)^{p^\sigma}, \tag{3}$$

for all $g \in G$. Therefore by invoking Lemma 2.6 we define the crossed pairing

$$(G \wedge G) \times G \longrightarrow C_{p^{\beta+\tau}}, \quad (u, g) \longmapsto rm + p^{\beta-\sigma}(r\binom{m}{2} + sm),$$

which induces a homomorphism $(G \wedge G) \otimes G \longrightarrow C_{p^{\beta+\tau}}$ and immediately it follows $a \otimes b \otimes a$ has order $p^{\beta+\tau}$, and is independent of the other five generators of $(G \wedge G) \otimes G$.

Now we work on the order of $a \otimes b \otimes b$. Taking $N = \langle b \otimes [a, b] \rangle \trianglelefteq G \otimes G$, it is easily seen that the map

$$N \times G \longrightarrow (G \otimes G) \otimes G, \quad ((b \otimes [a, b])^i, g) \longmapsto (g \otimes [a, b] \otimes b)^i$$

is a crossed pairing. So there exists a homomorphism $N \otimes G \longrightarrow (G \otimes G) \otimes G$ which maps $a \otimes [a, b] \otimes a$ to $a \otimes [a, b] \otimes b$. On the other hand as N and G act on each other trivially, [10, Proposition 2.4] implies that $N \otimes G \cong N \otimes G^{ab}$, whence $b \otimes [a, b] \otimes a$ has order p^σ in $N \otimes G$. Therefore from (3) we have that $(a \otimes b \otimes b)^{p^\beta} = (a \otimes [a, b] \otimes b)^{p^\sigma} = 1$, and the order of $a \otimes b \otimes b$ is indeed p^β. The independence follows by mapping to $\otimes^3 G^{ab}$, since its image has order p^β as well. As a consequence, the proof for the elements $a \otimes [a, b] \otimes a$ and $a \otimes [a, b] \otimes b$ follows directly from the relation (3). The remaind cases are obtained from the relations $(b \otimes [a, b] \otimes a)^{p^\sigma} = (b \otimes b \otimes a)^{p^\beta} = 1$, which are subsequences of Theorem 2.1, as well as the relations $|b \otimes b \otimes a| = |b \otimes b \otimes b| = p^\beta$. To show that the generators $a \otimes [a, b] \otimes a$, $a \otimes [a, b] \otimes b$, $b \otimes [a, b] \otimes a$, and $b \otimes [a, b] \otimes b$ are independent of each other, it needs to construct a suitable crossed pairing for each of them by using Lemma 2.6, and to apply the relation (3). □

Remark 3.2 According to Proposition 2.2, in case $\rho > \sigma$, the group $G \wedge G$ is generated by the elements $a \otimes b$, $(a \otimes b)^{-p^{\beta-\sigma}}(a \otimes [a, b])$, and $b \otimes [a, b]$. Analogously we construct the basis by taking our standard six generators (2), and proposing the following generators for $(G \wedge G) \otimes G$:

$$a \otimes b \otimes a,\ a \otimes b \otimes b,\ d_a = (a \otimes b \otimes a)^{-p^{\beta-\sigma}}(a \otimes [a, b] \otimes a),\ d_b = (a \otimes b \otimes b)^{-p^{\beta-\sigma}}(a \otimes [a, b] \otimes b),$$
$$b \otimes [a, b] \otimes a,\ b \otimes [a, b] \otimes b.$$

We will find the order of d_a and d_b. The relation (3) tells us that the orders of d_a and d_b divides p^σ. In addition the images of d_a and d_b under the homomorphism $\otimes^3 G \longrightarrow \otimes^3 G^{ab}$ have order $p^\beta / p^{\beta-\sigma} = p^\sigma$, which shows $|d_a| = |d_b| = p^\sigma$. The proof of Proposition 3.1 implies that these two are independent of the rest as well.

Proof of Theorem 1.3 As discussed at the beginning of this section, it is enough to determine the abelian invariants of $(G \wedge G) \otimes G$. Since it is generated by the six elements in (2), then Proposition 3.1 and Remark 3.2 yield that

$$(G \wedge G) \otimes G \cong \begin{cases} (C_{p^\beta})^2 \times (C_{p^\rho})^4 & if \quad \rho \leqslant \sigma \\ C_{p^{\beta+\tau}} \times C_{p^\beta} \times (C_{p^\sigma})^4 & if \quad \rho > \sigma \end{cases}$$

where the cyclic factors correspond to the subgroups generated by $a \otimes b \otimes a, a \otimes b \otimes b, a \otimes [a, b] \otimes a, a \otimes [a, b] \otimes b, b \otimes [a, b] \otimes a$, and $b \otimes [a, b] \otimes b$, respectively, when $\rho \leqslant \sigma$; and to the subgroups generated by $a \otimes b \otimes a$, $a \otimes b \otimes b, d_a, d_b, b \otimes [a, b] \otimes a$, and $b \otimes [a, b] \otimes b$, respectively, when $\rho > \sigma$. The next assertion follows easily, since the additional relations of $\wedge^3 G$ together with Lemma 2.4(ii) imply that $a \otimes [a, b] \otimes b = b \otimes [a, b] \otimes a$ modulo $\wedge^3 G$. □

3.2 The p=2 case

Let $G = G_2(\alpha, \beta, \gamma; \rho, \sigma)$ be the group with presentation as in Theorem 1.2. It is mentioned in [24] that in $p > 2$ case, the map $G \otimes G \longrightarrow G \wedge G$ always has a natural splitting, which identifies the exterior square with the subgroup $\nabla(G)$ of $G \otimes G$. In $p = 2$ case, although the above map does not split, but we always have an isomorphism $G \otimes G \cong \nabla(G) \times (G \wedge G)$. So, thanks to [9,24], we can state again the following decomposition

$$\otimes^3 G \cong (\nabla(G) \otimes G) \times ((G \wedge G) \otimes G).$$

Obviously by involving [10, Proposition 2.4] and [24, Theorem 50] we obtain

$$\nabla(G) \otimes G \cong \begin{cases} C_{2^\alpha} \times (C_{2^\beta})^5 & if \quad \rho < \sigma \\ C_{2^\alpha} \times C_{2^{min(\alpha,\beta+1)}} \times (C_{2^\beta})^4 & if \quad \sigma \leqslant \rho < \gamma \\ C_{2^\alpha} \times C_{2^{min(\alpha,\beta+1)}} \times (C_{2^\beta})^4 & if \quad \sigma < \rho = \gamma \\ C_{2^\alpha} \times (C_{2^\beta})^5 & if \quad \sigma = \rho = \gamma \end{cases}$$

Note that the generators of cyclic invariants are completely as same as the p odd case. By the above discussion, we know that in $p > 2$ case, $G \wedge G$ is isomorphic to the subgroup of $G \otimes G$ generated by $a \otimes b$, $a \otimes [a, b]$, and $b \otimes [a, b]$. But it is not the case in general when $p = 2$ and we should consider the image of these generators under the epimorphism $G \otimes G \longrightarrow G \wedge G$. The argument before of Proposition 45 in [24] shows that the last three generators of $G \otimes G$, which does not belong to $\nabla(G)$, are obtained by a suitable change of the basis elements $a \otimes a$, $b \otimes b$, $(a \otimes b)(b \otimes a)$, $a \otimes b$, $a \otimes [a, b]$, and $b \otimes [a, b]$. So, we expect the invariants of $(G \wedge G) \otimes G$ to follow the same pattern as for odd p, when keeping all the new considerations. We also anticipated most of the results through some sample computations in GAP.

Proposition 3.3 *Let $G = G_2(\alpha, \beta, \gamma; \rho, \sigma)$ be a 2-generator 2-group of class 2 with presentation as in Theorem 1.2. Then*

$$(G \wedge G) \otimes G \cong \begin{cases} (C_{2^\beta})^2 \times (C_{2^\rho})^4 & if \quad \rho < \sigma \\ C_{2^{\beta+\rho-\sigma}} \times C_{2^\beta} \times (C_{2^\sigma})^4 & if \quad \sigma \leqslant \rho < \gamma \\ C_{2^{\beta+\tau}} \times C_{2^\beta} \times (C_{2^\sigma})^4 & if \quad \sigma < \rho = \gamma \\ (C_{2^{\beta+\delta_{\sigma\beta}}})^2 \times (C_{2^{\gamma-\delta_{\alpha\gamma}}})^2 \times (C_{2^{\gamma-\delta_{\beta\gamma}}})^2 & if \quad \sigma = \rho = \gamma \end{cases}$$

Proof By Proposition 2.3 and the above discussion we can assume that $G \wedge G \cong \langle t_1, t_2, t_3 \rangle$, where $t_i's$ are a suitable combinations of the elements $a \otimes b$, $a \otimes [a, b]$ and $b \otimes [a, b]$ and are different in each case. Evidently the same result as in Lemma 2.6 tells us that $(G \wedge G) \otimes G$ is generated by the basis elements $t_i \otimes a$ and $t_i \otimes b$, $i = 1, 2, 3$. We are going to determine their orders in each case. As mentioned at the beginning of the proof of Proposition 3.1, it is enough to show that the order of generators are multiple of the claimed values in the assertion.

Suppose that $\rho < \sigma$. From [24] we have $t_1 = a \otimes b, t_2 = a \otimes [a, b]$, and $t_3 = b \otimes [a, b]$. So the proof follows as same as the proof of Proposition 3.1, the case $\rho \leqslant \sigma$. Now assume that $\sigma \leqslant \rho < \gamma$. In this case the elements t_i's are as in the case $\rho < \sigma$ with exception that when $\sigma < \rho < \gamma$, then $t_2 = (a \otimes b)^{-2^{\beta-\sigma}} (b \otimes [a, b])^{2^{\beta-1-\sigma}} (a \otimes [a, b])$. By finding a suitable crossed pairing, one can show that $|a \otimes [a, b] \otimes b| = 2^\rho$. On the other hand applying the relation $(a \otimes b)^{2^\beta} = (a \otimes [a, b])^{2^\sigma}$ of Theorem 2.1 yields $(a \otimes b \otimes a)^{2^\beta} = (a \otimes [a, b] \otimes a)^{2^\sigma}$, and consequently $|t_1 \otimes a| = 2^{\beta+\rho-\sigma}$, as claimed. Similarly we can show that $|t_1 \otimes b| = 2^\beta$. As in Remark 3.2 for p odd case, we can do a same argument to find the order of elements $t_2 \otimes a$ and $t_2 \otimes b$. The proof for the generators $t_3 \otimes a$ and $t_3 \otimes b$ can be obtained by constructing a required crossed pairing $(G \wedge G) \times G \longrightarrow C_{2^\sigma}$.

Assume $\sigma < \rho = \gamma$. In this case we know that $t_1 = a \otimes b$, $t_2 = (a \otimes b)^{-2^{\beta-\sigma}} (b \otimes [a, b])^{2^{\beta-1-\sigma}} (a \otimes [a, b])$, and $t_3 = b \otimes [a, b]$, and the assertion follows by a similar method. Finally if $\sigma = \rho = \gamma$ and $\gamma < \beta$, then $t_1 = a \otimes b$, $t_2 = a \otimes [a, b]$, $t_3 = b \otimes [a, b]$. We claim that $|a \otimes [a, b] \otimes a] = 2^\sigma$. By a same identity as in Lemma 2.6, consider the crossed pairing $\Phi : (G \wedge G) \times G \longrightarrow C_{2^\sigma}$ which maps (u, g) to $r\binom{m}{2} + sm$. Then the induced map Φ^* maps $a \otimes [a, b] \otimes a$ to the class of 1 modulo C_{2^σ}. Hence the order of $a \otimes [a, b] \otimes a$ is 2^σ, which is what we needed to establish. On the other hand we have that $(a \otimes b \otimes a)^{2^\beta} = (a \otimes [a, b] \otimes a)^{2^\sigma} = 1$, whence $|a \otimes b \otimes a| = 2^{\beta+\delta_{\sigma\beta}}$. Similarly $|a \otimes b \otimes b| = 2^{\beta+\delta_{\sigma\beta}}$. The order of the remained generators is obtained analogously.

Suppose that $\sigma = \rho = \gamma = \beta < \alpha$. Then $t_1 = a \otimes b$, $t_2 = a \otimes [a, b]$, and $t_3 = (a \otimes b)^{-2}(b \otimes [a, b])$. Also if $\sigma = \rho = \gamma = \beta = \alpha$, then $t_1 = a \otimes b$, $t_2 = (a \otimes b)^{-2}(a \otimes [a, b])$, $t_3 = (a \otimes b)^{-2}(b \otimes [a, b])$. In both cases the proof obtained by a same discussion as above. Note that the independence of the generators can be proved likewise the $p > 2$ case in the proof of Proposition 3.1. □

Proof of Theorem 1.4 The first assertion follows by a same argument as in the proof of Theorem 1.3, and by invoking Proposition 3.3. To prove the next assertion, recall from Definition 1.1 the group $\wedge^3 G$ is obtained from $(G \wedge G) \otimes G$ by imposing the additional relation $(a \wedge b) \otimes [a, b] = 1$. Suppose that $\sigma < \rho = \gamma$. Clearly Lemma 2.4(ii) implies that

$$\begin{aligned} t_2 \otimes a &= (a \otimes b \otimes a)^{-2^{\beta-\sigma}} (b \otimes [a, b] \otimes a)^{2^{\beta-1-\sigma}} \\ &= (a \otimes b \otimes a)^{-2^{\beta-\sigma}} (a \otimes [a, b] \otimes b)^{2^{\beta-1-\sigma}} \ (mod \wedge^3 G) \end{aligned}$$

Hence $t_2 \otimes a$ is dependent on $t_1 \otimes a$, $t_2 \otimes b$, and $t_3 \otimes b$ in $\wedge^3 G$. Therefore $\wedge^3 G \cong C_{2^{\beta+\tau}} \times C_{2^\beta} \times (C_{2^\sigma})^3$. The remained cases can be proved analogously. □

Since the structure of the triple exterior product $\wedge^3 G$ is known, we can easily describe the second nilpotent multiplier $\mathcal{M}^{(2)}(G)$, as the next aim of this paper, by computing the quotient group $\gamma_3^\sharp(G) = \wedge^3 G/\tau(G)$. In fact, these all coincide as we see in the following result.

Proposition 3.4 *Let G be a 2-generator nilpotent group of class 2. Then $\tau(G) = \langle 1 \rangle$ and*

$$\mathcal{M}^{(2)}(G) \cong \wedge^3 G.$$

Proof Recall that $\tau(G)$ is generated by the elements $\langle x, y, z \rangle$ for all $x, y, z \in G$. Since G is nilpotent of class two, then

$$\begin{aligned} \langle x, y, z \rangle &= (x \wedge y \wedge {}^y z)(y \wedge z \wedge {}^z x)(z \wedge x \wedge {}^x y) \\ &= (x \wedge y \wedge z)(y \wedge z \wedge x)(z \wedge x \wedge y)(x \wedge y \wedge [y, z])(y \wedge z \wedge [z, x])(z \wedge x \wedge [x, y]). \end{aligned}$$

We claim that $\tau(G) = \langle 1 \rangle$. Let $G = \langle a, b \rangle$, for some elements $a, b \in G$. Then

$$\langle a, b, a \rangle = (a \wedge b \wedge a)(b \wedge a \wedge a)(a \wedge a \wedge b)(a \wedge b \wedge [b, a])(b \wedge a \wedge [a, a])(a \wedge a \wedge [a, b]).$$

Since $(a \wedge b)(b \wedge a) = 1 = a \wedge a$, it is clear that $(a \wedge b \wedge a)(b \wedge a \wedge a) = 1$ and $a \wedge a \wedge b = 1 = a \wedge a \wedge [a, b]$. In addition $a \wedge b \wedge [b, a] = 1$ modulo $\wedge^3 G$. Hence $\langle a, b, a \rangle = 1$.

Now consider the element $\langle a, [a, b], b \rangle = (a \wedge [a, b] \wedge b)([a, b] \wedge b \wedge a)(b \wedge a \wedge [a, b])([a, b] \wedge b \wedge [b, a])$. From Lemma 2.4$(ii)$ it follows that $a \wedge [a, b] \wedge b = b \wedge [a, b] \wedge a]$ modulo $\wedge^3 G$. Also $[a, b] \wedge b \wedge [b, a] = 1$ by [3, Theorem 3.3], which implies that $\langle a, [a, b], b \rangle = 1$. Using a same discussion shows that all the generators of $\tau(G)$ are identity, which concludes the claim. Therefore the isomorphism $\mathcal{M}^{(2)}(G) \cong \wedge^3 G$ is a direct consequence of the diagram (1). □

Now, we are going to prove Theorem 1.5. The property of capability can be determined by computing a central subgroup of G called the exterior center $Z^\wedge(G)$, which is defined in [15] as $Z^\wedge(G) = \{g \in G \mid g \wedge h = 1_{G \wedge G} \ \forall\, h \in G\}$. It is shown [15] that the group G is capable if and only if $Z^\wedge(G)$ is trivial. Also, the group $Z^\wedge(G)$ is coincide with $Z^*(G)$, called the epicenter of G, which is defined [8] in a different manner. A similar criterion is given in [21] in order to decide whether a group is 2-capable or not:

A group G is 2-capable if and only if $Z^{\sharp^2}(G)$ is trivial,

in which $Z^{\sharp^2}(G) = \{g \in G \mid g \wedge x \wedge y = 1_{\gamma_3^\sharp(G)} \ \forall\, x, y \in G\}$ is a characteristic subgroup of G contained in $Z_2(G)$, and is precisely the group denoted by $Z_2^*(G)$ called [11] the second term of upper epicentral series of G.

Proof of Theorem 1.5 Evidently by definition 2-capability implies capability. We show that any 2-generator capable p-group $G = G_p(\alpha, \beta, \gamma; \rho, \sigma)$ of class 2 is 2-capable as well. Since $\gamma_3^\sharp(G) \cong \wedge^3 G$ by diagram (1) and Proposition 3.4, the subgroup $Z^{\sharp^2}(G)$ reduces to the subgroup $Z^{\wedge^2}(G) = \{g \in G \mid g \wedge x \wedge y = 1_{\wedge^3 G} \ \forall\, x, y \in G\}$.

So we must prove that $Z^{\wedge^2}(G)$ is trivial. Let $g = a^m b^n [a,b]^q$ be an arbitrary element in $Z^{\wedge^2}(G)$. The tensor calculus can be used to get

$$g \wedge b \wedge a = (a \wedge b \wedge a)^m (a \wedge [a,b] \wedge a)^{\binom{m}{2}} (b \wedge [a,b] \wedge a)^{-q} = 1, \tag{4}$$

$$g \wedge a \wedge a = (a \wedge b \wedge a)^{-n} (a \wedge [a,b] \wedge a]^{mn-q} (b \wedge [a,b] \wedge a]^{\binom{n}{2}} = 1, \tag{5}$$

$$g \wedge a \wedge b = (a \wedge b \wedge a)^{-n} (a \wedge [a,b] \wedge b)^{-q} (b \wedge [a,b] \wedge a)^{-mn} (b \wedge [a,b] \wedge b)^{-\binom{n}{2}}. \tag{6}$$

First assume $p > 2$. By the capability conditions [24, Theorem 63] we have that $\alpha - \beta = \rho - \sigma$ and $\gamma = \rho$. When $\rho \leqslant \sigma$, it is shown in the proof of Proposition 3.1 that the three elements in (4) are independent of each other. Therefore (4) implies that m and q are a multiple of p^β and p^ρ, respectively. On the other hand we see in the proof of Theorem 63 in [24] that $\alpha = \beta$. Besides the relation (5) implies n is a multiple of p^β, which concludes that $g = 1$, as required.

Assume $\sigma > \rho$. In this case the relation $(a \otimes b \otimes a)^{p^\beta} = (a \otimes [a,b] \otimes a)^{p^\sigma}$ may affect our discussion. If $p^\beta \nmid m$, the relation (4) gives $m = p^{\beta+\tau} = p^\alpha$, and if $p^\beta | m$ then as $\sigma + \tau \leqslant \beta$, it follows that $(a \wedge [a,b] \wedge a)^{\binom{m}{2}} = 1$. Hence again (4) implies $(a \wedge b \wedge a)^m = 1$ and consequently $m = p^\alpha$. Therefore for each case we get $g = b^n [a,b]^q$. Now by applying the relation (5) we obtain $(a \wedge b \wedge a)^n (a \wedge [a,b] \wedge a)^{-q} = 1$. If $(a \wedge b \wedge a)^n = (a \wedge [a,b] \wedge a)^q = 1$, then n and q must be a multiple of $p^{\beta+\tau}$ and $p^{\sigma+\tau}$, respectively. Thus $g = 1$. Otherwise, we have $n = kp^\beta$ and $q = kp^\sigma$ for some $1 \leqslant k \leqslant p^\tau$. So $g = [a,b]^{2kp^\sigma}$. From this we have that $g \wedge a \wedge a = ([a,b] \wedge a \wedge a)^{2kp^\sigma} = 1$ which is in contrary to the order of $a \wedge [a,b] \wedge a$ in Theorem 1.3.

Suppose $p = 2$. In this case the capability conditions are given in [24, Theorem 67]. Also it is discussed in the proof of Theorem 67 in [24] that either $\rho = \sigma = \gamma$ and $\alpha = \beta$ or one of the conditions (70)-(72) hold. Let $\sigma < \rho = \gamma$. The canonical homomorphism $\wedge^3 G \longrightarrow \wedge^3 G^{ab}$ together with (5) imply that 2^β divides n. Also from (6) we deduce q is a multiple of 2^σ. Take $n = u2^\beta$ and $q = v2^\sigma$ for some integers u, v. Then $g = a^m [a,b]^{2^\sigma (u+v)}$. If 2^ρ divides $2^\sigma (u+v)$, we get $g = a^m$, otherwise by using the equation $(a \wedge [a,b])^{u2^\sigma} = (a \wedge b)^n (b \wedge [a,b])^{\binom{n}{2}}$ (the general expansion formula can be found in [4]), we get $(a \wedge [a,b] \wedge a)^{u2^\sigma} = (a \wedge b \wedge a)^n (b \wedge [a,b] \wedge a)^{\binom{n}{2}}$. Therefore the relation (5) concludes that $(a \wedge [a,b] \wedge a)^{-2^\sigma (u+v)} (b \wedge [a,b] \wedge a)^{2\binom{n}{2}} = 1$, whence $2^\sigma \mid 2^\rho (u+v)$, a contradiction. Now the relation (4) implies $(a \wedge b \wedge a)^m (a \wedge [a,b] \wedge a)^{\binom{m}{2}} = 1$. By mapping to $\wedge^3 G^{ab}$ it follows that $m = u2^\beta$ for some $1 \leqslant u < 2^\tau$. If $(a \wedge b \wedge a)^m = 1$, then $g = 1$. Otherwise if $\rho \leqslant \beta - 1$, then $(a \wedge b \wedge a)^m = 1$ a contradiction, else $\beta = \rho$ and we have that $(a \wedge [a,b] \wedge a)^{u(2^\sigma + 2^{\rho-1})} = 1$, which is again a contradiction. Because from the proof of Theorem 67 in [24], $\beta > \sigma + 1$ which implies that $\sigma < \rho - 1$.

Finally assume $\sigma = \rho = \gamma$. In this case the capability conditions imply $\sigma = \rho = \gamma = \beta$ and $\alpha = \gamma + 1$. Likewise the previous case we have that n and q are multiple of 2^β and 2^σ, respectively. Thus $g = a^m$. Moreover the relation (4) concludes that 2^β divides m. On the other hand the equation $(a \wedge b)^m (b \wedge [a,b])^{\binom{m}{2}} = 1$ together with (4) imply that $(b \wedge [a,b] \wedge a)^{-\binom{m}{2}} (a \wedge [a,b] \wedge a)^{\binom{m}{2}} = 1$. Therefore $(a \wedge [a,b] \wedge a)^{\binom{m}{2}} = 1$ which is in contrary to the order of $a \wedge [a,b] \wedge a$ in Theorem 1.4. The remaind case $\rho = \sigma = \gamma, \alpha = \beta$, can be discussed similarly. This finishes the proof. □

4 GAP Results

Our investigation of this topic started with initial guidance from the GAP [26] computational algebra system. At the beginning, we collected experimental material by creating finite 2-generator p-groups of class 2. Let $G = G_p(\alpha, \beta, \gamma; \rho, \sigma)$ be any 2-generator 2-group of class 2 with presentation as in Theorem 1.2. For illustration, consider the group $G = G_2(3,3,2;2,2)$. The following GAP commands create a finitely presented group isomorphic to G. We suppress most of the output for brevity with the double semicolons.

```
gap> F:=FreeGroup("a","b");;
gap> a:=F.1;;b:=F.2;;
gap> R:=[a^(2^3)/Comm(a,b)^(2^2), b^(2^3)/Comm(a,b)^(2^2), Comm(a,b)^(2^2),
> Comm(a,b)^a/Comm(a,b), Comm(a,b)^b/Comm(a,b)];;
gap> G:=F/R;
<fp group on the generators [ a, b ]>
```

Most of the calculations of this paper has been implemented in GAP as part of a package called HAP. It has methods for some calculations in elementary algebraic topology and the cohomology of groups. We illustrate this implementation on the group $G = G_2(3,3,2;2,2)$. The following commands construct the crossed module

$id : G \longrightarrow G$, and then compute the tensor square $G \otimes G \longrightarrow G$, the triple tensor product $\otimes^3 G \longrightarrow G$, the exterior square $G \wedge G \longrightarrow G$, and the triple exterior product $\wedge^3 G \longrightarrow G$, respectively.

```
gap> K:=CrossedModuleByNormalSubgroup(G,G);;
gap> T2:=NonabelianTensorProduct(K,K);;
gap> T3:=NonabelianTensorProduct(T2,K);;
gap> AbelianInvariants(Source(T3!.map));
[ 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 8 ]
gap> E2:=NonabelianExteriorProduct(K,K);;
gap> E3:=NonabelianExteriorProduct(E2,K);;
gap> AbelianInvariants(Source(E3!.map));
[ 4, 4, 4, 8, 8 ]
```

The following commands return the second nilpotent multiplier $\mathcal{M}^{(2)}(G)$, and the order of the second term of upper epicentral series $Z_2^*(G)$, respectively, which show that $\mathcal{M}^{(2)}(G) \cong \wedge^3 G$ and the group $G = G_2(3, 3, 2; 2, 2)$ is 2-capable.

```
gap> BaerInvariant(G,2);
[ 4, 4, 4, 8, 8 ]
gap> Order(UpperEpicentralSeries(G,2));
1
```

Declarations

Competing Interests On behalf of all authors, the corresponding author states that there is no conflict of interest.

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