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ORIGINAL RESEARCH

On series involving sine, cosine, and k-colored partition function

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Abstract In this paper, we explore sine and cosine functions and also a relation involving these functions to establish identities involving convergent series in terms of the k-colored partition function and the set of the integer compositions of a positive integer n. Combinatorial interpretations are provided and some special cases are studied.

Keywords Trigonometric Functions · Convergent Series · Compositions · k-Colored Partition Function

Mathematics Subject Classification Primary 40B05 · 40C15 Secondary 17A42

1 Introduction

A partition of a positive integer n is a way of writing n as a sum of positive integers [more details in [2–6]]. In the literature, there are many ways to discuss the results of partition functions. From the point of view of basic hypergeometric series, it is still unclear how to interpret some q-series identities using combinatorial properties.

Our starting point is a perspective of inequalities involving k-colored partition functions discussed by Chern et al, [7]. More precisely, the authors extended the k-colored partition function multiplicatively to a function on k-colored partitions.

In this context, we will exhibit some identities involving k-colored partitions, many coming from sine and cosine functions. More specifically, from two representations of trigonometric functions via infinite product and Taylor series we will find interesting relations hidden in well-known identities.

A k-colored partition of n is an integer partition of n wherein each part appears colored with one of k available colors. For example, if $k = 2$ and the colors are black and red, the 2-colored partitions of $n = 4$ are 4: 4, 3 + 1, 3 + 1, 3 + 1, 3 + 1, 2 + 2, 2 + 2, 2 + 2, 2 + 1 + 1, 2 + 1 + 1, 2 + 1 + 1, 2 + 1 + 1, 2 + 1 + 1, 2 + 1 + 1, 1 + 1 + 1 + 1, 1 + 1 + 1 + 1, 1 + 1 + 1 + 1, 1 + 1 + 1 + 1 and 1 + 1 + 1 + 1.

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Let $p_k(n)$ be the number of k-colored partitions of an integer n. The generating function for $p_k(n)$ is given by the next infinite product,

$$\sum_{n=0}^{\infty} p_k(n)q^n = \frac{1}{(q;q)_\infty^k}, \tag{1}$$

where the q-Pochhammer symbol is defined by

$$(a,q)_n = \begin{cases} (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), & \text{if } n > 0, \\ 1, & \text{if } n = 0 \end{cases}.$$

Taking the limit $n \to \infty$, we have

$$(a,q)_\infty = \lim_{n\to\infty} (a,q)_n.$$

An integer composition of a positive integer n is a set of tuples of positive integers $w_1, w_2, \ldots, w_n$ whose sum is equal to n. The set of integer composition of a positive integer n is denoted by C(n).

We present some trigonometric identities below which are key tools for getting some combinatorial identities.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right), \tag{2}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \prod_{n=1}^{\infty} \left(1 - \frac{4x^2}{(2n-1)^2\pi^2}\right), \tag{3}$$

$$\sin^2(x) = \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} \binom{2n}{2i+1}\right) \frac{(-1)^{n-1}}{(2n)!} x^{2n}, \tag{4}$$

and

$$\cos^2(x) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \binom{2n}{2i}\right) \frac{(-1)^n}{(2n)!} x^{2n}. \tag{5}$$

In the present study, we focus on exploring the sine and cosine functions and also an identity involving these functions. Our approach consists of replacing the argument of the sine and cosine functions with convergent series (1). As a combinatorial interpretation, these newly obtained identities are given in terms of k-colored partition function and the set of integer compositions of n.

The content of this paper is structured as follows. In Sect. 2, we study the convergence of (1) and establish new identities considering it as an argument for the sine and cosine functions. Section 3 is devoted to exploring a relation involving sine and cosine functions by substituting the convergent series (1). Furthermore, these identities are in terms of k-colored partition function and the set of integer compositions of n, then their related combinatorial interpretations are provided. Finally, we state some final considerations and perspectives.

2 The k-colored partition function, and sine and cosine functions

Recall the generating function for the number of k-colored partitions of an integer n, $p_k(n)$, given by

$$\sum_{n=0}^{\infty} p_k(n)q^n = \frac{1}{(q;q)_\infty^k}.$$

First, we will investigate the convergence of the following related series

$$\sum_{k=0}^{\infty} \frac{(-1)^k p_k(n)}{k!}. \tag{6}$$

In order to show the convergence of series (6), consider the inequality for k-colored partition function presented by Chern et al, [7], given as follows.

Theorem 1 *(Theorem 1.2, [7]) For any positive integers $a \geq b$, we have*

$$p_k(a)p_k(b) > p_k(a+b),$$

except for $(a, b, k) = (1, 1, 2), (2, 1, 2), (3, 1, 2), (1, 1, 3)$.

Thus, Theorem 1 permits us to establish the following lemma.

Lemma 1 *For $k \geq 2$, we have $p_k(n) \leq 2k^n$.*

Proof We proceed with the proof of the lemma by induction on n. For $n = 1$ we have $p_k(1) = k < 2k$, for all $k \geq 2$. Consider that the inequality is valid for n, or, $p_k(n) \leq 2k^n$. By using Theorem 1 and the inductive hypothesis we have

$$p_k(n+1) < p_k(1)p_k(n) \leq 2k^{n+1},$$

for all $k \geq 2$. □

Consider that the inequality is valid for n, i.e., $p_k(n) \leq 2k^n$, with $k \geq 2$. Using Lemma 1 we obtain $\dfrac{p_k(n)}{k!} \leq \dfrac{2k^n}{k!}$. Since $\displaystyle\sum_{k=0}^{\infty} \frac{k^n}{k!}$ is convergent, then the series (6) also converges.

The next theorem is devoted to establishing the identity

$$\sum_{k=0}^{n} p_1(k) \sum_{\{w_1,\ldots,w_m\} \in C(n-k)} \frac{(-1)^m p_2(w_1)\cdots p_2(w_m)}{(2m+1)!} \prod_{l \neq l_1,\ldots,l_m} \left(1 - \frac{1}{l^2\pi^2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k p_{2k+1}(n)}{(2k+1)!}.$$

whose proof relies on determining the coefficient of q^n in the following equality

$$\frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} \left(1 - \frac{1}{\pi^2 n^2 (q;q)_\infty^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(q;q)_\infty^{2n+1}}.$$

Theorem 2 *For a positive integer n, under the previous notations, the following identity is valid*

$$\sum_{k=0}^{n-1} p_1(k) \sum_{\{w_1,\ldots,w_m\} \in C(n-k)} \sum_{1 \leq l_1 < \ldots < l_m} \frac{(-1)^m p_2(w_1)\cdots p_2(w_m)}{l_1^2 \cdots l_m^2 \pi^{2m}} \prod_{l \neq l_1,\ldots,l_m} \left(1 - \frac{1}{l^2\pi^2}\right)$$
$$+ p_1(n)\sin(1) = \sum_{k=0}^{\infty} \frac{(-1)^k p_{2k+1}(n)}{(2k+1)!}. \tag{7}$$

Proof Consider the following identity obtained by substituting $x = \dfrac{1}{(q;q)_\infty}$ in Expression (2),

$$\frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} \left(1 - \frac{1}{\pi^2 n^2 (q;q)_\infty^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(q;q)_\infty^{2n+1}}, \tag{8}$$

for $|q| < 1$.

First, we need to find the coefficient of q^n, for $n > 0$, on the left side of Equation (8).

Expression (1) for $k = 1$ gives us

$$\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} p_1(n) q^n. \tag{9}$$

Then, the coefficient of q^t in $\dfrac{1}{(q;q)_\infty}$ is $p_1(t)$. Also for any t, the coefficient of q^{n-t} in

$$\prod_{n=1}^{\infty} \left(1 - \frac{1}{\pi^2 n^2 (q;q)_\infty^2}\right) \tag{10}$$

is given by

$$\sum_{\{w_1,\ldots,w_m\}\in C(n-t)}\sum_{1\le l_1<\ldots<l_m}\frac{(-1)^m p_2(w_1)\cdots p_2(w_m)}{l_1^2\cdots l_m^2\pi^{2m}}\prod_{l\neq l_1,\ldots,l_m}\left(1-\frac{1}{l^2\pi^2}\right).$$

In fact, the constant term of Expression (10) is given by $1-\frac{1}{n^2\pi^2}$, and by Expression (1) for $k=2$, we get

$$1-\frac{1}{\pi^2n^2(q;q)_\infty^2}=1-\frac{1}{n^2\pi^2}\sum_{n=0}^{\infty}p_2(n)q^n.$$

Therefore, for each choice of m products, where each one is taken w_i times as a power factor of q, and $w_1+w_2+\ldots+w_m=n-t$, we have a part of coefficient of q^{n-t} equal to

$$\sum_{1\le l_1<\ldots<l_m}\frac{(-1)^m p_2(w_1)\cdots p_2(w_m)}{l_1^2\cdots l_m^2\pi^{2m}}\prod_{l\neq l_1,\ldots,l_m}\left(1-\frac{1}{l^2\pi^2}\right).$$

Thus, summing over all integer composition of $n-t$, the coefficients of q^{n-t} is given by

$$\sum_{\{w_1,\ldots,w_m\}\in C(n-t)}\sum_{1\le l_1<\ldots<l_m}\frac{(-1)^m p_2(w_1)\cdots p_2(w_m)}{l_1^2\cdots l_m^2\pi^{2m}}\prod_{l\neq l_1,\ldots,l_m}\left(1-\frac{1}{l^2\pi^2}\right).$$

Specifically, if we take n in terms of $q's$ in (9) we have, as part of coefficient of q^n, the following expression

$$p_1(n)\prod_{n=1}^{\infty}\left(1-\frac{1}{n^2\pi^2}\right)=p_1(n)\sin(1).$$

Therefore, summing $p_1(n)\sin(1)$ to

$$\sum_{k=0}^{n-1}p_1(k)\sum_{\{w_1,\ldots,w_m\}\in C(n-t)}\sum_{1\le l_1<\ldots<l_m}\frac{(-1)^m p_2(w_1)\cdots p_2(w_m)}{l_1^2\cdots l_m^2\pi^{2m}}\prod_{l\neq l_1,\ldots,l_m}\left(1-\frac{1}{l^2\pi^2}\right),$$

we obtain the coefficient of q^n.

Conversely, the right side of Equation (8) give us the expansion

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!(q;q)_\infty^{2n+1}}=\frac{1}{(q;q)_\infty}-\frac{1}{3!(q;q)_\infty^3}+\frac{1}{5!(q;q)_\infty^5}-\ldots.$$

Then, the coefficient of q^n is equal to

$$\sum_{k=0}^{\infty}\frac{(-1)^k p_{2k+1}(n)}{(2k+1)!},$$

which proves the theorem. □

Analogous to Theorem 2, the next identity is obtained by analyzing Equation (3) for $x=\dfrac{1}{(q;q)_\infty}$.

Theorem 3 *For a positive integer n, under the previous notations, the following identity is valid*

$$\sum_{\{w_1,\ldots,w_m\}\in C(n)}\sum_{1\le l_1<\ldots<l_m}\frac{(-1)^m2^{2m}p_2(w_1)\cdots p_2(w_m)}{(2l_1-1)^2\cdots(2l_m-1)^2\pi^{2m}}\prod_{l\neq l_1,\ldots,l_m}\left(1-\frac{4}{(2l-1)^2\pi^2}\right)$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^k p_{2k}(n)}{(2k)!}$$

Proof The proof of Theorem 3 follows using the same method as in Theorem 2. Let the following related identity, for $|q| < 1$,

$$\prod_{n=1}^{\infty}\left(1-\frac{4}{(2n-1)^2\pi^2(q;q)_\infty^{2n}}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!(q;q)_\infty^{2n}}. \tag{11}$$

First, we need to find the coefficient of q^n, for $n > 0$, on the left side of Equation (11).
The coefficient of q^n in

$$\prod_{n=1}^{\infty}\left(1-\frac{4}{(2n-1)^2\pi^2(q;q)_\infty^{2n}}\right) \tag{12}$$

is given by

$$\sum_{\{w_1,\ldots,w_m\}\in C(n)}\sum_{1\le l_1<\ldots<l_m}\frac{(-1)^m 2^{2m}p_2(w_1)\cdots p_2(w_m)}{(2l_1-1)^2\cdots(2l_m-1)^2\pi^{2m}}\prod_{l\neq l_1,\ldots,l_m}\left(1-\frac{4}{(2l-1)^2\pi^2}\right).$$

In fact, the constant in each term of Expression (12) is

$$1-\frac{4}{(2n-1)^2\pi^2},$$

and by direct application of (1) with $k = 2$ follows

$$1-\frac{4}{(2n-1)^2\pi^2(q;q)_\infty^2}=1-\frac{2^2}{(2n-1)^2\pi^2}\sum_{n=0}^{\infty}p_2(n)q^n.$$

Therefore, for each choice of m products, where each one is taken w_i times as a power factor of q, and $w_1 + w_2 + \ldots + w_m = n$, the coefficient of q^n is equal to

$$\sum_{1\le l_1<\ldots<l_m}\frac{(-1)^m 2^{2m}p_2(w_1)\cdots p_2(w_m)}{(2l_1-1)^2\cdots(2l_m-1)^2\pi^{2m}}\prod_{l\neq l_1,\ldots,l_m}\left(1-\frac{4}{(2l-1)^2\pi^2}\right).$$

Then, for all composition of n, we obtain the coefficient of q^n in the expansion of the product

$$\sum_{\{w_1,\ldots,w_m\}\in C(n)}\sum_{1\le l_1<\ldots<l_m}\frac{(-1)^m 2^{2m}p_2(w_1)\cdots p_2(w_m)}{(2l_1-1)^2\cdots(2l_m-1)^2\pi^{2m}}\prod_{l\neq l_1,\ldots,l_m}\left(1-\frac{4}{(2l-1)^2\pi^2}\right).$$

Conversely, the right side of Equation (11) gives us the expansion

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!(q;q)_\infty^{2n}}=\frac{1}{(q;q)_\infty}-\frac{1}{2!(q;q)_\infty^2}+\frac{1}{4!(q;q)_\infty^4}-\ldots.$$

Then, the coefficient of q^n is equal to

$$\sum_{k=0}^{\infty}\frac{(-1)^k p_{2k}(n)}{(2k)!},$$

which proves the result. □

It seems to us that the results given in Theorem 2 and Theorem 3 are new and have never been established in the literature.

Remark 1 Since we have $e^{ix} = \cos(x) + i\sin(x)$, $x \in \mathbb{R}$, then for $|q| < 1$ we obtain

$$\sum_{n=0}^{\infty} \frac{i^n}{n!(q;q)_\infty^n} = \frac{i}{(q;q)_\infty} \prod_{n=1}^{\infty} \left(1 - \frac{1}{n^2\pi^2(q;q)_\infty^2}\right) + \prod_{n=1}^{\infty} \left(1 - \frac{4}{(2n-1)^2\pi^2(q;q)_\infty^2}\right). \tag{13}$$

Combining Theorems 2 and 3 and analyzing the coefficient of q^n in both sides of Expression (13) we get

$$i \sum_{k=0}^{n} p_1(k) \sum_{\substack{\{w_1,\ldots,w_m\}\in C(n-t)\\ 1\le l_1<\ldots<l_m}} \frac{(-1)^m p_2(w_1)\cdots p_2(w_m)}{(2m+1)!} \times \prod_{l\neq l_1,\ldots,l_m} \left(1 - \frac{1}{l^2\pi^2}\right)$$
$$\sum_{\substack{\{w_1,\ldots,w_m\}\in C(n)\\ 1\le l_1<\ldots<l_m}} \frac{(-1)^m p_2(w_1)\cdots p_2(w_m)}{(2m)!} \times \prod_{l\neq l_1,\ldots,l_m} \left(1 - \frac{4}{(2l-1)^2\pi^2}\right) = \sum_{k=1}^{\infty} \frac{i^k p_k(n)}{k!}.$$

Using the same techniques described in previous results, the search for the coefficient of q^n in Equations (4) and (5) gives us the next two results.

Theorem 4 *For a positive integer n, under the previous notations, the following identity is valid*

$$\sum_{s=0}^{n} p_2(s) \sum_{\{w_1,\ldots,w_m\}\in C(n-s)} \sum_{1\le l_1<\ldots<l_m} \prod_{1\le i\le m} \left(-\frac{2}{(l_i)^2\pi^2} p_2(w_i) + \frac{1}{(l_i)^4\pi^4} p_4(w_i)\right)$$
$$\times \prod_{\substack{l\neq\\ l_1,\ldots,l_m}} \left(1 - \frac{2}{l^2\pi^2} + \frac{1}{l^4\pi^4}\right) = \sum_{k=1}^{\infty} \left(\sum_{i=0}^{k-1} \binom{2k}{2i+1}\right) \frac{(-1)^{k-1} p_{2k}(n)}{(2k)!}.$$

Theorem 5 *For a positive integer n, under the previous notations, the following identity is valid*

$$\sum_{\{w_1,\ldots,w_m\}\in C(n)} \sum_{1\le l_1<\ldots<l_m} \prod_{1\le i\le m} \left(-\frac{8}{(2l_i-1)^2\pi^2} p_2(w_i) + \frac{16}{(2l_i-1)^4\pi^4} p_4(w_i)\right)$$
$$\times \prod_{\substack{l\neq\\ l_1,\ldots,l_m}} \left(1 - \frac{4}{(2l-1)^2\pi^2} + \frac{16}{(2l-1)^4\pi^4}\right) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \binom{2k}{2i}\right) \frac{(-1)^k p_{2k}(n)}{(2k)!}.$$

3 Identity involving the fundamental relation between sine and cosine functions

In this section, we will establish an identity exploring the fundamental relation involving sine and cosine functions by substituting on $x = \dfrac{1}{(q;q)_\infty}$. An interesting result is related to the search for the coefficient of q^n in the expression

$$\sin^2\left(\frac{1}{(q;q)_\infty}\right) + cos^2\left(\frac{1}{(q;q)_\infty}\right) = 1, \tag{14}$$

for all complex number q with $|q| < 1$.

Note that, by Expressions (2) and (3), Equation (14) can be rewritten as

$$\frac{1}{(q;q)_\infty^2}\prod_{n=1}^{\infty}\left(1-\frac{2}{n^2\pi^2(q;q)_\infty^2}+\frac{1}{n^4\pi^4(q;q)_\infty^4}\right)$$
$$+\prod_{n=1}^{\infty}\left(1-\frac{8}{(2n-1)^2\pi^2(q;q)_\infty^2}+\frac{16}{(2n-1)^4\pi^4(q;q)_\infty^4}\right)=1. \tag{15}$$

Furthermore, looking for the coefficient of q^0 on the left side of (15), we get the following equation,

$$\prod_{n=1}^{\infty}\left(1-\frac{2}{n^2\pi^2}+\frac{1}{n^4\pi^4}\right)+\prod_{n=1}^{\infty}\left(1-\frac{8}{(2n-1)^2\pi^2}+\frac{16}{(2n-1)^4\pi^4}\right)=1.$$

The next theorem is provided by analyzing the coefficient of q^n in (15).

Theorem 6 *For a positive integer n, under the previous notations, the following identity is valid*

$$\sum_{s=0}^{n}p_2(s)\sum_{J\in C(n-s)}\sum_{1\le l_1<\ldots<l_r}\prod_{1\le i\le r}\left(p_2(j_i)a(l_i)+p_4(j_i)(a(l_i))^2\right)\prod_{\substack{m\neq\\ l_1,\ldots,l_r}}\left(1-2a(m)+(a(m))^2\right)$$
$$=-\sum_{T\in C(n)}\sum_{1\le l_1<\ldots<l_r}\prod_{1\le i\le r}\left(p_2(t_i)b(l_i)+p_4(t_i)(b(l_i))^2\right)\prod_{\substack{m\neq\\ l_1,\ldots,l_r}}\left(1-8b(m)+16(b(m))^2\right),$$

where $J=\{j_1,j_2\ldots,j_r\}\in C(n-s)$, $T=\{t_1,t_2,\ldots,t_r\}\in C(n)$, $a(n)=\frac{1}{n^2\pi^2}$, and $b(n)=\frac{4}{(2n-1)^2\pi^2}$.

Proof Consider the left side of (15). For all $s\le n$, the coefficient of q^s in

$$\frac{1}{(q;q)_\infty^2}$$

gives us the number $p_2(s)$. Having fixed the coefficient of q^s, we just need to find the coefficient of q^{n-s} in

$$\prod_{n=1}^{\infty}\left(1-\frac{2}{n^2\pi^2(q;q)_\infty^2}+\frac{1}{n^4\pi^4(q;q)_\infty^4}\right). \tag{16}$$

First, consider the term $\frac{-2}{m^2\pi^2(q;q)_\infty^2}$ in (16). Taking $\{j_1,j_2,\ldots,j_r\}$ an integer composition of $n-s$, for the part j_1, for example, the coefficient of q^{j_1} in $\frac{-2}{m^2\pi^2(q;q)_\infty^2}$ is equal to $\frac{-2p_2(j_1)}{m^2\pi^2}$. Therefore, in order to find the coefficient of q^{n-s} in these terms, we need to take r choices in the infinite product (16) and consider the terms in parentheses that are not chosen to appear as a coefficient of some q^{j_i}. By the multiplicative principle, a portion of the coefficient of q^{n-s} is

$$\sum_{1\le l_1<\ldots<l_r}\prod_{1\le i\le r}p_2(j_i)a(l_i)\prod_{\substack{m\neq\\ l_1,\ldots,l_r}}\left(1-2a(m)+(a(m))^2\right),$$

with $a(n)=\frac{1}{n^2\pi^2}$.

Similarly, by considering the term $\frac{1}{m^4\pi^4(q;q)_\infty^4}$, the other portion of the coefficient of q^{n-s} is given by

$$\sum_{1\le l_1<\ldots<r}\prod_{1\le i\le r}p_4(j_i)(a(l_i))^2\prod_{\substack{m\neq\\ l_1,\ldots,l_r}}\left(1-2a(m)+(a(m))^2\right),$$

with $a(n)=\frac{1}{n^2\pi^2}$.

For fixed s, taking the sum in every composition $J = \{j_1, j_2, \dots, j_r\} \in C(n-s)$, we have

$$\sum_{J \in C(n-s)} \sum_{1 \le l_1 < \ldots < l_r} \prod_{1 \le i \le r} \left(p_2(j_i)a(l_i) + p_4(j_i)(a(l_i))^2\right) \prod_{\substack{m \ne \\ l_1,\ldots,l_r}} \left(1 - 2a(m) + (a(m))^2\right),$$

which is the coefficient of q^{n-s} in the expansion of (16).

By the additive principle, for all s, $0 \le s \le n$, we conclude that

$$\sum_{s=1}^{n} p_2(s) \sum_{J \in C(n-s)} \sum_{1 \le l_1 < \ldots < l_r} \prod_{1 \le i \le r} \left(p_2(j_i)a(l_i) + p_4(j_i)(a(l_i))^2\right) \prod_{\substack{m \ne \\ l_1,\ldots,l_r}} \left(1 - 2a(m) + (a(m))^2\right)$$

is the coefficient of q^n in (16).

In an analogous way, the contribution of the coefficient of q^n in

$$\prod_{n=1}^{\infty} \left(1 - \frac{8}{(2n-1)^2\pi^2} + \frac{16}{(2n-1)^4\pi^4}\right)$$

is given by

$$\sum_{T \in C(n)} \sum_{1 \le l_1 < \ldots < l_r} \prod_{1 \le i \le r} \left(p_2(t_i)b(l_i) + p_4(t_i)(b(l_i))^2\right) \prod_{\substack{m \ne \\ l_1,\ldots,l_r}} \left(1 - 8b(m) + 16(b(m))^2\right).$$

Since $n > 1$ then in the right side of Equation (15) the coefficient of q^n is equal to zero, and the result is verified. □

The following corollary is An interesting case of Theorem 6. This result is obtained considering the case $n = 1$.

Corollary 1 *Under the previous notations, the following identity is valid*

$$\begin{aligned}
&\prod_{n=1}^{\infty} \left(2 - \frac{4}{n^2\pi^2} + \frac{2}{n^4\pi^4}\right) = \\
&\sum_{n=1}^{\infty} \left(\frac{4}{n^2\pi^2} - \frac{4}{n^4\pi^4}\right) \prod_{m \ne n} \left(1 - \frac{2}{m^2\pi^2} + \frac{1}{m^4\pi^4}\right) \\
&+ \sum_{n=1}^{\infty} \left(\frac{16}{(2n-1)^2\pi^2} - \frac{64}{(2n-1)^4\pi^4}\right) \prod_{m \ne n} \left(1 - \frac{8}{(2m-1)^2\pi^2} + \frac{16}{(2m-1)^4\pi^4}\right).
\end{aligned}$$

4 Conclusion and perspectives

This study presented new results regarding the sine and cosine functions. We established some identities involving convergent series in terms of the number of k-colored partitions of n and the set of the integer compositions of n.

It seems to us that the established identities are new in the literature and the approach proposed here can be applied to all convergent q-series. Moreover, it is possible to obtain some identities involving combinatorial interpretation in terms of some classes of integer partitions.

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References

1. Abramowitz, M., Stegun, I. A., *Bernoulli and Euler Polynomials and the Euler-Maclaurin Formula*. Par.23.1 in Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, pp. 804–806, 1972.
2. Alegri, M., Santos, W.F., Silva, R. *Identities involving trigonometric, zeta and partition functions*. Contributions to Discrete Mathematics. Accepted.
3. Andrews, G.E., *The Theory of Partitions, Encyclopedia of Mathematics and Its Applications* RotaEditor, Vol. 2, G.-C., Addison-Wesley, Reading, 1976.
4. Andrews, G. E., *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*. Providence, RI: Amer. Math. Soc., p. 10, 1986.
5. Andrews, G.E., Askey, R., Roy, R., *Special Functions*, Cambridge University Press, London, 1999.
6. Andrews, G.E., Eriksson, K., *Integer Partitions*, Cambridge University Press, London, 2004.
7. Chern, S., Fu, S., Tang, D., *Some inequalities for k-colored partition functions*, Ramanujan J, 46, 713–725, 2018.
8. Eberlein, W.F., *On Euler's infinite product for the sine*. Journal of Mathematical Analysis and Applications. Volume 58, Issue 1, 147–151, 1977.
9. Fu, S., Tang, D. *On a generalized crank for k-colored partitions.* J.Number Theory 184, 485-497, 2018.
10. Heubach, S., Mansour, T., *Compositions of n with parts in a set*. Congressus Numerantium, 168, 33–51, 2004.
11. Lovejoy, J., Corteel, S., *Overpartitions*, Trans. Amer. Math. Soc. 356, 2004, 1623-1635.
12. Lovejoy, J. (2006). *Overpartitions Pairs*, Ann. Inst. Fourier (Grenoble), 56, 781–794.
13. Lovejoy, J. (2007). *Partitions and overpartitions with attached parts*, Arch. Math., 88, 316–322.
14. Schneider, R., *Partition zeta functions* , Research in Number Theory, vol. 2, article 9, 2016.
15. Sills, A. V., *Compositions, Partitions, and Fibonacci Numbers*. Fibonacci Quarterly 40, 348–354, 2011.
16. Zagier, D., *Evaluation of the multiple zeta values* $\zeta(2,\ldots,2,3,2,\ldots,2)$, Annals of Mathematics 175, 977–1000, 2012.