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ORIGINAL RESEARCH

A note on the normal complement problem in semisimple group algebras

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Abstract Let FG be the semisimple group algebra of a finite group G over a finite field F. In this article, we obtain a sufficient condition for which G does not have a normal complement in the unit group of FG. In particular, we have studied the normal complement problem for semisimple group algebras of dihedral groups, quaternion groups and groups of order p^n, where $n = 3, 4$ and p is an odd prime.

Keywords Semisimple group algebras · Normal complement · Wedderburn decomposition · Dihedral groups · Unit group

Mathematics Subject Classification 16U60 · 20C05

1 Introduction

Let G be a finite group of order n and F_q a finite field of order q such that the characteristic of F_q does not divide n. Therefore F_qG is a semisimple group algebra. From the Wedderburn decomposition, we have

$$F_qG \cong \bigoplus_{i=1}^{r} M_{n_i}(D_i),$$

where D_i is a division ring containing an isomorphic copy of F_q in its center and $M_{n_i}(D_i)$ denotes the ring of all $n_i \times n_i$ matrices over D_i. Since F_qG is finite, it implies that each D_i is finite and therefore $D_i = F_{q^{k_i}}$, a finite field with q^{k_i} elements for every i. Hence

$$F_qG \cong \bigoplus_{i=1}^{r} M_{n_i}(F_{q^{k_i}}).$$

Thus,

$$\mathbb{U}(F_qG) \simeq \prod_{i=1}^{r} GL(n_i, q^{k_i}).$$

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Let

$$\pi_i : \prod_{j=1}^{r} GL(n_j, q^{k_j}) \to GL(n_i, q^{k_i})$$

denote the i^{th} projection map. It is known that if $q > 3$, then the normal subgroups of $GL(n, q)$ are either central subgroups or contain $SL(n, q)$. It is also known that the only maximal normal subgroup of $SL(n, q)$ is the center $Z(SL(n, q))$ with order $\gcd(n, q-1)$. For details, see [1, Theorem 4.9].

One of the interesting topics in the theory of group rings is to understand the structure of their unit groups. To find the structure of the unit group, the normal complement problem plays an important role. The problem was posed by H. N. Ward [15] in 1960 which states that for a given p-group G and a field K containing p elements, does G have a normal complement in $\mathbb{U}(KG)$?, i.e. is there a normal subgroup N in $\mathbb{U}(KG)$ such that $\mathbb{U}(KG) = N \rtimes G$? In 1976, R. K. Dennis [3] posed the same problem for an arbitrary group ring RG. In this paper, we deal with the normal complement problem for semisimple group algebras. The Wedderburn decomposition of semisimple group algebras plays a crucial role in analysing the existence of a normal complement in their unit groups.

It is shown in [10] that any finite abelian p-group or a finite p-group of exponent p and nilpotency class 2 has a normal complement in the unit group of the corresponding group algebra, where the group algebra is defined over the field containing exactly p elements. Moreover, if $|F| = 2$ then D_8 and Q_8 have a normal complement in $\mathbb{U}(FD_8)$ and $\mathbb{U}(FQ_8)$, respectively whereas D_{16} does not exhibit a normal complement in $\mathbb{U}(FD_{16})$. Some more results in favor of the normal complement problem are given in [6]. Further, if F has exactly 2 elements then it has been proved in [5] that dihedral, semi-dihedral and quaternion groups of order $2^n,\ n > 3$ do not have a normal complement in the unit group of their corresponding group algebras. In [11], the normal complement problem for central elementary-by-abelian p-groups over the field F_p has been discussed. Recently in [13], the problem has been studied for the group algebras of A_4, S_4 and D_{4m} (m is odd) over a finite field of characteristic 2. Several results on this problem for modular group algebras of finite groups which are not p-groups can be found in [8,9].

There are infinite examples of abelian semisimple group algebras in which the problem has an affirmative answer, see [7, Example 1]. In the same article, it is shown that normal complement does not exist in the case of semisimple group algebras of metacyclic groups of order $p_1 p_2$, where p_1, p_2 are odd primes. The problem for semisimple group algebras of symmetric and alternating groups has been discussed in [12].

This article deals with the normal complement problem for the semisimple group algebras. In section 2, we have derived a sufficient condition for which a finite group G does not have a normal complement in $\mathbb{U}(F_q G)$ for a semisimple group algebra $F_q G$. Further, we have studied the normal complement problem for a class of simple and perfect groups. In section 3, the same problem for dihedral groups, quaternion groups, and groups of order $p^n,\ n = 3, 4$ has been discussed.

2 Normal complement problem

In this section, we have discussed the normal complement problem for semisimple group algebra $F_q G$, where G is a finite group of order n and $q > 3$. We provide a sufficient condition for which G does not have a normal complement in the corresponding unit group $\mathbb{U}(F_q G)$. Let

$$\phi : \mathbb{U}(F_q G) \to \prod_{i=1}^{r} GL(n_i, q^{k_i})$$

be an isomorphism. The following lemma plays a very crucial role in proving the main result:

Lemma 2.1 *If G has a normal complement in $\mathbb{U}(F_q G)$, then $\pi_i \phi(G)$ has a normal complement in $GL(n_i, q^{k_i})$ for $1 \leq i \leq r$.*

Proof Since G has a normal complement in $\mathbb{U}(F_q G)$, it implies that there exists an epimorphism $\theta : \mathbb{U}(F_q G) \to G$, fixing G. For $1 \leq i \leq r$, define $\bar{\theta} : GL(n_i, q^{k_i}) \to \pi_i \phi(G)$ by $\bar{\theta}(\pi_i \phi(u))) = \pi_i \phi(\theta(u))$ for every $u \in \mathbb{U}(F_q G)$. It is clear that $\bar{\theta}$ is an epimorphism and fixes $\pi_i \phi(G)$. Hence $\pi_i \phi(G)$ has a normal complement in $GL(n_i, q^{k_i})$. □

Let $\pi_i|_{\phi(G)}$ be the restriction map of π_i on $\phi(G)$ and K_i the kernel of $\pi_i|_{\phi(G)}$. Therefore

$$|\pi_i\phi(G)| = \frac{|\phi(G)|}{|K_i|} = \frac{n}{|K_i|}.$$

Let $F_q^\times$ denote the unit group of F_q. If G is abelian then $n_i = 1$ for $1 \leq i \leq r$ and so $\pi_i\phi(G)$ is a subgroup of $F_{q^{k_i}}^\times$. Hence n divides $(q^{k_i} - 1)|K_i|$ for $1 \leq i \leq r$. Further, if G is non-abelian, then we have the following result.

Theorem 2.1 *If n does not divide $(q^{k_i} - 1)|K_i|$ for some i, $1 \leq i \leq r$ then G does not have a normal complement in $\mathbb{U}(F_qG)$.*

Proof Assume that G has a normal complement in $\mathbb{U}(F_qG)$. Since G is non abelian, there exists atleast one i, $1 \leq i \leq r$ such that $n_i > 1$. By Lemma 2.1, we have that $\pi_i\phi(G)$ has normal complement N in $GL(n_i, q^{k_i})$ for every i with $n_i > 1$. Therefore

$$|N| = \frac{|GL(n_i, q^{k_i})||K_i|}{n}.$$

Since N is a normal subgroup of $GL(n_i, q^{k_i})$, either N is contained in the center $Z(GL(n_i, q^{k_i}))$ or $SL(n_i, q^{k_i}) \leq N$. If $N \leq Z(GL(n_i, q^{k_i}))$ then q divides n, which is a contradiction. Now, if $SL(n_i, q^{k_i}) \leq N$, we get that n divides $(q^{k_i} - 1)|K_i|$, which leads to a contradiction. □

Corollary 2.1 *Let G be a non-abelian simple group of order n. For every i, $1 \leq i \leq r$ with $n_i > 1$, if n does not divide $q^{k_i} - 1$ then G does not have normal complement in $\mathbb{U}(F_qG)$.*

Proof Note that the possible value for $|K_i|$ is the cardinality of some normal subgroup of G. Since G is simple, the possible value for $|K_i|$ is either 1 or n. If $|K_i| = n$ for every i with $n_i > 1$, then $|\pi_i\phi(G)| = 1$, which leads to a contradiction as $\phi(G)$ is a non abelian group. Therefore, there exists j such that $|K_j| = 1$. Since n does not divide $q^{k_j} - 1$, it implies that G does not have a normal complement in $\mathbb{U}(F_qG)$. □

Next, we deal with perfect groups.

Definition 1 A group G is said to be perfect if $G = G^{'}$, where G' is the commutator subgroup of G. For example, A_5, $SL(2, 5)$, etc.

Note that if G has a normal complement in $\mathbb{U}(F_qG)$ then $G^{'}$ has a normal complement in $\mathbb{U}^{'}(F_qG)$ (see [8, Theorem 2]). Since $G = G^{'}$, it implies that G has a normal complement in $\mathbb{U}^{'}(F_qG)$. Further, observe that

$$\mathbb{U}^{'}(F_qG) \overset{\phi}{\simeq} \prod_{i=1}^{s} SL(n_i, q^{k_i}),$$

where $n_i > 1$ for $1 \leq i \leq s$. Define $d_i = \gcd(n_i, q^{k_i} - 1)$.

Theorem 2.2 *Let G be a perfect group of order n. If n does not exceed $|SL(n_i, q^{k_i})|/d_i$ for some i, then G does not have a normal complement in $\mathbb{U}(F_qG)$.*

Proof Assume that G has a normal complement in $\mathbb{U}(F_qG)$, then G has a normal complement in $\mathbb{U}^{'}(F_qG)$ and therefore $\pi_i\phi(G)$ has normal complement N in $SL(n_i, q^{k_i})$ for every i, $1 \leq i \leq s$. But the only maximal normal subgroup of $SL(n_i, q^{k_i})$ is the center, having order d_i. Therefore $|N| \leq d_i$, where

$$|N| = \frac{|SL(n_i, q^{k_i})|.|K_i|}{n}.$$

Since $|K_i| > 1$ for every i, we have

$$n \geq \frac{|SL(n_i, q^{k_i})|}{d_i}.$$

□

Corollary 2.2 *SL(2,5) does not have normal complement in $\mathbb{U}(F_q SL(2,5))$ for $q > 5$.*

Proof By [14], we can see that the decomposition of $\mathbb{U}'$ has a factor $SL(6, q)$. As for $q > 5$, we have

$$120 < \frac{|SL(6,q)|}{6} < \frac{|SL(6,q)|}{\gcd(6, q-1)}$$

and therefore $SL(2, 5)$ does not have normal complement in $\mathbb{U}(F_q SL(2, 5))$. □

3 Normal complement problem for some class of groups

In this section, we have discussed the normal complement problem of semisimple group algebras of dihedral groups, quaternion groups, and groups of order p^n, $n = 3, 4$ respectively.

3.1 Dihedral group algebras

Let $F_q D_{2n}$ be the semisimple group algebra of the dihedral group of order D_{2n} over the field F_q. Let $D_{2n} = \langle a, b \mid a^n = b^2 = 1,\ b^{-1}ab = a^{-1}\rangle$. Define

$$S = \left\{ d \mid d \text{ divides } n,\ q^u \equiv -1 \left(\mathrm{mod}\ \frac{n}{d}\right) \text{ for some } u \geq 1 \right\}.$$

By [2, Subsection 4.3], the Wedderburn decomposition of $F_q D_{2n}$ is given by

$$F_q^{(2)} \bigoplus_{d \notin S} M_2(F_{q^{f_d}})^{\left(\frac{\varphi(\frac{n}{d})}{2f_d}\right)} \bigoplus_{d \in S, d \neq n} M_2(F_{q^{f_d/2}})^{\left(\frac{\varphi(\frac{n}{d})}{f_d}\right)} \quad \text{if } n \text{ is odd and}$$

$$F_q^{(4)} \bigoplus_{d \notin S} M_2(F_{q^{f_d}})^{\left(\frac{\varphi(\frac{n}{d})}{2f_d}\right)} \bigoplus_{d \in S, d \neq n, \frac{n}{2}} M_2(F_{q^{f_d/2}})^{\left(\frac{\varphi(\frac{n}{d})}{f_d}\right)} \quad \text{if } n \text{ is even,}$$

where f_d denotes the order of q modulo $\frac{n}{d}$, for every divisor d of n and $\varphi(n)$ denotes the Euler's phi function of n. The following result deals with the case when every divisor d of n is in S.

Theorem 3.1 *If $q > 3$ and n divides $1 + q^u$ for some integer $u \geq 1$, then D_{2n} does not have normal complement in $\mathbb{U}(F_q D_{2n})$.*

Proof Note that if n divides $1 + q^u$ for some integer $u \geq 1$, then all divisors d of n are in S and hence

$$\mathbb{U}(F_q D_{2n}) \overset{\phi_1}{\cong} \begin{cases} F_q^\times \times F_q^\times \times \prod_{d|n, d \neq n} GL(2, q^{\frac{f_d}{2}})^{\left(\frac{\varphi(\frac{n}{d})}{f_d}\right)} & \text{if } n \text{ is odd,} \\ F_q^\times \times F_q^\times \times F_q^\times \times F_q^\times \times \prod_{d|n, d \neq n, \frac{n}{2}} GL(2, q^{\frac{f_d}{2}})^{\left(\frac{\varphi(\frac{n}{d})}{f_d}\right)} & \text{if } n \text{ is even.} \end{cases}$$

Let L_i be the kernel of the restriction map $\pi_i|_{\phi_1(D_{2n})}$. It implies that D_{2n} has a normal subgroup of order $|L_i|$. Therefore $|L_i| = d$ or $2n$, where d is a divisor of n. If n is odd and $|L_i| = n$ or $2n$ for every $i > 2$, then $|\pi_i \phi_1(D_{2n})| = 2$ or 1 respectively. It follows that $\phi_1(D_{2n})$ is abelian, which leads to a contradiction. Thus, there exists some $j > 2$, such that $|L_j| = d'$, where d' is different from n and $2n$. If $2n$ divides $\left(q^{\frac{f_{d'}}{2}} - 1\right)d'$, then $q^{\frac{f_{d'}}{2}} \equiv 1 \pmod{\frac{n}{d'}}$, which is a contradiction because the order of q modulo $\frac{n}{d'}$ is $f_{d'}$. Hence from Theorem 2.1, we can say that D_{2n} does not have normal complement in $\mathbb{U}(F_q D_{2n})$. Similar way one can prove the result when n is even. □

Remark 1 In particular, if $q = 3$ and n is odd, then it is clear that 8 is not divisible by $\frac{n}{d}$ for any proper divisor d of n. Hence $3^2 \not\equiv 1 \pmod{\frac{n}{d}}$ and therefore $\frac{f_d}{2} > 1$ and so $q^{\frac{f_d}{2}} > 3$. Then the normal subgroups of $GL(2, q^{\frac{f_d}{2}})$ are either central subgroups or contain $SL(2, q^{\frac{f_d}{2}})$. Thus the above result holds. For example, D_{10} does not have normal complement in $\mathbb{U}(F_3 D_{10})$.

3.2 Quaternion group algebras

We have studied the normal complement problem in group algebras $F_q Q_{4n}$, where $q > 3$ and $\gcd(q, 4n) = 1$. Let $Q_{4n} = \langle a, b \mid a^{2n} = 1,\ b^2 = a^n,\ b^{-1}ab = a^{-1}\rangle$. Define

$$T = \{d|\ d \text{ divides } 2n,\ q^u \equiv -1 \left(\bmod \frac{2n}{d}\right) \text{ for some } u \geq 1\}.$$

By [2, subsection 4.4], the Wedderburn decomposition of $F_q Q_{4n}$ is given by
Case 1. $q \equiv 1 \mod 4$

$$F_q^{(4)} \bigoplus_{d \notin T} M_2(F_{q^{o_d}})^{\left(\frac{\varphi(\frac{2n}{d})}{2o_d}\right)} \bigoplus_{d \in T, d \neq n, 2n} M_2(F_{q^{o_d/2}})^{\left(\frac{\varphi(\frac{2n}{d})}{o_d}\right)}$$

Case 2. $q \equiv 3 \mod 4$

$$F_q^{(2)} \bigoplus F_{q^2} \bigoplus_{d \notin T} M_2(F_{q^{o_d}})^{\left(\frac{\varphi(\frac{2n}{d})}{2o_d}\right)} \bigoplus_{d \in T, d \neq n, 2n} M_2(F_{q^{o_d/2}})^{\left(\frac{\varphi(\frac{2n}{d})}{o_d}\right)} \quad \text{if } n \text{ is odd and}$$

$$F_q^{(4)} \bigoplus_{d \notin T} M_2(F_{q^{o_d}})^{\left(\frac{\varphi(\frac{2n}{d})}{2o_d}\right)} \bigoplus_{d \in T, d \neq n, 2n} M_2(F_{q^{o_d/2}})^{\left(\frac{\varphi(\frac{2n}{d})}{o_d}\right)} \quad \text{if } n \text{ is even,}$$

where o_d denotes the order of q modulo $\frac{2n}{d}$, for every divisor d of $2n$. The following result deals with the case when every divisor of $2n$ is in T.

Theorem 3.2 *If $2n$ divides $1 + q^u$ for some integer $u \geq 1$, then Q_{4n} does not have normal complement in $\mathbb{U}(F_q Q_{4n})$.*

Proof Assume that $2n$ divides $1 + q^u$ for some integer $u \geq 1$. So, the set T contains all divisors of $2n$ and hence

$$\mathbb{U}(F_q Q_{4n}) \overset{\phi_2}{\simeq} \begin{cases} F_q^{\times(4)} \times \prod\limits_{d \in T, d \neq n, 2n} GL(2, q^{\frac{o_d}{2}})^{\left(\frac{\varphi(\frac{2n}{d})}{o_d}\right)} & q \equiv 1 \mod 4 \\ F_q^{\times(2)} \times F_{q^2}^{\times} \times \prod\limits_{d \in T, d \neq n, 2n} GL(2, q^{\frac{o_d}{2}})^{\left(\frac{\varphi(\frac{2n}{d})}{o_d}\right)} & \text{if } n \text{ is odd and } q \equiv 3 \mod 4 \\ F_q^{\times(4)} \times \prod\limits_{d \in T, d \neq n, 2n} GL(2, q^{\frac{o_d}{2}})^{\left(\frac{\varphi(\frac{2n}{d})}{o_d}\right)} & \text{if } n \text{ is even and } q \equiv 3 \mod 4. \end{cases}$$

Let M_i be the kernel of the restriction map $\pi_i|_{\phi_2(Q_{4n})}$. It implies that Q_{4n} has a normal subgroup of order $|M_i|$. Therefore $|M_i| = d$ or $4n$, where d is a divisor of $2n$. If $q \equiv 1(mod\ 4)$ and $|M_i| = n,\ 2n$ or $4n$ for every $i > 4$, then $|\pi_i \phi_2(Q_{4n})| = 4,\ 2$ or 1 respectively. It follows that $\phi_2(Q_{4n})$ is abelian, which leads to a contradiction. Thus, there exist some $j > 4$, such that $|M_j| = d'$, where d' is different from $n,\ 2n$ and $4n$. If $4n$ divides $\left(q^{\frac{o_{d'}}{2}} - 1\right)d'$, then $q^{\frac{o_{d'}}{2}} \equiv 1 \pmod{\frac{2n}{d'}}$, which is a contradiction because the order of q modulo $\frac{2n}{d'}$ is $o_{d'}$. Hence from Theorem 2.1, we can say that Q_{4n} does not have a normal complement in $\mathbb{U}(F_q Q_{4n})$. The remaining cases follow in the same way. □

3.3 Group algebras of groups of order p^3

Here, we deal with group algebra $F_q G$, where G is a non-abelian group of order p^3 for an odd prime p such that $\gcd(p, q) = 1$. It is known [4] that

$$F_q G \cong F_q \bigoplus F_{q^f}{}^{((1+p)e)} \bigoplus M_p(F_{q^f})^{(e)},$$

where f denotes the order of q modulo p and $e = \frac{p-1}{f}$.

Theorem 3.3 *If p^3 does not divide $q^f - 1$, then G does not have a normal complement in $\mathbb{U}(F_qG)$.*

Proof Let $\phi_3 : \mathbb{U}(F_qG) \to F_q^\times \times \underbrace{F_{q^f}^\times \times \cdots \times F_{q^f}^\times}_{(1+p)e} \times \underbrace{GL(p, q^f) \times \cdots \times GL(p, q^f)}_{e}$ be an isomorphism and M_i be the kernel of the restriction map $\pi_i|_{\phi_3(G)}$. If $|\pi_i\phi_3(G)| < p^3$ for every $i > 1+(1+p)e$, then it follows that $\phi_3(G)$ is an abelian group, which is a contradiction. So, there exists $j > 1+(1+p)e$ such that $|\pi_j\phi_3(G)| = p^3$ and hence $|M_j| = 1$. Thus, if p^3 does not divide $q^f - 1$, then Theorem 2.1 implies that G does not have a normal complement in $\mathbb{U}(F_qG)$. □

3.4 Group algebras of groups of order p^4

Now, assume that G is a group of order p^4 for an odd prime p. The Wedderburn decomposition of the semisimple group algebra F_qG has been provided in [4]. The following table provides the set of groups of order p^4 which does not have normal complement in their unit group under certain conditions. Here f denotes the order of q modulo p and $e = \frac{p-1}{f}$.

G	Wedderburn decomposition	Sufficient condition
$\mathbb{G}_1$	$F_q \bigoplus F_{q^f}{}^{((1+p)e)} \bigoplus F_{q^{fp}}{}^{(pe)} \bigoplus M_p(F_{q^{fp}})^{(e)}$	$p^4 \nmid p(q^{fp} - 1)$
$\mathbb{G}_2$	$F_q \bigoplus F_{q^f}{}^{((1+p+p^2)e)} \bigoplus M_p(F_{q^{fp}})^{(e)}$	$p^4 \nmid p(q^{fp} - 1)$
$\mathbb{G}_3$	$F_q \bigoplus F_{q^f}{}^{((1+p)e)} \bigoplus F_{q^{fp}}{}^{(pe)} \bigoplus M_p(F_{q^f})^{(pe)}$	$p^4 \nmid p(q^f - 1)$
$\mathbb{G}_4$	$F_q \bigoplus F_{q^f}{}^{((1+p+p^2)e)} \bigoplus M_p(F_{q^f})^{(pe)}$	$p^4 \nmid p(q^f - 1)$
$\mathbb{G}_5$	$F_q \bigoplus F_{q^f}{}^{((1+p)e)} \bigoplus F_{q^{fp}}{}^{(pe)} \bigoplus M_p(F_{q^f})^{(pe)}$	$p^4 \nmid p(q^f - 1)$
$\mathbb{G}_6$	$F_q \bigoplus F_{q^f}{}^{((1+p)e)} \bigoplus M_p(F_{q^f})^{((1+p)e)}$	$p^4 \nmid p(q^f - 1)$
$\mathbb{G}_7$	$F_q \bigoplus F_{q^f}{}^{((1+p)e)} \bigoplus F_{q^f}{}^{(e)} \bigoplus M_p(F_{q^{fp}})^{(e)}$	$p^4 \nmid p(q^{fp} - 1)$
$\mathbb{G}_8$	$F_q \bigoplus F_{q^f}{}^{((1+p)e)} \bigoplus F_{q^f}{}^{(e)} \bigoplus M_p(F_{q^{fp}})^{(e)}$	$p^4 \nmid p(q^{fp} - 1)$
$\mathbb{G}_9$	$F_q \bigoplus F_{q^f}{}^{((1+p+p^2)e)} \bigoplus M_p(F_{q^f})^{(pe)}$	$p^4 \nmid p(q^f - 1)$
$\mathbb{G}_{10}$	$F_q \bigoplus F_{q^f}{}^{((1+p)e)} \bigoplus F_{q^f}{}^{(e)} \bigoplus M_p(F_{q^{fp}})^{(e)}$, $p = 3$	$p^4 \nmid p(q^{fp} - 1)$
	$F_q \bigoplus F_{q^f}{}^{((1+p)e)} \bigoplus M_p(F_{q^f})^{((1+p)e)}$, $p > 3$	$p^4 \nmid p(q^f - 1)$

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