Indian J Pure Appl Math (2026) 57:321–337
https://doi.org/10.1007/s13226-024-00533-3

ORIGINAL RESEARCH

On some representation numbers by $a\sum x_i^2 + b\sum x_i x_j$ representing one

Ick Sun Eum

Received: 8 December 2022 / Accepted: 1 January 2024 / Published online: 13 January 2024

Abstract Let $Q = a\sum x_i^2 + b\sum x_i x_j$ be an integral positive definite quadratic form of level N in $r(\geq 2)$ variables and $r_Q(n)$ the representation number by Q for nonnegative integers n. First, we provide a necessary and sufficient condition that Q represents one and find the exact value of $r_Q(1)$. Second, for such forms, we show that $r_Q(n)$ satisfies a certain congruence relation for infinitely many n. Finally, when $r(\geq 4)$ is even and $(-1)^{r/2}N$ is a fundamental discriminant, we classify the quadratic forms Q, which represent one, whose representation numbers $r_Q(n)$ satisfy a partially multiplicative relation $r_Q(p^2n)r_Q(1) = r_Q(p^2)r_Q(n)$ and provide closed formulas for $r_Q(n)$.

Keywords Representations by quadratic forms · Eisenstein series · Modular forms

Mathematics Subject Classification Primary 11E25; Secondary 11F11

1 Introduction

Let

$$Q(\mathbf{x}) = Q(x_1, x_2) = x_1^2 + x_2^2 + x_1x_2.$$

It is well-known that, for a nonnegative integer n, the representation number

$$r_Q(n) = \#\left\{ \mathbf{x} \in \mathbb{Z}^2 \mid Q(\mathbf{x}) = n \right\}$$

of n by Q is given by

$$r_Q(n) = 6\sum_{0<d|n} \left(\frac{-3}{d}\right)$$

(See [7, Theorem 17.1]). Here, $\left(\frac{-3}{d}\right)$ denotes the Kronecker symbol. Then one can readily check that $r_Q(1) = 6(\neq 0)$ and

$$r_Q(n) \equiv 0 \pmod{r_Q(1)} \tag{1.1}$$

Communicated by Eknath Ghate.

This work was supported by the Dongguk University Research Fund of 2022 and the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No. NRF-2020R1F1A1A01070647)

I. S. Eum
Department of Mathematics Education, Dongguk University WISE Campus, Gyeongju, Republic of Korea
E-mail: zandc@dongguk.ac.kr

for all positive integers n. Moreover, it can be easily verified that

$$r_Q(p^2n) = \frac{r_Q(p^2)r_Q(n)}{r_Q(1)} \tag{1.2}$$

for any prime $p \neq 3$ and any positive integer n prime to p. Here, we note that 3 is the level of Q. In general, for an integral positive definite quadratic form Q of level N in r variables, let

$$r_Q(n) = \#\left\{ \mathbf{x} \in \mathbb{Z}^r \mid Q(\mathbf{x}) = n \right\}$$

be the representation number of a nonnegative integer n by Q. Then we call $r_Q(n)$ *p-multiplicative* for a prime $p \nmid N$ if it satisfies the relation (1.2) for all positive integers n prime to p. In [1], the author provided some equivalent conditions for p-multiplicativeness of $r_Q(n)$ under a certain assumption on Q.

Now, let $r \geq 2$ be a positive integer. We will focus on the following integral positive definite quadratic form

$$Q(\mathbf{x}) = a\sum_{i=1}^{r} x_i^2 + b\sum_{1\leq i<j\leq r} x_i x_j,$$

which is a generalization of $x_1^2 + x_2^2 + x_1x_2$. Let N be the level of Q. In this paper, we shall obtain similar results as (1.1) and (1.2) for the above form Q. More precisely, we shall provide a necessary and sufficient condition that Q represents one and find the exact value of $r_Q(1)$ (Theorem 3.1). When r is odd and $b \geq 1$, we will assume that $rb + 1 \equiv 0 \pmod 4$. For such forms, we shall show that $r_Q(n)$ satisfies the following congruence relation

$$r_Q(n) \equiv 0 \pmod{r_Q(1)}$$

for any positive integer n with $\gcd(n, r(r-1)(r|b|+1)) = 1$ and $\left(\frac{2n(rb+1)}{p}\right) = -1$ for all odd prime $p|(r-1)$ (Theorem 3.5). Here, $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. Furthermore, when $r(\geq 4)$ is even and $(-1)^{r/2}N$ is a fundamental discriminant, we will classify the quadratic forms Q, which represent one, whose representation numbers $r_Q(n)$ are p-multiplicative and provide closed formulas for $r_Q(n)$ by using the results in [1] (Theorem 4.3).

2 Preliminaries

In this section, we recall some basic notions of modular forms, Eisenstein series and quadratic forms.

2.1 Modular forms and Eisenstein series

Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}\tau > 0\}$ be the upper-half plane. For positive integers k and N, put $\Gamma_0(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod N\}$, and for each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we define the slash operator $\cdot|_k\gamma$ on the functions $f(\tau)$ on $\mathbb{H}$ by

$$f(\tau)|_k\gamma = (c\tau + d)^{-k} f(\gamma \cdot \tau),$$

where γ acts on $\mathbb{H}$ as the fractional linear transformation $\gamma \cdot \tau = (a\tau + b)/(c\tau + d)$. For a Dirichlet character χ modulo N, we define a character χ of $\Gamma_0(N)$ to be

$$\chi(\gamma) = \chi(d) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

A holomorphic function $f(\tau)$ on $\mathbb{H}$ is a modular form of weight k for $\Gamma_0(N)$ associated with the character χ if

(i) $f(\tau)|_k\gamma = \chi(\gamma)f(\tau)$ for all $\gamma \in \Gamma_0(N)$,
(ii) $f(\tau)$ is holomorphic at every cusp.

In particular, since

$$f(\tau)|_k \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = f(\tau + 1) = f(\tau)$$

by (i), $f(\tau)$ has a Laurent series expansion with respect to $q = e^{2\pi i\tau}$ of the form

$$f(\tau) = \sum_{n=0}^{\infty} a(n)q^n \quad (a(n) \in \mathbb{C}),$$

which is called the Fourier expansion of $f(\tau)$. The $a(n)$ is said to be the n-th Fourier coefficient of $f(\tau)$. If a modular form vanishes at every cusp, then it is called a cusp form. We denote by $\mathcal{M}_k(N, \chi)$, $\mathcal{S}_k(N, \chi)$ and $\mathcal{E}_k(N, \chi)$ the space of all modular forms, cusp forms and Eisenstein series of weight k for $\Gamma_0(N)$ associated with the character χ, respectively. As is well-known, $\mathcal{M}_k(N, \chi) = \mathcal{E}_k(N, \chi) \oplus \mathcal{S}_k(N, \chi)$ and $\mathcal{M}_k(N, \chi) = \{0\}$ if $\chi(-1) \neq (-1)^k$ ([4, Theorem 2.1.7(1) and Lemma 4.3.2(1)]).

For a nonnegative integer k and a nontrivial primitive Dirichlet character χ modulo N, we let $B_{k,\chi}$ be a generalized Bernoulli number defined by

$$\sum_{a=1}^{N-1} \chi(a) \frac{xe^{ax}}{e^{Nx} - 1} = \sum_{k=0}^{\infty} B_{k,\chi} \frac{x^k}{k!}.$$

Then it is well-known that $B_{k,\chi} \neq 0$ if and only if $\chi(-1) = (-1)^k$ ([3, Chapter XIV, Corollary of Theorem 2.2 and Theorem 2.3]).

Proposition 2.1 *Denote by χ_0 the trivial character modulo 1. Let ψ_1 and ψ_2 be primitive Dirichlet characters modulo N_1 and N_2, respectively. Let $k(\geq 2)$ be an integer satisfying $\psi_1(-1)\psi_2(-1) = (-1)^k$. Define*

$$E_{k,\psi_1,\psi_2}(\tau) = b_0 + \sum_{n=1}^{\infty} \left(\sum_{d>0, d|n} \psi_1(n/d)\psi_2(d)d^{k-1} \right) q^n \in \mathbb{C}[[q]] \quad (\tau \in \mathbb{H}),$$

where

$$b_0 = \begin{cases} 0 & \text{if } L > 1 \\ -B_{k,\psi_1\psi_2}/2k & \text{if } L = 1 \end{cases}$$

and we set $\psi_i(h) = 0$ if $\gcd(h, N_i) \neq 1$ for $i = 1, 2$. Then except for the case when $k = 2$ and $\psi_1 = \psi_2 = \chi_0$, $E_{k,\psi_1,\psi_2}(\tau)$ defines an element of $\mathcal{M}_k(N_1N_2, \psi_1\psi_2)$ which is called an Eisenstein series.

Proof See [4, Theorem 4.7.1] or [6, Theorem 5.8]. □

Proposition 2.2 *Let $k(\geq 2)$ be an integer and χ a nontrivial primitive Dirichlet character modulo N with $\chi(-1) = (-1)^k$. Let*

$$\begin{aligned} X_{k,\chi} = \{(\psi_1, \psi_2) \mid\ & \psi_1 \text{ and } \psi_2 \text{ are primitive Dirichlet characters modulo } N_1 \text{ and } N_2, \\ & \text{respectively}, \\ & N_1N_2 = N \text{ and } \psi_1\psi_2 = \chi\}. \end{aligned}$$

Then $\{E_{k,\psi_1,\psi_2}(\tau) \mid (\psi_1, \psi_2) \in X_{k,\chi}\}$ is a basis of $\mathcal{E}_k(N, \chi)$. Furthermore, we have

$$\dim_{\mathbb{C}} \mathcal{E}_k(N, \chi) = \prod_{\substack{p|N \\ p \text{ prime}}} 2.$$

Proof See [4, Theorem 4.7.2] or [6, Theorem 5.9]. For the dimension formula, see [2, Remark 2.3]. □

2.2 Quadratic forms and Theta functions

For an integer k and a positive integer N such that $(-1)^k N \equiv 0$ or $1 \pmod 4$, let $\chi_{(-1)^k N}$ be the Dirichlet character defined by the Kronecker symbol, that is,

$$\chi_{(-1)^k N}(\cdot) = \left(\frac{(-1)^k N}{\cdot}\right).$$

Then $(-1)^k N$ is a fundamental discriminant if and only if $\chi_{(-1)^k N}$ is a primitive Dirichlet character modulo N ([5, Theorem 9.13]). Now, let $r \geq 2$ be an even positive integer and $k = r/2$. Let Q be an integral positive definite quadratic form in r variables and $A = \left(\frac{\partial^2 Q}{\partial x_i \partial x_j}\right)$ the Hessian matrix of Q so that

$$Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} \quad \text{for } \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} \in \mathbb{Z}^r.$$

Further, let N be the level of Q, which is the smallest positive integer such that NA^{-1} is integral and has even diagonal entries. For a nonnegative integer n, let

$$r_Q(n) = \#\{\mathbf{x} \in \mathbb{Z}^r \mid Q(\mathbf{x}) = n\}$$

be the representation number of n by Q. We say that Q represents n if $r_Q(n) \neq 0$. Denote the generating function $\Theta_Q(\tau)$ of $r_Q(n)$ by

$$\Theta_Q(\tau) = \sum_{\mathbf{x} \in \mathbb{Z}^{2k}} e^{2\pi i Q(\mathbf{x})\tau} = \sum_{n=0}^{\infty} r_Q(n) q^n \quad (\tau \in \mathbb{H}),$$

which is called the theta function associated with Q. Then it is well-known that the theta function $\Theta_Q(\tau)$ belongs to $\mathcal{M}_k(N, \chi_{(-1)^k \det(A)})$ (See [4, Corollary 4.9.5(3)]). We say that r_Q is *p-multiplicative* for a prime $p \nmid N$ if it satisfies the relation (1.2) for all positive integers n prime to p. In [1], the author provided some necessary and sufficient conditions that $r_Q(n)$ is p-multiplicative as follows.

Proposition 2.3 *Let $k \geq 2$ and Q be an integral positive definite quadratic form in $2k$ variables of level N. Assume that $\chi_{(-1)^k N}$ is the primitive character satisfying $\chi_{(-1)^k N} = \chi_{(-1)^k \det(A)}$. Then the following are equivalent:*

(i) r_Q is p-multiplicative for some prime p with $p \nmid N$.
(ii) $\Theta_Q(\tau)$ can be expressed as a sum of Eisenstein series.
(iii) r_Q is p-multiplicative for any prime p with $p \nmid N$.

When $\dim_{\mathbb{C}} \mathcal{S}_k(N, \chi_{(-1)^k N}) = 1$, then $(i) \sim (iii)$ are equivalent to the following statement:

(iv) For some prime $p \nmid N$, we have

$$\frac{r_Q(p^2)}{r_Q(1)} = 1 + \chi_{(-1)^k N}(p) p^{k-1} + p^{2(k-1)}.$$

Proof See [1, Theorem 3.1 and Corollary 3.2]. □

Remark 2.4 Notation being as in Proposition 2.3. From the proof of [1, Theorem 3.1], one may see that if r_Q is p-multiplicative for some prime p with $p \nmid N$, then

$$\frac{r_Q(p^2)}{r_Q(1)} = 1 + \chi_{(-1)^k N}(p) p^{k-1} + p^{2(k-1)}.$$

Proposition 2.5 *Let $k \geq 2$ and Q be an integral positive definite quadratic form in $2k$ variables of level N. Let A be the Hessian matrix of Q and assume that $N = \det(A)$ and $(-1)^k N$ is a fundamental discriminant, that is, $\chi_{(-1)^k N}$ is a primitive Dirichlet character. Suppose that*

$$\dim_{\mathbb{C}} \mathcal{E}_k(N, \chi_{(-1)^k N}) = \dim_{\mathbb{C}} \mathcal{M}_k(N, \chi_{(-1)^k N}) - \dim_{\mathbb{C}} \mathcal{S}_k(N, \chi_{(-1)^k N}) = 2.$$

Then we may express $\Theta_Q(\tau) \in \mathcal{M}_k(N, \chi_{(-1)^k N})$ as

$$\Theta_Q(\tau) = c_1 E_{k,\chi_0,\chi_{(-1)^k N}}(\tau) + c_2 E_{k,\chi_{(-1)^k N},\chi_0}(\tau) + a \text{ cusp form},$$

where

$$c_1 = -\frac{2k}{B_{k,\chi_{(-1)^k N}}} \quad \text{and} \quad c_2 = N^{k-1} i^{-k^2+k} \left(-\frac{2k}{B_{k,\chi_{(-1)^k N}}} \right).$$

When $\dim_{\mathbb{C}} \mathcal{M}_k(N, \chi_{(-1)^k N}) = 2$, then we have

$$r_Q(n) = -\frac{2k}{B_{k,\chi_{(-1)^k N}}} \sum_{d>0, d|n} \left(\chi_{(-1)^k N}(d) + N^{k-1} i^{-k^2+k} \chi_{(-1)^k N}(n/d) \right) d^{k-1}.$$

Proof See [2, the proof of Theorem 6.2 and Remark 6.5]. □

3 The exact value of $r_Q(1)$ and a congruence relation of $r_Q(n)$

For the rest of the paper, we will assume that $r \geq 2$ is a positive integer and

$$Q(\mathbf{x}) = Q(x_1, \cdots, x_r) = a \sum_{i=1}^{r} x_i^2 + b \sum_{1 \leq i < j \leq r} x_i x_j \quad (a, b \in \mathbb{Z})$$

is an integral positive definite quadratic form in r variables. Then its Hessian matrix

$$A = \left(\frac{\partial^2 Q}{\partial x_i \partial x_j} \right) = \begin{pmatrix} 2a & b & b & \cdots & b \\ b & 2a & b & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & 2a \end{pmatrix}$$

is a symmetric positive definite integral matrix with even diagonal entries. From the positive definiteness, we get

$$a \geq 1, \quad 2a \pm b \geq 1 \quad \text{and} \quad \det(A) = (2a - b)^{r-1}(2a + (r-1)b) \geq 1. \tag{3.1}$$

We will provide an equivalent condition for $r_Q(1) \neq 0$ and give a formula for $r_Q(1)$ as follows.

Theorem 3.1 *Let r and Q be as above. Then we have*

$$r_Q(1) \neq 0 \quad \text{if and only if} \quad \begin{cases} b \geq 1 \ \text{ and } \ 2a - b = 1, \\ b = 0 \ \text{ and } \ a = 1, \\ b \leq -1, \ r = 2 \ \text{ and } \ 2a + b = 1. \end{cases}$$

Thus, b should be odd if $b \neq 0$. For each case, we can find the exact value of $r_Q(1)$ as follows:

$$r_Q(1) = \begin{cases} r(r-1), & \text{if } \ b \geq 3 \ \text{ and } \ 2a - b = 1, \\ r(r+1), & \text{if } \ b = 1 \ \text{ and } \ a = 1, \\ 2r, & \text{if } \ b = 0 \ \text{ and } \ a = 1, \\ 6, & \text{if } \ b = -1, \ r = 2 \ \text{ and } \ a = 1, \\ 2, & \text{if } \ b \leq -3, \ r = 2 \ \text{ and } \ 2a + b = 1. \end{cases}$$

Proof When $b = 0$, then

$$Q(\mathbf{x}) = a\sum_{i=1}^{r} x_i^2 = 1$$

has an integral solution if and only if $a = 1$. In this case, one may easily see that all solutions are of the form

$$\mathbf{x} = (\pm 1, 0, \cdots, 0),\ (0, \pm 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, \pm 1).$$

Thus, we get $r_Q(1) = 2r \neq 0$. Consider the case when $b \geq 1$. Then one may verify that

$$1 = Q(\mathbf{x}) = a\sum_{i=1}^{r} x_i^2 + b\sum_{1\leq i<j\leq r} x_i x_j = \frac{2a-b}{2}\sum_{i=1}^{r} x_i^2 + \frac{b}{2}\left(\sum_{i=1}^{r} x_i\right)^2$$

has an integral solution if and only if $2a - b = 1$ because $2a - b \geq 1$ from (3.1) and $b \geq 1$. Suppose that $2a - b = 1$ and let $\mathbf{u} = (u_1, \cdots, u_r) \in \mathbb{Z}^r$ be a solution of $Q(\mathbf{x}) = 1$. Multiplying by 2 on both sides of $Q(\mathbf{u}) = 1$, we have

$$2 = (2a-b)\sum_{i=1}^{r} u_i^2 + b\left(\sum_{i=1}^{r} u_i\right)^2 = \sum_{i=1}^{r} u_i^2 + b\left(\sum_{i=1}^{r} u_i\right)^2. \tag{3.2}$$

Note that b should be odd. If $b \geq 3$, then by (3.2), we get

$$\sum_{i=1}^{r} u_i^2 = 2 \quad \text{and} \quad \sum_{i=1}^{r} u_i = 0$$

and the solution vectors $\mathbf{u}$ should be of the form

$$\mathbf{u} = (\pm 1, \mp 1, 0, \cdots, 0),\ (\pm 1, 0, \mp 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, \pm 1, \mp 1),$$

which implies that

$$r_Q(1) = \frac{r!}{1!1!(r-2)!} = r(r-1) \neq 0.$$

Here, $m!$ denotes the factorial of a nonnegative integer m. When $b = 1$, then $a = 1$ since $2a - b = 1$. From (3.2), we have

$$\sum_{i=1}^{r} u_i = 0 \quad \text{or} \quad \sum_{i=1}^{r} u_i = \pm 1.$$

When $\sum_{i=1}^{r} u_i = 0$, then the same argument as in the case $b \geq 3$ shows that there are $r(r-1)$ solutions. When $\sum_{i=1}^{r} u_i = \pm 1$, then we have $\sum_{i=1}^{r} u_i^2 = 1$ by (3.2). As in the case $b = 0$, there are $2r$ solutions satisfying $\sum_{i=1}^{r} u_i^2 = 1$. Thus we get

$$r_Q(1) = r(r-1) + 2r = r(r+1) \neq 0.$$

The remaining case is when $b \leq -1$. Since $2a + b \geq 1$ from (3.1), $b \leq -1$ and the form

$$\sum_{i=1}^{r} x_i^2 - 2\sum_{1\leq i<j\leq r} x_i x_j$$

is semi-positive definite, one may derive that

$$1 = Q(\mathbf{x}) = \frac{2a+b}{2}\sum_{i=1}^{r} x_i^2 - \frac{b}{2}\left(\sum_{i=1}^{r} x_i^2 - 2\sum_{1\leq i<j\leq r} x_i x_j\right) \tag{3.3}$$

has an integral solution if and only if $2a + b = 1$. Suppose that $2a + b = 1$. Since $2a - b \geq 1$ and $\det(A) \geq 1$ from (3.1), we have $2a + (r-1)b \geq 1$. Since $2a + b = 1$, we find that

$$1 \leq 2a + (r-1)b = 2a + b + (r-2)b = 1 + (r-2)b,$$

which shows that $0 \leq (r-2)b$. Since $b \leq -1$, we should have $r = 2$. Now let $\mathbf{u} = (u_1, u_2) \in \mathbb{Z}^2$ be a solution of $Q(\mathbf{x}) = 1$. Then we have

$$2 = 2 \cdot 1 = 2Q(\mathbf{u}) = u_1^2 + u_2^2 - b\left((u_1^2 + u_2^2) - 2u_1u_2\right) = u_1^2 + u_2^2 - b(u_1 - u_2)^2.$$

If $b \leq -3$, then we have $u_1^2 + u_2^2 = 2$ and $u_1 = u_2$. Thus the only solution vectors are $\mathbf{u} = \pm(1, 1)$ and we have

$$r_Q(1) = 2 \neq 0.$$

When $b = -1$, then $a = 1$ since $2a + b = 1$. Thus Q is of the form

$$Q(\mathbf{x}) = x_1^2 + x_2^2 - x_1x_2.$$

It is easy to verify that the only solution vectors are $\mathbf{u} = (\pm 1, 0),\ (0, \pm 1),\ (\pm 1, \pm 1)$ and we get

$$r_Q(1) = 6 \neq 0.$$

□

From now on, we always assume that Q represents one.

Remark 3.2 Let N be the level of Q. From Theorem 3.1 and (3.1), one may easily obtain the followings.

(i) If $b \neq 0$, then b should be odd since $2a - b = 1$ or $2a + b = 1$.
(ii) When $b \geq 1$, we have $2a - b = 1$, $\det(A) = rb + 1$ and

$$A^{-1} = \frac{1}{rb+1}\begin{pmatrix} (r-1)b+1 & -b & -b \cdots & -b \\ -b & (r-1)b+1 & -b \cdots & -b \\ \vdots & \vdots & \vdots \ddots & \vdots \\ -b & -b & -b \cdots & (r-1)b+1 \end{pmatrix}.$$

In this case, we have $N = rb + 1$ or $N = 2(rb + 1)$ if r is even or odd, respectively.

For a positive integer m and a prime p, we denote by $v_p(m)$ the highest power of p dividing m. Let S_r be the symmetric group on $\{1, 2, \cdots, r\}$ and $X(n) = \{\mathbf{x} \in \mathbb{Z}^r \mid Q(\mathbf{x}) = n\}$ for each positive integer n. For $\mathbf{u} = (u_1, u_2, \cdots, u_r) \in X(n)$ and $\sigma \in S_r$, let $\sigma(\mathbf{u}) = (u_{\sigma(1)}, u_{\sigma(2)}, \cdots, u_{\sigma(r)}) \in X(n)$. Before we introduce a congruence relation of $r_Q(n)$, we need the following lemmas.

Lemma 3.3 *For* $\mathbf{u}, \mathbf{v} \in X(n)$, *define* $\mathbf{u} \sim \mathbf{v}$ *if there is* $\sigma \in S_r$ *such that* $\mathbf{u} = \pm\sigma(\mathbf{v})$. *Then* $\sim$ *is an equivalence relation on* $X(n)$.

Proof It is immediate from the definition of Q. □

Lemma 3.4 *Let* m *and* h *be positive integers with* $m \geq h$. *Then we have*

$$\frac{m}{\gcd(m,h)} \Bigg| \binom{m}{h}.$$

Here, $\binom{m}{h}$ *denotes the binomial coefficient.*

Proof This lemma is well-known, but for the completeness of the paper, we give proof below. Let x and y be integers such that $mx + hy = \gcd(m, h)$. Then we have

$$\frac{\gcd(m,h)}{m}\binom{m}{h} = \frac{mx+hy}{m}\binom{m}{h} = x\binom{m}{h} + y\frac{h}{m}\binom{m}{h} = x\binom{m}{h} + y\binom{m-1}{h-1} \in \mathbb{Z}.$$

This proves our lemma. □

Theorem 3.5 *Let $r \geq 2$ be a positive integer. Suppose that $rb+1 \equiv 0 \pmod 4$ when r is odd and $b \geq 1$. Assume that Q represents one. Let n be a positive integer such that $\gcd(n, r(r-1)(r|b|+1)) = 1$ and $\left(\frac{2n(rb+1)}{p}\right) = -1$ for all odd prime $p|(r-1)$ (if there are any). Then we have the following congruence relation*

$$r_Q(n) \equiv 0 \pmod{r_Q(1)}. \tag{3.4}$$

Proof Let n be a positive integer such that $\gcd(n, r(r-1)(rb+1)) = 1$ and $\left(\frac{2n(rb+1)}{p}\right) = -1$ for all odd prime $p|(r-1)$. Note that n should be odd. Let $\mathbf{u} = (u_1, u_2, \cdots, u_r) \in \mathbb{Z}^r$ be a solution of $Q(\mathbf{x}) = n$, that is, $\mathbf{u} \in X(n)$. Consider the equivalence relation defined in Lemma 3.3 and denote by $[\mathbf{u}]$ the equivalence class of $\mathbf{u}$. From Lemma 3.3, after a suitable rearrangement (if necessary), we may write the solution vector $\mathbf{u}$ as

$$\mathbf{u} = (\underbrace{y_1, y_1, \cdots, y_1}_{r_1}, \underbrace{y_2, y_2, \cdots, y_2}_{r_2}, \cdots, \underbrace{y_l, y_l, \cdots, y_l}_{r_l})$$

where $l, r_i \in \mathbb{Z}^+$ for $1 \leq i \leq l$, $r = r_1 + r_2 + \cdots + r_l$ and $y_i \neq y_j$ for $1 \leq i \neq j \leq l$. Then we have

$$\#[\mathbf{u}] = \frac{r!}{r_1!r_2!\cdots r_l!} \quad \text{or} \quad 2 \cdot \frac{r!}{r_1!r_2!\cdots r_l!} \tag{3.5}$$

if $\mathbf{u} = -\sigma(\mathbf{u})$ for some $\sigma \in S_r$ or not, respectively. This can be easily proven by the fact that $[\mathbf{u}] = \{\sigma(\mathbf{u}) \mid \sigma \in S_r\} \cup \{-\sigma(\mathbf{u}) \mid \sigma \in S_r\}$ and

$$\#\{\sigma(\mathbf{u}) \mid \sigma \in S_r\} = \#\{-\sigma(\mathbf{u}) \mid \sigma \in S_r\} = \frac{r!}{r_1!r_2!\cdots r_l!}.$$

Note that the equivalence classes form a partition on $X(n)$ and $r_Q(n) = \#X(n)$. Therefore, it suffices to show that

$$\#[\mathbf{u}] \equiv 0 \pmod{r_Q(1)} \tag{3.6}$$

for all $\mathbf{u} \in X(n)$. When $b = -1$, then we get the congruence (3.4) from (1.1) since the forms $x_1^2 + x_2^2 + x_1x_2$ and $x_1^2 + x_2^2 - x_1x_2$ are equivalent over $\mathbb{Z}$. When $b \leq -3$, we have $r = 2$, $2a + b = 1$ and $r_Q(1) = 2$ by Theorem 3.1. Thus, we get

$$2n = 2Q(\mathbf{u}) = u_1^2 + u_2^2 - b(u_1 - u_2)^2.$$

Note that we cannot have $u_1 = -u_2$. Indeed, if $u_1 = -u_2$, then we get

$$2n = u_1^2 + u_2^2 - b(u_1 - u_2)^2 = 2(-2b+1)u_1^2,$$

which shows that $(-2b+1)|n$. This contradicts our assumption that $1 = \gcd(n, r(r-1)(r|b|+1)) = \gcd(n, 2(-2b+1))$. Therefore, one may easily verify that there are only two possibilities $[\mathbf{u}] = \{\pm(u_1, u_1)\}$ or $[\mathbf{u}] = \{\pm(u_1, u_2), \ \pm(u_2, u_1)\}$. Thus, we obtain that $\#[\mathbf{u}] \equiv 0 \pmod{r_Q(1)}$ for all $\mathbf{u} \in X(n)$ since $r_Q(1) = 2$ and $\#[\mathbf{u}] = 2$ or 4. Now, we suppose that $b \geq 1$. Then we have $2a - b = 1$ by Theorem 3.1. Thus we get

$$2n = 2Q(\mathbf{u}) = (2a-b)\sum_{i=1}^{r} u_i^2 + b\left(\sum_{i=1}^{r} u_i\right)^2 = \sum_{i=1}^{r} u_i^2 + b\left(\sum_{i=1}^{r} u_i\right)^2 = \sum_{i=1}^{l} r_i y_i^2 + b\left(\sum_{i=1}^{l} r_i y_i\right)^2. \tag{3.7}$$

Note that $l \geq 2$. Indeed, if $l = 1$, then $r = r_1$ and we obtain from (3.7) that

$$2n = ry_1^2 + br^2y_1^2 = r(rb+1)y_1^2 \equiv 0 \pmod{rb+1},$$

which leads a contradiction because $rb + 1 \geq 2 \cdot 1 + 1 = 3$ and $\gcd(n, rb+1) = 1$. To prove (3.6), we first claim that

$$p^{v_p(r(r-1))} \,\Big|\, \frac{r!}{r_1!r_2!\cdots r_l!} \tag{3.8}$$

for all odd prime p with $p|r(r-1)$. Here, we note that

$$\frac{r!}{r_1!r_2!\cdots r_l!} = \binom{r}{r_1}\cdot\frac{(r-r_1)!}{r_2!r_3!\cdots r_l!} = \frac{r}{r_1}\cdot\binom{r-1}{r_1-1}\cdot\frac{(r-r_1)!}{r_2!r_3!\cdots r_l!} \tag{3.9}$$

$$= \frac{r}{r_1}\cdot\binom{r-1}{r_1-1}\cdot\binom{r-r_1}{r_2}\cdot\frac{(r-r_1-r_2)!}{r_3!r_4!\cdots r_l!}. \tag{3.10}$$

Now, let $d = \gcd(r_1, r_2, \cdots, r_l)$. Then from (3.7) we get

$$2n = \sum_{i=1}^{l} r_i y_i^2 + b\left(\sum_{i=1}^{l} r_i y_i\right)^2 \equiv 0 \pmod{d}.$$

Since $\gcd(n, r) = 1$ and $d|r = r_1 + \cdots + r_l$, we have $d|2$. Let p be an odd prime divisor of $r(r-1)$. Since $d|2$, we have $p \nmid r_i$ for some $1 \le i \le l$. Without loss of generality, we may assume that $r_i = r_1$. When $p|r$, then we get $p \nmid \gcd(r, r_1)$ and, therefore, we have

$$p^{v_p(r)} \,\Big|\, \frac{r}{\gcd(r, r_1)} \quad \text{and} \quad p^{v_p(r)} \,\Big|\, \binom{r}{r_1}$$

by Lemma 3.4. From (3.9), we obtain that

$$p^{v_p(r)} \,\Big|\, \frac{r!}{r_1!r_2!\cdots r_l!}. \tag{3.11}$$

Suppose that $p|(r-1)$. When $p \nmid (r_1 - 1)$, then $p \nmid \gcd(r-1, r_1-1)$ and we get

$$p^{v_p(r-1)} \,\Big|\, \frac{r-1}{\gcd(r-1, r_1-1)} \quad \text{and} \quad p^{v_p(r-1)} \,\Big|\, \binom{r-1}{r_1-1}$$

by Lemma 3.4. Since $p \nmid r_1$, from (3.9), we obtain that

$$p^{v_p(r-1)} \,\Big|\, \frac{r!}{r_1!r_2!\cdots r_l!}.$$

When $p|(r_1-1)$, assume that $p|r_j$ for all $2 \le j \le l$. Then we get from (3.7) that

$$2n = \sum_{i=1}^{l} r_i y_i^2 + b\left(\sum_{i=1}^{l} r_i y_i\right)^2 \equiv (1+b)y_1^2 \equiv (rb+1)y_1^2 \pmod{p}$$

since $p|(r-1)$. If $p|(rb+1) = (r|b|+1)$, then we have $p|n$, which contradicts for $\gcd(n, r(r-1)(r|b|+1)) = 1$. Therefore, we get $p \nmid (rb+1)$ and $\left(\frac{2n(rb+1)}{p}\right) = 1$. This is a contradiction to our hypothesis that $\left(\frac{2n(rb+1)}{p}\right) = -1$. Thus, we have $p \nmid r_j$ for some $2 \le j \le l$ and, again, we may assume that $r_j = r_2$. Since $p \nmid r_2$, we get

$$\begin{aligned} &v_p\left(\frac{r-1}{\gcd(r-1, r_1-1)}\cdot\frac{r-r_1}{\gcd(r-r_1, r_2)}\right) \\ &= v_p(r-1) + \underbrace{v_p(r-r_1) - v_p(\gcd(r-1, r_1-1))}_{\ge 0} - v_p(\gcd(r-r_1, r_2)) \\ &\text{since } \gcd(r-1, r_1-1) = \gcd(r-1, r-r_1) \\ &\ge v_p(r-1) \quad \text{since } p \nmid r_2 \text{ and } v_p(\gcd(r-r_1, r_2)) = 0, \end{aligned}$$

which shows that

$$p^{v_p(r-1)} \,\Big|\, \frac{r-1}{\gcd(r-1, r_1-1)}\cdot\frac{r-r_1}{\gcd(r-r_1, r_2)}.$$

Since $p \nmid r_1$, we derive from Lemma 3.4 and (3.10) that

$$p^{v_p(r-1)} \left| \binom{r-1}{r_1-1}\binom{r-r_1}{r_2} \right. \text{ and } p^{v_p(r-1)} \left| \frac{r!}{r_1!r_2!\cdots r_l!}. \right. \tag{3.12}$$

From (3.11) and (3.12), we obtain that

$$p^{v_p(r(r-1))} \left| \frac{r!}{r_1!r_2!\cdots r_l!} \right.$$

since $\gcd(r, r-1) = 1$. Thus, we have (3.8). To verify (3.6), it remains to show that

$$2^{v_2(r(r-1))} \mid \#[\mathbf{u}]. \tag{3.13}$$

Suppose that r is even. When the r_i are all even, then we have $2|r_i$ and $4 \nmid r_i$ for some $1 \le i \le l$ because $d = \gcd(r_1, \cdots, r_l)|2$. We may assume that $r_i = r_1$. Then we get $2|\gcd(r, r_1)$, $4 \nmid \gcd(r, r_1)$ and

$$2^{v_2(r)-1} \left| \frac{r}{\gcd(r, r_1)}, \right.$$

which shows from Lemma 3.4 and (3.9) that

$$2^{v_2(r)-1} \left| \binom{r}{r_1} \right. \text{ and } 2^{v_2(r)-1} \left| \frac{r!}{r_1!r_2!\cdots r_l!}. \right.$$

In this case, we will show that $\mathbf{u} \neq -\sigma(\mathbf{u})$ for all $\sigma \in S_r$. Suppose that $\mathbf{u} = -\sigma(\mathbf{u})$ for some $\sigma \in S_r$. Then for each $1 \le i \le l$, there is a unique $1 \le j \le l$ such that $y_i = -y_j$ and $r_i = r_j$. Note that $i = j$ occurs if and only if l is odd, and in this case, we should have $y_i = 0$. We may assume that $y_l = y_i = 0$ when l is odd. Put $h = (l-1)/2$ or $h = l/2$ if l is odd or even, respectively. From Lemma 3.3, we may assume that $y_{2k-1} = -y_{2k}$ for $1 \le k \le h$. Then we get $y_{2k-1}^2 = y_{2k}^2$. Thus we derive from (3.7) and the above that $\sum_{i=1}^{l} r_i y_i = 0$ and

$$2n = \sum_{i=1}^{l} r_i y_i^2 + b\left(\sum_{i=1}^{l} r_i y_i\right)^2 = \sum_{k=1}^{h} 2r_{2k-1} y_{2k-1}^2.$$

Since the r_i are all even, the above equality shows that n is even, which is impossible because n is odd. Thus, we have $\mathbf{u} \neq -\sigma(\mathbf{u})$ for all $\sigma \in S_r$ and

$$2^{v_2(r)} = 2 \cdot 2^{v_2(r)-1} \left| 2 \cdot \frac{r!}{r_1!r_2!\cdots r_l!} = \#[\mathbf{u}] \right.$$

by (3.5). When one of the r_i is odd, we may assume that r_1 is odd. Then we have $2 \nmid \gcd(r, r_1)$ and

$$2^{v_2(r)} \left| \frac{r}{\gcd(r, r_1)}, \right.$$

which implies from Lemma 3.4 and (3.9) that

$$2^{v_2(r)} \left| \binom{r}{r_1} \right. \text{ and } 2^{v_2(r)} \left| \frac{r!}{r_1!r_2!\cdots r_l!}. \right.$$

Now, suppose that r is odd. From the hypothesis of our Theorem, we have $rb + 1 \equiv 0 \pmod 4$. Since $r = r_1 + \cdots + r_l$ is odd, at least one of the r_i is odd. In the case when only one of the r_i is odd and the rest are even, say r_l is odd, we claim that $2|r_i$ and $4 \nmid r_i$ for some $1 \le i \le l-1$. If $4|r_i$ for all $1 \le i \le l-1$, then $r \equiv r_l \pmod 4$ and, from (3.7), we get

$$2n = \sum_{i=1}^{l} r_i y_i^2 + b\left(\sum_{i=1}^{l} r_i y_i\right)^2 \equiv r_l(r_l b + 1)y_l^2 \equiv r(rb+1)y_l^2 \equiv 0 \pmod 4$$

since $br+1 \equiv 0 \pmod 4$. This is a contradiction since n is odd. Therefore, $2|r_i$ and $4 \nmid r_i$ for some $1 \le i \le l-1$ and we may assume that $r_i = r_1$. Then we have $2 \nmid (r_1 - 1)$ and $2 \nmid \gcd(r-1, r_1 - 1)$. Thus we get

$$2^{v_2(r-1)} \,\Big|\, \frac{r-1}{\gcd(r-1, r_1-1)}.$$

Since $2|r_1$ and $4 \nmid r_1$, Lemma 3.4 and (3.9) show that

$$2^{v_2(r-1)} \,\Big|\, \binom{r-1}{r_1-1} \quad \text{and} \quad 2^{v_2(r-1)-1} \,\Big|\, \frac{r!}{r_1!r_2!\cdots r_l!}.$$

As in the case when r is even, we shall prove that $\mathbf{u} \neq -\sigma(\mathbf{u})$ for all $\sigma \in S_r$. Suppose that $\mathbf{u} = -\sigma(\mathbf{u})$ for some $\sigma \in S_r$. Since r_l is odd and the rest of the r_i are even, we obtain that l is odd, $y_l = 0$ and, for each $1 \le i \le l-1$, there is a unique $1 \le j \le l-1$ with $i \neq j$ such that $y_i = -y_j$ and $r_i = r_j$. Put $h = (l-1)/2$. From Lemma 3.3, we may assume that $y_{2k-1} = -y_{2k}$ for $1 \le k \le h$. Thus we derive from (3.7) and the above that $\sum_{i=1}^{l} r_i y_i = 0$ and

$$2n = \sum_{i=1}^{l} r_i y_i^2 + b\left(\sum_{i=1}^{l} r_i y_i\right)^2 = \sum_{k=1}^{h} 2r_{2k-1} y_{2k-1}^2.$$

Since the $r_1, \cdots, r_h$ are even, n must be even, which is a contradiction for n is odd. Thus, we have $\mathbf{u} \neq -\sigma(\mathbf{u})$ for all $\sigma \in S_r$ and

$$2^{v_2(r-1)} = 2 \cdot 2^{v_2(r-1)-1} \,\Big|\, 2 \cdot \frac{r!}{r_1!r_2!\cdots r_l!} = \#[\mathbf{u}].$$

When at least two of the r_i are odd, we may assume that r_1 and r_2 are odd. Then we deduce that

$$\begin{aligned} &v_2\left(\frac{r-1}{\gcd(r-1, r_1-1)} \cdot \frac{r-r_1}{\gcd(r-r_1, r_2)}\right) \\ &\quad = v_2(r-1) + \underbrace{v_2(r-r_1) - v_2(\gcd(r-1, r_1-1))}_{\ge 0} - v_2(\gcd(r-r_1, r_2)) \\ &\quad \text{since } \gcd(r-1, r_1-1) = \gcd(r-1, r-r_1) \\ &\quad \ge v_2(r-1) \quad \text{since } 2 \nmid r_2 \text{ and } v_2(\gcd(r-r_1, r_2)) = 0. \end{aligned}$$

Thus we have

$$2^{v_2(r-1)} \,\Big|\, \frac{r-1}{\gcd(r-1, r_1-1)} \cdot \frac{r-r_1}{\gcd(r-r_1, r_2)}.$$

Since $2 \nmid r_1$, we derive from Lemma 3.4 and (3.10) that

$$2^{v_2(r-1)} \,\Big|\, \binom{r-1}{r_1-1}\binom{r-r_1}{r_2} \quad \text{and} \quad 2^{v_2(r-1)} \,\Big|\, \frac{r!}{r_1!r_2!\cdots r_l!}.$$

Combining the above results, we get (3.13). From (3.8) and (3.13), we obtain that

$$\#[\mathbf{u}] \equiv 0 \pmod{r(r-1)} \tag{3.14}$$

and thus, we have

$$r_Q(n) \equiv 0 \pmod{r(r-1)}. \tag{3.15}$$

When $b \ge 3$, we have $r_Q(1) = r(r-1)$ by Theorem 3.1. Thus, we get the desired one. When $b = 1$, we have $r_Q(1) = r(r+1)$ by Theorem 3.1 and $1 = \gcd(n, r(r-1)(rb+1)) = \gcd(n, r(r-1)(r+1))$. Put $u_{r+1} = u_1 + u_2 + \cdots + u_r$ and $\mathbf{v} = (u_1, \cdots, u_r, u_{r+1})$. Then $\mathbf{v}$ becomes a solution of $2n = x_1^2 + \cdots x_r^2 + x_{r+1}^2$ by (3.7). Again, we may write $\mathbf{v}$ as

$$\mathbf{v} = (\underbrace{v_1, v_1, \cdots, v_1}_{s_1}, \underbrace{v_2, v_2, \cdots, v_2}_{s_2}, \cdots, \underbrace{v_m, v_m, \cdots, v_m}_{s_m})$$

where $r+1 = s_1 + s_2 + \cdots + s_m$ with $v_i \neq v_j$ for $1 \leq i \neq j \leq m$. By using the same idea as above, one may deduce that

$$p^{v_p(r+1)} \left| \frac{(r+1)!}{s_1!s_2!\cdots s_m!} \right.$$

for all odd prime $p|(r+1)$ and

$$2^{v_2(r+1)-1} \left| \frac{(r+1)!}{s_1!s_2!\cdots s_m!} \right.$$

when r is odd. Thus, we get

$$r_Q(n) \equiv 0 \pmod{r+1} \quad \text{or} \quad r_Q(n) \equiv 0 \pmod{(r+1)/2} \tag{3.16}$$

if r is even or odd, respectively. Since $\gcd(r+1, r(r-1)) = \gcd(r+1, r-1) = 1$ or 2 if r is even or odd, respectively, from (3.15) and (3.16), we have

$$r_Q(n) \equiv 0 \pmod{(r+1)r(r-1)} \quad \text{or} \quad r_Q(n) \equiv 0 \pmod{(r+1)r(r-1)/2}$$

if r is even or odd, respectively. This proves our congruence (3.4) because $r_Q(1) = r(r+1)$.

The remaining case is when $b = 0$. In this case, we have $a = 1$ and $r_Q(1) = 2r$ from Theorem 3.1 and

$$n = Q(\mathbf{u}) = \sum_{i=1}^{l} r_i y_i^2.$$

Since $1 = \gcd(n, r(r-1)(rb+1)) = \gcd(n, r(r-1))$ and $r = r_1 + \cdots + r_l$, we have $d = \gcd(r_1, \cdots, r_l) = 1$. Let p be a prime divisor of r. Since $d = 1$, we have $p \nmid r_i$ for some $1 \leq i \leq l$ and we may assume that $r_i = r_1$. Then we have $p \nmid \gcd(r, r_1)$. By Lemma 3.4, we get

$$p^{v_p(r)} \left| \frac{r}{\gcd(r, r_1)} \right. \quad \text{and} \quad p^{v_p(r)} \left| \binom{r}{r_1} \right..$$

From (3.9), we obtain that

$$p^{v_p(r)} \left| \frac{r!}{r_1!r_2!\cdots r_l!} \right.. \tag{3.17}$$

Assume that $\sigma(\mathbf{u}) = -\mathbf{u}$ for some $\sigma \in S_r$. Let $h = l/2$ or $h = (l-1)/2$ if l is even or odd, respectively. By applying the same method as the above, we may assume that $y_{2k-1} = -y_{2k}$ and $r_{2k-1} = r_{2k}$ for $1 \leq k \leq h$ and $y_l = 0$ when l is odd. Then we get

$$n = \sum_{i=1}^{l} r_i y_i^2 = \sum_{k=1}^{h} 2r_{2k-1} y_{2k-1}^2,$$

which implies a contradiction since n is odd. Thus, we have $\sigma(\mathbf{u}) \neq -\mathbf{u}$ for all $\sigma \in S_r$. By (3.17) and (3.5), we have

$$\#[\mathbf{u}] = 2 \cdot \frac{r!}{r_1!r_2!\cdots r_l!} \equiv 0 \pmod{2r}.$$

Hence, we get the desired one. □

Remark 3.6 From the proof of Theorem 3.5, one may observe the followings.

(1) For any odd positive integers n with $\gcd(n, r) = 1$, we have

$$r_Q(n) \equiv 0 \pmod{r}.$$

(2) When $b = 1$, then the congruence

$$r_Q(n) \equiv 0 \pmod{r_Q(1)}$$

holds for any positive integers n with $\gcd(n, r(r+1)) = 1$.

4 p-multiplicative representation numbers

In this section, we let $r \geq 4$ be even, $k = r/2$ and $b \geq 1$. Then $\det(A) = rb + 1$ is the level of Q by Remark 3.2 and $\Theta_Q(\tau)$ belongs to $\mathcal{M}_k(rb+1, \chi_{(-1)^k(rb+1)})$. We begin this section with the following two lemmas.

Lemma 4.1 *The value of $r_Q(4)$ is given as follows.*

(i) When $r = 4, 6$, we have

$$r_Q(4) = \begin{cases} 60, \text{ if } (r,b) = (4,1), \\ 36, \text{ if } (r,b) = (4,3), \\ 36, \text{ if } (r,b) = (4,5), \\ 20, \text{ if } (r,b) = (4,7), \\ 12, \text{ if } r = 4 \text{ and } b \geq 9, \end{cases} \quad \text{and} \quad r_Q(4) = \begin{cases} 882, \text{ if } (r,b) = (6,1), \\ 450, \text{ if } (r,b) = (6,3), \\ 390, \text{ if } (r,b) = (6,5), \\ 282, \text{ if } (r,b) = (6,7), \\ 270, \text{ if } r = 6 \text{ and } b \geq 9, \end{cases}$$

(ii) When $r \geq 8$ and $b \geq 9$, we have

$$\begin{aligned} r_Q(4) &= r(r-1) + \frac{1}{3}r(r-1)(r-2)(r-3)(r-4) \\ &\quad + \frac{1}{(4!)^2}r(r-1)(r-2)(r-3)(r-4)(r-5)(r-6)(r-7). \end{aligned}$$

(iii) When $r \geq 8$ and $b = 7$, we have

$$\begin{aligned} r_Q(4) &= 2r + r(r-1) + \frac{1}{3}r(r-1)(r-2)(r-3)(r-4) \\ &\quad + \frac{1}{(4!)^2}r(r-1)(r-2)(r-3)(r-4)(r-5)(r-6)(r-7). \end{aligned}$$

(iv) When $r \geq 8$ and $b = 5$, we have

$$\begin{aligned} r_Q(4) &= r(r-1)(r-2) + r(r-1) + \frac{1}{3}r(r-1)(r-2)(r-3)(r-4) \\ &\quad + \frac{1}{(4!)^2}r(r-1)(r-2)(r-3)(r-4)(r-5)(r-6)(r-7). \end{aligned}$$

(v) When $r \geq 8$ and $b = 3$, we have

$$\begin{aligned} r_Q(4) &= 3r(r-1) + \frac{1}{2}r(r-1)(r-2)(r-3)(r-4) \\ &\quad + \frac{1}{(4!)^2}r(r-1)(r-2)(r-3)(r-4)(r-5)(r-6)(r-7). \end{aligned}$$

(vi) When $r \geq 8$ and $b = 1$, we have

$$\begin{aligned} r_Q(4) &= r(r-1) + \frac{1}{3}r(r-1)(r-2)(r-3)(r-4) \\ &\quad + \frac{1}{(4!)^2}r(r-1)(r-2)(r-3)(r-4)(r-5)(r-6)(r-7) \\ &\quad + 2r + \frac{5}{3}r(r-1)(r-2)(r-3) + \frac{1}{72}r(r-1)(r-2)(r-3)(r-4)(r-5)(r-6). \end{aligned}$$

Proof One may easily verify case (i). We only prove (ii) since the rest cases can be obtained in the same manner. Suppose that $r \geq 8$ and $b \geq 9$. Let $\mathbf{u} = (u_1, \cdots, u_r) \in \mathbb{Z}^r$ be a solution of $Q(\mathbf{x}) = 4$. Further, we let $\sim$ be the equivalence relation as in Lemma 3.3 and $[\mathbf{u}]$ the equivalence class of $\mathbf{u}$. Multiplying by 2 on both sides of $Q(\mathbf{u}) = 4$, we get

$$8 = \sum_{i=1}^{r} u_i^2 + b\left(\sum_{i=1}^{r} u_i\right)^2.$$

Since $b \geq 9$, we have

$$\sum_{i=1}^{r} u_i^2 = 8 \quad \text{and} \quad \sum_{i=1}^{r} u_i = 0.$$

Since $r \geq 8$ and $8 = 4+4 = 4+1+1+1+1 = 1+1+1+1+1+1+1+1$, we obtain that

$$X(4) = [(2, -2, 0, \cdots, 0)] \cup [(2, 1, -1, -1, -1, 0, \cdots, 0)] \cup [(1, 1, 1, 1, -1, -1, -1, -1, 0, \cdots, 0)].$$

Hence we have

$$\begin{aligned} r_Q(4) &= \#X(4) \\ &= \#[(2, -2, 0, \cdots, 0)] + \#[(2, 1, -1, -1, -1, 0, \cdots, 0)] + \#[(1, 1, 1, 1, -1, -1, -1, -1, 0, \cdots, 0)] \\ &= r(r-1) + \frac{1}{3} r(r-1)(r-2)(r-3)(r-4) + \frac{1}{(4!)^2} r(r-1)(r-2)(r-3)(r-4)(r-5)(r-6)(r-7). \end{aligned}$$

□

Lemma 4.2 *If $r \geq 8$, then we have*

$$\frac{r_Q(4)}{r_Q(1)} \neq 1 + \chi_{(-1)^k(rb+1)}(2) 2^{k-1} + 2^{2(k-1)}.$$

Proof We only provide proof of the case when $r \geq 8$ and $b \geq 9$ since the other cases can be proved in the same manner. Assume that

$$\frac{r_Q(4)}{r_Q(1)} = 1 + \chi_{(-1)^k(rb+1)}(2) 2^{k-1} + 2^{2(k-1)}. \tag{4.1}$$

Since $b \geq 9$, by Theorem 3.1 and Lemma 4.1, we have $r_Q(1) = r(r-1)$ and

$$\frac{r_Q(4)}{r_Q(1)} = 1 + \frac{1}{3}(r-2)(r-3)(r-4) + \frac{1}{(4!)^2}(r-2)(r-3)(r-4)(r-5)(r-6)(r-7). \tag{4.2}$$

Substituting $r = 2k$ and combining (4.1) and (4.2), we obtain an inequality

$$\frac{1}{3}(2k-2)(2k-3)(2k-4) + \frac{1}{(4!)^2}(2k-2)(2k-3)(2k-4)(2k-5)(2k-6)(2k-7) \geq 2^{2(k-1)} - 2^{k-1} \tag{4.3}$$

since $\chi_{(-1)^k(rb+1)}(2) = \pm 1$. Put

$$\begin{aligned} f(x) = 2^{2(x-1)} - 2^{x-1} &- \frac{1}{3}(2x-2)(2x-3)(2x-4) \\ &- \frac{1}{(4!)^2}(2x-2)(2x-3)(2x-4)(2x-5)(2x-6)(2x-7). \end{aligned}$$

Then one may verify that for $x \geq 2$, $f(x) > 0$. Since $k = r/2 \geq 8/2 = 4$, we have $f(k) > 0$, that is,

$$\frac{1}{3}(2k-2)(2k-3)(2k-4) + \frac{1}{(4!)^2}(2k-2)(2k-3)(2k-4)(2k-5)(2k-6)(2k-7) < 2^{2(k-1)} - 2^{k-1},$$

which contradicts the inequality (4.3). This proves our lemma. □

Now we can prove our main theorem.

Theorem 4.3 *Let $r(\geq 4)$ be even and $b \geq 1$. Suppose that $(-1)^k(rb+1)$ is a fundamental discriminant. Then r_Q is p-multiplicative for all prime $p \nmid (rb+1)$ if and only if*

$$(r, b) = (4, 1),\ (4, 3),\ (4, 5),\ (6, 1).$$

For each case, we can provide the close formula for $r_Q(n)$ as follows:

$$r_Q(n) = -5 \sum_{d>0, d|n} (\chi_5(d) - 5\chi_5(n/d))\, d \quad \textit{when } (r, b) = (4, 1),$$

$$r_Q(n) = -\sum_{d>0, d|n} (\chi_{13}(d) - 13\chi_{13}(n/d))\, d \quad \textit{when } (r, b) = (4, 3),$$

$$r_Q(n) = -\frac{1}{2} \sum_{d>0, d|n} (\chi_{21}(d) - 21\chi_{21}(n/d) + 3\chi_{-3}(n/d)\chi_{-7}(d) - 7\chi_{-7}(n/d)\chi_{-3}(d))\, d \quad \textit{when } (r, b) = (4, 5),$$

$$r_Q(n) = -\frac{7}{8} \sum_{d>0, d|n} (\chi_{-7}(d) - 49\chi_{-7}(n/d))\, d^2 \quad \textit{when } (r, b) = (6, 1),$$

for every positive integer n.

Proof Let $N = \det(A) = rb + 1$ which is the level of Q. Since $r \geq 4$ is even, $k \geq 2$ is an integer and N is odd. Since 2 is a prime with $2 \nmid N$, by Proposition 2.3, it suffices to check whether r_Q is 2-multiplicative or not. If r_Q is 2-multiplicative, then from Remark 2.4, we should have

$$\frac{r_Q(4)}{r_Q(1)} = 1 + \chi_{(-1)^k N}(2)2^{k-1} + 2^{2(k-1)}. \tag{4.4}$$

By Lemma 4.2, (4.4) does not hold if $r \geq 8$. Suppose that $r = 4$ or 6. Then by Theorem 3.1, we have

$$r_Q(1) = \begin{cases} 20, \text{ if } & r = 4 \text{ and } b = 1, \\ 12, \text{ if } & r = 4 \text{ and } b \geq 3, \\ 42, \text{ if } & r = 6 \text{ and } b = 1, \\ 30, \text{ if } & r = 6 \text{ and } b \geq 3. \end{cases}$$

When $r = 4$, then $k = 2$ and $(-1)^k N \equiv 4b + 1 \equiv 5 \pmod 8$ since b is odd. Thus we have

$$1 + \chi_{(-1)^k N}(2)2^{k-1} + 2^{2(k-1)} = 1 + \chi_{4b+1}(2)2 + 2^2 = 3.$$

When $r = 6$, then $k = 3$ and $(-1)^k N \equiv -(6b+1) \equiv 1 \text{ or } 5 \pmod 8$ if $b \equiv 1 \text{ or } 3 \pmod 4$, respectively. Thus we get

$$1 + \chi_{(-1)^k N}(2)2^{k-1} + 2^{2(k-1)} = 1 + \chi_{-(6b+1)}(2)2^2 + 2^4 = \begin{cases} 21, \text{ if } & b \equiv 1 \pmod 4, \\ 13, \text{ if } & b \equiv 3 \pmod 4. \end{cases}$$

From the above and Lemma 4.1 (i), one may easily check that (4.4) holds if and only if

$$(r, b) = (4, 1),\ (4, 3),\ (4, 5),\ (6, 1).$$

Note that $(-1)^k N = 5, 13, 21, -7$ if $(r, b) = (4, 1),\ (4, 3),\ (4, 5),\ (6, 1)$, respectively. Thus $(-1)^k N$ is a fundamental discriminant. From Proposition 2.2 and the dimension formula given by Cohen and Oesterlé ([6, §6.3]), we find that

$$\begin{aligned} \dim_{\mathbb{C}} \mathcal{M}_2(5, \chi_5) = \dim_{\mathbb{C}} \mathcal{E}_2(5, \chi_5) = 2, &\quad \text{when } (r, b) = (4, 1), \\ \dim_{\mathbb{C}} \mathcal{M}_2(13, \chi_{13}) = \dim_{\mathbb{C}} \mathcal{E}_2(13, \chi_{13}) = 2, &\quad \text{when } (r, b) = (4, 3), \\ \dim_{\mathbb{C}} \mathcal{M}_2(21, \chi_{21}) = \dim_{\mathbb{C}} \mathcal{E}_2(21, \chi_{21}) = 4, &\quad \text{when } (r, b) = (4, 5). \end{aligned}$$

Therefore, if $(r, b) = (4, 1),\ (4, 3),\ (4, 5)$, then the theta function $\Theta_Q(\tau)$ is a sum of Eisenstein series and by Proposition 2.3, r_Q is p-multiplicative for all prime $p \nmid N = (rb + 1)$. Since

$$-\frac{4}{B_{2,\chi_5}} = -5, \quad -\frac{4}{B_{2,\chi_{13}}} = -1 \quad \text{and} \quad \dim_{\mathbb{C}} \mathcal{M}_2(5, \chi_5) = \dim_{\mathbb{C}} \mathcal{M}_2(13, \chi_{13}) = 2,$$

from Proposition 2.5, we deduce that

$$r_Q(n) = -5 \sum_{d>0, d|n} (\chi_5(d) - 5\chi_5(n/d))\, d \quad \text{when } (r, b) = (4, 1),$$
$$r_Q(n) = -\sum_{d>0, d|n} (\chi_{13}(d) - 13\chi_{13}(n/d))\, d \quad \text{when } (r, b) = (4, 3)$$

for all positive integers n. Consider the case when $(r, b) = (4, 5)$. Since

$$\Theta_Q(\tau) \in \mathcal{M}_2(21, \chi_{21}) = \mathcal{E}_2(21, \chi_{21})$$

and

$$\left\{E_{2,\chi_0,\chi_{21}}(\tau),\ E_{2,\chi_{21},\chi_0}(\tau),\ E_{2,\chi_{-3},\chi_{-7}}(\tau),\ E_{2,\chi_{-7},\chi_{-3}}(\tau)\right\}$$

is a basis of $\mathcal{E}_2(21, \chi_{21})$ by Proposition 2.2, we have

$$\Theta_Q(\tau) = c_1 E_{2,\chi_0,\chi_{21}}(\tau) + c_2 E_{2,\chi_{21},\chi_0}(\tau) + c_3 E_{2,\chi_{-3},\chi_{-7}}(\tau) + c_4 E_{2,\chi_{-7},\chi_{-3}}(\tau) \tag{4.5}$$

for some $c_1, c_2, c_3, c_4 \in \mathbb{C}$. Since

$$\begin{aligned}
\Theta_Q(\tau) &= 1 + 12q + 6q^2 + 32q^3 + \cdots \\
E_{2,\chi_0,\chi_{21}}(\tau) &= -2 + q - q^2 + q^3 + \cdots \\
E_{2,\chi_{21},\chi_0}(\tau) &= \quad q + q^2 + 3q^3 + \cdots \\
E_{2,\chi_{-3},\chi_{-7}}(\tau) &= \quad q + q^2 - 3q^3 + \cdots \\
E_{2,\chi_{-7},\chi_{-3}}(\tau) &= \quad q - q^2 - q^3 + \cdots,
\end{aligned}$$

by comparing the first four coefficients of both sides of (4.5), we derive that

$$\Theta_Q(\tau) = -\frac{1}{2} E_{2,\chi_0,\chi_{21}}(\tau) + \frac{21}{2} E_{2,\chi_{21},\chi_0}(\tau) - \frac{3}{2} E_{2,\chi_{-3},\chi_{-7}}(\tau) + \frac{7}{2} E_{2,\chi_{-7},\chi_{-3}}(\tau).$$

Thus, by comparing the n-th coefficients of both sides, we get

$$r_Q(n) = -\frac{1}{2} \sum_{d>0, d|n} (\chi_{21}(d) - 21\chi_{21}(n/d) + 3\chi_{-3}(n/d)\chi_{-7}(d) - 7\chi_{-7}(n/d)\chi_{-3}(d))\, d.$$

The remaining case is when $(r, b) = (6, 1)$. Again, from Proposition 2.2 and the dimension formula given by Cohen and Oesterlé, one may find that

$$\dim_{\mathbb{C}} \mathcal{M}_3(-7, \chi_{-7}) = 3, \quad \dim_{\mathbb{C}} \mathcal{E}_3(-7, \chi_{-7}) = 2 \quad \text{and} \quad \dim_{\mathbb{C}} \mathcal{S}_3(-7, \chi_{-7}) = 1.$$

Since $\dim_{\mathbb{C}} \mathcal{S}_3(-7, \chi_{-7}) = 1$ and r_Q satisfies (4.4), r_Q is p-multiplicative for all prime $p \nmid N = 7$ and $\Theta_Q(\tau) \in \mathcal{M}_3(-7, \chi_{-7})$ can be expressed as a sum of Eisenstein series by Proposition 2.3. Since

$$-\frac{6}{B_{3,\chi_{-7}}} = -\frac{7}{8} \quad \text{and} \quad \dim_{\mathbb{C}} \mathcal{E}_3(-7, \chi_{-7}) = 2,$$

from Proposition 2.5, we obtain that

$$\Theta_Q(\tau) = -\frac{7}{8} E_{3,\chi_0,\chi_{-7}}(\tau) + \frac{7^3}{8} E_{8,\chi_{-7},\chi_0}(\tau).$$

Hence, we have a closed formula

$$r_Q(n) = -\frac{7}{8} \sum_{d>0, d|n} (\chi_{-7}(d) - 49\chi_{-7}(n/d))\, d^2$$

for all positive integers n. This proves our theorem. □

References

1. I. S. Eum, On the representation numbers satisfying partially multiplicative relations, J. Number Theory 224 (2021) 1–12.
2. I. S. Eum, J. K. Koo, D. H. Shin, On some applications of Eisenstein series, Publ. Math. Debrecen 85(1-2) (2014) 73–91.
3. S. Lang, Introduction to Modular Forms, Grundlehren der mathematischen Wissenschaften, No. 222, Springer-Verlag, Berlin-New York, 1976.
4. T. Miyake, Modular Forms, Springer-Verlag, Berlin, 1989.
5. H. L. Montgomery, R. C. Vaughan, Multiplicative number theory I. Classical Theory, Cambridge Studies in Advanced Mathematics, 97, Cambridge University Press, Cambridge, 2007.
6. W. Stein, Modular forms, a Computational approach, with an appendix by P.E. Gunnells, Grad. Studies in Math., vol. 79, Amer. Math. Soc., Providence, RI, 2007.
7. K. S. Williams, Number Theory in the Spirit of Liouville, Cambridge University Press, New York, 2011.