



Ramanujan-type congruences for partition k -tuples with 5-cores

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Abstract We prove several Ramanujan-type congruences modulo powers of 5 for partition k -tuples with 5-cores, for $k = 2, 3, 4$. We also prove some new infinite families of congruences modulo powers of primes for k -tuples with p -cores, where p is a prime.

Keywords Integer partitions · Ramanujan-type congruences

Mathematics Subject Classification 11P81 · 11P83

1 Introduction

A partition of a positive integer n is a finite non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ such that $\sum_{i=1}^k \lambda_i = n$. The number of partitions of n is denoted by $p(n)$. Euler found the generating function of $p(n)$, given by

$$\sum_{n \geq 0} p(n)q^n = \frac{1}{(q; q)_{\infty}},$$

where

$$(a; q)_{\infty} := \prod_{i \geq 0} (1 - aq^i), \quad |q| < 1.$$

For example, the five partitions of 4 are

$$4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1.$$

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$$p(5n + 4) \equiv 0 \pmod{5}, \quad p(7n + 5) \equiv 0 \pmod{7}, \quad \text{and} \quad p(11n + 6) \equiv 0 \pmod{11}.$$

The Ferrers-Young diagram of a partition is a pattern of dots with λ_i dots in the i -th row for the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of n . The hook number of a dot is the number of dots directly below and to the right of the dot, including the dot itself. A partition is called a t -core partition for $t \geq 2$ if the partition has no hook number divisible by t . We further define a partition k -tuple of n to be the k -tuple of partitions $(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$, where $\sum_{i=1}^k \Lambda_i = n$.

Let $A_{t,k}(n)$ be the number of partition k -tuples of n with t -cores and its generating function is given by

These partitions and the generating functions are the objects of study in this paper.

$$A_{5.2}(5^\alpha n + 5^\alpha - 2) \equiv 0 \pmod{5^\alpha}, \quad \alpha \geq 1. \quad (1.2)$$
$$A_{5,3}(5^\alpha n + 5^\alpha - 3) \equiv 0 \pmod{5^\alpha}, \quad \alpha \geq 1. \quad (1.3)$$

Theorem 11 *For all $n > 0$ and $\alpha > 1$, we have*

For instance, putting $\alpha = 4$ and 5, we have the following congruences:

and

$$A_{5\ 4}(5^6n + 15621) \equiv 0 \pmod{5^9}.$$

Theorem 12 *Let $p \geq 5$ be a prime and let $r \in \mathbb{N}$ with $1 \leq r \leq p - 1$, be such that $24r + 1$ is a quadratic nonresidue modulo p . Then, for all $n \geq 0$, $i \geq 1$ and $N \geq 1$, we have*

$$A_{p, p^N i-1}(pn+r) \equiv 0 \pmod{p^N}.$$



For example, if we take $p = 5$, then $24r + 1$ is quadratic nonresidue modulo 5 when $r = 3$ and 4. Hence, the above theorem implies that

$$A_{5,5^N i-1}(5n+r) \equiv 0 \pmod{5^N}, \quad \text{for } n \geq 0, i \geq 1, N \geq 1 \text{ and } r \in \{3, 4\}.$$

Theorem 13 Let $p \geq 5$ be a prime and let $r \in \mathbb{N}$ with $1 \leq r \leq p-1$, be such that $8r+1$ is a quadratic nonresidue modulo p . Then, for all $n \geq 0, i \geq 1$ and $N \geq 1$, we have

$$A_{p,p^N i-3}(pn+r) \equiv 0 \pmod{p^N}.$$

For instance, if we take $p = 5$, then $8r+1$ is quadratic nonresidue modulo 5 when $r = 2$ and 4. Hence, the above theorem implies that

$$A_{5,5^N i-3}(5n+r) \equiv 0 \pmod{5^N}, \quad \text{for } n \geq 0, i \geq 1, N \geq 1 \text{ and } r \in \{2, 4\}.$$

Theorem 14 Let $p \geq 5$ be a prime and let $r \in \mathbb{N}$ with $1 \leq r \leq p-1$, be the unique value such that $8r+1 \equiv 0 \pmod{p}$. Then, for all $n \geq 0$ and $i \geq 1$, we have

$$A_{p,pi-3}(pn+r) \equiv 0 \pmod{p}.$$

For example, if we take $p = 5$, then $8r+1 \equiv 0 \pmod{5}$ for $r = 3$. Hence, the above theorem implies that

$$A_{5,5i-3}(5n+3) \equiv 0 \pmod{5}, \quad \text{for all } n \geq 0 \text{ and } i \geq 1.$$

We also prove some new individual congruences for $A_{5,t}(n)$ for some specific values of t .

Theorem 15 For all $n \geq 0$, the following congruences are true:

$$A_{5,2}(25n+23) \equiv 0 \pmod{5^2}, \tag{1.5}$$

$$A_{5,2}(125n+123) \equiv 0 \pmod{5^3}, \tag{1.6}$$

$$A_{5,3}(25n+22) \equiv 0 \pmod{5}, \tag{1.7}$$

$$A_{5,3}(125n+122) \equiv 0 \pmod{5^2}, \tag{1.8}$$

$$A_{5,4}(25n+21) \equiv 0 \pmod{5^5}, \tag{1.9}$$

$$A_{5,4}(125n+121) \equiv 0 \pmod{5^6}. \tag{1.10}$$

Next, we present a congruence result for $A_{t,k}(n)$ modulo powers of primes, which can be also viewed as an existence result for infinite families of congruences.

Theorem 16 Let p be a prime, $k \geq 1, j \geq 0, N \geq 1, M \geq 1$, and r be integers such that $1 \leq r \leq p^M - 1$. If for all $n \geq 0$,

$$A_{p,k}(p^M n + r) \equiv 0 \pmod{p^N},$$

then for all $n \geq 0$ we have

$$A_{p,p^{M+N-1}i+k}(p^M n + r) \equiv 0 \pmod{p^N}.$$

The following is an easy corollary.



Corollary 17 For all $i \geq 0$ and $n \geq 0$, we have

$$A_{5,5^2i+2}(25n+23) \equiv 0 \pmod{5}, \quad (1.11)$$

$$A_{5,5^3i+2}(25n+23) \equiv 0 \pmod{5^2}, \quad (1.12)$$

$$A_{5,5^5i+2}(125n+123) \equiv 0 \pmod{5^3}, \quad (1.13)$$

$$A_{5,5^2i+3}(25n+22) \equiv 0 \pmod{5}, \quad (1.14)$$

$$A_{5,5^4i+3}(125n+122) \equiv 0 \pmod{5^2}, \quad (1.15)$$

$$A_{5,5i+4}(5n+3, 4) \equiv 0 \pmod{5}, \quad (1.16)$$

$$A_{5,5^3i+4}(25n+21) \equiv 0 \pmod{5^2}, \quad (1.17)$$

$$A_{5,5^2i+4}(25n+21) \equiv 0 \pmod{5}, \quad (1.18)$$

$$A_{5,5^3i+4}(25n+21) \equiv 0 \pmod{5^2}, \quad (1.19)$$

$$A_{5,5^4i+4}(25n+21) \equiv 0 \pmod{5^3}, \quad (1.20)$$

$$A_{5,5^5i+4}(25n+21) \equiv 0 \pmod{5^4}, \quad (1.21)$$

$$A_{5,5^6i+4}(25n+21) \equiv 0 \pmod{5^5}, \quad (1.22)$$

$$A_{5,5^8i+4}(125n+121) \equiv 0 \pmod{5^6}. \quad (1.23)$$

Proof The proofs of the above congruences follow from Theorem 15 and Theorem 16 and are similar in nature. Hence, here we only present the proof of (1.17).

The case for $i = 0$ is true by (1.9). Using Theorem 16, and the case for $i = 0$, we deduce that

$$\sum_{n \geq 0} A_{5,125i+4}(25n+21)q^n \equiv \sum_{n \geq 0} A_{5,4}(25n+21)q^n \equiv 0 \pmod{25},$$

which completes the proof. $\square$

The rest of the paper is organized as follows: in Section 2 we state some preliminary results that we require for our proofs, Theorem 11 is then proved in Section 3, Theorems 12, 13 and 14 are proved in Section 4, Theorem 15 is proved in Section 5, Theorem 16 is proved in Section 6 and finally we close the paper with some concluding remarks and conjectures in Section 7.

2 Preliminary Results

For the sake of brevity, we use the notation $f_k := (q^k; q^k)_\infty$ throughout the rest of the paper. We also let

$$R(q) = \frac{(q; q^5)_\infty (q^4; q^5)_\infty}{(q^2; q^5)_\infty (q^3; q^5)_\infty}.$$

First, we recall the following 5-dissections of $\frac{1}{f_1}$ and f_1 . We have [2, Eq. (7.4.14)]

$$\begin{aligned} \frac{1}{f_1} = \frac{f_{25}^5}{f_5^6} & \left(\frac{1}{R^4(q^5)} + \frac{q}{R^3(q^5)} + \frac{2q^2}{R^2(q^5)} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5 R(q^5) + 2q^6 R^2(q^5) \right. \\ & \left. - q^7 R^3(q^5) + q^8 R^4(q^5) \right), \end{aligned} \quad (2.1)$$

and [2, Theorem 7.4.4]

$$f_1 = f_{25} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right). \quad (2.2)$$



Lemma 21 Let $\sum_{n \geq 0} P_4(n)q^n = \frac{1}{f_1^4}$. Then we have

$$\sum_{n \geq 0} P_4(5n+1)q^n = 4 \frac{f_5^2}{f_1^6} + 550q \frac{f_5^8}{f_1^{12}} + 12500q^2 \frac{f_5^{14}}{f_1^{18}} + 78125q^3 \frac{f_5^{20}}{f_1^{24}}.$$

Proof Using equation (2.1), extracting the terms involving q^{5n+1} , dividing by q and then replacing q^5 by q , we arrive at

$$\sum_{n \geq 0} P_4(5n+1)q^n = \frac{f_5^{20}}{f_1^{24}} \left(\frac{4}{R^{15}(q)} + \frac{418q}{R^{10}(q)} + \frac{1840q^2}{R^5(q)} + 1015q^3 - 1840q^4 R^5(q) + 418q^5 R^{10}(q) - 4q^6 R^{15}(q) \right). \quad (2.3)$$

We use the following formula [2, Theorem 7.4.4]

$$\frac{1}{R(q)^5} - 11q - q^2 R(q)^5 = \frac{f_1^6}{f_5^6}, \quad (2.4)$$

to obtain from equation (2.3)

$$\sum_{n \geq 0} P_4(5n+1)q^n = 4 \frac{f_5^2}{f_1^6} + 550q \frac{f_5^8}{f_1^{12}} + 12500q^2 \frac{f_5^{14}}{f_1^{18}} + 78125q^3 \frac{f_5^{20}}{f_1^{24}}. \quad (2.5)$$

□

Lemma 22 If $\sum_{n \geq 0} Q_4(n)q^n = f_5^2 f_1^{14}$, then we have $\sum_{n \geq 0} Q_4(5n+4)q^n = -15625q^2 f_5^{14} f_1^2$.

Proof Using equation (2.2), we obtain

$$\begin{aligned} \sum_{n \geq 0} Q_4(n)q^n &= f_5^2 f_{25}^{14} (1/R^{14}(q) - 14q/R^{13}(q) + 77q^2/R^{12}(q) - 182q^3/R^{11}(q) + 910q^5/R^9(q) \\ &\quad - 1365q^6/R^8(q) - 1430q^7/R^7(q) + 5005q^8/R^6(q) - 10010q^{10}/R^4(q) \\ &\quad + 3640q^{11}/R^3(q) + 14105q^{12}/R^2(q) - 6930q^{13}/R(q) - 15625q^{14} \\ &\quad + 6930q^{15}R(q) + 14105q^{16}R^2(q) - 3640q^{17}R^3(q) - 10010q^{18}R^4(q) \\ &\quad + 5005q^{20}R^6(q) + 1430q^{21}R^7(q) - 1365q^{22}R^8(q) - 910q^{23}R^9(q) \\ &\quad + 182q^{25}R^{11}(q) + 77q^{26}R^{12}(q) + 14q^{27}R^{13}(q) + q^{28}R^{14}(q)). \end{aligned}$$

Extracting the terms involving q^{5n+4} and then dividing by q^4 and replacing q^5 by q , we get

$$\sum_{n \geq 0} Q_4(5n+4)q^n = -15625q^2 f_5^{14} f_1^2.$$

□

Lemma 23 If $\sum_{n \geq 0} Q_5(n)q^n = q f_5^8 f_1^8$, then we have $\sum_{n \geq 0} Q_5(5n+4)q^n = -125q f_5^8 f_1^8$.

Lemma 24 If $\sum_{n \geq 0} Q_6(n)q^n = q^2 f_5^{14} f_1^2$, then we have $\sum_{n \geq 0} Q_6(5n+4)q^n = -f_5^2 f_1^{14}$.

The proofs of Lemma 23 and Lemma 24 are exactly similar to the proof of Lemma 22. So, we leave them to the reader.

We also recall that

$$f_1 = \sum_{m=-\infty}^{\infty} (-1)^m q^{m(3m-1)/2}, \quad (2.6)$$



and [1, p.39, Entry 24(iii)]

$$f_1^3 = \sum_{m \geq 0} (-1)^m (2m+1) q^{m(m+1)/2}. \quad (2.7)$$

We know that for all primes p and integers $k \geq 1$, we have

$$f_1^{p^k} \equiv f_p^{p^{k-1}} \pmod{p^k}. \quad (2.8)$$

3 Proof of Theorem 11

We prove Theorem 11 using elementary q -series techniques, reminiscent of the proof of the result of Majid and Fathima [3]. But, before that we need the following result.

Theorem 31 *For all integers $\alpha \geq 0$, we have*

$$\sum_{n \geq 0} A_{5,4}(5^{\alpha+1}n + 5^{\alpha+1} - 4)q^n = A_\alpha f_5^2 f_1^{14} + B_\alpha q f_5^8 f_1^8 + C_\alpha q^2 f_5^{14} f_1^2 + D_\alpha q^3 \sum_{n \geq 0} A_{5,4}(n)q^n, \quad (3.1)$$

where $A_0 = 4$, $B_0 = 550$, $C_0 = 12500$, $D_0 = 78125$, and for any integer $n \geq 1$, A_n , B_n , C_n , and D_n are defined as

$$A_n = -C_{n-1} + 4D_{n-1}, \quad (3.2)$$

$$B_n = -125B_{n-1} + 550D_{n-1}, \quad (3.3)$$

$$C_n = -15625A_{n-1} + 12500D_{n-1}, \quad (3.4)$$

$$D_n = D_0^{n+1}. \quad (3.5)$$

Proof From equation (1.1), we have

$$\sum_{n \geq 0} A_{5,4}(n)q^n = \frac{f_5^{20}}{f_1^4}.$$

From equation (2.5), we have

$$\begin{aligned} \sum_{n \geq 0} A_{5,4}(5n+1)q^n &= 4f_5^2 f_1^{14} + 550q f_5^8 f_1^8 + 12500q^2 f_5^{14} f_1^2 + 78125q^3 \frac{f_5^{20}}{f_1^4} \\ &= 4f_5^2 f_1^{14} + 550q f_5^8 f_1^8 + 12500q^2 f_5^{14} f_1^2 + 78125q^3 \sum_{n \geq 0} A_{5,4}(n)q^n. \end{aligned} \quad (3.6)$$

Equation (3.6), is the case for $\alpha = 0$. Now assume that the result holds for all values up to $\alpha + 1$ ($\alpha \geq 0$). Replacing n by $5n + 4$, and by using Lemmas 22, 23, 24, and equation (3.6), we have

$$\begin{aligned} &\sum_{n \geq 0} A_{5,4}(5^{\alpha+2}n + 5^{\alpha+2} - 4)q^n \\ &= A_\alpha (-15625q^2 f_1^2 f_5^{14}) + B_\alpha (-125q f_1^8 f_5^8) + C_\alpha (-f_1^{14} f_5^2) + D_\alpha (4f_5^2 f_1^{14} + 550q f_5^8 f_1^8 \\ &\quad + 12500q^2 f_5^{14} f_1^2 + 78125q^3 \sum_{n \geq 0} A_{5,4}(n)q^n) \\ &= (-C_\alpha + 4D_\alpha) f_1^{14} f_5^2 + (-125B_\alpha + 550D_\alpha) q f_1^8 f_5^8 \\ &\quad + (-15625A_\alpha + 12500D_\alpha) q^2 f_1^2 f_5^{14} + 78125q^3 D_\alpha \sum_{n \geq 0} A_{5,4}(n)q^n \\ &= A_{\alpha+1} f_1^{14} f_5^2 + B_{\alpha+1} q f_1^8 f_5^8 + C_{\alpha+1} q^2 f_1^2 f_5^{14} + D_{\alpha+1} q^3 \sum_{n \geq 0} A_{5,4}(n)q^n. \end{aligned}$$

Hence, the result is true by induction. $\square$



We can finally prove Theorem 11 now.

Proof of Theorem 11 From equations (3.2), (3.3), (3.4) and (3.5), we see that

$$\begin{array}{llll} A_1 \equiv 0 \pmod{5^5}, & B_1 \equiv 0 \pmod{5^6}, & C_1 \equiv 0 \pmod{5^6}, & D_1 \equiv 0 \pmod{5^7}, \\ A_2 \equiv 0 \pmod{5^6}, & B_2 \equiv 0 \pmod{5^7}, & C_2 \equiv 0 \pmod{5^7}, & D_2 \equiv 0 \pmod{5^8}, \\ \vdots & \vdots & \vdots & \vdots \\ A_\alpha \equiv 0 \pmod{5^{\alpha+4}}, & B_\alpha \equiv 0 \pmod{5^{\alpha+5}}, & C_\alpha \equiv 0 \pmod{5^{\alpha+5}}, & D_\alpha \equiv 0 \pmod{5^{\alpha+6}}. \end{array}$$

Now, it is easy to see that equation (3.1) implies Theorem 11. $\square$

4 Proofs of Theorems 12, 13 and 14

Proof of Theorem 12 From the generating function of $A_{p,p^{N_i-1}}(n)$ and equation (2.8), we have

$$\sum_{n \geq 0} A_{p,p^{N_i-1}}(n)q^n = \frac{f_p^{p(p^{N_i-1})}}{f_1^{p^{N_i-1}}} \equiv \frac{f_p^{p(p^{N_i-1})}}{f_p^{p^{N-1}i}} f_1 \pmod{p^N}.$$

With the help of equation (2.6), we obtain

$$\sum_{n \geq 0} A_{p,p^{N_i-1}}(n)q^n \equiv \frac{f_p^{p(p^{N_i-1})}}{f_p^{p^{N-1}i}} \left(\sum_{m=-\infty}^{\infty} (-1)^m q^{m(3m-1)/2} \right) \pmod{p^N}.$$

For some m and n , we are interested in finding out whether $m(3m-1)/2 = pn + r$. This is equivalent to asking whether $24pn + 24r + 1 = (6m-1)^2$, which implies $24r + 1 \equiv (6m-1)^2 \pmod{p}$. However $24r + 1$ is a quadratic nonresidue modulo p . It follows that

$$A_{p,p^{N_i-1}}(pn + r) \equiv 0 \pmod{p^N}.$$

$\square$

Proof of Theorem 13 Like before, we have

$$\sum_{n \geq 0} A_{p,p^{N_i-3}}(n)q^n = \frac{f_p^{p(p^{N_i-3})}}{f_1^{p^{N_i-3}}} \equiv \frac{f_p^{p(p^{N_i-3})}}{f_p^{p^{N-1}i}} f_1^3 \pmod{p^N}.$$

With the help of equation (2.7), we obtain

$$\sum_{n \geq 0} A_{p,p^{N_i-3}}(n)q^n \equiv \frac{f_p^{p(p^{N_i-3})}}{f_p^{p^{N-1}i}} \left(\sum_{m \geq 0} (-1)^m (2m+1) q^{m(m+1)/2} \right) \pmod{p^N}. \quad (4.1)$$

For some m and n , we are interested in finding out whether $m(m+1)/2 = pn + r$. This is equivalent to asking whether $8pn + 8r + 1 = (2m+1)^2$, which implies $8r + 1 \equiv (2m+1)^2 \pmod{p}$. However $8r + 1$ is a quadratic nonresidue modulo p . It follows that

$$A_{p,p^{N_i-3}}(pn + r) \equiv 0 \pmod{p^N}.$$

$\square$

Proof of Theorem 14 Due to equations (2.7) and (4.1), we must determine whether $pn + r = m(m+1)/2$ for some integers m and n . Completing the square and considering the result modulo p gives $(2m+1)^2 \equiv 8r+1 \equiv 0 \pmod{p}$. Therefore, p divides $(2m+1)^2$, implying that p divides $2m+1$. Since the coefficient of $q^{m(m+1)/2}$ in the series representation in equation (2.7) is exactly $2m+1$, it follows that the coefficient we are interested in is congruent to 0 modulo p . $\square$



5 Proof of Theorem 15

The proofs of the congruences are similar in nature. So, we only present proofs of (1.9) and (1.10). For others, we just give the corresponding generating functions.

First, we prove (1.9). We have,

$$\sum_{n \geq 0} A_{5,4}(n)q^n = \frac{f_5^{20}}{f_1^4} = f_5^{20} \sum_{n \geq 0} P_4(n)q^n.$$

Then, extracting the terms involving q^{5n+1} and dividing by q and then replacing q^5 by q , we obtain

$$\sum_{n \geq 0} A_{5,4}(5n+1)q^n = f_1^{20} \sum_{n \geq 0} P_4(5n+1)q^n.$$

With the help of Lemma 21, we get

$$\begin{aligned} \sum_{n \geq 0} A_{5,4}(5n+1)q^n &= f_1^{20} \left(4 \frac{f_5^2}{f_1^6} + 550q \frac{f_5^8}{f_1^{12}} + 12500q^2 \frac{f_5^{14}}{f_1^{18}} + 78125q^3 \frac{f_5^{20}}{f_1^{24}} \right) \\ &= 4f_5^2 f_1^{14} + 550q f_5^8 f_1^8 + 12500q^2 f_5^{14} f_1^2 + 78125q^3 \frac{f_5^{20}}{f_1^4}. \end{aligned}$$

Using equations (2.1) and (2.2), extracting the terms involving q^{5n+4} , dividing by q and then replacing q^5 by q , we arrive at

$$\begin{aligned} \sum_{n \geq 0} A_{5,4}(25n+21)q^n &= 5^5 \left(-4f_1^{14} f_5^2 + 100 \frac{f_5^{20}}{f_1^4 R^{15}(q)} + q \left(10450 \frac{f_5^{20}}{f_1^4 R^{10}(q)} - 22f_1^8 f_5^8 \right) \right. \\ &\quad + q^2 \left(46000 \frac{f_5^{20}}{f_1^4 R^5(q)} - 62500 f_1^2 f_5^{14} \right) + 25375q^3 \frac{f_5^{20}}{f_1^4} \\ &\quad \left. - 46000q^4 \frac{f_5^{20} R^5(q)}{f_1^4} + 10450q^5 \frac{f_5^{20} R^{10}(q)}{f_1^4} - 100q^6 \frac{f_5^{20} R^{15}(q)}{f_1^4} \right), \end{aligned}$$

which on usage of (2.4) reduces to

$$\sum_{n \geq 0} A_{5,4}(25n+21)q^n = 5^5 \left(96f_1^{14} f_5^2 + 13728q f_1^8 f_5^8 + 312480q^2 f_1^2 f_5^{14} + 1953125q^3 \frac{f_5^{20}}{f_1^4} \right),$$

which implies (1.9).

Proceeding in a similar way, we can also deduce

$$\begin{aligned} \sum_{n \geq 0} A_{5,4}(125n+121)q^n &= 5^6 \left(1500004 f_1^{14} f_5^2 + 214500550q f_1^8 f_5^8 + 4882512500q^2 f_1^2 f_5^{14} \right. \\ &\quad \left. + 30517578125q^3 \frac{f_5^{20}}{f_1^4} \right) \end{aligned}$$

which proves (1.10).



Now, we give the generating functions which will complete the proofs of the other congruences stated in the result:

$$\begin{aligned}\sum_{n \geq 0} A_{5,2}(25n+23)q^n &= 5^2 \left(48f_1^4 f_5^4 + 625q \frac{f_5^{10}}{f_1^2} \right), \\ \sum_{n \geq 0} A_{5,2}(125n+123)q^n &= 5^3 \left(1202f_1^4 f_5^4 + 15625q \frac{f_5^{10}}{f_1^2} \right), \\ \sum_{n \geq 0} A_{5,3}(25n+22)q^n &= 5 \left(5838f_1^9 f_5^3 + 233250q f_1^3 f_5^9 + 1953125q^2 \frac{f_5^{15}}{f_1^3} \right), \\ \sum_{n \geq 0} A_{5,3}(125n+122)q^n &= 5^2 \left(3643791f_1^9 f_5^3 + 145754625q f_1^3 f_5^9 + 1220703125q^2 \frac{f_5^{15}}{f_1^3} \right).\end{aligned}$$

This completes the proof of Theorem 15.

6 Proof of Theorem 16

Without loss of generality, we may assume that $r = \sum_{j=0}^{M-1} p^j r_j$ for $0 \leq r_j \leq p-1$, as $\sum_{j=0}^{M-1} p^j r_j$ can take any value between 1 and $p^M - 1$. For integers $M \geq 1$ (sufficiently large) and $N \geq 1$, we have

$$\begin{aligned}\sum_{n \geq 0} A_{p,p^{M+N-1}i+k}(n)q^n &= \frac{f_p^{p(p^{M+N-1}i+k)}}{f_1^{p^{M+N-1}i+k}} = \frac{f_p^{p^{M+N}i}}{f_1^{p^{M+N-1}i}} \sum_{n=0}^{M-1} A_{p,k}(n)q^n \\ &\equiv f_p^{p^{M+N-2}(p^2-1)i} \sum_{n=0}^{M-1} A_{p,k}(n)q^n \pmod{p^N}.\end{aligned}$$

Extracting the terms that involve q^{pn+r_0} from the above identity, we obtain

$$\begin{aligned}\sum_{n \geq 0} A_{p,p^{M+N-1}i+k}(pn+r_0)q^n &\equiv f_1^{p^{M+N-2}(p^2-1)i} \sum_{n=0}^{M-1} A_{p,k}(pn+r_0)q^n \\ &\equiv f_p^{p^{M+N-3}(p^2-1)i} \sum_{n=0}^{M-1} A_{p,k}(pn+r_0)q^n \pmod{p^N}.\end{aligned}$$

Now, extracting the terms that involve q^{pn+r_1} from the above identity, we obtain

$$\begin{aligned}\sum_{n \geq 0} A_{p,p^{M+N-1}i+k}(p^2n+r_0+pr_1)q^n &\equiv f_1^{p^{M+N-3}(p^2-1)i} \sum_{n=0}^{M-1} A_{p,k}(p^2n+r_0+pr_1)q^n \\ &\equiv f_p^{p^{M+N-4}(p^2-1)i} \sum_{n=0}^{M-1} A_{p,k}(p^2n+r_0+pr_1)q^n \pmod{p^N}.\end{aligned}$$

From the above identity, we extract the terms that contain q^{pn+r_2} , and from the resulting identity, we again extract the terms that contain q^{pn+r_3} and so on. It can be seen that after the M -th extraction using this iterative scheme, we arrive at

$$\begin{aligned}\sum_{n \geq 0} A_{p,p^{M+N-1}i+k}(p^Mn+r_0+pr_1+\cdots+p^{M-1}r_{M-1})q^n \\ \equiv f_1^{p^{N-1}(p^2-1)i} \sum_{n \geq 0} A_{p,k}(p^Mn+r_0+pr_1+\cdots+p^{M-1}r_{M-1})q^n \pmod{p^N}.\end{aligned} \quad (6.1)$$



Therefore, if we assume that $A_{p,k}(p^M n + r_0 + pr_1 + \cdots + p^{M-1}r_{M-1}) = A_{p,k}(p^M n + r) \equiv 0 \pmod{p^N}$, then from the above identity, we have

$$A_{p,p^{M+N-1}i+k}(p^M n + r) \equiv 0 \pmod{p^N}.$$

This completes the proof of Theorem 16.

We have the following easy corollary, which follows from equation (6.1) when $M = 1$.

Corollary 61 *Let p be a prime, $k \geq 1$, $j \geq 0$, $N \geq 1$, and r be integers such that $1 \leq r \leq p - 1$. Then for all $n \geq 0$, we have*

$$\sum_{n \geq 0} A_{p,p^N i+k}(pn + r)q^n \equiv f_1^{p^{N-1}(p^2-1)i} \sum_{n \geq 0} A_{p,k}(pn + r)q^n \pmod{p^N}.$$

7 Concluding Remarks

- (1) We have found several congruences modulo powers of 5, individual as well as infinite families similar to those stated in Theorem 16, for $A_{5,k}(n)$ for higher values of k . The proofs of these are routine exercises similar to the proofs of (1.9) and (1.10) hence, they are not proved here. For instance, the following congruences are true:

$$A_{5,6}(25n + 14, 19, 24) \equiv 0 \pmod{5^2}, \quad (7.1)$$

$$A_{5,6}(125n + 119) \equiv 0 \pmod{5^3}, \quad (7.2)$$

$$A_{5,7}(25n + 13, 18, 23) \equiv 0 \pmod{5^2}, \quad (7.3)$$

$$A_{5,7}(125n + 118) \equiv 0 \pmod{5^3}, \quad (7.4)$$

$$A_{5,5^2 i+6}(25n + 14, 19, 24) \equiv 0 \pmod{5}, \quad (7.5)$$

$$A_{5,5^3 i+6}(25n + 14, 19, 24) \equiv 0 \pmod{5^2}, \quad (7.6)$$

$$A_{5,5^5 i+6}(125n + 119) \equiv 0 \pmod{5^3}, \quad (7.7)$$

$$A_{5,5^2 i+7}(25n + 13, 18, 23) \equiv 0 \pmod{5}, \quad (7.8)$$

$$A_{5,5^3 i+7}(25n + 13, 18, 23) \equiv 0 \pmod{5^2}, \quad (7.9)$$

$$A_{5,5^5 i+7}(125n + 118) \equiv 0 \pmod{5^3}. \quad (7.10)$$

- (2) Looking at the sequence of results in equations (1.2), (1.3) and (1.9), it suggests the possibility of there being other such structures modulo higher powers of 5, and more generally for $A_{t,k}(n)$ modulo higher powers of t . Experiments also suggest some sort of cyclic behaviour, which we do not conjecture here, but it would be interesting to know what can be said about $A_{5,5i+3}(5^\alpha n + 5^\alpha - 3) \pmod{5^\alpha}$ and $A_{5,5i+2}(5^\alpha n + 5^\alpha - 2) \pmod{5^\alpha}$?
- (3) Experiments suggest some additional infinite family of congruences modulo powers of 5 which are stronger results than those indicated by Theorem 16. We present them here as conjectures:

$$A_{5,5i+1}(25n + 24) \equiv 0 \pmod{5^2}, \quad (7.11)$$

$$A_{5,5^2 i+2}(25n + 23) \equiv 0 \pmod{5^2}, \quad (7.12)$$

$$A_{5,5^3 i+2}(125n + 123) \equiv 0 \pmod{5^3}, \quad (7.13)$$

$$A_{5,5^3 i+3}(125n + 122) \equiv 0 \pmod{5^2}, \quad (7.14)$$

$$A_{5,5^4 i+4}(125n + 121) \equiv 0 \pmod{5^5}, \quad (7.15)$$

$$A_{5,5^5 i+4}(125n + 121) \equiv 0 \pmod{5^6}, \quad (7.16)$$

$$A_{5,5^2 i+6}(25n + 14, 19) \equiv 0 \pmod{5^2}, \quad (7.17)$$

$$A_{5,5^2 i+6}(125n + 119) \equiv 0 \pmod{5^3}, \quad (7.18)$$

$$A_{5,5^2 i+7}(25n + 13, 18, 23) \equiv 0 \pmod{5^2}, \quad (7.19)$$

$$A_{5,5^3 i+7}(125n + 118) \equiv 0 \pmod{5^3}. \quad (7.20)$$



- (4) Many other congruences modulo higher powers of 5 may exist for higher subsequences. It may be interesting to study those congruences.

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