Indian J Pure Appl Math (2026) 57:355–363
https://doi.org/10.1007/s13226-024-00538-y

ORIGINAL RESEARCH

15 Order types in 36 packages

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Received: 10 February 2023 / Accepted: 8 January 2024 / Published online: 2 February 2024

Abstract We study the behaviour of trajectories of real homeomorphisms h. There are exactly 15 order types among them. For each h let T_h be the collection of all order types of h-trajectories. This T_h will be called a package (of order types). We prove that $| T_h |\leq 11$ always. As h varies over the set of all homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$, we ask how many sets can arise as T_h? We prove that the answer is exactly 36. An example is given at the end to show how these packages are helpful for conjugacy classification.

Keywords Forcing relation · Order type · Homeomorphism

1 Introduction

The subject of dynamics is all about the study of behaviour of orbits. For any real homeomorphism f we denote the identity map on $\mathbb{R}$ by f^0 and the map $f \circ f^{n-1}$ by f^n, $\forall n \in \mathbb{R}$. The sequence $\{f^n(x)\}$ is known as f-***trajectory*** of x. Two trajectories, say f-trajectories of x and y, are said to be of ***same order type***(of same orbit type or of same type simply) if there exists an order isomorphism $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\phi(f^n(x)) = f^n(y), \ \forall n \in \mathbb{N}$ [9].

Two sequences are said to be order isomorphic if there is an order isomorphism (may not be extendable to whole $\mathbb{R}$) between their ranges, i.e.,

Definition 1 [4] Two sequences (a_n) and (b_n) are said to be order isomorphic if

$$a_m < a_n \iff b_m < b_n \ \forall m, n \in \mathbb{N}$$

From the following example, we can see that the terms 'same order type' and 'order isomorphic' are not the same, but former is the stronger than the latter, as mentioned in [9].

Example 1 The sequences $\{\frac{(-1)^n}{n}\}$ and $\{(1 + \frac{1}{n})(-1)^n\}$ are order isomorphic but they are of different order types since first is convergent, whereas second has two limit points.

The set of all order types of a map is used implicitly in [2] & [3] to understand the drawbacks of a population model. Set of all order types of the map $f(x) = -x^3$ is discussed in [5].

Communicated by S.G. Dani.

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Table 1 Notations and order types

S.No.	notation	order-type	Example
1	k	Constant sequence	0,0,0,…
2	ic	Increasing convergent	$\{\frac{-1}{n}\}$
3	dc	Decreasing convergent	$\{\frac{1}{n}\}$
4	iu	Increasing unbounded	$\{n\}$
5	du	Decreasing unbounded	$\{-n\}$
6	t	Two-cycle	$\{(-1)^n\}$
7			$\{(-1)^{n+1}\}$
8	cf	Convergent flipping	$\{\frac{(-1)^n}{n}\}$
9			$\{\frac{(-1)^{n+1}}{n}\}$
10	fi	Flipping inward	$\{(1+\frac{1}{n})(-1)^n\}$
11			$\{(1+\frac{1}{n})(-1)^{n+1}\}$
12	fo	Flipping outward	$\{(1-\frac{1}{n})(-1)^n\}$
13			$\{(1-\frac{1}{n})(-1)^{n+1}\}$
14	uf	Unbounded flipping	$\{n(-1)^n\}$
15			$\{n(-1)^{n+1}\}$

In [6] the set of all order types of the map $f(x) = x^2$ is used to understand the dynamics of that map. The dynamics of a map can be well understood by studying the set of all order type exhibited by it. Table 1 lists 15 order types available for real homeomorphisms. Let $\mathbb{T}$ be the set of these order types. While giving notations for them, we do not distinguish two types if each is a tail of the other. In this way, ten notations are given.

Let us denote the collection of all these ten order types as $\mathbb{S}$. That is,

$$\mathbb{S} = \{k, ic, dc, iu, du, t, fi, fo, cf, uf\}.$$

Let $\mathbb{H}$ be the collection of all homeomorphisms on $\mathbb{R}$.

Definition 2 For $h \in \mathbb{H}$, define

$$T_h = \{\tau \in \mathbb{T} \mid h \text{ has a trajectory of type } \tau\}.$$

And

$$S_h = \{\tau \in \mathbb{S} \mid h \text{ has a trajectory of type } \tau\}.$$

For example

Example 2 Let $I : \mathbb{R} \longrightarrow \mathbb{R}$ be such that

$$I(x) = x$$

Then $S_I = T_I = \{k\}$

Example 3 Let $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ be such that

$$\rho(x) = x + 1$$

and $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\sigma(x) = x - 1$$

Then $S_\rho = T_\rho = \{iu\}$ and $S_\sigma = T_\sigma = \{du\}$

Example 4 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = sinx + x$. All the f-trajectories are showed in Table 2 below.
$S_f = T_f = \{k, ic, dc\}$

Table 2 Order types for $f(x) = sinx + x$

position of x	Trajectory	Trajectory type
$x = n\pi, n \in \mathbb{Z}$	constant sequence $\{n\pi\}$	k
$x \in (2n\pi, (2n+1)\pi),\ n \in \mathbb{Z}$	increasing and converging to $(2n+1)\pi$	ic
$x \in ((2n-1)\pi, 2n\pi),\ n \in \mathbb{Z}$	decreasing and converging to $(2n+1)\pi$	dc

Table 3 Order types for $g(x) = sinx - x$

position of x	Trajectory	Trajectory type
$x = 0$	constant sequence $\{0\}$	k
$x = n\pi,\ n \in \mathbb{Z},\ n \neq 0$	Two cycle $\{n\pi, -n\pi\}$	t
$x \in (-\pi, \pi) \setminus \{0\}$	flipping converging to 0	cf
$x \in (2n\pi, (2n+1)\pi), n \in \mathbb{Z}$	flipping inward asymptotic to two cycle $\{2n\pi, 2n\pi\}$	fi
$x \in ((2n-1)\pi, 2n\pi), n \in \mathbb{Z}, n \neq 0$	flipping outward asymptotic to two cycle $\{2n\pi, -2n\pi\}$	fo

It gives, $S_f = T_f = k, ic, dc$

Example 5 Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$g(x) = sinx - x$$

then the g-trajectories are shown in table 3 below.

So, $S_g = \{k, t, cf, fi, fo\}$ But T_g contains four elements more than S_g.

In Section 3, we will use Example 4 and 5 to construct homeomorphism for some packages.

Since $\mathbb{S}$ is a finite set a natural question can be raised.

Question 1 As h varies over $\mathbb{H}$ how many distinct S_h are possible?

We will prove that the answer to this question is 36.

Note that there are 1024 subsets of $\mathbb{S}$ but only 36 of them are qualifying to be S_h, for some $h \in \mathbb{H}$. The number 36 in this context is somewhat special, because it is stable in the following two senses;

1. If we count as per the known definition of order type, there are 15 order types available for self-homeomorphisms of $\mathbb{R}$ (in place of 10 in our counting). But this does not change the number of packages available. The same answer 36 remains.
2. If we enlarge the class of maps under study to, say, $\mathcal{F} = \{f : \mathbb{R} \longrightarrow \mathbb{R} \mid f \text{ is injective continuous}\}$, then it is (not obvious but) true that not only the answer 36 but also the sets S_h themselves remain the same.

   To be precise, we have:

$$\max_{h\in\mathbb{H}} \mid S_h \mid = 6$$

$$\max_{h\in\mathbb{H}} \mid T_h \mid = 11$$

$$\mid \bigcup_{h\in\mathbb{H}} S_h \mid = 10$$

$$\mid \bigcup_{h\in\mathbb{H}} T_h \mid = 15$$

and

$$\mid \{S_h \mid h \in \mathbb{H}\} \mid = \mid \{T_h \mid h \in \mathbb{H}\} \mid = 36$$

In our proof we use these three kinds of results,

(a) Some order types force some others
(b) Some order types cannot coexist with some others
(c) some pairs of order types, force some other order types (even though individually they do not force it.)

These lead us to further research on these for general maps.

2 The number of packages is not more than 36

In this section first we will prove that the set $\mathbb{S}$, mentioned above, is the set of all possible order types of real homeomorphisms, then will give some consequences of intermediate value theorem to study the forcing relation between these order types.

Theorem 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and increasing. Then each of its trajectories is of one of the following types:*

1. *constant sequence (k)*
2. *Increasing bounded (ic)*
3. *Decreasing bounded (dc)*
4. *Increasing unbounded (iu)*
5. *Decreasing unbounded (du)*

Proof Since $x, f(x) \in \mathbb{R}$, exactly one of the followings is true:

- $x = f(x)$
- $x < f(x)$
- $x > f(x)$

If $x = f(x)$ clearly the f-trajectory of x is a constant sequence.

Let $x < f(x)$. This implies that $f(x) < f^2(x)$ since f is increasing. By induction $f^n(x) < f^{n+1}(x),\ \forall n \in \mathbb{N}$. This proves that the f-trajectory of x is increasing.

A similar argument shows that if $x > f(x)$ then the f-trajectory of x is decreasing. □

Remark 1 If no fixed point of f exists to the right of x and $x < f(x)$, then the f-trajectory of x is increasing unbounded.

Similarly, if no fixed point of f is to the left of x and $x > f(x)$, then the f-trajectory of x is unbounded decreasing.

Theorem 2 *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and decreasing. Then each of its trajectories is of one of the following types:*

1. *constant sequence (k)*
2. *a two cycle (t)*
3. *flipping inward asymptotic to a two cycle(fi)*
4. *flipping outward asymptotic to a two cycle (fo)*
5. *convergent flipping (cf)*
6. *unbounded flipping (uf)*

Proof Since $x, f^2(x) \in \mathbb{R}$ exactly one of the followings is true:

- $x = f^2(x)$
- $x < f^2(x)$
- $x > f^2(x)$

If $x = f^2(x)$, clearly the f-trajectory of x is either a constant sequence or a 2-cycle.

Let $x < f^2(x)$. This gives $f(x) > f^3(x)$ since f is decreasing. The function f^2 is increasing; hence, by mathematical induction $f^{2n}(x) < f^{2n+2}(x)$ and $f^{2n-1}(x) > f^{2n+1}(x)$. This proves that the f^2-trajectory of x is increasing and the f^2-trajectory of $f(x)$ is decreasing.

Being a decreasing continuous function, f must posses a fixed point, say p. Then $f(p) = p$. Without loss of generality, let $x < p \Rightarrow f(x) > f(p) = p \Rightarrow f^2(x) < p$. Again by induction, $f^{2n}(x) < p$, $f^{2n-1}(x) > p$. This proves that the f-trajectory of x is a flipping. Depending on the position of x, it will be either inward, outward, convergent, or unbounded.

If it is bounded but not convergent, then the subsequences of even terms and odd terms, being monotonic, will be convergent. As $\{f^n(x)\}$ is not convergent, those sequences will converge to two different points, and the trajectory will be asymptotic to a 2-cycle. □

The fact that every bijective continuous function on $\mathbb{R}$ is monotonic implies that each trajectory of a homeomorphism on $\mathbb{R}$ can have one of the above mentioned ten types only, i.e., $\mathbb{S}$ is the set of **all** possible order types of real homeomorphisms.

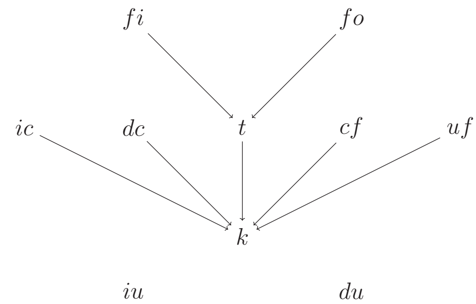

Fig. 1 A Hasse diagram for forcing relation

Lemma 1 *Let $h \in \mathbb{H}$. Then no element of $P = \{ic, dc, iu, du\}$ can coexist with any element of $Q = \{t, fi, fo, cf, uf\}$ in S_h.*

Proof From the above Theorems 1 and 2, it is clear that if f is increasing, then $S_h \cap Q = \phi$ and if it is decreasing, then $S_h \cap P = \phi$. This completes the proof. □

Corollary 1 *For $h \in \mathbb{H}$, $0 <| S_h |< 7$.*

The next definition is a formal continuation of the above study of coexistence, which plays an important role in the counting of packages (S_h).

Definition 3 An order type τ_1 is said to be forcing an order type τ_2 if every real continuous function having a trajectory of type τ_1 must have a trajectory of type τ_2.

Lemma 2 *The following Hasse diagram shows forcing relation between those order types. Where in the diagram $\tau_1 \longrightarrow \tau_2$ means order type τ_1 forces order type τ_2.*

Proof We will give proof for bounded flipping. Same proof can be given for fi forcing t and fo forcing t. Let $\{f^n(x)\}$ be a bounded flipping. This implies that the subsequences $\{f^{2n}(x)\}$ and $\{f^{2n-1}(x)\}$ are monotone bounded sequences. Therefore, they are convergent. Since $\{f^n(x)\}$ is not convergent, they will converge to two different limits, say r and s, respectively. That is

$$\begin{aligned}
&\{f^{2n}(x)\} \longrightarrow r \quad and \quad \{f^{2n-1}(x)\} \longrightarrow s \\
&\quad \Rightarrow f(f^{2n}(x)) \longrightarrow f(r) \quad and \quad \{f^{2n-1}(x)\} \longrightarrow f(s) \\
&\quad \Rightarrow f(r) = s \quad and \quad f(s) = r
\end{aligned}$$

which proves our claim.

A convergent trajectory of a continuous function converges to a fixed point of that function. This proves that ic forces k, dc forces k, and cf forces k.

Let us prove now that t forces k. Let $f(s) = r$ and $f(r) = s$. Without loss of generality assume that $r < s$. This means $f(s) < s$ and $f(r) > r$. This implies that there is a fixed point of f between r and s. This proves the required forcing.

To prove uf forces k Suppose that the f trajectory of x is a flipping around a. Without loss of generality, let $x < a$. Then $f^2(x) < x < a < f(x)$. Say $f(x) = r$. This implies that $f(r) < r$ and $f(x) > x$. This implies that there is a fixed point of f between x and r as required.

To prove that no other forcing is there, refer to the Examples 1 to 5 and 19 to 23 in the list of S_h given at the end of Section 3. □

Corollary 2 *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous. Then between any two consecutive terms of a flipping of f, there is a fixed point of f. That is, that flipping is around a fixed point.*

Lemma 3 *For $h \in \mathbb{H}$*

1. $\{iu, du\} \subset S_h \Rightarrow k \in S_h$
2. $\{cf, uf\} \subset S_h \Rightarrow t \in S_h$

Proof (1) If f-trajectory of x is increasing unbounded and that of y is decreasing unbound, then $x < f(x)$ and $y > f(y)$. This implies that f has a fixed point between x and y, completing the proof.

(2) Since a f-trajectory is flipping, f is decreasing function; hence it has a fixed point, say p. Let x be a (first or second) term of convergent flipping and y be a (first or second) term of unbounded flipping such that $x, \ y < p$.

$\Rightarrow f^2$-trajectory of x is increasing and that of y is decreasing.

$\Rightarrow f^2(x) > x$ and $f^2(y) < y$.

$\Rightarrow f^2$ has a fixed point between x and y.

Its f-trajectory is a 2-cycle. □

With this theory, we are ready to provide the answer to Question 1 as below.

In the light of Lemma 1, it is clear that for any $h \in \mathbb{H}$, $S_h \subset P = \{k, ic, dc, iu, du\}$ or $S_h \subset Q = \{k, t, cf, fo, fi, uf\}$ depending on whether h is increasing or decreasing, respectively. So, let us count the number of S_h for increasing h first.

Since ic forces k and dc forces k, the number of subsets containing ic of P can be S_h for some h is 2^3, i.e, If $\mathcal{P}$ denotes power set, then

$$| \{S_h \mid ic \in S_h\} |=| \mathcal{P}(\{dc, iu, du\}) |= 2^3$$

And the number of subsets containing dc of P that can be S_h for some h is 2^3, i.e.,

$$| \{S_h \mid dc \in S_h\} |=| \mathcal{P}(\{ic, iu, du\}) |= 2^3.$$

The number of subsets containing both ic and dc of P that can be S_h for some h is 2^2, i.e.,

$$| \{S_h \mid dc \in S_h \& ic \in S_h\} |=| \mathcal{P}(\{iu, du\}) |= 2^2$$

The number of subsets containing neither ic nor dc of P that can be S_h for some h is 2^3, i.e.,

$$| \{S_h \mid dc \notin S_h \ \& \ ic \notin S_h\} |=| \mathcal{P}(\{k, iu, du\}) |= 2^3.$$

Out of which ϕ and $\{iu, du\}$ (Lemma 3 part 1) should be rejected. So, the total number of subsets of P that can be S_h for some h is

$$2^3 + 2^3 - 2^2 + 2^3 - 2 = 18$$

Now let us do the same for decreasing h.

By Lemma 2, fi forces t and t forces k. So the number of subsets containing fi of Q that can be S_h for some h is 2^3, i.e.,

$$| \{S_h \mid fi \in S_h\} |=| \mathcal{P}(\{fo, cf, uf\}) |= 2^3.$$

Again by Lemma 2, fo forces t and t forces k. So, the number of subsets containing fo of Q that can be S_h for some h is 2^3, i.e.,

$$| \{S_h \mid fo \in S_h\} |=| \mathcal{P}(\{fi, cf, uf\}) |= 2^3$$

And the number of subsets containing both fi and fo of Q that can be S_h for some h is 2^2, i.e.,

$$| \{S_h \mid fi \in S_h \ \& \ fo \in S_h\} |=| \mathcal{P}(\{cf, uf\}) |= 2^2$$

The number of subsets containing neither fi nor fo of Q that can be S_h for some h is 2^3, i.e.,

$$| \{S_h \mid fi \notin S_h \ \& \ fo \notin S_h\} |=| \mathcal{P}(\{t, cf, uf\}) |= 2^3$$

Out of which $\{k\}$ (since every decreasing homeomorphism has exactly one fixed point) and $\{cf, uf\}$ (Lemma 3 part 2) should be rejected. So, total number of subsets of P that can be S_h for some h is

$$2^3 + 2^3 - 2^2 + 2^3 - 2 = 18$$

3 The number of packages is not less than 36

The above argument shows that the number of distinct S_h is not more than 36. We will prove that it is exactly 36 by giving a list of them and a way to construct h for each S_h. For doing so, the two examples below are helpful.

Example 6 Define $f : \mathbb{R} \longrightarrow \mathbb{R}$, such that

$$f(x) = \begin{cases} x + sinx & \text{if } \mid x \mid \leq 2\pi; \\ 2x - 2\pi & \text{if } x > 2\pi; \\ 2x + 2\pi & \text{if } x < -2\pi. \end{cases}$$

Then $S_f = P = \{k, ic, dc, iu, du\}$

Example 7 Define $g : \mathbb{R} \longrightarrow \mathbb{R}$, such that

$$g(x) = \begin{cases} \sin x - x & \text{if } \mid x \mid \leq 4\pi; \\ -2x + 4\pi & \text{if } x > 4\pi; \\ -2x - 4\pi & \text{if } x < -4\pi. \end{cases}$$

Then $S_g = Q = \{k, t, cf, fi, fo, uf\}$

Here is a list of all 36 S_h with ideas for constructing a homeomorphism for each of them.
Refer to Example 1 and Example 2 for

1. {k}
2. {iu}
3. {du}

To construct a homeomorphism h for $S_h \subset P$ such that $k \in S_h$ let $\tau_1, \tau_2 ... \tau_n \in P \setminus S_h$. Then define $A = \{x \in \mathbb{R} \mid f\text{-trajectory of } x \text{ is of one of the types } \tau_1, \tau_2, ...\tau_n\}$ and $h : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$h(x) = \begin{cases} x & \text{if } x \in A; \\ f(x) & \text{if } x \notin A, \end{cases}$$

where f is as defined in Example 6. Then $S_h = P \setminus \{\tau_1, \tau_2, \ldots, \tau_n\}$
With this idea, we can construct homeomorpnisms for each of the following sets:

4. {k, dc}
5. {k, ic}
6. {k, ic, dc}
7. {k, iu}
8. {k, ic, iu}
9. {k, dc, iu}
10. {k, ic, dc, iu}
11. {k, du}
12. {k, ic, du}
13. {k, dc, du}
14. {k, ic, dc, du}
15. {k, iu, du}
16. {k, ic, iu, du}
17. {k, dc, iu, du}

and the Example 6 as it is gives for

18. {k, ic, dc, iu, du}

Now for decreasing h we have 18 number of S_h namely

19. {k, cf} with $h : \mathbb{R} \longrightarrow \mathbb{R}$ such that $h(x) = -\frac{x}{2}$
20. {k, uf} with $h : \mathbb{R} \longrightarrow \mathbb{R}$ such that $h(x) = -2x$

To construct a homeomorphism h for $S_h \subset Q$ such that $t \in S_h$ and so, $k \in S_h$. Let $\tau_1, \tau_2 ... \tau_n \in Q \setminus S_h$. Then define $A = \{x \in \mathbb{R} \mid f\text{-trajectory of } x \text{ is of one of the types } \tau_1,\ \tau_2,\ ...\tau_n\}$ and $h : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$h(x) = \begin{cases} -x & \text{if } x \in A; \\ g(x) & \text{if } x \notin A, \end{cases}$$

where g is as defined in Example 7. Then $S_h = Q \setminus \{\tau_1,\ \tau_2, \ldots,\ \tau_n\}$

With this idea we can construct homeomorpnisms for each of following sets

21. {k, t}
22. {k, t, fi}
23. {k, t, fo}
24. {k, t, cf, fo}
25. {k, t, cf}
26. {k, t, cf, fi}
27. {k, t, fo, fi}
28. {k, t, uf}
29. {k, t, cf, uf}
30. {k, t, fo, uf}
31. {k, t, fi, uf}
32. {k, t, cf, fi, uf}
33. {k, t, fo, fi, uf}
34. {k, t, cf, fo, fi}
35. {k, t, cf, fo, uf}
36. {k, t, cf, fo, fi, uf}

For each one of these 36, as remarked already, we can construct a real homeomorphism h such that S_h is same as the given set.

4 Further results

In this section we will consider the packages of general continuous maps also (not only homeomorphisms). No proofs are included here. Some results presented here are already proved elsewhere.

4.1 Result on counting conjugucy classes

Our first result is an application of this notion using the following terminology. Two real continuous maps, f and g, are said to be ***topologically conjugate*** if there exists a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $f \circ h = h \circ g$ [6]. If this h is increasing, then f and g are said to be ***order conjugate*** [1].

By calculating the package for each, we have proved elsewhere that there are exactly 18 conjugucy classes of antisymmetric polynomials of degree less than three [7].

4.2 Results on cardinality of packages

It is natural to study the cardinality of the package. In [9], the possible order types for the map f whose S_f is finite are characterized. It is proved there that such an order type is eventually one of the order types in $\mathbb{S}$. In [8], we studied the cyclic patterns whose presence in the package assures that the package is uncountable.

4.3 Resut on complete conjugucy classification

A fixed point p of f is said to be ***attracting*** if $\exists\, \delta > 0$ such that $\forall x \in (p - \delta, p + \delta)$ the f-trajectory of x, $\{f^n(x)\}$ converges to p [3] [6]. If p is an attracting fixed point for f^{-1} then it is known as the ***repelling*** fixed point of f. A fixed point is said to be ***neutral*** if it is neither attracting nor repelling. A 2-cycle $\{s, t\}$ is said to be attracting, repelling, or neutral if s and t are of that type of fixed points of f^2.

Remark 2 If f and g are order conjugate, then $S_f = S_g$.

The following theorem gives an application of packages in conjugucy classification.

Theorem 3 *Let $\mathcal{H} \subset \mathbb{H}$ be such that no member of $\mathcal{H}$ has a neutral fixed point or neutral 2-cycle. Then $f,\ g \in \mathcal{H}$ are topologically conjugate if and only if*

1. *f and g have the same number of periodic points.*
2. *$S_f = S_g$.*

4.4 Connection with topological entropy

Also it is expected that the topological entropy increases with the package, i.e., for interval maps $f,\ g$ if $S_f \subset S_g$ then the topological entropy $\eta(f) \leq \eta(g)$. Though it seems to be a correct statement, it remains to provide a proof.

4.5 An open question

We conclude with a question whose answer will supplement this result. Consider the collection $\mathcal{C}$ of continuous maps from $\mathbb{R}$ to $\mathbb{R}$ that are homeomorphisms on their range.

Question 2 As h varies over $\mathcal{C}$, how many distinct S_h are possible?

Acknowledgements The second author acknowledges the Council of Scientific & Industrial Research, India, for financial support (Ref. No. 09/1045(0042)/2020-EMR-I).

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