Indian J Pure Appl Math (2026) 57:364–374
https://doi.org/10.1007/s13226-024-00552-0

ORIGINAL RESEARCH

Closed formulas for 2-domination and 2-outer-independent domination numbers of rooted product graphs

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Received: 20 February 2023 / Accepted: 19 January 2024 / Published online: 17 February 2024

Abstract A set of vertices $W \subseteq V(G)$ of a graph G with no isolated vertex is said to be a 2-dominating set of G if every vertex in $V(G) \setminus W$ is adjacent to at least two vertices in W. In addition, if $V(G) \setminus W$ is an independent set, then W is said to be a 2-outer-independent dominating set of G. The 2-domination number (2-outer-independent domination number) of G is the minimum cardinality among all 2-dominating sets (2-outer-independent dominating sets) of G. In this paper we study the above two parameters for the case of the rooted product graphs. In particular, we obtain closed formulas (three possible expressions) for each of these domination parameters, and we also characterise the graphs that satisfy each of these formulas. As a consequence of the study, we obtain the corresponding formulas for the corona product graphs.

Keywords 2-domination · 2-outer-independent domination · Rooted product graphs · Corona product graphs

Mathematics Subject Classification 05C69 · 05C76

1 Introduction

Domination in graphs is one of the most researched topics in the area of graph theory. A set $W \subseteq V(G)$ is a *dominating set* of a graph G if every vertex in $V(G) \setminus W$ is adjacent to at least one vertex in W. Let $\mathcal{D}(G)$ be the set of the dominating sets of G. The *domination number* of G is defined as $\gamma(G) = \min\{|W| : W \in \mathcal{D}(G)\}$. A $\gamma(G)$-*set* is a set $W \in \mathcal{D}(G)$ such that $|W| = \gamma(G)$. A similar agreement will be assumed when referring to optimal sets associated to other parameters used in this article. For introductory reading related to this parameter (as well as its main variants), we suggest the excellent books [8] and [9].

Fink and Jacobson [4,5] generalized the concept of dominating sets in graphs. A set of vertices $W \subseteq V(G)$ of a graph G is said to be a k-dominating set (kDS) of G if every vertex in $V(G) \setminus W$ is adjacent to at least k vertices in W. In addition, if $V(G) \setminus W$ is an independent set, then W is said to be a k-outer-independent dominating set (kOIDS) of G. Let $\mathcal{D}_k(G)$ and $\mathcal{D}_k^{oi}(G)$ be the sets of the kDSs and kOIDSs of G, respectively. The *k-domination*

Communicated by Shariefuddin Pirzada.

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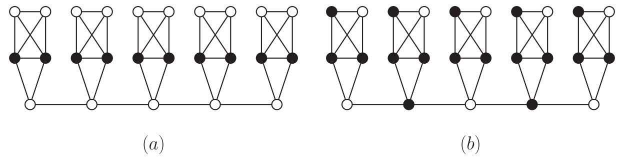

Fig. 1 The set of black-coloured vertices forms a $\gamma_2(P_5 \circ_v H)$-set (a) and a $\gamma_2^{oi}(P_5 \circ_v H)$-set (b), respectively

number (resp., *k-outer-independent domination number*) of G is defined as $\gamma_k(G) = \min\{|W| : W \in \mathcal{D}_k(G)\}$ (resp., $\gamma_k^{oi}(G) = \min\{|W| : W \in \mathcal{D}_k^{oi}(G)\}$).

In [7], Hansberg and Volkmann summarise the main results obtained up to 2020 concerning k-domination, with special emphasis on the case $k = 2$. Moreover, the k-outer-independent domination number of a graph has been studied for the cases in which $k \leq 2$. The parameter $\gamma_2^{oi}(G)$ was introduced in [12] and studied further in [13–15,17], while for $k = 1$, the 1OIDS is the well-known vertex cover set, and hence the *vertex cover number* is denoted as $\beta(G) = \gamma_1^{oi}(G)$.

Godsil and Mckay [6] introduced the concept of rooted product graphs. Let G be a graph of order n and let H be a graph with root vertex $v \in V(H)$. The *rooted product graph* $G \circ_v H$ is the graph obtained by taking one copy of G and n copies of H, and identifying the i^{th}-vertex of G with the root v in the i^{th}-copy of H. For every vertex $u \in V(G)$, the graph $H_u \cong H$ will denote the copy of H in $G \circ_v H$ containing the vertex u. Throughout the article, we will assume that G and H are graphs with no isolated vertex. In recent years, several domination parameters have been studied in rooted product graphs. For instance, we cite the following works [1–3,10,11,16].

In the Figure 1, we show two copies of the rooted product graph $P_5 \circ_v H$, where H is the graph obtained from the complete graph K_5 and eliminating exactly two edges incident to a root vertex $v \in V(K_5)$. For this graph, we have that $\gamma_2(P_5 \circ_v H) = 10$ and $\gamma_2^{oi}(P_5 \circ_v H) = 17$. The *corona product graph* $G \odot H'$ is a particular case of a rooted product graph $G \circ_v H$, where $H = K_1 + H'$ (by considering v as the vertex of K_1 and $K_1 + H'$ as the *join graph* of K_1 and H'), i.e., $G \odot H' \cong G \circ_v (K_1 + H')$.

In this paper we study the 2-domination number and the 2-outer-independent domination number in rooted product graphs. A basic problem in the study of a domination parameter in any product of graphs is obtaining closed formulas or sharp bounds and expressing these in terms of parameters of the graphs involved in the product. In such a sense, this work settles the previous problem. In particular, we obtain closed formulas for the two aforementioned parameters in the rooted product graphs and characterise the graphs that satisfy each of the formulas obtained. As a consequence of the study, we deduce the corresponding formulas for the corona product graphs.

2 The 2-domination number of rooted product graphs

To give the main results of this section, we need to establish some preliminary lemmas. As usual, given a graph H and a vertex $v \in V(H)$, $N(v)$ and $H - v$ denote the open neighbourhood of v in H and the subgraph of H induced by $V(H) \setminus \{v\}$, respectively.

Lemma 1 *Let H be a graph. The following statements hold for any vertex $v \in V(H)$.*

(i) $\gamma_2(H - v) \geq \gamma_2(H) - 1$.
(ii) *If $N(v) = V(H) \setminus \{v\}$, then $\gamma_2(H - v) = \gamma_2(H) - 1$ if and only if $H \cong K_2$.*

Proof Let D' be a $\gamma_2(H - v)$-set. Observe that $D = D' \cup \{v\} \in \mathcal{D}_2(H)$. Hence, $\gamma_2(H) - 1 \leq |D| - 1 = |D' \cup \{v\}| - 1 = |D'| = \gamma_2(H - v)$, which completes the proof of (i).

From now on, let us assume that $N(v) = V(H) \setminus \{v\}$. Suppose that $\gamma_2(H - v) = \gamma_2(H) - 1$. If $|N(v) \cap D'| = |D'| \geq 2$, then $D' \in \mathcal{D}_2(H)$, which implies that $\gamma_2(H) \leq |D'| = \gamma_2(H - v) = \gamma_2(H) - 1$, a contradiction. Therefore, $1 \leq |D'| = |N(v) \cap D'| \leq 1$, and as a consequence, $\gamma_2(H - v) = |D'| = 1$. Hence $H - v \cong K_1$, which implies that $H \cong K_2$, as desired. The other implication is straightforward, which completes the proof. □

Lemma 2 *Let G and H be two graphs. If G has order n and $v \in V(H)$, then the following statements hold.*

(i) $\gamma_2(G \circ_v H) \leq n\gamma_2(H)$.
(ii) *If $\gamma_2(H - v) = \gamma_2(H) - 1$, then the following statements hold.*

(a) $\gamma_2(G \circ_v H) \leq \gamma_2(G) + n(\gamma_2(H) - 1)$.
(b) *If* $N(v) \cap D \neq \emptyset$ *for some* $\gamma_2(H - v)$*-set* D*, then* $\gamma_2(G \circ_v H) \leq \gamma(G) + n(\gamma_2(H) - 1)$.

Proof Let $S \subseteq V(G \circ_v H)$ such that $S \cap V(H_u)$ is a $\gamma_2(H_u)$-set for every $u \in V(G)$. It is straightforward that $S \in \mathcal{D}_2(G \circ_v H)$. Hence,

$$\gamma_2(G \circ_v H) \leq |S| = \sum_{u \in V(G)} |S \cap V(H_u)| = \sum_{u \in V(G)} \gamma_2(H_u) = n\gamma_2(H),$$

which completes the proof of (i).

From now on we assume that $\gamma_2(H-v) = \gamma_2(H)-1$. Let $S' \subseteq V(G \circ_v H) \setminus V(G)$ such that $S' \cap (V(H_u) \setminus \{u\})$ is a $\gamma_2(H_u - u)$-set for every $u \in V(G)$. From any $\gamma_2(G)$-set X', it follows that $X' \cup S' \in \mathcal{D}_2(G \circ_v H)$. Hence,

$$\begin{aligned}\gamma_2(G \circ_v H) \leq |X' \cup S'| &= |X'| + \sum_{u \in V(G)} |S' \cap (V(H_u) \setminus \{u\})| \\ &= |X'| + \sum_{u \in V(G)} \gamma_2(H_u - u) \\ &= \gamma_2(G) + n(\gamma_2(H) - 1),\end{aligned}$$

which completes the proof of (ii)-(a).

Finally, we also suppose that there exists a $\gamma_2(H - v)$-set D such that $N(v) \cap D \neq \emptyset$. Now, let us consider a set $W \subseteq V(G \circ_v H) \setminus V(G)$, where $W \cap (V(H_u) \setminus \{u\})$ is the subset of $V(H_u) \setminus \{u\}$ induced by D for every $u \in V(G)$. From any $\gamma(G)$-set X, it follows that $X \cup W \in \mathcal{D}_2(G \circ_v H)$. Hence,

$$\begin{aligned}\gamma_2(G \circ_v H) \leq |X \cup W| &= |X| + \sum_{u \in V(G)} |W \cap (V(H_u) \setminus \{u\})| \\ &= |X| + n|D| \\ &= \gamma(G) + n(\gamma_2(H) - 1),\end{aligned}$$

which completes the proof. □

Lemma 3 *The following statements hold for any* $\gamma_2(G \circ_v H)$*-set* W *and any vertex* $u \in V(G)$.

(i) $|W \cap V(H_u)| \geq \gamma_2(H) - 1$.
(ii) *If* $|W \cap V(H_u)| = \gamma_2(H) - 1$*, then* $u \notin W$ *and* $|N(u) \cap W \cap V(H_u)| \leq 1$*, and as a consequence,* $\gamma_2(H - v) = \gamma_2(H) - 1$.

Proof Let W be any $\gamma_2(G \circ_v H)$-set and we fix any vertex $u \in V(G)$. Notice that $(W \cap V(H_u)) \cup \{u\} \in \mathcal{D}_2(H_u)$. Hence,

$$\gamma_2(H) - 1 = \gamma_2(H_u) - 1 \leq |(W \cap V(H_u)) \cup \{u\}| - 1 \leq |W \cap V(H_u)|,$$

which completes the proof of (i).

From now on, let us suppose that $|W \cap V(H_u)| = \gamma_2(H) - 1$. If $u \in W$ or $|N(u) \cap W \cap V(H_u)| \geq 2$, then we have that $W \cap V(H_u) \in \mathcal{D}_2(H_u)$, which implies that $\gamma_2(H) = \gamma_2(H_u) \leq |W \cap V(H_u)| = \gamma_2(H) - 1$, a contradiction. Therefore, $u \notin W$ and $|N(u) \cap W \cap V(H_u)| \leq 1$, as desired. Since $u \notin W$, it follows that $W \cap V(H_u) \in \mathcal{D}_2(H_u - u)$. Hence, $\gamma_2(H - v) = \gamma_2(H_u - u) \leq |W \cap V(H_u)| = \gamma_2(H) - 1$. Finally, from Lemma 1-(i) we deduce that $\gamma_2(H - v) = \gamma_2(H) - 1$, which completes the proof. □

The following theorem provides closed formulas for the 2-domination number of $G \circ_v H$.

Theorem 4 *Let* G *be a graph of order* n *and let* H *be any graph. For any vertex* $v \in V(H)$,

$$\gamma_2(G \circ_v H) \in \{\gamma(G) + n(\gamma_2(H) - 1),\ \gamma_2(G) + n(\gamma_2(H) - 1),\ n\gamma_2(H)\}.$$

Proof Let W be any $\gamma_2(G \circ_v H)$-set. By Lemma 3-(i) we have that $|W \cap V(H_u)| \geq \gamma_2(H) - 1$ for any vertex $u \in V(G)$. Considering the above fact, we next define the subsets V^- and $V^{\geq}$ of $V(G)$, which satisfy that $V^- \cup V^{\geq} = V(G)$.

$$\begin{aligned} V^- =& \{u \in V(G) : |W \cap V(H_u)| = \gamma_2(H) - 1\}, \\ V^{\geq} =& \{u \in V(G) : |W \cap V(H_u)| \geq \gamma_2(H)\}. \end{aligned}$$

Next, we analyse the following two cases.
Case 1: $V^- = \emptyset$. In this case we have that $V(G) = V^{\geq}$, which implies that

$$\gamma_2(G \circ_v H) = |W| = \sum_{u \in V(G)} |W \cap V(H_u)| = \sum_{u \in V^{\geq}} |W \cap V(H_u)| \geq n\gamma_2(H). \tag{1}$$

Therefore, Lemma 2-(i) leads to $\gamma_2(G \circ_v H) = n\gamma_2(H)$.
Case 2: $V^- \neq \emptyset$. In this case we have that

$$\begin{aligned} \gamma_2(G \circ_v H) = |W| &= \sum_{u \in V^{\geq}} |W \cap V(H_u)| + \sum_{u \in V^-} |W \cap V(H_u)| \\ &\geq \sum_{u \in V^{\geq}} \gamma_2(H) + \sum_{u \in V^-} (\gamma_2(H) - 1) \\ &= |V^{\geq}| + \sum_{u \in V(G)} (\gamma_2(H) - 1) \\ &= |V^{\geq}| + n(\gamma_2(H) - 1). \end{aligned} \tag{2}$$

By Lemma 3-(ii) it follows that $u \in V(G) \setminus W$ and that $|N(u) \cap W \cap V(H_u)| \leq 1$ for every vertex $u \in V^-$. Since $|N(u) \cap W| \geq 2$ for every $u \in V(G) \setminus W$, it is easy to see that $W \cap V(G) \subseteq V^{\geq}$ is a dominating set of G. Hence, $\gamma(G) \leq |W \cap V(G)| \leq |V^{\geq}|$, which leads to $\gamma_2(G \circ_v H) \geq \gamma(G) + n(\gamma_2(H) - 1)$ by inequality chain (2). Moreover, observe that Lemma 3-(ii) leads to $\gamma_2(H - v) = \gamma_2(H) - 1$, and by Lemma 2-(ii)-(a) it follows that $\gamma_2(G \circ_v H) \leq \gamma_2(G) + n(\gamma_2(H) - 1)$.

In summary, we have that

$$\gamma(G) + n(\gamma_2(H) - 1) \leq \gamma_2(G \circ_v H) \leq \gamma_2(G) + n(\gamma_2(H) - 1). \tag{3}$$

We only need to prove that $\gamma_2(G \circ_v H)$ only can take the previous extreme values. For this purpose, we next analyse the following two subcases.

Subcase 2.1: There exists a vertex $u \in V^-$ such that $|N(u) \cap W \cap V(H_u)| = 1$. Let us consider a set $W' \subseteq V(G \circ_v H)$ where $W' \cap V(H_x)$ is the subset of $V(H_x)$ induced by $W \cap V(H_u)$ for every vertex $x \in V(G)$. From any $\gamma(G)$-set X, it follows that $X \cup W' \in \mathcal{D}_2(G \circ_v H)$. Hence,

$$\begin{aligned} \gamma_2(G \circ_v H) \leq |X \cup W'| &= |X| + \sum_{x \in V(G)} |W' \cap V(H_x)| \\ &= |X| + \sum_{x \in V(G)} |W \cap V(H_u)| \\ &= \gamma(G) + n(\gamma_2(H) - 1). \end{aligned}$$

Thus, by the previous upper bound and the lower bound given in inequality chain (3) we have that $\gamma_2(G \circ_v H) = \gamma(G) + n(\gamma_2(H) - 1)$, as desired.

Subcase 2.2: $|N(u) \cap W \cap V(H_u)| = 0$ for every vertex $u \in V^-$. As $|N(u) \cap W| \geq 2$ for every vertex $u \in V^-$, it follows that $W \cap V(G) \subseteq V^{\geq}$ is a 2-dominating set of G. Hence, $\gamma_2(G) \leq |W \cap V(G)| \leq |V^{\geq}|$, which leads to $\gamma_2(G \circ_v H) \geq \gamma_2(G) + n(\gamma_2(H) - 1)$ by inequality chain (2). Thus, by the upper bound given in the inequality chain (3) it follows that $\gamma_2(G \circ_v H) = \gamma_2(G) + n(\gamma_2(H) - 1)$, as desired.

Therefore, the proof is complete. $\square$

Now, we proceed to characterise the graphs with $\gamma_2(G \circ_v H) = \gamma(G) + n(\gamma_2(H) - 1)$.

Theorem 5 *Let G and H be two graphs. If G has order n and $v \in V(H)$, then the following statements are equivalent.*

(i) $\gamma_2(G \circ_v H) = \gamma(G) + n(\gamma_2(H) - 1)$.
(ii) $\gamma_2(H - v) = \gamma_2(H) - 1$, *and in addition,* $\gamma_2(G) = \gamma(G)$ *or there exists a* $\gamma_2(H - v)$*-set* D *such that* $N(v) \cap D \neq \emptyset$.

Proof We first assume that $\gamma_2(G \circ_v H) = \gamma(G) + n(\gamma_2(H) - 1)$. Let W be a $\gamma_2(G \circ_v H)$-set, and consider the sets V^- and $V^{\geq}$ defined in the proof of Theorem 4. By Lemma 3-(i) and the fact that $\gamma(G) < n$, we deduce that $V^- \neq \emptyset$. Let $u \in V^-$ such that $|N(u) \cap W \cap V(H_u)|$ is maximum among all vertices in V^-. By Lemma 3-(ii) we have the following conditions.

- $\gamma_2(H - v) = \gamma_2(H) - 1$.
- $u \notin W$ and $|N(u) \cap W \cap V(H_u)| \leq 1$.

If $|N(u) \cap W \cap V(H_u)| = 0$, then it follows that $|N(x) \cap W \cap V(H_x)| = 0$ for every $x \in V^-$. Proceeding as in Subcase 2.2 of the proof of Theorem 4 we obtain that $|W| = \gamma_2(G) + n(\gamma_2(H) - 1)$, which implies that $\gamma_2(G) = \gamma(G)$. Now, we suppose that $|N(u) \cap W \cap V(H_u)| = 1$. Notice that $W \cap V(H_u)$ is a $\gamma_2(H_u - u)$-set such that $N(u) \cap W \cap V(H_u) \neq \emptyset$. As $H_u - u \cong H - v$, it follows that there exists a $\gamma_2(H - v)$-set D such that $N(v) \cap D \neq \emptyset$, as desired.

Conversely, we assume that (ii) holds. By Theorem 4 and Lemma 2-(ii)-(a) we have that $\gamma_2(G \circ_v H) \in \{\gamma(G) + n(\gamma_2(H) - 1),\ \gamma_2(G) + n(\gamma_2(H) - 1)\}$. If $\gamma_2(G) = \gamma(G)$, then $\gamma_2(G \circ_v H) = \gamma(G) + n(\gamma_2(H) - 1)$, as desired. Finally, if there exists a $\gamma_2(H - v)$-set D such that $N(v) \cap D \neq \emptyset$, then $\gamma_2(G \circ_v H) \leq \gamma(G) + n(\gamma_2(H) - 1)$ by Lemma 2-(ii)-(b), which implies that $\gamma_2(G \circ_v H) = \gamma(G) + n(\gamma_2(H) - 1)$. Therefore, the proof is complete. □

We next proceed to characterise the graphs with $\gamma_2(G \circ_v H) = \gamma_2(G) + n(\gamma_2(H) - 1)$. Observe that the case $\gamma(G) = \gamma_2(G)$ is excluded because it was discussed in Theorem 5.

Theorem 6 *Let G be a graph of order n such that $\gamma(G) < \gamma_2(G)$ and let H be a graph and $v \in V(H)$. Then* $\gamma_2(G \circ_v H) = \gamma_2(G) + n(\gamma_2(H) - 1)$ *if and only if one of the following conditions holds.*

(i) $\gamma_2(H - v) = \gamma_2(H) - 1$ *and* $N(v) \cap D = \emptyset$ *for any* $\gamma_2(H - v)$*-set* D.
(ii) $\gamma_2(H - v) \geq \gamma_2(H)$ *and* $\gamma_2(G) = n$.

Proof We first assume that $\gamma_2(G \circ_v H) = \gamma_2(G) + n(\gamma_2(H) - 1)$. By Lemma 1-(i) it follows that $\gamma_2(H - v) \geq \gamma_2(H) - 1$. Now, we analyse the next two cases.
Case 1: $\gamma_2(H - v) = \gamma_2(H) - 1$. By Lemma 2-(ii)-(a) and Theorem 4 we have that $\gamma_2(G \circ_v H) \in \{\gamma(G) + n(\gamma_2(H) - 1), \gamma_2(G) + n(\gamma_2(H) - 1)\}$. As $\gamma(G) < \gamma_2(G)$, Theorem 5 leads to $N(v) \cap D = \emptyset$ for any $\gamma_2(H - v)$-set D. Hence, (i) follows.
Case 2: $\gamma_2(H - v) \geq \gamma_2(H)$. Let W be a $\gamma_2(G \circ_v H)$-set, and consider the sets V^- and $V^{\geq}$ defined in the proof of Theorem 4. By Lemma 3-(ii) we deduce that $V^- = \emptyset$. From this previous condition and the fact that $\gamma_2(G) \leq n$ we have that

$$n\gamma_2(H) \leq \sum_{u \in V^+} |W \cap V(H_u)| = |W| = \gamma_2(G \circ_v H) = \gamma_2(G) + n(\gamma_2(H) - 1) \leq n\gamma_2(H).$$

The inequality chain above leads to $\gamma_2(G) = n$. Hence, (ii) follows.

On the other side, if condition (i) holds then Lemma 2-(ii)-(a) and Theorem 4 lead to $\gamma_2(G \circ_v H) \in \{\gamma(G) + n(\gamma_2(H) - 1), \gamma_2(G) + n(\gamma_2(H) - 1)\}$. Since $\gamma(G) < \gamma_2(G)$, it follows by Theorem 5 that $\gamma_2(G \circ_v H) \neq \gamma(G) + n(\gamma_2(H) - 1)$, which implies that $\gamma_2(G \circ_v H) = \gamma_2(G) + n(\gamma_2(H) - 1)$, as desired. Finally, if condition (ii) holds then by Theorem 5 we have that $\gamma_2(G \circ_v H) \neq \gamma(G) + n(\gamma_2(H) - 1)$, which implies that $\gamma_2(G \circ_v H) \in \{\gamma_2(G) + n(\gamma_2(H) - 1), n\gamma_2(H)\}$ by Theorem 4. Since $\gamma_2(G) = n$, it follows that $\gamma_2(G \circ_v H) = \gamma_2(G) + n(\gamma_2(H) - 1)$, which completes the proof. □

Finally, we proceed to characterise the graphs with $\gamma_2(G \circ_v H) = n\gamma_2(H)$. Observe that the case $\gamma_2(G) = n$ is excluded because it was discussed in Theorem 6.

Theorem 7 *Let G be a graph of order n such that $\gamma_2(G) < n$ and let H be a graph and $v \in V(H)$. Then the following statements are equivalent.*

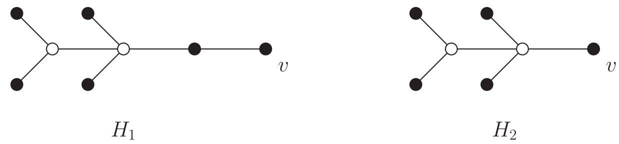

Fig. 2 The set of black-coloured vertices forms a $\gamma_2(H_i)$-set, for $i \in \{1, 2\}$

(i) $\gamma_2(G \circ_v H) = n\gamma_2(H)$.
(ii) $\gamma_2(H - v) \geq \gamma_2(H)$.

Proof We first assume that $\gamma_2(G \circ_v H) = n\gamma_2(H)$. By Lemma 1-(i) it follows that $\gamma_2(H - v) \geq \gamma_2(H) - 1$. If $\gamma_2(H - v) = \gamma_2(H) - 1$, then by Lemma 2-(ii)-(a) and the fact that $\gamma_2(G) < n$ we have that $\gamma_2(G \circ_v H) \leq \gamma_2(G) + n(\gamma_2(H) - 1) < n\gamma_2(H)$, a contradiction. Therefore, $\gamma_2(H - v) \geq \gamma_2(H)$, as desired.

Conversely, we assume that (ii) holds. As $\gamma_2(G) < n$, it follows by Theorems 5 and 6 that $\gamma_2(G \circ_v H) \notin \{\gamma(G) + n(\gamma_2(H) - 1), \gamma_2(G) + n(\gamma_2(H) - 1)\}$, which implies that $\gamma_2(G \circ_v H) = n\gamma_2(H)$ by Theorem 4. Therefore, the proof is complete. □

Next, we show that the three possible values of $\gamma_2(G \circ_v H)$ obtained in Theorem 4 (and characterised in Theorems 5, 6 and 7) are realisable.

Example 1 Let G be a graph of order n. Let H_1 and H_2 be the graphs given in Figure 2 and let H_3 be the graph obtained from the complete graph K_5 and eliminating exactly two edges incident to a root vertex $v \in V(K_5)$ (see Figure 1). Then the three possible values of $\gamma_2(G \circ_v H)$ are described below.

- $\gamma_2(G \circ_v H_1) = \gamma(G) + 5n = \gamma(G) + n(\gamma_2(H_1) - 1)$.
- $\gamma_2(G \circ_v H_2) = \gamma_2(G) + 4n = \gamma_2(G) + n(\gamma_2(H_2) - 1)$.
- $\gamma_2(G \circ_v H_3) = 2n = n\gamma_2(H_3)$.

As defined above, the corona product graph $G \odot H$ can be seen as a rooted product graph, i.e., $G \odot H \cong G \circ_v (K_1 + H)$. The following theorem provides closed formulas for the 2-domination number of $G \odot H$.

Theorem 8 *For any graph G of order n and any graph H,*

$$\gamma_2(G \odot H) = \begin{cases} n + \gamma(G) & \text{if } H \cong K_1, \\ n\gamma_2(K_1 + H) & \text{otherwise.} \end{cases}$$

Proof Let $K_1 + H$ be the join graph of K_1 and H, and let $V(K_1) = \{v\}$. Observe that $G \odot H \cong G \circ_v (K_1 + H)$. We first suppose that $H \cong K_1$. Hence, $K_1 + H \cong K_2$, which implies that $\gamma_2(K_1) = 1 = \gamma_2(K_2) - 1$. Therefore, Theorem 5 leads to the following equality chain.

$$\gamma_2(G \odot K_1) = \gamma_2(G \circ_v K_2) = \gamma(G) + n(\gamma_2(K_2) - 1) = \gamma(G) + n.$$

From now on we consider that $H \not\cong K_1$. Hence $K_1 + H \not\cong K_2$, which implies that $\gamma_2(H) \neq \gamma_2(K_1 + H) - 1$ by Lemma 1-(ii). In addition, Lemma 1-(i) leads to $\gamma_2(H) \geq \gamma_2(K_1 + H)$. Hence, by Theorem 5 we deduce that $\gamma_2(G \odot H) \neq \gamma(G) + n(\gamma_2(K_1 + H) - 1)$. Therefore, one of the following conditions holds by Theorem 4.

(a) $\gamma(G) < \gamma_2(G)$ and $\gamma_2(G \odot H) = \gamma_2(G) + n(\gamma_2(K_1 + H) - 1)$.
(b) $\gamma_2(G \odot H) = n\gamma_2(K_1 + H)$.

If (b) holds, then we are done. Finally, if (a) holds, then by Theorem 6 and the fact that $\gamma_2(H) \geq \gamma_2(K_1 + H)$ we deduce that $\gamma_2(G) = n$. Therefore, $\gamma_2(G \odot H) = \gamma_2(G) + n(\gamma_2(K_1 + H) - 1) = n\gamma_2(K_1 + H)$, which completes the proof. □

3 The 2-outer-independent domination number of rooted product graphs

To give the main results of this section, we need to establish the following useful lemma.

Lemma 9 *Let G and H be two graphs. If G has order n and $v \in V(H)$, then the following statements hold.*

(i) $|W \cap V(H_u)| \geq \gamma_2^{oi}(H) - 1$ *for any $\gamma_2^{oi}(G \circ_v H)$-set W and any $u \in V(G)$.*

(ii) $\gamma_2^{oi}(G \circ_v H) \leq \beta(G) + n\gamma_2^{oi}(H)$.
(iii) *If* $v \in S$ *for some* $\gamma_2^{oi}(H)$*-set* S*, then* $\gamma_2^{oi}(G \circ_v H) \leq n\gamma_2^{oi}(H)$.

Proof Let W be any $\gamma_2^{oi}(G \circ_v H)$-set and we fix any vertex $u \in V(G)$. It is easy to see that $(W \cap V(H_u)) \cup \{u\} \in \mathcal{D}_2^{oi}(H_u)$. Hence,

$$\gamma_2^{oi}(H) - 1 = \gamma_2^{oi}(H_u) - 1 \leq |(W \cap V(H_u)) \cup \{u\}| - 1 \leq |W \cap V(H_u)|,$$

which completes the proof of (i).

We next proceed to prove (ii). Let S be a $\gamma_2^{oi}(H)$-set. Now, let us consider a set $W \subseteq V(G \circ_v H)$, where $W \cap V(H_u)$ is the subset of $V(H_u)$ induced by S for every $u \in V(G)$. From any $\beta(G)$-set X, it follows that $X \cup W \in \mathcal{D}_2^{oi}(G \circ_v H)$. Hence,

$$\gamma_2^{oi}(G \circ_v H) \leq |X \cup W| \leq |X| + \sum_{u \in V(G)} |W \cap V(H_u)| = \beta(G) + n|S| = \beta(G) + n\gamma_2^{oi}(H),$$

which completes the proof of (ii).

Finally, we proceed to prove (iii). If we assume that $v \in S$, then it is straightforward that $W \in \mathcal{D}_2^{oi}(G \circ_v H)$. Hence, $\gamma_2^{oi}(G \circ_v H) \leq |W| = \sum_{u \in V(G)} |W \cap V(H_u)| = n|S| = n\gamma_2^{oi}(H)$, which completes the proof. $\square$

The following theorem provides closed formulas for the 2-outer-independent domination number of rooted product graphs.

Theorem 10 *Let* G *be a graph of order* n *and let* H *be any graph. For any vertex* $v \in V(H)$,

$$\gamma_2^{oi}(G \circ_v H) \in \{\beta(G) + n(\gamma_2^{oi}(H) - 1),\ n\gamma_2^{oi}(H),\ \beta(G) + n\gamma_2^{oi}(H)\}.$$

Proof Let W be any $\gamma_2^{oi}(G \circ_v H)$-set. By Lemma 9-(i) we have that $|W \cap V(H_u)| \geq \gamma_2^{oi}(H) - 1$ for any vertex $u \in V(G)$. Considering the previous fact, we next define the following subsets of $V(G)$.

$$\begin{aligned} V^- &= \{u \in V(G) : |W \cap V(H_u)| = \gamma_2^{oi}(H) - 1\}, \\ V^= &= \{u \in V(G) : |W \cap V(H_u)| = \gamma_2^{oi}(H)\}, \\ V^+ &= \{u \in V(G) : |W \cap V(H_u)| \geq \gamma_2^{oi}(H) + 1\}. \end{aligned}$$

Observe that $V(G) = V^- \cup V^= \cup V^+$. Next, let us analyse the following possible cases.

Case 1: $V^- \neq \emptyset$. If there exists a vertex $u \in V^- \cap W$, then $W \cap V(H_u) \in \mathcal{D}_2^{oi}(H_u)$, which contradicts the fact that $|W \cap V(H_u)| = \gamma_2^{oi}(H) - 1$. Therefore, $V^- \subset V(G) \setminus W$. Observe that V^- is an independent set of G because $V(G) \setminus W$ is also an independent set of G. Hence, $V^= \cup V^+ = V(G) \setminus V^-$ is a vertex cover of G, which implies that $\beta(G) \leq |V^= \cup V^+| = |V^=| + |V^+|$. Therefore,

$$\begin{aligned} \gamma_2^{oi}(G \circ_v H) &= \sum_{u \in V^+} |W \cap V(H_u)| + \sum_{u \in V^=} |W \cap V(H_u)| + \sum_{u \in V^-} |W \cap V(H_u)| \\ &\geq \sum_{u \in V^+} (\gamma_2^{oi}(H) + 1) + \sum_{u \in V^=} \gamma_2^{oi}(H) + \sum_{u \in V^-} (\gamma_2^{oi}(H) - 1) \\ &\geq 2|V^+| + |V^=| + \sum_{u \in V(G)} (\gamma_2^{oi}(H) - 1) \\ &\geq |V^+| + |V^=| + n(\gamma_2^{oi}(H) - 1) \\ &\geq \beta(G) + n(\gamma_2^{oi}(H) - 1). \end{aligned} \tag{4}$$

Now, we proceed to prove that $\gamma_2^{oi}(G \circ_v H) \leq \beta(G) + n(\gamma_2^{oi}(H) - 1)$. Let $u \in V^- \subset V(G) \setminus W$. Now, let us consider a set $W'' \subseteq V(G \circ_v H)$ where $W'' \cap V(H_x)$ is the subset of $V(H_x)$ induced by $W \cap V(H_u)$ for every vertex $x \in V(G)$. From any $\beta(G)$-set X, it follows that $X \cup W'' \in \mathcal{D}_2^{oi}(G \circ_v H)$. Hence,

$$\begin{aligned}\gamma_2^{oi}(G \circ_v H) &\leq |X \cup W''| = |X| + \sum_{x \in V(G)} |W'' \cap V(H_x)| \\ &= \beta(G) + \sum_{x \in V(G)} |W \cap V(H_u)| \\ &= \beta(G) + n(\gamma_2^{oi}(H) - 1).\end{aligned} \tag{5}$$

Therefore, $\gamma_2^{oi}(G \circ_v H) = \beta(G) + n(\gamma_2^{oi}(H) - 1)$ by inequality chains (4) and (5).

Case 2: $V^- = \emptyset$. In this case, we have that $V(G) = V^= \cup V^+$. If $V^+ = \emptyset$, then the following equality chain holds.

$$\gamma_2^{oi}(G \circ_v H) = |W| = \sum_{u \in V^=} |W \cap V(H_u)| = \sum_{u \in V^=} \gamma_2^{oi}(H) = \sum_{u \in V(G)} \gamma_2^{oi}(H) = n\gamma_2^{oi}(H).$$

From now on, we assume that $V^+ \neq \emptyset$. This implies that $|W| > n\gamma_2^{oi}(H)$. By Lemma 9-(ii) we have that $|W| \leq \beta(G) + n\gamma_2^{oi}(H)$ and as $\beta(G) < n$, it follows that $V^= \neq \emptyset$. If there exists $u \in V^= \cap W$, then $W \cap V(H_u)$ is a $\gamma_2^{oi}(H_u)$-set containing vertex u, which implies by Lemma 9-(iii) that $|W| \leq n\gamma_2^{oi}(H)$, a contradiction. Therefore, $V^= \subset V(G) \setminus W$, which implies that $V^=$ is an independent set of G because $V(G) \setminus W$ is also an independent set of G. Hence, $V^+ = V(G) \setminus V^=$ is a vertex cover of G, and as a consequence, $\beta(G) \leq |V^+|$. Thus,

$$\begin{aligned}\gamma_2^{oi}(G \circ_v H) &= \sum_{u \in V^+} |W \cap V(H_u)| + \sum_{u \in V^=} |W \cap V(H_u)| \\ &\geq \sum_{u \in V^+} (\gamma_2^{oi}(H) + 1) + \sum_{u \in V^=} \gamma_2^{oi}(H) \\ &\geq |V^+| + \sum_{u \in V(G)} \gamma_2^{oi}(H) \\ &\geq \beta(G) + n\gamma_2^{oi}(H).\end{aligned}$$

Therefore, $\gamma_2^{oi}(G \circ_v H) = \beta(G) + n\gamma_2^{oi}(H)$, which completes the proof. □

Now, we proceed to characterise the graphs with $\gamma_2^{oi}(G \circ_v H) = \beta(G) + n(\gamma_2^{oi}(H) - 1)$. Before, we recall that $N[v] = N(v) \cup \{v\}$ and $\mathcal{L}(H) = \{v \in V(H) : |N(v)| = 1\}$.

Theorem 11 *Let G and H be two graphs. If G has order n and $v \in V(H)$, then the following statements are equivalent.*

(i) $\gamma_2^{oi}(G \circ_v H) = \beta(G) + n(\gamma_2^{oi}(H) - 1)$.
(ii) *$v \in \mathcal{L}(H)$ and there exists a $\gamma_2^{oi}(H)$-set S such that $N[v] \subseteq S$.*

Proof We first assume that $\gamma_2^{oi}(G \circ_v H) = \beta(G) + n(\gamma_2^{oi}(H) - 1)$. Let W be a $\gamma_2^{oi}(G \circ_v H)$-set, and consider the sets V^-, $V^=$ and V^+ defined in the proof of Theorem 10. By Lemma 9-(i) and the fact that $\beta(G) < n$, we have that $V^- \neq \emptyset$. Let $u \in V^-$. If $u \in W$ or $|N(u) \cap V(H_u)| \geq 2$, then it is straightforward that $W \cap V(H_u) \in \mathcal{D}_2^{oi}(H_u)$. Hence, $\gamma_2^{oi}(H) = \gamma_2^{oi}(H_u) \leq |W \cap V(H_u)| = \gamma_2^{oi}(H) - 1$, a contradiction. Therefore, $u \in V(G) \setminus W$ and $u \in \mathcal{L}(H_u)$. The condition $u \in V(G) \setminus W$ leads to $N(u) \cap V(H_u) \subseteq W$, which implies that $S = (W \cap V(H_u)) \cup \{u\}$ is a $\gamma_2^{oi}(H_u)$-set such that $N[u] \cap V(H_u) \subseteq S$. Therefore, condition (ii) holds since $H \cong H_u$.

On the other side, let us assume that the condition (ii) holds, i.e., $v \in \mathcal{L}(H)$ and there exists a $\gamma_2^{oi}(H)$-set S such that $N[v] \subseteq S$. Let us consider a set $W' \subseteq V(G \circ_v H)$ where $W' \cap V(H_u)$ is the subset of $V(H_u)$ induced by $S \setminus \{v\}$ for every $u \in V(G)$. From any $\beta(G)$-set X, it follows that $X \cup W' \in \mathcal{D}_2^{oi}(G \circ_v H)$. Hence,

$$\begin{aligned}\gamma_2^{oi}(G \circ_v H) \leq |X \cup W'| &= |X| + \sum_{u \in V(G)} |W' \cap V(H_u)| \\ &= \beta(G) + \sum_{u \in V(G)} |S \setminus \{v\}| \\ &= \beta(G) + n(\gamma_2^{oi}(H) - 1).\end{aligned}$$

Therefore, by Theorem 10 we have that $\gamma_2^{oi}(G \circ_v H) = \beta(G) + n(\gamma_2^{oi}(H) - 1)$, which completes the proof. □

We next proceed to characterise the graphs with $\gamma_2^{oi}(G \circ_v H) = \beta(G) + n\gamma_2^{oi}(H)$.

Theorem 12 *Let G and H be two graphs. If G has order n and $v \in V(H)$, then the following statements are equivalent.*

(i) $\gamma_2^{oi}(G \circ_v H) = \beta(G) + n\gamma_2^{oi}(H)$.
(ii) *$v \notin S$ for every $\gamma_2^{oi}(H)$-set S.*

Proof We first assume that $\gamma_2^{oi}(G \circ_v H) = \beta(G) + n\gamma_2^{oi}(H)$. If there exists a $\gamma_2^{oi}(H)$-set S' such that $v \in S'$, then by Lemma 9-(iii) we have that $\gamma_2^{oi}(G \circ_v H) \leq n\gamma_2^{oi}(H)$, which is a contradiction due to the fact that $\beta(G) > 0$. Therefore, $v \notin S$ for every $\gamma_2^{oi}(H)$-set S, as desired.

On the other side, let us assume that (ii) holds, i.e., $v \notin S$ for every $\gamma_2^{oi}(H)$-set S. Let W be a $\gamma_2^{oi}(G \circ_v H)$-set, and consider the sets V^-, $V^=$ and V^+ defined in the proof of Theorem 10. If there exists a vertex $u \in V^-$, then the set $(W \cap V(H_u)) \cup \{u\}$ is a $\gamma_2^{oi}(H_u)$-set containing vertex u, which contradicts condition (ii) because $H \cong H_u$. Hence, $V^- = \emptyset$. Now, if $V^+ = \emptyset$, then $V(G) = V^=$ and as a consequence, there exists a vertex $u \in V^= \cap W$ as $V(G) \setminus W$ is an independent set and G is not an edgeless graph. Observe that $W \cap V(H_u)$ is a $\gamma_2^{oi}(H_u)$-set containing vertex u, which again contradicts condition (ii) because $H \cong H_u$. Hence, $V^+ \neq \emptyset$ and as a consequence, we have that

$$\begin{aligned}\gamma_2^{oi}(G \circ_v H) = |W| &= \sum_{u \in V^+} |W \cap V(H_u)| + \sum_{u \in V^=} |W \cap V(H_u)| \\ &\geq \sum_{u \in V^+} (\gamma_2^{oi}(H) + 1) + \sum_{u \in V^=} \gamma_2^{oi}(H) \\ &= |V^+| + \sum_{u \in V(G)} \gamma_2^{oi}(H) \\ &> n\gamma_2^{oi}(H).\end{aligned}$$

Therefore, by Theorem 10 we have that $\gamma_2^{oi}(G \circ_v H) = \beta(G) + n\gamma_2^{oi}(H)$, which completes the proof. □

Finally, we proceed to characterise the graphs with $\gamma_2^{oi}(G \circ_v H) = n\gamma_2^{oi}(H)$. As shown in Theorem 10, there are three possible expressions for the 2-outer-independent domination number of $G \circ_v H$. Thus, for the case of the equality $\gamma_2^{oi}(G \circ_v H) = n\gamma_2^{oi}(H)$, the corresponding characterisation is obtained by negating the equivalences associated to the previous results (Theorems 11 and 12).

Theorem 13 *Let G and H be two graphs. If G has order n and $v \in V(H)$, then $\gamma_2^{oi}(G \circ_v H) = n\gamma_2^{oi}(H)$ if and only if the next conditions hold.*

(i) *There exists a $\gamma_2^{oi}(H)$-set S such that $v \in S$.*
(ii) *$v \notin \mathcal{L}(H)$ or $N[v] \nsubseteq S'$ for every $\gamma_2^{oi}(H)$-set S'.*

Next, we show that the three possible values of $\gamma_2^{oi}(G \circ_v H)$ obtained in Theorem 10 (and characterised in Theorems 11, 12 and 13) are realisable.

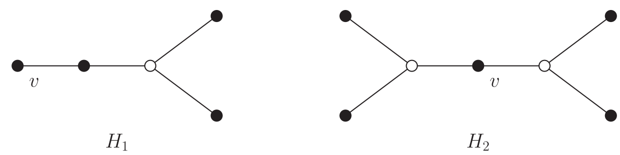

Fig. 3 The set of black-coloured vertices forms a $\gamma_2^{oi}(H_i)$-set, for $i \in \{1, 2\}$

Example 2 Let G be a graph of order n. Let H_1 and H_2 be the graphs given in Figure 3 and let H_3 be the graph obtained from the complete graph K_5 and eliminating exactly two edges incident to a root vertex $v \in V(K_5)$ (see Figure 1). Then the three possible values of $\gamma_2^{oi}(G \circ_v H)$ are described below.

- $\gamma_2^{oi}(G \circ_v H_1) = \beta(G) + 3n = \beta(G) + n(\gamma_2^{oi}(H_1) - 1)$.
- $\gamma_2^{oi}(G \circ_v H_2) = 5n = n\gamma_2^{oi}(H_2)$.
- $\gamma_2^{oi}(G \circ_v H_3) = \beta(G) + 3n = \beta(G) + n\gamma_2^{oi}(H_3)$.

The following theorem provides closed formulas for the 2-outer-independent domination number of the corona product graph $G \odot H$. Recall that $G \odot H$ can be seen as a rooted product graph, i.e., $G \odot H \cong G \circ_v (K_1 + H)$. We use the notation N_r for the edgeless graph of order r.

Theorem 14 *Let G be a graph of order n. For any graph H of order r,*

$$\gamma_2^{oi}(G \odot H) = \begin{cases} nr + \beta(G) & \text{if } H \cong N_r, \\ n\gamma_2^{oi}(K_1 + H) & \text{otherwise.} \end{cases}$$

Proof Let $K_1 + H$ be the join graph of K_1 and H, and let $V(K_1) = \{v\}$. Observe that $G \odot H \cong G \circ_v (K_1 + H)$. Now, we analyse the following two cases.
Case 1: $H \cong N_r$. We first consider that $r = 1$. In this case, we have that $K_1 + H \cong K_2$. Therefore, Theorem 11 leads to $\gamma_2^{oi}(G \odot H) = \gamma_2^{oi}(G \circ_v K_2) = \beta(G) + n(\gamma_2^{oi}(K_2) - 1) = \beta(G) + n$, as desired. Now, we consider that $r \geq 2$. Hence, $K_1 + H$ is isomorphic to a star graph of order $r + 1$. Hence, $V(K_1 + H) \setminus \{v\}$ is the unique $\gamma_2^{oi}(K_1 + H)$-set. Therefore, Theorem 12 leads to $\gamma_2^{oi}(G \odot H) = \gamma_2^{oi}(G \circ_v (K_1 + H)) = \beta(G) + n\gamma_2^{oi}(K_1 + H) = \beta(G) + nr$, as desired.
Case 2: $H \not\cong N_r$. In this case, we have that $v \notin \mathcal{L}(K_1 + H)$. By Theorem 11 we deduce that $\gamma_2^{oi}(G \odot H) = \gamma_2^{oi}(G \circ_v (K_1 + H)) \neq \beta(G) + n(\gamma_2^{oi}(K_1 + H) - 1)$, and by Theorem 10 it follows that

$$\gamma_2^{oi}(G \odot H) = \gamma_2^{oi}(G \circ_v (K_1 + H)) \in \{n\gamma_2^{oi}(K_1 + H), \beta(G) + n\gamma_2^{oi}(K_1 + H)\}.$$

Let us suppose that $\gamma_2^{oi}(G \odot H) = \beta(G) + n\gamma_2^{oi}(K_1 + H)$. Hence, Theorem 12 leads to $v \notin S'$ for every $\gamma_2^{oi}(K_1 + H)$-set S'. Let S be a $\gamma_2^{oi}(K_1 + H)$-set. As $v \notin S$, it follows that $V(H) = N(v) \subseteq S$. Hence $S = V(H)$, which implies that $\gamma_2^{oi}(K_1 + H) = |V(H)| = r$. As $K_1 + H$ is not a star graph, there exists a vertex $x \in V(H)$ such that $N(x) \cap V(H) \neq \emptyset$. Hence, it is easy to see that $V(K_1 + H) \setminus \{x\} \in \mathcal{D}_2^{oi}(K_1 + H)$. As $|V(K_1 + H) \setminus \{x\}| = r$, it follows that $V(K_1 + H) \setminus \{x\}$ is a $\gamma_2^{oi}(K_1 + H)$-set containing vertex v, a contradiction. Thus, $\gamma_2^{oi}(G \odot H) \neq \beta(G) + n\gamma_2^{oi}(K_1 + H)$, which implies that $\gamma_2^{oi}(G \odot H) = n\gamma_2^{oi}(K_1 + H)$, as desired. □

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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