Indian J Pure Appl Math (2026) 57:409–418
https://doi.org/10.1007/s13226-024-00582-8

ORIGINAL RESEARCH

On some super-congruences for the coefficients of analytic solutions of certain differential equations

Guo-Shuai Mao · Hao Zhang

Received: 1 March 2023 / Accepted: 26 March 2024 / Published online: 3 April 2024

Abstract In this paper, we prove some congruences involving the coefficients $\{A_n\}_{n=0,1,2,...}$ of the analytic solution $y_0(z) = \sum_{n=0}^{\infty} A_n z^n$ of certian differential eqution $\mathcal{D}y = 0$ normalized by the condition $y_0(0) = A_0 = 1$, where $\mathcal{D}$ is a 4th-order linear differential operator.

Keywords Calabi-Yau equations · Harmonic numbers · Congruences

Mathematics Subject Classification Primary: 11A07 · Secondry: 05A10 · 35M99

1 Introduction

Recall that the Bernoulli numbers $\{B_n\}_{n\geq 0}$ and Euler numbers $\{E_n\}_{n\geq 0}$ are defined as follows:

$$B_0 = 1, \ \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad (n = 2, 3, \ldots),$$

$$\frac{2}{e^x + e^{-x}} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!} \quad \left(|x| < \frac{\pi}{2}\right).$$

The differential equations are of the following form

$$\mathcal{D}_{n,k}\, y := \left(\sum_{i=0}^{k} z^i P_i(\theta) \right) y = 0,$$

here $\theta := z\frac{\partial}{\partial z}$ and the $P_i(\theta)$ are polynomials with integral coefficients and order n. Independently of the number of terms we can rewrite the differential equation $\mathcal{D}_{n,k}\, y = 0$ in the following way

$$y^{(n)} + a_{n-1}(z) y^{(n-1)} + \cdots + a_2(z) y^{''} + a_1(z) y^{'} + a_0(z) y = 0,$$

Communicated by B. Sury.

This research was supported by the National Natural Science Foundation of China (Grant Nos. 12001288, 12071208)

G.-S. Mao (✉)· H. Zhang
Department of Mathematics, Nanjing University of Information Science and Technology, Nanjing 210044, People's Republic of China
E-mail: maogsmath@163.com

H. Zhang
E-mail: 1037352761@qq.com

where $a_i(z)$ are rational functions of z.

In [2], the authors define Calabi-Yau type equations to be differential equations of the form $\mathcal{D}_{4,k}y = 0$ satisfying certain extra conditions.

In [1], there are a lot of Calabi-Yau type differential equations, and the authors listed the corresponding coefficients of the analytic solution to the differential equations. For example, $A_n = \sum_{k=0}^{n}\binom{n}{k}^3\binom{n+k}{k}\binom{2n-k}{n}$ are coefficients of the analytic solution of the differential equation

$$\begin{aligned}\mathcal{D}y = (23^2\theta^4 &- 23z(921\theta^4 + 2046\theta^3 + 1644\theta^2 + 621\theta + 92) - z^2(380851\theta^4 \\ &+ 1328584\theta^3 + 1772673\theta^2 + 1033528\theta + 221168) - 2z^3(475861\theta^4 + 1310172\theta^3 \\ &+ 1028791\theta^2 + 208932\theta - 27232) - 2^2\cdot 17z^4(8873\theta^4 + 14020\theta^3 + 5139\theta^2 \\ &- 1664\theta - 976) + 2^3\cdot 3\cdot 17^2z^5(\theta+1)^2(3\theta+2)(3\theta+4))y = 0.\end{aligned}$$

In this paper, we first investigate congruences for the more general numbers $A_n^{(a)} = \sum_{k=0}^{n}\binom{n}{k}^a\binom{n+k}{k}\binom{2n-k}{n}$ of A_n, where a is a nonnegative integer.

Theorem 1.1 *Let $p > 3$ be a prime and let a be a nonnegative integer. Then modulo p^4 we have*

$$A_{p-1}^{(a)} \equiv \begin{cases} -2p - 6p^2 - 18p^3 & \text{if } 2|a, \\ -2p - p^2(6 - 4q_p(2)) + p^3(-18 + 12q_p(2) + (2a-6)q_p(2)^2) & \text{if } 2\nmid a, \end{cases}$$

where $q_p(2) = (2^{p-1} - 1)/p$ is the Fermat quotient.

Remark 1.1 In particular, we have

$$A_{p-1} \equiv -2p - p^2(6 - 4q_p(2)) + p^3(-18 + 12q_p(2)) \pmod{p^4}.$$

Next we will investigate congruences for another type of numbers $B_n^{(a)}$ which are defined as follows:

$$B_n^{(a)} = \sum_{k=0}^{n}\binom{n}{k}^a\binom{2k}{k}^2\binom{2n-2k}{n-k}^2,$$

where a is a nonnegative integer and $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. When $a = 0$, $B_n^{(a)}$ are involved in the coefficients $\binom{2n}{n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}^2$ of the analytic solution of the differential equation

$$\mathcal{D}y = (\theta^4 - 2^4z(2\theta+1)^2(2\theta^2+2\theta+1) + 2^{10}z^2(\theta+1)^2(2\theta+1)(2\theta+3))y = 0.$$

When $a = 1$, $B_n^{(a)}$ are the coefficients of the analytic solution of the differential equation

$$\begin{aligned}\mathcal{D}y = (9\theta^4 &- 12z(64\theta^4 + 80\theta^3 + 73\theta^2 + 33\theta + 6) + 2^7z^2(194\theta^4 + 440\theta^3 + 527\theta^2 \\ &+ 315\theta + 75) - 2^{12}z^3(94\theta^4 + 288\theta^3 + 397\theta^2 + 261\theta + 66) + 2^{17}z^4(22\theta^4 \\ &+ 80\theta^3 + 117\theta^2 + 77\theta + 19) - 2^{23}z^5(\theta+1)^4)y = 0.\end{aligned}$$

Theorem 1.2 *Let $p > 3$ be a prime and let a be a nonnegative integer. Then*
(1) *If a is even,*

$$B_{p-1}^{(a)} \equiv 4^{(p-1)(a+4)} - \left(\frac{3}{2} + \frac{5a}{3}\right)p^3B_{p-3} \pmod{p^4}.$$

(2) *If a is odd,*

$$B_{p-1}^{(a)} \equiv (-1)^{(p-1)/2}4^{(p-1)(a+4)} + p^2E_{p-3} \pmod{p^3}.$$

Remark 1.2 In particular, we have

$$B_{p-1}^{(0)} = \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{2p-2-2k}{p-1-k}^2 \equiv 256^{p-1} - \frac{3}{2} p^3 B_{p-3} \pmod{p^4}$$

and

$$B_{p-1}^{(3)} = \sum_{k=0}^{p-1} \binom{p-1}{k}^3 \binom{2k}{k}^2 \binom{2p-2-2k}{p-1-k}^2 \equiv (-1)^{\frac{p-1}{2}} 4^{7(p-1)} + p^2 E_{p-3} \pmod{p^3}.$$

These two kinds of numbers are involved in the coefficients of analytic solutions of the differential equations in [1].

We will give the proof of Theorem 1.1 in Section 3, and Section 4 is devoted to the proof of Theorem 1.2.

2 Preliminary results

Let m be a positive integer and let $(a_1, a_2, \ldots, a_m) \in (\mathbb{N})^m = \underbrace{\mathbb{N} \times \mathbb{N} \cdots \times \mathbb{N}}_{m\ times}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. For any $n \geq m$, we define the alternating multiple harmonic sum as

$$H(a_1, a_2, \ldots, a_m; n) = \sum_{1 \leq k_1 < k_2 < \ldots < k_m \leq n} \prod_{i=1}^{m} \frac{\operatorname{sign}(a_i)^{k_i}}{k_i^{|a_i|}}.$$

The integers m and $\sum_{i=1}^m |a_i|$ are respectively the depth and the weight of the harmonic sum. As a matter of convenience, we remember $H(1; n)$ as H_n.

A lot of mathematicians worked on the harmonic numbers, one may consult [4,6–10,12–14]. For example, in 1862, J. Wolstenholme [15] showed that

$$H_{p-1} \equiv 0 \pmod{p^2} \text{ and } H(2, p-1) \equiv 0 \pmod{p}.$$

We know several nonalternating harmonic sums modulo a power of a prime as follows:
(i) ([5]) for $a, r > 0$ and for any prime $p > ar + 2$

$$H(\{a\}^r; p-1) \equiv \begin{cases} (-1)^r \frac{a(ar+1)}{2(ar+2)} p^2 B_{p-ar-2} \pmod{p^3} & \text{if } ar \text{ is odd}, \\ (-1)^{r-1} \frac{a}{ar+1} p B_{p-ar-1} \pmod{p^2} & \text{if } ar \text{ is even}. \end{cases}$$

(ii) ([9]) for a positive integer a and for any prime $p > a + 2$, we have

$$H\left(a; \frac{p-1}{2}\right) \equiv \begin{cases} -2q_p(2) + pq_p(2)^2 - \frac{2}{3} p^2 q_p(2)^3 - \frac{7}{12} p^2 B_{p-3} \pmod{p^3} & \text{a=1}, \\ -\frac{2^a-2}{a} B_{p-a} \pmod{p} & \text{if } 2 \nmid a > 1, \\ \frac{a(2^{a+1}-1)}{2(a+1)} p B_{p-a-1} \pmod{p^2} & \text{if } 2 \mid a. \end{cases}$$

Lemma 2.1 [3] *Let $p > 3$ be a prime. Then*

$$\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} \left(4^{p-1} + \frac{p^3}{12} B_{p-3}\right) \pmod{p^4}.$$

Lemma 2.2 *For any prime $p > 3$, we have*

$$\sum_{k=1}^{(p-1)/2} \frac{H_{2k}}{k^2} \equiv \frac{3}{2} B_{p-3} \pmod{p}. \tag{2.1}$$

Proof Noting that

$$\sum_{k=1}^{(p-1)/2}\frac{H_{2k}}{k^2}=4\sum_{\substack{j=1\\2|j}}^{p-1}\frac{H_j}{j^2}=2\sum_{j=1}^{p-1}\frac{H_j}{j^2}+2\sum_{j=1}^{p-1}\frac{(-1)^jH_j}{j^2},$$

then in view of [14, (2.2)] and [11, (5.4)], we have

$$\sum_{j=1}^{p-1}\frac{(-1)^jH_j}{j^2}\equiv-\frac{B_{p-3}}{4}\pmod p \text{ and } \sum_{j=1}^{p-1}\frac{H_j}{j^2}\equiv B_{p-3}\pmod p.$$

Therefore, the proof of Lemma 2.2 is finished. □

Lemma 2.3 *Let p be an odd prime. Then for all $k\in\{1,2,\ldots,p-1\}$ we have*

$$k^2\binom{2k}{k}^2\binom{2(p-k)}{p-k}^2\equiv 4p^2(1-4pH_{k-1}+4pH_{2k-1-\lfloor 2k/p\rfloor p})\pmod{p^4}. \tag{2.2}$$

Proof *Case* 1. If $1\le k\le(p-1)/2$, then

$$\begin{aligned}k^2\binom{2k}{k}^2\binom{2(p-k)}{p-k}^2&\equiv p^2k^2\binom{2k}{k}^2\frac{(p-2k)!^2(1+pH_{p-2k})^2(k-1)!^2(1-pH_{k-1})^2}{(p-k)!^2}\\&\equiv p^2\binom{2k}{k}^2\frac{1+2pH_{p-2k}-2pH_{k-1}}{\binom{p-k}{k}^2}\pmod{p^4},\end{aligned}$$

while

$$\begin{aligned}\binom{p-k}{k}^2&=\frac{(p-k)^2\cdots(p-2k+1)^2}{k!^2}\equiv\frac{k^2\cdots(2k-1)^2(1-p(H_{2k-1}-H_{k-1}))^2}{k!^2}\\&\equiv\frac14\binom{2k}{k}^2(1-2p(H_{2k-1}-H_{k-1}))\pmod{p^2}.\end{aligned}$$

This, with $H_{p-2k}\equiv H_{2k-1}\pmod p$ yields the result.
Case 2. If $(p+1)/2\le k\le p-1$, then we have

$$\begin{aligned}k^2\binom{2k}{k}^2\binom{2(p-k)}{p-k}^2&\equiv k^2\binom{2k}{k}^2\frac{(2k)^2\cdots(p+k-1)^2}{(p-k)!^2}\left(1-2p\sum_{j=2k}^{p+k-1}\frac1j\right)^2\\&\equiv p^2\binom{2k}{k}^2\frac{(p+k-1)^2\cdots(p+1)^2(p-1)^2\cdots(p-k+1)^2}{(2k-1)!^2}\left(1-4p\sum_{j=2k}^{p+k-1}\frac1j\right)\\&\equiv p^2k^2\binom{2k}{k}^2\frac{(k-1)!^2(1+pH_{k-1})^2(k-1)!^2(1-pH_{k-1})^2}{(2k-1)!^2}\left(1-4p\sum_{j=2k}^{p+k-1}\frac1j\right)\\&\equiv 4p^2\left(1-4p\sum_{j=2k}^{p+k-1}\frac1j\right)\equiv 4p^2\left(1-4p\sum_{j=2k-p}^{k-1}\frac1j\right)\pmod{p^4}.\end{aligned}$$

Now the proof of Lemma 2.3 is complete. □

3 Proof of Theorem 1.1

Lemma 3.1 *Let $p > 3$ be a prime and let a be a nonnegative integer. Then*

$$\sum_{k=1}^{p-2} \frac{(-1)^{ak}}{k(k+1)} \equiv \begin{cases} p+2 \pmod{p^2} & \text{if } a \equiv 0 \pmod 2, \\ p+2-4q_p(2)+2pq_p(2)^2 \pmod{p^2} & \text{if } a \equiv 1 \pmod 2, \end{cases} \tag{3.1}$$

and

$$\sum_{k=1}^{p-2} \frac{(-1)^{ak}}{k^2(k+1)^2} \equiv \begin{cases} -6 \pmod p & \text{if } a \equiv 0 \pmod 2, \\ -6+8q_p(2) \pmod p & \text{if } a \equiv 1 \pmod 2. \end{cases} \tag{3.2}$$

Proof *Case* 1. $a \equiv 0 \pmod 2$.

$$\sum_{k=1}^{p-2} \frac{(-1)^{ak}}{k(k+1)} = \sum_{k=1}^{p-2} \frac{1}{k} - \sum_{k=1}^{p-2} \frac{1}{k+1} = 1 - \frac{1}{p-1} \equiv p+2 \pmod{p^2},$$

in view of (i), we have

$$\begin{aligned} \sum_{k=1}^{p-2} \frac{(-1)^{ak}}{k^2(k+1)^2} &= \sum_{k=1}^{p-2} \left(\frac{1-2k}{k^2} + \frac{2(k+1)+1}{(k+1)^2} \right) \\ &= 2H(2; p-1) - \frac{1}{(p-1)^2} - 1 - 2 \left(\sum_{k=1}^{p-2} \frac{1}{k} - \sum_{k=2}^{p-1} \frac{1}{k} \right) \\ &\equiv -2 - 2\left(1 - \frac{1}{p-1}\right) \equiv -6 \pmod p. \end{aligned}$$

Case 2. $a \equiv 1 \pmod 2$.

In the light of (i) and (ii), we have

$$\begin{aligned} \sum_{k=1}^{p-2} \frac{(-1)^{ak}}{k(k+1)} &= \sum_{k=1}^{p-2} \frac{(-1)^k}{k} - \sum_{k=1}^{p-2} \frac{(-1)^k}{k+1} = 2\sum_{k=1}^{p-1} \frac{(-1)^k}{k} + \frac{p-2}{p-1} \\ &= 2\left(H\left(1; \frac{p-1}{2}\right) - H(1; p-1) \right) + p + 2 \equiv p+2-4q_p(2)+2pq_p(2)^2 \pmod{p^2} \end{aligned}$$

and

$$\begin{aligned} \sum_{k=1}^{p-2} \frac{(-1)^{ak}}{k^2(k+1)^2} &= \sum_{k=1}^{p-2} (-1)^k \left(\frac{1-2k}{k^2} + \frac{2(k+1)+1}{(k+1)^2} \right) \\ &= \sum_{k=1}^{p-2} \frac{(-1)^k}{k^2} - \sum_{k=2}^{p-1} \frac{(-1)^k}{k^2} - 2\left(\sum_{k=1}^{p-2} \frac{(-1)^k}{k} + \sum_{k=2}^{p-1} \frac{(-1)^k}{k} \right) \\ &\equiv -6 - 4H\left(1; \frac{p-1}{2}\right) \equiv -6 + 8q_p(2) \pmod p. \end{aligned}$$

Now the proof of Lemma 3.1 is complete. □

Lemma 3.2 *For any nonnegative integer a and prime $p > 2$, we have*

$$\sum_{k=1}^{p-2} \frac{(-1)^{ak} H_k}{k(k+1)} \equiv \begin{cases} 0 \pmod p & \text{if } a \equiv 0 \pmod 2, \\ 2q_p(2)^2 \pmod p & \text{if } a \equiv 1 \pmod 2, \end{cases} \tag{3.3}$$

$$\sum_{k=1}^{p-2} \frac{(-1)^{ak} H_{k+1}}{k(k+1)} \equiv \begin{cases} 3 \pmod p & \text{if } a \equiv 0 \pmod 2, \\ 3+2q_p(2)^2-4q_p(2) \pmod p & \text{if } a \equiv 1 \pmod 2, \end{cases} \tag{3.4}$$

Proof It is easy to see that

$$\sum_{k=1}^{p-2}\frac{(-1)^{ak}H_k}{k+1}=\sum_{j=1}^{p-2}\frac{(-1)^{a(p-1-j)}H_{p-1-j}}{p-1-j+1}\equiv-\sum_{k=1}^{p-2}\frac{(-1)^{ak}H_k}{k}\pmod p. \tag{3.5}$$

Case 1. $a\equiv 0 \pmod 2$.

$$\begin{aligned}\sum_{k=1}^{p-2}\frac{(-1)^{ak}H_k}{k}&=\sum_{k=1}^{p-2}\frac{H_k}{k}=\sum_{k=0}^{p-3}\frac{H_{k+1}}{k+1}=\sum_{k=0}^{p-3}\frac{H_k}{k+1}+\sum_{k=1}^{p-2}\frac{1}{k^2}\\&=-\sum_{k=1}^{p-2}\frac{H_k}{k}+1+\sum_{k=1}^{p-2}\frac{1}{k^2}\equiv-\sum_{k=1}^{p-2}\frac{H_k}{k}+H(2;p-1)\pmod p.\end{aligned}$$

In view of (i) and (3.5), we have

$$2\sum_{k=1}^{p-2}\frac{H_k}{k}\equiv H(2;p-1)\equiv 0\pmod p. \tag{3.6}$$

Hence

$$\sum_{k=1}^{p-2}\frac{(-1)^{ak}H_k}{k(k+1)}=\sum_{k=1}^{p-2}\frac{H_k}{k(k+1)}=\sum_{k=1}^{p-2}\frac{H_k}{k}-\sum_{k=1}^{p-2}\frac{H_k}{k+1}\equiv 2\sum_{k=1}^{p-2}\frac{H_k}{k}\equiv 0\pmod p.$$

Case 2. $a\equiv 1 \pmod 2$.
It is known that

$$\binom{p-1}{k-1}\equiv(-1)^{k-1}(1-pH_{k-1})\pmod{p^2}. \tag{3.7}$$

Then by (i) and (ii), we have

$$\begin{aligned}p\sum_{k=1}^{p-1}\frac{(-1)^{k-1}H_{k-1}}{k}&\equiv\sum_{k=1}^{p-1}\frac{(-1)^{k-1}-\binom{p-1}{k-1}}{k}=\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}-\frac{1}{p}\sum_{k=1}^{p-1}\binom{p}{k}\\&=-\left(H\left(1;\frac{p-1}{2}\right)-H(1;p-1)\right)-\frac{2^p-2}{p}\\&\equiv -H\left(1;\frac{p-1}{2}\right)-2q_p(2)\equiv -pq_p(2)^2\pmod{p^2},\end{aligned}$$

this means that

$$\sum_{k=1}^{p-1}\frac{(-1)^{k-1}H_{k-1}}{k}\equiv -q_p(2)^2\pmod p. \tag{3.8}$$

This, with (i) and (ii) yield that

$$\begin{aligned}\sum_{k=1}^{p-2}\frac{(-1)^kH_k}{k}&=\sum_{k=1}^{p-2}\frac{(-1)^k}{k^2}+\sum_{k=1}^{p-2}\frac{(-1)^kH_{k-1}}{k}\equiv\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}+\sum_{k=1}^{p-1}\frac{(-1)^kH_{k-1}}{k}\\&=\left(\frac{1}{2}H\left(2;\frac{p-1}{2}\right)-H(2;p-1)\right)-\sum_{k=1}^{p-1}\frac{(-1)^{k-1}H_{k-1}}{k}\equiv q_p(2)^2\pmod p.\end{aligned} \tag{3.9}$$

By (3.5) and (3.9), we have

$$\sum_{k=1}^{p-2}\frac{(-1)^{ak}H_k}{k(k+1)}=\sum_{k=1}^{p-2}\frac{(-1)^kH_k}{k}-\sum_{k=1}^{p-2}\frac{(-1)^kH_k}{k+1}\equiv 2\sum_{k=1}^{p-2}\frac{(-1)^kH_k}{k}\equiv 2q_p(2)^2\pmod p.$$

Now the proof of (3.3) is finished.
Now we turn to prove (3.4), it is easy to check that

$$\begin{aligned}
&\sum_{k=1}^{p-2}\frac{(-1)^{ak}H_{k+1}}{k(k+1)}-\sum_{k=1}^{p-2}\frac{(-1)^{ak}H_k}{k(k+1)}\\
&=\sum_{k=1}^{p-2}\frac{(-1)^{ak}}{k(k+1)^2}=\sum_{k=1}^{p-2}(-1)^{ak}\left(\frac{1}{k}-\frac{1}{k+1}-\frac{1}{(k+1)^2}\right)\\
&=\sum_{k=1}^{p-2}\frac{(-1)^{ak}}{k}-(-1)^a\sum_{k=2}^{p-1}\frac{(-1)^{ak}}{k}-(-1)^a\sum_{k=2}^{p-1}\frac{(-1)^{ak}}{k^2}\\
&\equiv 3+(1-(-1)^a)\sum_{k=1}^{p-1}\frac{(-1)^{ak}}{k}-(-1)^a\sum_{k=1}^{p-1}\frac{(-1)^{ak}}{k^2} \pmod p.
\end{aligned}$$

If $a\equiv 0 \pmod 2$. By (i) and (3.3), we have

$$\sum_{k=1}^{p-2}\frac{(-1)^{ak}H_{k+1}}{k(k+1)}\equiv 3 \pmod p.$$

If $a\equiv 1 \pmod 2$. By (i), (ii) and (3.3), we have

$$\begin{aligned}
&\sum_{k=1}^{p-2}\frac{(-1)^{ak}H_{k+1}}{k(k+1)}-2q_p(2)^2-3\equiv(1-(-1)^a)\sum_{k=1}^{p-1}\frac{(-1)^{ak}}{k}-(-1)^a\sum_{k=1}^{p-1}\frac{(-1)^{ak}}{k^2}\\
&\equiv 2\left(H\left(1;\frac{p-1}{2}\right)-H(1;p-1)\right)+\left(H\left(2;\frac{p-1}{2}\right)-H(2;p-1)\right)\\
&\equiv -4q_p(2) \pmod p.
\end{aligned}$$

Now we finish the proof of (3.4). □

Lemma 3.3 *Let $p>3$ be a prime, then for all integers $k\in\{1,2,\ldots,p-2\}$, we have*

$$\binom{p-1+k}{k}\binom{2p-2-k}{p-1-k}\equiv\frac{(p-k^2-k)p^2(1+pH_k+pH_{k+1})}{k^2(k+1)^2} \pmod{p^4}.$$

Proof We know that

$$\begin{aligned}
&\binom{p-1+k}{k}\binom{2p-2-k}{p-1-k}=\frac{(p-1+k)\cdots(p+1)p}{k!}\,\frac{(2p-2-k)\cdots p\cdots(p-k)}{(p-1)!}\\
&=\frac{p^2(1+p)\cdots(k-1+p)(-1+p)\cdots(-k+p)(p+1)\cdots(p+p-2-k)}{k!(p-1)!}\\
&\equiv\frac{p^2(-1)^k k!(p+1)\cdots(p+p-2-k)}{(p+k)(p-1)!}\equiv\frac{p^2(-1)^k k!(p-2-k)!(1+H_{p-2-k})}{(p+k)(p-1)!}\\
&=\frac{p^2(-1)^k k!(p-1-k)!(1+pH_{p-2-k})}{(p+k)(p-1-k)(p-1)!}=\frac{p^2(-1)^k(1+pH_{p-2-k})}{(p+k)(p-1-k)\binom{p-1}{k}} \pmod{p^4},
\end{aligned}$$

this, with $H_{p-1-k}\equiv H_k \pmod p$ and (3.7) yields that

$$\begin{aligned}
\binom{p-1+k}{k}\binom{2p-2-k}{p-1-k}&\equiv\frac{p^2(1+pH_{k+1})}{(p+k)(p-1-k)(1-pH_k)}\\
&\equiv\frac{(p-k^2-k)p^2(1+pH_k+pH_{k+1})}{k^2(k+1)^2} \pmod{p^4}.
\end{aligned}$$

Now the proof of Lemma 3.3 is finished. □

Proof of Theorem 1.1. By (3.7) and Lemma 3.3, we have

$$A_{p-1}^{(a)}-2\binom{2p-2}{p-1}\equiv\sum_{k=1}^{p-2}(-1)^{ak}(1-apH_k)\frac{(p-k^2-k)p^2(1+pH_k+pH_{k+1})}{k^2(k+1)^2}.$$

It is known that $\binom{2p}{p}\equiv 2 \pmod{p^3}$, (cf., [15]). So we have

$$\binom{2p-2}{p-1}=\frac{p}{2(2p-1)}\binom{2p}{p}\equiv\frac{p}{2p-1}\equiv -p-2p^2-4p^3 \pmod{p^4}. \tag{3.10}$$

This, with Lemma 3.1 and Lemma 3.2 yields the desired result.
Now we finish the proof of Theorem 1.1. □

4 Proof of Theorem 1.2

Proof of Theorem 1.2. By (3.7) and

$$\binom{2k}{k}\binom{2p-2-2k}{p-1-k}\equiv 0 \pmod{p}\quad \text{for } k\in\{0,1,\ldots,p-1\}\setminus\left\{\frac{p-1}{2}\right\},$$

we have

$$\begin{aligned}
B_{p-1}^{(a)}-\binom{p-1}{\frac{p-1}{2}}^{a+4}-2\binom{2p-2}{p-1}^2&=\sum_{\substack{1\le k\le p-2\\ k\ne\frac{p-1}{2}}}\binom{p-1}{k}^a\binom{2k}{k}^2\binom{2p-2-2k}{p-1-k}^2\\
&\equiv\sum_{\substack{1\le k\le p-2\\ k\ne\frac{p-1}{2}}}(-1)^{ak}(1-apH_k)\binom{2k}{k}^2\binom{2p-2-2k}{p-1-k}^2\\
&\equiv\sum_{\substack{1\le k\le p-2\\ k\ne\frac{p-1}{2}}}(-1)^{ak}(1-apH_k)\frac{(p-k)^2}{4(2p-1-2k)^2}\binom{2k}{k}^2\binom{2p-2k}{p-k}^2 \pmod{p^4}.
\end{aligned}$$

In view of Lemma 2.3, we have the following congruence modulo p^4,

$$\begin{aligned}
&B_{p-1}^{(a)}-\binom{p-1}{(p-1)/2}^{a+4}-2\binom{2p-2}{p-1}^2\\
&\equiv\sum_{k=1}^{(p-3)/2}(-1)^{ak}(1-apH_k)\frac{(p-k)^2}{(2p-1-2k)^2}\frac{p^2}{k^2}(1+4pH_{2k-1}-4pH_{k-1})\\
&+\sum_{k=(p+1)/2}^{p-2}(-1)^{ak}(1-apH_k)\frac{(p-k)^2}{(2p-1-2k)^2}\frac{p^2}{k^2}(1+4pH_{2k-p-1}-4pH_{k-1}).
\end{aligned}$$

Replacing k by $p-1-k$ in the second sum and noting that $H_{p-1-k} \equiv H_k \pmod p$, we have

$$\begin{aligned}
&B_{p-1}^{(a)} - \binom{p-1}{(p-1)/2}^{a+4} - 2\binom{2p-2}{p-1}^2 \\
&\equiv \sum_{k=1}^{(p-3)/2} (-1)^{ak}(1-apH_k)\frac{(p-k)^2}{(2p-1-2k)^2}\frac{p^2}{k^2}(1+4pH_{2k-1}-4pH_{k-1}) \\
&+ \sum_{k=1}^{(p-3)/2} (-1)^{ak}(1-apH_k)\frac{(k+1)^2}{(2k+1)^2}\frac{p^2(p+1+k)^2}{(k+1)^4}(1+4pH_{2k+2}-4pH_{k+1}) \\
&\equiv 2p^2\sum_{k=1}^{(p-3)/2}\frac{(-1)^{ak}}{(2k+1)^2} - 2p^3\sum_{k=1}^{(p-3)/2}\frac{(-1)^{ak}}{k(2k+1)^3} + 2p^3\sum_{k=1}^{(p-3)/2}\frac{(-1)^{ak}}{(k+1)(2k+1)^2} \\
&+ p^3\sum_{k=1}^{(p-3)/2}\frac{(-1)^{ak}}{(2k+1)^2}\left(8H_{2k}-8H_k-2aH_k+\frac{2}{k}-\frac{2}{k+1}+\frac{4}{2k+1}\right) \pmod{p^4}.
\end{aligned}$$

Then with some simple computation we have

$$\begin{aligned}
&B_{p-1}^{(a)} - \binom{p-1}{(p-1)/2}^{a+4} - 2\binom{2p-2}{p-1}^2 \\
&\equiv \frac{p^2}{2}\sum_{k=1}^{\frac{p-3}{2}}\frac{(-1)^{a((p-1)/2-k)}}{k^2} - \frac{p^3}{2}\sum_{k=1}^{\frac{p-3}{2}}\frac{(-1)^{a((p-1)/2-k)}}{k^3} \\
&+ \frac{p^3}{2}\sum_{k=1}^{\frac{p-3}{2}}\frac{(-1)^{a((p-1)/2-k)}}{k^2}((a+4)(H_k-H_{(p-1)/2})-(2a+4)H_{2k}) \pmod{p^4},
\end{aligned}$$

where we used $H_{(p-1)/2-k} \equiv 2H_{2k} - H_k + H_{(p-1)/2} \pmod p$.

Proof of (1) Now a is even. Then

$$\begin{aligned}
&B_{p-1}^{(a)} - \binom{p-1}{(p-1)/2}^{a+4} - 2\binom{2p-2}{p-1}^2 \\
&\equiv \frac{p^2}{2}\left(H\left(2;\frac{p-1}{2}\right)-\frac{4}{(p-1)^2}\right) - \frac{p^3}{2}\left(H\left(3;\frac{p-1}{2}\right)-\frac{8}{(p-1)^3}\right) \\
&+ \frac{p^3}{2}\sum_{k=1}^{(p-1)/2}\frac{1}{k^2}((a+4)(H_k-H_{(p-1)/2})-(2a+4)H_{2k}) \pmod{p^4},
\end{aligned}$$

in the light of (ii), we know that

$$\begin{aligned}
&B_{p-1}^{(a)} - \binom{p-1}{(p-1)/2}^{a+4} - 2\binom{2p-2}{p-1}^2 \\
&\equiv -2p^2 - 8p^3 + \frac{13}{6}p^3B_{p-3} + \frac{p^3(a+4)}{2}\sum_{k=1}^{(p-1)/2}\frac{H_k}{k^2} - (a+2)p^3\sum_{k=1}^{(p-1)/2}\frac{H_{2k}}{k^2},
\end{aligned}$$

then by Lemma 2.2 and [6, Lemma3.2] we have

$$B_{p-1}^{(a)} - \binom{p-1}{\frac{p-1}{2}}^{a+4} - 2\binom{2p-2}{p-1}^2 \equiv -2p^2 - 8p^3 - \frac{11}{6}p^3B_{p-3} - \frac{7a}{4}p^3B_{p-3} \pmod{p^4}.$$

Therefore with (3.10) and Lemma 2.1 we immediately get the desired result. □

Proof of (2) Now a is odd. Then

$$\begin{aligned} B_{p-1}^{(a)} - \binom{p-1}{(p-1)/2}^{a+4} - 2\binom{2p-2}{p-1}^2 &\equiv \frac{p^2}{2}\sum_{k=1}^{(p-3)/2}\frac{(-1)^k}{k^2} \\ &= \frac{p^2}{2}\left(\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^2} - \frac{4(-1)^{(p-1)/2}}{(p-1)^2}\right) \\ &\equiv \frac{p^2}{2}\left(\frac{1}{2}H\left(2;\lfloor p/4\rfloor\right) - H\left(2;\frac{p-1}{2}\right) - 4(-1)^{(p-1)/2}\right) \pmod{p^3}. \end{aligned}$$

With the help of [10, Corollary 3.7] and (ii), we have

$$B_{p-1}^{(a)} - \binom{p-1}{(p-1)/2}^{a+4} - 2\binom{2p-2}{p-1}^2 \equiv -2p^2 + p^2E_{p-3} \pmod{p^3}.$$

At last by (3.10) and Lemma 2.1 we immediately get the desired result.
Now the proof of Theorem 1.2 is complete. □

Acknowledgements The authors would like to thank the anonymous referee for his/her useful suggestions on this paper.

References

1. G. Almkvist, C. van Enckevort, D. van Straten and W. Zudilin, *Tables of Calabi-Yau equations*, arXiv:math/0507430v2.
2. G. Almkvist and W. Zudilin, Differential equations, mirror maps and zeta values, Mirror Symmetry V, (N. Yui, S.-T. Yau, and J. D. Lewis, eds.), Proceedings of BIRS workshop on Calabi-Yau Varieties and Mirror Symmetry (December 6-C11, 2003), AMS/IP Stud. Adv. Math. **38**, Amer. Math. Soc. International Press, Providence, RI (2007), 481–515; math.NT/0402386 (2004).
3. L. Calitz, *A theorem of Glaisher*, Canadian J. Math. **5** (1953), 306–316.
4. H.-Q. Cao and H. Pan, *Note on some congruences of Lehmer*, J. Number Theory **129** (2009), no.8, 1813–1819.
5. M. E. Hoffman, *Quasi-symmetric functions and mod p multiple hamonic sums*, Kyushu. J. Math. **69** (2015), 345–366.
6. G.-S. Mao, *Proof of some congruences conjectured by Z.-W. Sun*, Int. J. Number Theory **13** (2017), No.8, 1983–1993.
7. G.-S. Mao, *Proof of two conjectural supercongruences involving Catalan-Larcombe-French numbers*, J. Number Theory **179** (2017), 88–96.
8. G.-S. Mao and Z.-W. Sun, *Two congruences involving harmonic numbers with applications*, Int. J. Number Theory **12** (2016), no.02, 527-539.
9. Z.-H. Sun, *Congruences concerning Bernoulli numbers and Bernoulli polynomials*, Discrete. Appl. Math. **105** (2000), 193–223.
10. Z.-H. Sun, *Congruences involving Bernoulli and Euler numbers*, J. Number Theory **128** (2008), 280–312.
11. Z.-W. Sun and R. Tauraso, *New congruences for central binomial cofficients*, Adv. in Appl. Math. **45** (2010), no.1, 125–148
12. Z.-W. Sun, *Super congruences and Euler numbers*, Sci. China Math. **54** (2011), 2509–2535.
13. Z.-W. Sun, *A new series for π^3 and related congruences*, Internat. J. Math. **26** (2015), no.8, 1550055 (23 pages).
14. Z.-W. Sun and L.-L. Zhao, *Arithmetic theory of harmonic numbers (II)*, Colloq. Math. **130** (2013), no.1, 67–78.
15. J. Wolstenholme, *On certain properties of prime numbers*, Quart. J. Math. **5** (1862), 35–39.