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ORIGINAL RESEARCH

On extension of isometries between the positive unit spheres of $c_0(\Gamma)$

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Abstract Let Γ, Δ be two index sets and let $S^+_{c_0(\Gamma)} = \{x = (x_\gamma)_{\gamma\in\Gamma} \in c_0(\Gamma) : \|x\| = 1; x_\gamma \geq 0, \ \forall\, \gamma \in \Gamma\}$. In this paper, we show that every surjective isometry $f : S^+_{c_0(\Gamma)} \to S^+_{c_0(\Delta)}$ can be extended to a linear surjective isometry from $c_0(\Gamma)$ onto $c_0(\Delta)$.

Keywords Isometries · Positive unit spheres · $c_0(\Gamma)$-spaces · Tingley's problem

Mathematics Subject Classification Primary 46B04 · 46B20

1 Introduction

Throughout this paper, X, Y denote real Banach spaces, S_X, S_Y denote their unit spheres. A mapping f from X to Y is said to be an isometry if $\|f(x) - f(y)\| = \|x - y\|$ for all $x, y \in X$. The study on the properties of isometric mappings between Banach spaces has become very active since the important Mazur-Ulam theorem [17] which states that every surjective isometry between real Banach spaces is affine.

In 1972, Mankiewicz [16] first investigated the isometric extension problem and generalized the Mazur-Ulam's theorem as follows: Every surjective isometry between the connected open subsets or the closed convex bodies of two Banach spaces can be extended to whole space as a surjective affine isometry. In 1987, Tingley [30] introduced the isometric extension problem between the unit spheres of two Banach spaces which is nowadays

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called the Tingley's problem: Whether the surjective isometry $f : S_X \to S_Y$ can be extended to whole space as a linear surjective isometry ?

Tingley's problem attracts a large number of mathematicians attention, and many positive answers on this topic have been obtained (see [1–11,15,25–29,31]). Based on a variant Tingley's problem raised by Peralta [23], Leung, Ng and Wong [14] proposed the following isometric extension problem between the positive unit spheres of two ordered Banach spaces.

Problem 1.1 *Let X, Y be two ordered Banach spaces with generating cones, $S_X^+ = \{x \in X : x \geq 0, \|x\| = 1\}$ be the positive unit sphere of X. Suppose that $f : S_X^+ \to S_Y^+$ is a surjective isometry. Does there exists a linear surjective isometry $\tilde{f} : X \to Y$ such that $\tilde{f}|_{S_X^+} = f$?*

Recently, there are some positive results about Problem 1.1. For instance, it has positive answer when $X = Y$ is the space $C_p(H)$ of the Schatten p-class operators on a complex Hilbert space H for $p \in [1, \infty]$ (see [18,19,21,22,24]). For non-commutative L^p-spaces $X = L^p(M)$ and $Y = L^p(N)$ associated with two von Neumann algebras M and N, the case of $p = 1$ was proved by Lau, Ng, Wong and Mori (see [12,20]), the case of $p = 2$ was proved by Leung, Ng and Wong [13]. In 2021, Leung, Ng and Wong [14] furthermore presented positive answers to Problem 1.1 when X, Y are two classical L^p-spaces with $p \in [1, \infty]$ or X, Y are two $C(K)$-spaces where K is a compact Hausdorff space.

Let Γ be an index set, $c_0(\Gamma) = \{x = (x_\gamma)_{\gamma\in\Gamma} : \ \forall\, \delta > 0, \ \{\gamma : |x_\gamma| \geq \delta\}$ is a finite set$\}$. In this paper, we study the isometric extension problem between the positive unit spheres of $c_0(\Gamma)$ spaces, and present another positive answer to Problem 1.1.

2 Main results

Let Γ, Δ be two index sets, $c_0(\Gamma)^+ = \{x = (x_\gamma)_{\gamma\in\Gamma} \in c_0(\Gamma) : x_\gamma \geq 0, \ \forall\, \gamma \in \Gamma\}$ be the positive cone of $c_0(\Gamma)$, and $S^+_{c_0(\Gamma)} = \{x \in c_0(\Gamma)^+ : \|x\| = 1\}$ be the positive unit sphere of $c_0(\Gamma)$. In this paper, we will prove that every surjective isometry $f : S^+_{c_0(\Gamma)} \to S^+_{c_0(\Delta)}$ can be extended to a linear surjective isometry from $c_0(\Gamma)$ onto $c_0(\Delta)$. To begin with, we present some notations.

For $x \in c_0(\Gamma)^+$ with $\|x\| = 1$, let

$$\begin{aligned} P(x) &= \{y \in S^+_{c_0(\Gamma)} : \|x - y\| = 1\}, \\ I(x) &= \{\gamma \in \Gamma : x_\gamma = 1\}, \\ O(x) &= \{\gamma \in \Gamma : x_\gamma = 0\}. \end{aligned}$$

For any $\gamma \in \Gamma$, let $e_\gamma = \{\xi_{\gamma'} : \xi_\gamma = 1, \xi_{\gamma'} = 0, \gamma' \neq \gamma, \gamma' \in \Gamma\}$.

We first establish some lemmas.

Lemma 2.1 *Let $x \in S^+_{c_0(\Gamma)}$, then*

$$P(x) = \left(\bigcup_{\alpha\in I(x)} \{y \in S^+_{c_0(\Gamma)} : \alpha \in O(y)\}\right) \bigcup \left(\bigcup_{\gamma\in O(x)} \{y \in S^+_{c_0(\Gamma)} : \gamma \in I(y)\}\right).$$

Proof For $x, h \in S^+_{c_0(\Gamma)}$, we have

$$\begin{aligned} h \in P(x) &\Leftrightarrow \|x - h\| = 1; \\ &\Leftrightarrow \exists\, \beta \in \Gamma, |(x-h)_\beta| = |x_\beta - h_\beta| = 1, \\ &\Leftrightarrow \exists\, \beta \in \Gamma, \{x_\beta = 1, h_\beta = 0\} \text{ or } \{x_\beta = 0, h_\beta = 1\}, \\ &\Leftrightarrow \exists\, \beta \in \Gamma, \{\beta \in I(x) \cap O(h)\} \text{ or } \{\beta \in O(x) \cap I(h)\}, \\ &\Leftrightarrow h \in \left(\bigcup_{\alpha\in I(x)} \{y \in S^+_{c_0(\Gamma)} : \alpha \in O(y)\}\right) \bigcup \left(\bigcup_{\gamma\in O(x)} \{y \in S^+_{c_0(\Gamma)} : \gamma \in I(y)\}\right). \end{aligned}$$

□

Lemma 2.2 *Let $x, y \in S^+_{c_0(\Gamma)}$, then $P(x) \subset P(y)$ if and only if $I(x) \subset I(y)$ and $O(x) \subset O(y)$.*

Proof The sufficiency follows from Lemma 2.1. We only need to prove the necessity. Suppose, to the contrary, that $I(x) \not\subset I(y)$ or $O(x) \not\subset O(y)$.

Case 1. $I(x) \not\subset I(y)$. Then there exists $\gamma \in \Gamma$ such that $\gamma \in I(x)$ and $\gamma \notin I(y)$. Let $z = \sum_{i \in I(y)} e_i$. Note that $I(y)$ is a finite set, we have $z \in S^+_{c_0(\Gamma)}$. It is easy to observe that $z \in P(x)$ but $z \notin P(y)$. This contradicts $P(x) \subset P(y)$.

Case 2. $O(x) \not\subset O(y)$. Then there exists $\gamma \in \Gamma$ such that $\gamma \in O(x)$ and $\gamma \notin O(y)$. Let $z' = \sum_{i \in I(y) \cup \{\gamma\}} e_i$. Since $I(y) \cup \{\gamma\}$ is a finite set, we have $z' \in S^+_{c_0(\Gamma)}$. Note that $z' \in P(x)$ but $z' \notin P(y)$. This contradicts $P(x) \subset P(y)$. □

Lemma 2.3 *Let $x, y \in S^+_{c_0(\Gamma)}$, then $I(x) \cap I(y) = \emptyset$ if and only if there is no $z \in S^+_{c_0(\Gamma)}$ such that $P(z) \subset P(x) \cap P(y)$.*

Proof “$\Leftarrow$” Suppose, to the contrary, that $I(x) \cap I(y) \neq \emptyset$. Then we have $z = \frac{1}{2}(x + y) \in S^+_{c_0(\Gamma)}$ and $P(z) \subset P(x) \cap P(y)$. This is a contradiction.

“$\Rightarrow$” Suppose, to the contrary, that there exists $z \in S^+_{c_0(\Gamma)}$ such that $P(z) \subset P(x) \cap P(y)$, then $I(z) \subset I(x) \cap I(y)$. Since $I(x) \cap I(y) = \emptyset$, we have $I(z) = \emptyset$. This contradicts $z \in S^+_{c_0(\Gamma)}$.

□

Lemma 2.4 *Suppose that $f : S^+_{c_0(\Gamma)} \to S^+_{c_0(\Delta)}$ is a surjective isometry, $x, y \in S^+_{c_0(\Gamma)}$, then $P(x) \subset P(y)$ if and only if $P(f(x)) \subset P(f(y))$.*

Proof Since f is a surjective isometry, we just need to prove that $f(P(x)) = P(f(x))$. For $x \in S^+_{c_0(\Gamma)}$, we have

$$\begin{aligned} f(P(x)) &= \{f(h) : h \in S^+_{c_0(\Gamma)}, \|x - h\| = 1\}, \\ &= \{f(h) : h \in S^+_{c_0(\Gamma)}, \|f(x) - f(h)\| = 1\} = P(f(x)). \end{aligned}$$

□

Lemma 2.5 *Suppose that $f : S^+_{c_0(\Gamma)} \to S^+_{c_0(\Delta)}$ is a surjective isometry, $x, y \in S^+_{c_0(\Gamma)}$, then $I(x) \cap I(y) = \emptyset$ if and only if $I(f(x)) \cap I(f(y)) = \emptyset$.*

Proof Note that f is surjective, by Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned} I(x) \cap I(y) = \emptyset &\Leftrightarrow \forall\, z \in S^+_{c_0(\Gamma)},\ P(z) \not\subset P(x) \cap P(y), \\ &\Leftrightarrow \forall\, f(z) \in S^+_{c_0(\Delta)},\ P(f(z)) \not\subset P(f(x)) \cap P(f(y)), \\ &\Leftrightarrow I(f(x)) \cap I(f(y)) = \emptyset. \end{aligned}$$

□

Lemma 2.6 *Suppose that $f : S^+_{c_0(\Gamma)} \to S^+_{c_0(\Delta)}$ is a surjective isometry, $x \in S^+_{c_0(\Gamma)}$, then $I(x) \cup O(x) = \Gamma$ if and only if $I(f(x)) \cup O(f(x)) = \Delta$.*

Proof Note that f is surjective, due to Lemma 2.2 and Lemma 2.4, we obtain

$$\begin{aligned} I(x) \cup O(x) = \Gamma &\Leftrightarrow \text{For any } y \in S^+_{c_0(\Gamma)} \text{ with } P(x) \subset P(y), \text{ we have } y = x, \\ &\Leftrightarrow \text{For any } f(y) \in S^+_{c_0(\Delta)} \text{ with } P(f(x)) \subset P(f(y)), \\ &\quad\ \text{we have } f(y) = f(x), \\ &\Leftrightarrow I(f(x)) \cup O(f(x)) = \Delta. \end{aligned}$$

□

Lemma 2.7 *Suppose that $f : S^+_{c_0(\Gamma)} \to S^+_{c_0(\Delta)}$ is a surjective isometry, then there exists a unique bijective mapping $\tau : \Gamma \to \Delta$ such that $f(e_\gamma) = e_{\tau(\gamma)}$.*

Proof Fix $\gamma \in \Gamma$, for any $\alpha \in \Gamma$, $\alpha \neq \gamma$,we have $I(e_\alpha) \cap I(e_\gamma) = \emptyset$. According to Lemma 2.5, $I(f(e_\alpha)) \cap I(f(e_\gamma)) = \emptyset$. We assert that $I(f(e_\gamma))$ is a singleton subset. Since $f(e_\gamma) \in S^+_{c_0(\Delta)}$, $I(f(e_\gamma)) \neq \emptyset$. Assume that $I(f(e_\gamma))^\sharp \geq 2$, where $\sharp$ denotes the cardinality of the set, then there exist $\mu \neq \nu \in I(f(e_\gamma)) \subset \Delta$. Note that $I(e_\mu) \subset I(f(e_\gamma))$, we have $I(e_\mu) \cap I(f(e_\alpha)) = \emptyset$. Due to Lemma 2.5, $I(f^{-1}(e_\mu)) \cap I(e_\alpha) = \emptyset$. Since $f^{-1}(e_\mu) \in S^+_{c_0(\Gamma)}$, $I(f^{-1}(e_\mu)) \neq \emptyset$. By the arbitrariness of α, we obtain $I(f^{-1}(e_\mu)) = \{\gamma\}$. Note that $I(e_\mu) \cup O(e_\mu) = \Delta$, by Lemma 2.6, $I(f^{-1}(e_\mu)) \cup O(f^{-1}(e_\mu)) = \Gamma$, this implies that $f^{-1}(e_\mu) = e_\gamma$. By the same argument as above, we have $f^{-1}(e_\nu) = e_\gamma$. This contradicts f^{-1} is injective, and $I(f(e_\gamma))$ is a singleton subset. In the sequel, we will blur the distinction between the element γ and the singleton subset $\{\gamma\}$, and this should not lead to confusion.

For $\gamma \in \Gamma$, define

$$\tau(\gamma) = I(f(e_\gamma)),$$

then $\tau : \Gamma \to \Delta$ is well-defined. Since $I(e_\gamma) \cup O(e_\gamma) = \Gamma$, by Lemma 2.6, $I(f(e_\gamma)) \cup O(f(e_\gamma)) = \Delta$, this implies that

$$f(e_\gamma) = e_{\tau(\gamma)}.$$

In what follows, we will show that τ is bijective. For $\gamma_1 \neq \gamma_2 \in \Gamma$, we have $I(e_{\gamma_1}) \cap I(e_{\gamma_2}) = \emptyset$. Due to Lemma 2.5, $I(f(e_{\gamma_1})) \cap I(f(e_{\gamma_2})) = \emptyset$, i.e., $\tau(\gamma_1) \neq \tau(\gamma_2)$, this implies that τ is injective. Since f is a surjective isometry, $f^{-1} : S^+_{c_0(\Delta)} \to S^+_{c_0(\Gamma)}$ is also a surjective isometry. By the same argument as above, for $\mu \in \Delta$, define

$$\tau^{'}(\mu) = I(f^{-1}(e_\mu)),$$

then $\tau^{'} : \Delta \to \Gamma$ is well-defined, and $f^{-1}(e_\mu) = e_{\tau^{'}(\mu)}$. It is easy to observe that $\tau(\tau^{'}(\mu)) = I(f(e_{\tau^{'}(\mu)}) = I(e_\mu) = \mu$, this implies that τ is surjective. The uniqueness of τ is obvious, and the proof is complete. □

Lemma 2.8 *Suppose that $f : S^+_{c_0(\Gamma)} \to S^+_{c_0(\Delta)}$ is a surjective isometry, then there exists a unique bijective mapping $\tau : \Gamma \to \Delta$ such that*

$$x_\gamma = f(x)_{\tau(\gamma)}, \text{ for all } x \in S^+_{c_0(\Gamma)},\ \gamma \in \Gamma.$$

Proof Let $\tau : \Gamma \to \Delta$ be as in Lemma 2.7, and let $x \in S^+_{c_0(\Gamma)}$, $\gamma \in \Gamma$. The proof is divided into three cases.

Case 1. $x_\gamma = 1$. Since $\gamma \in I(x)$, $I(e_\gamma) \cap I(x) \neq \emptyset$. According to Lemma 2.5 and Lemma 2.7, we obtain that $I(e_{\tau(\gamma)}) \cap I(f(x)) \neq \emptyset$. Thus, $f(x)_{\tau(\gamma)} = 1 = x_\gamma$.

Case 2. $x_\gamma = 0$. Let $Q(\gamma) = \{y \in S^+_{c_0(\Gamma)} : \gamma \in I(y)\}$ and $Q(\tau(\gamma)) = \{z \in S^+_{c_0(\Delta)} : \tau(\gamma) \in I(z)\}$. It follows from Case 1 that $Q(\tau(\gamma)) = f(Q(\gamma))$. By the proof of Lemma 2.4, we have $P(f(x)) = f(P(x))$. Since $Q(\gamma) \subset P(x)$, $Q(\tau(\gamma)) = f(Q(\gamma)) \subset f(P(x)) \subset P(f(x))$. Therefore, $f(x)_{\tau(\gamma)} = 0 = x_\gamma$.

Case 3. $x_\gamma \in (0, 1)$. Define

$$y_\alpha = \begin{cases} x_\alpha & \alpha \neq \gamma, \\ 0 & \alpha = \gamma. \end{cases}$$

Then $y = (y_\alpha)_{\alpha\in\Gamma} \in S^+_{c_0(\Gamma)}$ and $y + e_\gamma \in S^+_{c_0(\Gamma)}$. Note that

$$\|y - x\| = x_\gamma,\ \|y + e_\gamma - x\| = 1 - x_\gamma.$$

Since f is an isometry, we have

$$|(f(y) - f(x))_{\tau(\gamma)}| \leq \|f(y) - f(x)\| = \|y - x\| = x_\gamma,$$

and

$$|(f(y + e_\gamma) - f(x))_{\tau(\gamma)}| \leq \|f(y + e_\gamma) - f(x)\| = \|y + e_\gamma - x\| = 1 - x_\gamma.$$

By Case 1 and Case 2, we have $f(y)_{\tau(\gamma)} = 0$ and $f(y + e_\gamma)_{\tau(\gamma)} = 1$. Therefore, $f(x)_{\tau(\gamma)} = x_\gamma$ and the proof is complete. □

Now, we are ready to show the main result of this paper.

Theorem 2.9 *Suppose that $f : S^+_{c_0(\Gamma)} \to S^+_{c_0(\Delta)}$ is a surjective isometry, then f can be extended to a linear surjective isometry from $c_0(\Gamma)$ onto $c_0(\Delta)$.*

Proof For $x \in c_0(\Gamma)^+$, define

$$\bar{f}(x) = \begin{cases} 0 & x = 0, \\ \|x\| f(\frac{x}{\|x\|}) & x \neq 0. \end{cases}$$

Then $\bar{f} : c_0(\Gamma)^+ \to c_0(\Delta)^+$ is well-defined. Since f is surjective, $\bar{f}$ is also surjective. In what follows, we show that $\bar{f}$ is an additive isometry.

By Lemma 2.8, there exists a unique bijective mapping $\tau : \Gamma \to \Delta$ such that

$$x_\gamma = f(x)_{\tau(\gamma)}, \text{ for all } x \in S^+_{c_0(\Gamma)}, \ \gamma \in \Gamma.$$

Let $\vartheta = \tau^{-1} : \Delta \to \Gamma$, then we have

$$f(x)_\mu = x_{\vartheta(\mu)}, \text{ for all } x \in S^+_{c_0(\Gamma)}, \ \mu \in \Delta.$$

We assert that for every $x \in c_0(\Gamma)^+$, $\mu \in \Delta$, then

$$\bar{f}(x)_\mu = x_{\vartheta(\mu)}.$$

When $x = 0$, it is obviously true. Assume that $x \neq 0$, by the definition of $\bar{f}$, we have

$$\bar{f}(x)_\mu = \|x\| f\left(\frac{x}{\|x\|}\right)_\mu = \|x\| \left(\frac{x}{\|x\|}\right)_{\vartheta(\mu)} = x_{\vartheta(\mu)}.$$

For $x, y \in c_0(\Gamma)^+$, we get

$$\|\bar{f}(x) - \bar{f}(y)\| = \sup_{\mu \in \Delta} |\bar{f}(x)_\mu - \bar{f}(y)_\mu| = \sup_{\mu \in \Delta} |x_{\vartheta(\mu)} - y_{\vartheta(\mu)}| = \|x - y\|,$$

this implies that $\bar{f}$ is an isometry. For every $\mu \in \Delta$, we obtain

$$f(x+y)_\mu = (x+y)_{\vartheta(\mu)} = x_{\vartheta(\mu)} + y_{\vartheta(\mu)} = f(x)_\mu + f(y)_\mu,$$

this implies that $\bar{f}(x+y) = \bar{f}(x) + \bar{f}(y)$. Thus, $\bar{f}$ is an additive isometry.

Note that $c_0(\Gamma)^+$ is the reproducing cone of $c_0(\Gamma)$, i.e., $c_0(\Gamma)^+ - c_0(\Gamma)^+ = c_0(\Gamma)$. For every $x = (x_\gamma)_{\gamma \in \Gamma} \in c_0(\Gamma)$, let $m(\gamma) = \max\{x(\gamma), 0\}$, $n(\gamma) = \max\{-x(\gamma), 0\}$, then $m = (m_\gamma)_{\gamma \in \Gamma}, n = (n_\gamma)_{\gamma \in \Gamma} \in c_0(\Gamma)^+$ and $x = m - n \in c_0(\Gamma)^+ - c_0(\Gamma)^+ = c_0(\Gamma)$. Define

$$\tilde{f}(x) = \bar{f}(m) - \bar{f}(n),$$

then $\tilde{f} : c_0(\Gamma) \to c_0(\Delta)$ is well-defined. The additivity of $\tilde{f}$ follows from the additivity of $\bar{f}$. For $x, y \in c_0(\Gamma)$, there exist $m^1, n^1, m^2, n^2 \in c_0(\Gamma)^+$ such that $x = m^1 - n^1$, $y = m^2 - n^2$. Thus,

$$\begin{aligned} \|\tilde{f}(x) - \tilde{f}(y)\| &= \|(\bar{f}(m^1) - \bar{f}(n^1)) - (\bar{f}(m^2) - \bar{f}(n^2))\| \\ &= \|(\bar{f}(m^1) + \bar{f}(n^2)) - (\bar{f}(n^1) + \bar{f}(m^2))\| \\ &= \|\bar{f}(m^1 + n^2) - \bar{f}(n^1 + m^2)\| \\ &= \|(m^1 + n^2) - (n^1 + m^2)\| \\ &= \|(m^1 - n^1) - (m^2 - n^2)\| \\ &= \|x - y\|. \end{aligned}$$

This shows that $\tilde{f}$ is an isometry. Combining with the additivity of $\tilde{f}$, we have $\tilde{f}$ is a linear isometry satisfying

$$\tilde{f}|_{S^+_{c_0(\Gamma)}} = f.$$

It is easy to see that $\tilde{f}$ is surjective. The proof is complete. □

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