Indian J Pure Appl Math (2026) 57:434–444
https://doi.org/10.1007/s13226-024-00560-0

ORIGINAL RESEARCH

# Hardy–Littlewood maximal function on Locally compact Abelian groups

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Received: 19 April 2023 / Accepted: 6 February 2024 / Published online: 29 February 2024


**Abstract** In this paper we investigate a boundedness of the Hardy-Littlewood maximal operator $M$ in the variable Lebesgue spaces in the context locally compact abelian group. We show that the local Muckenhoupt condition implies the local boundedness of $M$.

**Keywords** Locally compact abelian groups · maximal function · variable $L^p$ spaces · Muckenhoupt weigts

**Mathematics Subject Classification** 42B25 · 46E30 · 43A70

## 1 Introduction

The purpose of this paper is to investigate the Hardy-Littlewood maximal operator on $L^{p(\cdot)}(G)$ space, when $G$ is a locally compact abelian group (LCA group). Suppose that, $G$ is a locally compact abelian group equipped with a nontrivial and regular measure $\mu$, such that $\mu(K) > 0$ for all compact $K \subset G$. Notice that $\mu$ does not need to be Haar measure, because we do not assume $\mu$ to be translation invariant. Through this work we shall assume that measure $\mu$ has this property. The general assumption on the group will be that it admits a sequence of neighborhoods of 0 with certain properties that we will describe in the next definition (see, [5], Section 2.1).

**Definition 1.1** A collection $\{U_i\}_{i\in\mathbb{Z}}$ is a covering family for $G$ if:
(1) $\{U_i\}_{i\in\mathbb{Z}}$ is an increasing base of relatively compact neighborhoods of 0, $\cup_{k\in\mathbb{Z}}U_k = G$, and $\cap_{k\in\mathbb{Z}}U_k = \{0\}$;
(2) there exists a positive constant $c_D$ and a mapping $\theta : \mathbb{Z} \to \mathbb{Z}$ such that for all $k \in \mathbb{Z}$ and all $x \in G$
a) $k < \theta(k)$,
b) $U_k - U_k \subset U_{\theta(k)}$,
c) $\mu(x + U_{\theta(k)}) \le c_D\mu(x + U_k)$.

We will refer to the condition c) as the doubling property of measure $\mu$ with respect to $\theta$ and we will call $c_D$ the doubling constant. Note that, $c_D \ge 1$ is required by the reason of $U_k \subset U_{\theta(k)}$.

Among the well known groups satisfying Definition 1.1 are the groups $\mathbb{R}$, $\mathbb{Z}$, torus $\mathbb{T}$, $p$-adic group $\mathbb{Q}_p$ and finite products of these groups (for details see [5]). Moreover, if $G$ is a locally compact abelian group with

Communicated by E. K. Narayanan.

This work was supported by Shota Rustaveli National Science Foundation of Georgia FR-22 17770

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equipped nontrivial Haar measure and with an increasing sequence $\{U_k\}_{k\in\mathbb{Z}}$ of compact open subgroups, such that

$$\bigcup_{k\in\mathbb{Z}} U_k = G, \quad \bigcap_{k\in\mathbb{Z}} U_k = \{0\},$$

then Definition 1.1 satisfies if and only if $\sup_{k\in\mathbb{Z}} |U_k/U_{k-1}| < \infty$, and one may take $\theta(k) = k+1$.

We will call any set of the form $x + U_k$, $x \in G$, $k \in \mathbb{Z}$ a base set and the collection of all base sets will be denoted by $\mathcal{B}$. For any natural number $n$ we denote $\theta^n(k) = \theta(\theta^{n-1}(k)), n > 1,\ \theta^1(k) = \theta(k),\ k \in \mathbb{Z}$. For $V = x + U_j$ base set we denote $V^* = x + U_{\theta(k)}$ and $V^{**} = x + U_{\theta^2(k)}$. Observe that we can assume without loss of generality the base sets to be symmetric. This is not a restriction at all because one can always consider the new family of base sets formed by the difference sets $U_i - U_i$ which increases the doubling constant from $c_D$ to $c_D^2$. We denote $2U_k := U_k - U_k = U_k + U_k$. We also may assume that the function $\theta$ be non-decreasing. Indeed if we replace $\theta$ by

$$\widetilde{\theta}(k) = \min\{l \in \mathbb{Z} :\ l > k \text{ with } U_k - U_k \subset U_l\},$$

we obtain that $\widetilde{\theta}$ function is non-decreasing, $\widetilde{\theta}(k) \le \theta(k),\ k \in \mathbb{Z}$ and for all $x \in G$ and $k \in \mathbb{Z}$

$$\mu(U_{\widetilde{\theta}(k)}) \le \mu(U_{\theta(k)}) \le c_D \mu(U_k).$$

The notion of base sets allows to define a direct similar of the Hardy-Littlewood maximal function (uncentered):

$$Mf(x) = \sup_{x\in U\in\mathcal{B}} \frac{1}{\mu(B)} \int_B |f| d\mu, \quad f \in L^1_{loc}(G).$$

Likewise we may define the centered maximal function for $f \in L^1_{loc}(G)$

$$\mathcal{M}f(x) = \sup_{k\in\mathbb{Z}} \frac{1}{\mu(x+U_k)} \int_{x+U_k} |f| d\mu.$$

Note that uncentered maximal operator is comparable to the centered maximal operator (see [11], Lemma 1). Accurately, for $f \in L^1_{loc}(G)$ we have

$$\mathcal{M}f \le Mf \le c_D^2 \mathcal{M}f. \tag{1.1}$$

The boundedness of $M$ in $L^p(G)$, when measure $\mu$ is regular (not necessary translation invariant) with doubling property with respect to $\theta$ was considered in [11]. In this paper proved a weighted norm inequalities for Hardy-Littlewood maximal function for LCA group following Lerner's approach.

**Theorem 1.2** *( [11]) Let $1 < p \le \infty$. Then the maximal operator $M$ is bounded in $L^p(G)$. Furthermore, for $f \in L^1(G)$, we have*

$$\mu(\{x \in G : Mf(x) > t\}) \le \frac{C}{t} \|f\|_1, \quad t > 0. \tag{1.2}$$

Given a function $p(\cdot)\colon G \to [1,\infty)$, denote by $L^{p(\cdot)}(G)$ the space of measurable functions $f$ such that for some $\lambda > 0$,

$$\int_G |f(x)/\lambda|^{p(x)} d\mu(x) < \infty,$$

with (Luxemburg) norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 :\ \int_G |f(x)/\lambda|^{p(x)} d\mu(x) \le 1\right\}.$$

These spaces have been intensively studied for the past twenty years: for detailed histories and references see monographs [1], [2].

Given a measurable function $p(\cdot)\colon G \to [1,\infty)$, let us denote $p_- = \operatorname{ess\,inf}_{x\in G} p(x)$ and $p_+ = \operatorname{ess\,sup}_{x\in G} p(x)$. By $p'_+$ and $p'_-$ we denote $(p')_+$ and $(p')_-$. For the variable exponent $p(\cdot)$ we define the conjugate exponent by the formula $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

One of the main problems of Harmonic Analysis on variable Lebesgue spaces is the problem of the boundedness of the Hardy-Littlewood maximal operator $M$ on the space $L^{p(\cdot)}$. For a detailed history of this problem in the Euclidean setting may found in monographs Cruz-Uribe and Fiorenza [1] and Diening et.al. [2].

**Definition 1.3** We say that exponent $p(\cdot)$, $1 < p_- \le p_+ < \infty$ satisfies $\mathcal{A}_{loc}(G)$ condition if there exists a positive constant $C > 0$ such that, for any base set $U \in \mathcal{B}$

$$\frac{1}{\mu(U)}\|\chi_U\|_{p(\cdot)}\|\chi_U\|_{p'(\cdot)} \le C. \tag{1.3}$$

Note that condition (1.3) would interpreted as a full analogue of the classical Muckenhoupt $A_p$ condition in the context of $L^{p(\cdot)}$ spaces (see for Euclidean setting [7], [9]).

In case $X = \mathbb{R}^n$ and $\mu$ Lebesgue measure Kopaliani [7], showed that

$$M\colon L^{p(\cdot)}(\mathbb{R}^n) \to L^{p(\cdot)}(\mathbb{R}^n)$$

provided $p(\cdot)$ is constant and bounded outside of 1 and $\infty$ large ball if and only if $p(\cdot) \in \mathcal{A}_{loc}(\mathbb{R}^n)$. However, it has been shown in [8] (see, also [2, Theorem 5.3.4] and [1, Example 4.51]) by a counterexample that without additional condition at infinity does not imply that $M\colon L^{p(\cdot)}(\mathbb{R}^n) \to L^{p(\cdot)}(\mathbb{R}^n)$.

A. Lerner [9], in Euclidean setting ($X = \mathbb{R}^n$, $\mu$-Lebesgue measure), proved the following theorem.

**Theorem 1.4** *Let $1 < p_- \le p_+ < \infty$, $p \in \mathcal{A}_{loc}(\mathbb{R}^n)$ and $E \subset \mathbb{R}^n$ be a measurable set of positive finite measures. Then there exists a constant $c > 0$ depending on $p, n$ and $E$ such that, for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,*

$$\|(Mf)\chi_E\|_{p(\cdot)} \le c\|f\|_{p(\cdot)}.$$

In this paper we will prove an analogue of the above theorem for $L^{p(\cdot)}(G)$ setting. Namely we prove the following theorem.

**Theorem 1.5** *Suppose $1 < p_- \le p_+ < \infty$ and $p \in \mathcal{A}_{loc}(G)$. Let $E \subset G$ be a $\mu$-measurable set such that $\chi_E \in L^{p(\cdot)}(G)$. Then there exists a constant $c > 0$ depending on $p$ and $E$ such that for any $f \in L^{p(\cdot)}(G)$,*

$$\|(Mf)\chi_E\|_{p(\cdot)} \le c\|f\|_{p(\cdot)}.$$

The paper is organized as follows. In Section 2 we have some preliminary results. We define the $A_\infty(G)$ class of weights and study some of their properties. In Section 3 we will prove the local boundedness of $M$ when $p(\cdot) \in \mathcal{A}_{loc}(G)$. We use C, c as an absolute positive constants, which may have different values in different occurrences.

## 2 Preliminary results

Some basic properties of covering family follow directly from Definition 1.1. For instance (i) for every $x \in G$ and $k \in \mathbb{Z}$ it holds $\mu(x + U_k) > 0$; (2) the interiors of the base sets $U_k$ cover $G$, i.e., $\cup_{k\in\mathbb{Z}} U'_k = G$. In particular, for every compact $K \subset G$ there is $k \in \mathbb{Z}$ such that $K \subset U_k$ (see [11], Proposition 1).

In [6], Theorem 44.18 it is proven a version of the Lebesgue differentiation theorem with respect to Haar measure for LCA groups having a $D'$ sequence (see [6], Definition 44.10). Note that the result is still true which the obvious changes for measures with are not translation invariant. Thus, since a covering family is in particular $D'$ sequence, we have that the Lebesgue Differentiation theorem holds in our context (see for details [10], Remak 2.6).

For a given $V \in \mathcal{B}$, we will denote by $j(V)$ the maximum integer such that $V = x + U_{j(V)}$ for some $x \in G$. Such a number exists (see for details [10], Remark 2.3).

Let $U \in \mathcal{B}$ be a fixed base set and $k = j(U)$. The local base $\mathcal{B}_U$ is defined as

$$\mathcal{B}_U = \{y + U_j : y \in U,\ j \le k\}.$$

We also defined the enlarged set $\widehat{U}$ by the formula

$$\widehat{U} = \bigcup_{V \in \mathcal{B}_U} V.$$

We have the following geometric properties of $\widehat{U}$ (see [10]): For any $z \in U$, $\widehat{U} \subset z + U_{\theta^2(k)}$ and as a consequence of this property, we have

$$\mu(\widehat{U}) \leq \mu(z + U_{\theta^2(k)}) \leq C_D^2 \mu(z + U_k). \tag{2.1}$$

Any covering family has the so-called engulfing property:

**Lemma 2.1** *( [10], Lemma 2.2) Let $U$, $V$ be two base sets such that $U = x + U_i$ and $V = y + U_j$ with $i \leq j$ and $x, y \in G$. If $U \cap V \neq \emptyset$, then $x + U_i \subset y + U_{\theta^2(j)}$.*

We need following covering lemma. Note that Lemma 2.2 has proven in [5] in the case of an translation invariant measure $\mu$. For regular measure $\mu$ with doubling property with respect to $\theta$ Lemma 2.2 proved in [11].

**Lemma 2.2** *(see [11], Lemma 2) Let $E$ be a subset of $G$ and $k : E \to \mathbb{Z}$ a mapping bounded from above, such that for every $k_0 \in \mathbb{Z}$ the set $\{x \in E : k(x) \geq k_0\}$ is relatively compact in $G$. Then there is a sequence $(x_n) \subset E$, finite or infinite, such that*
*(i) the sequence $(k_n) := (k(x_n))$ is decreasing,*
*(ii) the sets $x_n + U_{k_n}$ are pairwise disjoint and*
*(iii) $E \subset \bigcup(x_n + 2U_{k_n})$.*

We now define the local maximal function as follows

$$M_U f(x) = \sup_{x \in V \in \mathcal{B}_U} \frac{1}{\mu(V)} \int_V |f(y)| d\mu$$

for any $x \in \widehat{U}$ and $M_U f(x) = 0$ otherwise.

Consider for a fixed $U \in \mathcal{B}$, the level set for the local maximal function acting on a function $w$ at sale $\lambda > 0$ :

$$\Omega_\lambda = \{x \in \widehat{U} : \; M_U w(x) > \lambda\}.$$

A basic method for investigating level sets is the Calderón-Zygmund decomposition of the set $\Omega_\lambda$.

**Lemma 2.3** *( [10], Lemma 2.5) Let $U \in \mathcal{B}$ be a fixed base set in $G$ and $w$ be a nonnegative and integrable function supported on $\widehat{U}$ and $\lambda > w_{\widehat{U}}$. If $\Omega_\lambda$ is nonempty, there exists a finite or countable index set $Q$ and family $\{y_i + U_{\alpha_i}\}_{i \in Q}$ of pairwise disjoint base sets from $\mathcal{B}_U$ such that:*
*(a) The sequence $\{\alpha_i\}_{i \in Q}$ is decreasing.*
*(b) $\bigcup_{i \in Q} y_i + U_{\alpha_i} \subset \Omega_\lambda \subset \bigcup_{i \in Q} y_i + U_{\theta^2(\alpha_i)}$.*
*(c) For any $i \in Q$, we have that*

$$\lambda < \frac{1}{\mu(y_i + U_{\alpha_i})} \int_{y_i + U_{\alpha_i}} w d\mu. \tag{2.2}$$

*(d) Given $r > \alpha_i$ for some $i \in Q$, then*

$$\frac{1}{\mu(y_i + U_r)} \int_{y_i + U_r} w d\mu \leq c_D^2 \lambda. \tag{2.3}$$

From part (b) of Lemma 2.3 and Lebesgue differentiation theorem we obtain that

$$w(x) \leq \lambda \;\text{ for }\; x \in \widehat{U} \setminus \bigcup_{i \in Q} y_i + U_{\theta^2(\alpha_i)}. \tag{2.4}$$

A nonnegative, locally integrable function is called a weight. Below we will use the standard notations: for any measurable set $E$ we denote

$$sw(E) = \int_E w d\mu, \;\text{ and }\; f_E = \frac{1}{\mu(E)} \int_E f d\mu.$$

It is well known fact that the class $A_\infty$ weights in Euclidean case can be defined in many equivalent ways. Most of these results can be found in [3]. We introduce in the context LCA group the following definition of $A_\infty$ weights.

**Definition 2.4** We say that a weight $w$ satisfies $A_\infty$ condition ($w \in A_\infty(G)$) if there are positive constants $\alpha, \beta < 1$ such that whenever $E$ is a measurable set of a base set $V$

$$\frac{\mu(E)}{\mu(V)} < \alpha \quad \text{implies} \quad \frac{w(E)}{w(V)} < \beta. \tag{2.5}$$

In [10] the authors use another definition $A_\infty$ weights for LCA groups. In Euclidean setting this class of weights was originally introduced by Fujii [4] and later by Wilson [12]. In this paper the authors obtained sharp quantitative weighted inequalities for the Hardy-Littlewood maximal operator in context of LCA groups, obtaining an improved version of so-called Buckley's theorem. The authors obtained reverse Hölder inequality for Muckenhoupt $A_\infty$ weights and provide a valid version of the "open property" for Muckenhoupt $A_p$ weights.

We prove following version of reverse Hölder inequality.

**Theorem 2.5** *(Weak reverse Hölder inequality) Let $w \in A_\infty(G)$. Then there are positive constants $c$ and $\delta$ such that for every base set $V$*

$$\left(\frac{1}{\mu(\widehat{V})}\int_{\widehat{V}} w^{1+\delta} d\mu\right)^{\frac{1}{1+\delta}} \le c\frac{1}{\mu(\widehat{V})}\int_{\widehat{V}} w d\mu.$$

**Corollary 2.6** *Let $w \in A_\infty(G)$. Then there are positive constants $c$ and $\delta$ such that for every base set $V$*

$$\left(\frac{1}{\mu(V)}\int_{V} w^{1+\delta} d\mu\right)^{\frac{1}{1+\delta}} \le c\frac{1}{\mu(\widehat{V})}\int_{\widehat{V}} w d\mu. \tag{2.6}$$

*Proof.* By (2.1) this is obvious.

**Lemma 2.7** *Let $w \in A_\infty(G)$. Then there are constants $0 < b, c < 1$ such that for every base set $V \in \mathcal{B}$*

$$\mu(\{x \in V : w(x) \le bw_V\}) \le c\mu(V). \tag{2.7}$$

*Proof* First observe that condition (2.5) is equivalent to that, there are positive constants $\alpha', \beta' < 1$, such that whenever $E$ is a measurable set of a $V$ base set

$$\frac{w(E)}{w(V)} < \alpha' \quad \text{implies} \quad \frac{\mu(E)}{\mu(V)} < \beta'. \tag{2.8}$$

Let $E = \{x \in V : w(x) > bw_V\}$ where $b \in (0, 1)$ is going to be chosen now and let $E' = V \setminus E = \{x \in V : w(x) \le bw_V\}$. Then $w(E') \le bw_V\mu(E') \le bw(V)$. Next if we take $b = \beta'$ we have that $\mu(E') \le \alpha'\mu(V)$. This yields (2.7). □

**Lemma 2.8** *Let $w \in A_\infty(G)$. There are positive constants $C$ and $b$ such that for every base set $V \in \mathcal{B}$ and for every $\lambda > w_{\widehat{V}}$*

$$w(\{x \in \widehat{V} : w(x) > \lambda\}) \le C\lambda\mu(\{x \in \widehat{V} : w(x) > b\lambda\}). \tag{2.9}$$

*Proof* Let $\lambda > w_{\widehat{V}}$. We may consider the Calderón-Zygmund decomposition of the set $\Omega_\lambda = \{x \in \widehat{V} : M_V w(x) > \lambda\}$. By Lemma 2.3 we find a family of pairwise disjoint base sets $\{V_i\}_{i\in Q} = \{y_i + U_{\alpha_i}\}_{i\in Q}$ from $\mathcal{B}_V$ satisfying

$$\bigcup_{i\in Q} V_i \subset \Omega_\lambda \subset \bigcup_{i\in Q} V_i^{**}$$

such that

$$\lambda < \frac{1}{\mu(V_i)}\int_{V_i} w d\mu, \quad \text{and} \quad \frac{1}{\mu(V_i^{**})}\int_{V_i^{**}} w d\mu \le c_D^2\lambda$$

for each $i$. Using (2.7) we obtain

$$\mu(V_i) \le \frac{1}{1-\alpha}\mu(\{x \in V_i : w(x) > bw_{V_i}\}).$$

By these properties of the $V_i$ sets combined with (2.4) we have

$$
\begin{aligned}
& w(\{x \in \widehat{V} : w(x) > \lambda\}) \le \sum_{i\in Q} w(V_i^{**}) \le c_D^2 \lambda \sum_{i\in Q} \mu(V_i^{**}) \le c_D^4 \lambda \sum_{i\in Q} \mu(V_i) \\
& \le \frac{c_D^4 \lambda}{1-c} \sum_{i\in Q} \mu(\{x \in V_i : w(x) > b w_{V_i}\}) \le C\lambda \sum_{i\in Q} \mu(\{x \in V_i : w(x) > b\lambda\}) \\
& \le C\lambda \mu(\{x \in \widehat{V} : w(x) > b\lambda\}) \le C\lambda \mu(\{x \in \widehat{V} : w(x) > b\lambda\}).
\end{aligned}
$$

□

*Proof of Theorem 2.5* For arbitrary base set $V$ and positive $\delta$ we apply (2.9) to obtain estimate

$$
\begin{aligned}
& \frac{1}{\mu(\widehat{V})} \int_{\widehat{V}} w^{1+\delta} d\mu = \frac{1}{\mu(\widehat{V})} \int_{\widehat{V}} w^{\delta} w d\mu = \frac{\delta}{\mu(\widehat{V})} \int_0^\infty \lambda^{\delta-1} w(\{x \in \widehat{V} : w(x) > \lambda\}) d\lambda \\
& = \frac{\delta}{\mu(\widehat{V})} \int_0^{w_{\widehat{V}}} \lambda^{\delta-1} w(\{x \in \widehat{V} : w(x) > \lambda\}) d\lambda + \frac{\delta}{\mu(\widehat{V})} \int_{w_{\widehat{V}}}^\infty \lambda^{\delta-1} w(\{x \in \widehat{V} : w(x) > \lambda\}) d\lambda \\
& \le \frac{w(\widehat{V})}{\mu(\widehat{V})} \delta \int_0^{w_{\widehat{V}}} \lambda^{\delta-1} d\lambda + \frac{C\delta}{\mu(\widehat{V})} \int_{w_{\widehat{V}}}^\infty \lambda^{\delta} w(\{x \in \widehat{V} : w(x) > b\lambda\}) d\lambda \\
& \le (w_{\widehat{V}})^{1+\delta} + \frac{C\delta}{b^{1+\delta}\mu(\widehat{V})} \int_{b w_{\widehat{V}}}^\infty \lambda^{\delta} w(\{x \in \widehat{V} : w(x) > \lambda\}) d\lambda \\
& \le (w_{\widehat{V}})^{1+\delta} + \frac{C\delta}{b^{1+\delta}\mu(\widehat{V})} \int_{\widehat{V}} w^{1+\delta} d\mu.
\end{aligned}
$$

If we choose $\delta$ small enough such that $\frac{C\delta}{b^{1+\delta}} < 1$ the last term can be absorbed by the first term of the string of inequalities. □

**Lemma 2.9** *Let $w \in A_\infty(G)$. Then there are positive constants c and $\rho$ such that for any base set V and any measurable set E contained in V*

$$
w(E) \le c \left( \frac{\mu(E)}{\mu(V)} \right)^{\rho} w(\widehat{V}). \tag{2.10}
$$

*Proof* Let $E \subset V$. If we use Hölder inequality with $r = 1+\delta$, (2.1) and (2.6), we obtain

$$
\begin{aligned}
& \frac{w(E)}{\mu(V)} = \frac{1}{\mu(V)} \int_V \chi_E w d\mu \le \left( \frac{1}{\mu(V)} \int_V w^r \right)^{1/r} \left( \frac{\mu(E)}{\mu(V)} \right)^{1/r'} \\
& \le \frac{C}{\mu(\widehat{V})} \int_{\widehat{V}} w d\mu \left( \frac{\mu(E)}{\mu(V)} \right)^{1/r'}.
\end{aligned}
$$

This implies (2.10) with $\rho = 1/r'$.

Below we need the local variant of Lemma 2.9.

**Definition 2.10** For fixed base set $U = x + U_k$ we say that a weight $w$ satisfies weak $A_\infty(U)$ condition ($w \in WA_\infty(U)$) if there are positive constants $\alpha, \beta < 1$ such that for any base set $V \in \mathcal{B}_U$ and for any measurable subset $E \subset V$

$$
\frac{\mu(E)}{\mu(V)} < \alpha \quad \text{implies} \quad \frac{w(E)}{w(V)} < \beta.
$$

**Lemma 2.11** *If a weight w satisfies weak $A_\infty(U)$ condition, then there are constant $C > 0$ and $\gamma > 0$ such that*

$$
w(E) \le C \left( \frac{\mu(E)}{\mu(V)} \right)^{\gamma} w(\widehat{V}) \tag{2.11}
$$

*for each base set $V \subset U$ and each measurable set $E \subset V$.*

*Proof* Proof is the same as in Lemma 2.9. □

## 3 Proof of Theorem 1.5

Given a base set $V \in \mathcal{B}$ define the averaging operator $A_V$ by

$$A_V f(x) = \frac{1}{\mu(V)} \int_V f d\mu \, \chi_V(x).$$

**Proposition 3.1** *Given an exponent $p(\cdot)$, $1 < p_- \le p_+ < \infty$, there exists a constant $C > 0$ such that for any base set $V \in \mathcal{B}$*

$$\|A_V f\|_{p(\cdot)} \le C \|f\|_{p(\cdot)} \tag{3.1}$$

*if and only if $p(\cdot) \in \mathcal{A}_{loc}(G)$.*

The proof of Proposition 3.1 is essentially the same as for averaging operator defined by cubes for Euclidean setting (see for example [1], Proposition 4.47).

Lemma 3.2 shows that the condition $p(\cdot) \in \mathcal{A}_{loc}(G)$ is actually sufficient for modular inequality. Similar estimate for the case $L^{p(\cdot)}(\mathbb{R}^n)$ was obtained by Kopaliani [7].The proof in [7] is based on some concepts from convex analysis. Lerner in [9] gave a different and simple proof. In this paper our approach is based on the adaptation of Lerner's proof [9].

**Lemma 3.2** *Given exponent $p(\cdot)$ such that $1 < p_- \le p_+ < \infty$, suppose $p(\cdot) \in \mathcal{A}_{loc}(G)$. Let $f \in L^{p(\cdot)}(G)$. If there exists a base set $V \in \mathcal{B}$ and constants $c_1, c_2 > 0$ such that $|f|_V \ge c_1$ and $\|f\|_{p(\cdot)} \le c_2$, where $c_1, c_2 > 0$, then there exists a constant $c$ depending only on $p(\cdot), c_1, c_2$ such that*

$$\int_V (|f|_V)^{p(x)} d\mu(x) \le c \int_V |f(x)|^{p(x)} d\mu(x).$$

*Proof* Using the condition $p_+ < \infty$ we may consider only the case $c_1 = c_2 = 1$. Since $p'_+ < \infty$, by continuity there exists $\alpha > 0$ such that

$$\int_V \alpha^{p'(y)-1} d\mu(y) = \int_V |f(x)| d\mu(x). \tag{3.2}$$

Since $|f|_V \ge 1$, we have $\alpha \ge 1$. By generalized Hölder inequality

$$\int_V f(x) d\mu(x) \le 2 \|f\|_{p(\cdot)} \|\chi_V\|_{p'(\cdot)}$$

we get $\int_V \alpha^{p'(y)-1} d\mu(y) \le 2\|\chi_V\|_{p'(\cdot)}$ and consequently,

$$\alpha \le c / \|\chi_V\|_{p'(\cdot)}. \tag{3.3}$$

Given this value $\alpha$, we have that

$$\int_V (|f|_V)^{p(x)} d\mu(x) = \int_V \left( \frac{1}{\mu(V)} \int_V \alpha^{p'(y)-1} d\mu(y) \right)^{p(x)} d\mu(x) \tag{3.4}$$

$$= \left( \frac{1}{\mu(V)} \int_V \left( \frac{1}{\mu(V)} \int_V \alpha^{p'(y)-p'(x)} d\mu(y) \right)^{p(x)-1} d\mu(x) \right) \int_V \alpha^{p'(y)} d\mu(y).$$

For each $x \in V$ partition $V$ into $E_1(x) = \{y \in V : \; p'(y) > p'(x)\}$ and $E_2(x) = V \backslash E_1(x)$. Using (3.3) and the estimate $\alpha \ge 1$, we obtain

$$\begin{aligned} \int_V \alpha^{p'(y)-p'(x)} d\mu(y) &= \int_{E_1(x)} \alpha^{p'(y)-p'(x)} d\mu(y) + \int_{E_2(x)} \alpha^{p'(y)-p'(x)} d\mu(y) \\ &\le c (\|\chi_V\|_{p'(\cdot)})^{p'(x)} + \mu(V). \end{aligned}$$

In view of $p(\cdot) \in A_{loc}(G)$, we have

$$\frac{1}{\mu(V)} \int_V \left( \frac{1}{\mu(V)} \int_V \alpha^{p'(y)-p'(x)} d\mu(y) \right)^{p(x)-1} d\mu(x) \tag{3.5}$$

$$\begin{aligned}
&\le c \frac{1}{\mu(V)} \int_V \left( \frac{1}{\mu(V)} (\|\chi_V\|_{p'(\cdot)})^{p'(x)} + 1 \right)^{p(x)-1} d\mu(x) \\
&\le c + c \frac{1}{\mu(V)} \int_V \left( \frac{1}{\mu(V)} (\|\chi_V\|_{p'(\cdot)})^{p'(x)} \right)^{p(x)-1} d\mu(x) \\
&\le c + c \int_V \left( \frac{\|\chi_V\|_{p'(\cdot)}}{\mu(V)} \right)^{p(x)} d\mu(x) \\
&\le c + c \int_V \left( \frac{1}{\|\chi_V\|_{p(\cdot)}} \right)^{p(x)} d\mu(x) \le c.
\end{aligned}$$

Further,

$$\int_V \alpha^{p'(y)} d\mu(y) = 2\alpha \int_V |f(x)| d\mu(x) - \int_V \alpha^{p'(y)} d\mu(y) \tag{3.6}$$

$$\begin{aligned}
&\le 2\alpha \int_{\{y \in V : 2\alpha |f(y)| > \alpha^{p'(y)}\}} |f(y)| d\mu(y) \\
&\le c \int_V |f(y)|^{p(y)} d\mu(y).
\end{aligned}$$

From (3.4), (3.5) and (3.6) we obtain desired estimate. □

**Corollary 3.3** *Let $1 < p_- \le p_+ < \infty$ and $p(\cdot) \in \mathcal{A}_{loc}(G)$. Suppose for base set $V$ we have $\xi_1 \le t \le \xi_2/\|\chi_V\|_{p(\cdot)}$, where $\xi_1, \xi_2 > 0$. Then $t^{p(x)} \in WA_\infty(V)$ with constant in inequality (2.11) depending only on $p(\cdot)$, $\xi_1, \xi_2$.*

*Proof* Let $V_0 \subset \widehat{V}$. Let $E \subset V_0$ be any measurable subset with $\mu(E) \ge \mu(V_0)/2$. Define $f = t\chi_E$. Then

$$|f|_{V_0} = \frac{1}{\mu(V_0)} \int_{V_0} t\chi_E(x) d\mu(x) = t \frac{\mu(E)}{\mu(V_0)} \ge \frac{\xi_1}{2},$$

$$\|f\|_{p(\cdot)} = t\|\chi_E\|_{p(\cdot)} \le \xi_2 \frac{\|\chi_E\|_{p(\cdot)}}{\|\chi_V\|_{p(\cdot)}} \le \xi_2.$$

Therefore, $f$ satisfies the hypotheses of Lemma 3.2 with $c_1 = \xi_1/2$, $c_2 = \xi_2$ and there exists a constant $c$ depending only on $p(x), \xi_1, \xi_2$ such that

$$\frac{1}{2^{p_+}} \int_{V_0} t^{p(x)} d\mu(x) \le c \int_E t^{p(x)} d\mu(x).$$

Consequently, there exists constant $C > 1$ such that for any subset $E \subset V_0$, $\mu(E) \ge \mu(V_0)/2$ we have

$$\int_{V_0} t^{p(x)} d\mu(x) \le C \int_E t^{p(x)} d\mu(x). \tag{3.7}$$

Define $d\nu(x) = t^{p(x)} d\mu(x)$. From (3.7) we obtain that for any subset $E \subset V_0$, $\mu(E) < \mu(V_0)/2$ we have $\nu(E) \le (1 - 1/C)\nu(V_0)$. Consequently $t^{p(x)} \in WA_\infty(V)$.

**Corollary 3.4** *Let $1 < p_- \le p_+ < \infty$ and $p(\cdot) \in \mathcal{A}_{loc}(G)$. Then there are constants $C, \gamma > 0$ such that for any base set $V$ and measurable subset $E \subset V$*

$$\frac{\int_E t^{p(x)} d\mu(x)}{\int_V t^{p(x)} d\mu(x)} \le C \left( \frac{\mu(E)}{\mu(V)} \right)^\gamma. \tag{3.8}$$

*Proof* From Theorem 2.11 we obtain that there are constant $C, \gamma > 0$ such that

$$\int_E t^{p(x)} d\mu(x) \leq C \left(\frac{\mu(E)}{\mu(V)}\right)^{\gamma} \int_{\widehat{V}} t^{p(x)} d\mu(x) \tag{3.9}$$

for each base set $V$ and subset $E \subset V$.

Note that $V \subset \widehat{V} \subset V^{**}$ and $\mu(V) \leq \mu(\widehat{V}) \leq \mu(V^{**}) \leq c_D^2 \mu(V)$ (see (2.1)). As in proof of Corollary 3.3 we may find constants $C_1, C_2 > 0$ such that

$$C_1 \int_V t^{p(x)} d\mu(x) \leq \int_{\widehat{V}} t^{p(x)} d\mu(x) \leq C_2 \int_V t^{p(x)} d\mu(x).$$

From (3.9) we obtain desired inequality.

*Proof of Theorem 1.5.* We consider only the case $\mu(G) = \infty$, the case $\mu(G) < \infty$ needs minor modification. Suppose $f \in L^{p(\cdot)}(G)$ and $\chi_E \in L^{p(\cdot)}(G)$. We can prove the desired inequality for a non-negative function $f$ which are bounded and have a compact support.

Suppose $f \in L^{p(\cdot)}(G)$ and $\|f\|_{p(\cdot)} \leq 1$. It is sufficient to proof that there exists a positive constant $C$ (independent of $f$) such that for any nonnegative function $g \in L^{p'(\cdot)}(G)$, with support on $E$, $\|g\|_{p'(\cdot)} \leq 1$

$$\int_G Mf(x)g(x)d\mu \leq C. \tag{3.10}$$

Let $A = c_D^2 + 1$. For each integer $k$, set

$$\Omega_k = \{x \in G \ : \ \mathcal{M}f(x) > A^k\}.$$

Define $D_k = \Omega_k \backslash \Omega_{k+1}$. Let $F_k$ be an arbitrary compact subset of $D_k$. We will prove that

$$\int_{\cup F_k} Mf(x)g(x)d\mu \leq C. \tag{3.11}$$

By simple limiting argument from (3.11) we obtain (3.10).

Let $\mu(F_k) > 0$. define $l : F_k \to \mathbb{Z}$ function

$$l(x) = \max \left\{k \in \mathbb{Z} : \ \frac{1}{\mu(x+U_k)} \int_{x+U_k} |f| d\mu > \lambda \right\}.$$

This mapping is certainly well-defined, note also that the mapping $l$ is bounded from above (see more details in [11] proof of Theorem 3).

Without loss of generality we may assume that $\mu(F_k) > 0$ for all $k \in \mathbb{Z}$. For $x \in F_k$ there exists a base set $V\alpha_x = x + U_{\alpha_x}$, such that $|f|_{V_{\alpha_x}} > A^k$. We may select from collection $\{V_{\alpha_x}, x \in F_k\}$, finite or infinite of pairwise disjoint base sets $V_{k_j}$, such that $F_k \subset \bigcup_j V_{k_j}^{**}$,(see Lemma 2.2).

Define the sets

$$E_1^k = V_{k_1}^{**} \cap F_k, \ E_j^k = (V_{k_j}^{**} \backslash \bigcup_{s<j} V_{k_s}^{**}) \cap F_k, \ j > 1.$$

Note that the sets $E_j^k$ are pairwise disjoint and $\cup_j E_j^k = F_k$.

Define

$$Tg(x) = \sum_{k \in \mathbb{Z}}^{\infty} \sum_j \left(\frac{1}{\mu(V_{k_j})} \int_{E_j^k} g d\mu\right) \chi_{V_{k_j}}(x).$$

Using the above definition, we get

$$\int_{\cup_k F_k} (Mf)(x)g(x)d\mu \leq A^{k+1} \sum_{k \in \mathbb{Z}}^{\infty} \sum_j \int_{E_j^k} g d\mu \leq A \sum_{k=1}^{\infty} \sum_j f_{V_{k_j}} \int_{E_j^k} g d\mu$$

$$= A \int_G f T g \le 2A \|f\|_{p(\cdot)} \|Tg\|_{p'(\cdot)},$$

and consequently for proving(3.11), it is sufficient to show that $\|Tg\|_{p'(\cdot)} \le C$.

Note that $V_{k_j} \subset \Omega_k = \cup_{l=0}^{\infty} D_{k+l}$ and hence $Tg = \sum_{l=0}^{\infty} T_l g$, where

$$T_l g(x) = \sum_{k\in\mathbb{Z}}^{\infty} \sum_j a_{j,k}(g) \chi_{V_{k_j} \cap D_{k+l}}(x), \quad (l = 0, 1, ...)$$

where $\alpha_{j,k}(g) = \frac{1}{\mu(V_{k_j})} \int_{E_j^k} g d\mu$.

Let $\mathcal{I}_1 = \{(j,k) \ : \ \alpha_{j,k}(g) > 1\}$ and $\mathcal{I}_2 = \{(j,k) \ : \ \alpha_{j,k}(g) \le 1\}$.

Note that the condition $p \in \mathcal{A}_{loc}(G)$ is equivalent to $p' \in \mathcal{A}_{loc}(G)$. The last condition implies that, for any base set $V$ $\|\chi_{V^{**}}\|_{p'(\cdot)} \le C \|\chi_V\|_{p'(\cdot)}$ (it is sufficient take use (3.1) inequality for $p'(\cdot)$, $U = V^{**}$ and $f = \chi_V$ ).

We have

$$\alpha_{j,k}(g) \le \frac{2}{\mu(V_{k_j})} \|\chi_{E_j^k}\|_{p(\cdot)} \|g \chi_{E_j^k}\|_{p'(\cdot)} \le \frac{2}{\mu(V_{k_j})} \|\chi_{V_{k_j}^{**}}\|_{p(\cdot)}$$
$$\le \frac{C}{\|\chi_{V_{k_j}^{**}}\|_{p'(\cdot)}} \le \frac{C}{\|\chi_{V_{k_j}}\|_{p'(\cdot)}}.$$

Let $(j,k) \in \mathcal{I}_1$. Then by Corollary 3.4 $\alpha_{j,k}(g)^{p'(x)} \in WA_\infty(V_{k_j})$ and by Lemma 3.2,

$$\int_{V_{k_j} \cap D_{k+l}} \alpha_{j,k}(g)^{p'(x)} d\mu \le C \left( \frac{\mu(V_{k_j} \cap D_{k+l})}{\mu(V_{k_j})} \right)^\gamma \int_{V_{k_j}} \alpha_{j,k}(g)^{p'(x)} d\mu$$

$$\le C \left( \frac{\mu(V_{k_j} \cap D_{k+l})}{\mu(V_{k_j})} \right)^\gamma \int_{E_j^k} g(x)^{p'(x)} d\mu. \tag{3.12}$$

If $(j,k) \in \mathcal{I}_2$, then we have

$$\int_{V_{k_j} \cap D_{k+l}} \alpha_{j,k}(g)^{p'(x)} d\mu \le \int_{V_{k_j} \cap D_{k+l}} \alpha_{j,k}(g) d\mu$$

$$= \frac{\mu(V_{k_j} \cap D_{k+l})}{\mu(V_{k_j})} \int_{E_j^k} g(x) d\mu. \tag{3.13}$$

We need estimate $\mu(V_{k_j} \cap D_{k+l})$ for $l \ge 2$. Let $x \in V_{k_j}$ and $V$ be an arbitrary base set such that $x \in V$. Observe that either $V \subset V_{k_j}^{**}$ or $V_{k_j} \subset V^{**}$ (see Lemma 2.1). If the second inclusion holds, then $V^{**} \cap D_k \ne \emptyset$ and hence

$$|f|_V \le c_D^2 |f|_{V^{**}} \le c_D^2 \cdot A^{k+1} \le A^{k+l} \ (l \ge 2).$$

Therefore, if $|f|_V > A^{k+l}$, $(l \ge 2)$ then $V \subset V_{k_j}^{**}$. From this and from weak type property of $M$ (see (1.2)), we get

$$\mu(V_{k_j} \cap D_{k+l}) \le \mu\{x \in V_{k_j} \ : \ M(f \chi_{V_{k_j}^{**}})(x) > A^{k+l}\}$$

$$\le \frac{C}{A^{k+l}} \int_{V_{k_j}^{**}} |f| d\mu \le C \frac{\mu(V_{k_j})}{A^{k+l}} |f|_{V_{k_j}^{**}} \le C \frac{A^{k+1}}{A^{k+l}} \mu(V_{k_j}) \le \frac{C}{A^l} \mu(V_{k_j}). \tag{3.14}$$

By estimates (3.12),(3.13),(3.14), when $l \geq 2$ we obtain

$$\begin{aligned}\int_G (T_l g(x))^{p'(x)} d\mu &= \sum_{k\in\mathbb{Z}}^{\infty} \sum_j \int_{V_{k_j}\cap D_{k+l}} \alpha_{j,k}(g)^{p'(x)} d\mu \\ &\leq CA^{-l\gamma} \sum_{(j,k)\in\mathcal{I}_1} \int_{E_j^k} g(x)^{p'(x)} d\mu + CA^{-l} \sum_{(j,k)\in\mathcal{I}_2} \int_{E_j^k} g(x) d\mu \\ &\leq CA^{-l\alpha} \left( \int_G g(x)^{p'(x)} d\mu + \int_G g(x) d\mu \right).\end{aligned}$$

Where $\alpha = \min\{1, \gamma\}$.

Using the fact that $\|g\|_1 \leq 2\|\chi_E\|_{p'(\cdot)}$, and $\int_G g(x)^{p'(x)} d\mu \leq 1$ we obtain

$$\|T_l g\|_{p'(\cdot)} \leq CA^{-l\alpha/p'_+} \quad (l \geq 2).$$

For $l = 0, 1$ if we use a trivial estimation $\mu(V_{k_j} \cap D_{k+l}) \leq \mu(V_{k_j})$, analogously will be obtained the estimation $\|T_l g\|_{p'(\cdot)} \leq C$. Finally, we obtain

$$\|Tg\|_{p'(\cdot)} \leq \sum_{l=0}^{\infty} \|T_l g\|_{p'(\cdot)} \leq C.$$

□

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