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ORIGINAL RESEARCH

Conformal Ricci almost solitons with certain soliton vector fields

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Abstract The object of the present article is to characterize conformal Ricci almost solitons on Riemannian manifolds with soliton vector fields as torqued vector fields and infinitesimal harmonic transformations. We also deduce a triviality condition under integral inequality.

Keywords Solitons · Riemannian manifolds · Torqued vector fields · Quasi-Einstein manifolds · Conformal Ricci almost soliton

Mathematics Subject Classification 53C21 · 53C24

1 Introduction

Barros and Ribeiro [1] constructed compact Ricci almost solitons on the standard sphere $S^n(c)$ with constant sectional curvature c. Let $i : S^n \to R$ be a height function defined by $i(x) =< x, a >$. If g_0 is the canonical metric of $S^n(c)$ considered as a hepersurface of R^{n+1}, then $(S^n(c), V, g_0, \lambda)$ is a Ricci almost soliton with $V(x) = \nabla i(x)$ and $\lambda = (n-1) - i$.

Fischer [9], defined conformal Ricci flow in the context of relativistic mechanics and theory of gravitation. Conformal Ricci flow is a variation of classical Ricci flow that modifies the unit volume constraint of the evolution equation to a scalar curvature constraint. Conformal Ricci flow is vector field sum of Ricci flow and conformal flow equations. The conformal Ricci flow equation is analogous to Navier-Stokes equation of fluid mechanics. A fixed point, upto diffeomorphism and scaling of a conformal Ricci flow equation is known as conformal Ricci soliton. In [7], the present authors with Falcitelli studied conformal Ricci soliton in the perspective of Kenmotsu manifolds. Generalizing the soliton constant of a conformal Ricci soliton by a smooth function, we get the notion of conformal Ricci almost soliton. The details will turn up the next section. Ricci almost solitons have been studied in the papers [10] and [11].

The soliton vector field of a Ricci almost soliton plays a pivotal role to determine the geometry of the soliton. A Ricci almost soliton becomes trivial [2] when soliton vector field ξ is Killing in case of a manifold of dimension greater than two. Furthermore, if the soliton vector field is non-conformal Killing and dimension of the manifold is greater than three then a Ricci almost soliton becomes isometric to a Euclidean sphere S^n.

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The above situation inspires us to pose the question that what will be the phenomenon when a conformal Ricci almost soliton will admit some non-Killing and non-conformal vector fields like torqued vector fields introduced by Chen [4,5] as soliton vector fields. The motive of the present paper is to analyze such situations and related analogue.

In the next section, we write up the preliminaries. Section 3 contains the analysis of conformal Ricci almost solitons carrying the soliton vector fields as a torqued vector fields. The case when the soliton vector field is an infinitesimal harmonic transformation has been taken care of in Section 4. A triviality condition associated with integral inequality has been deduced in the last section.

2 Preliminaries

The theory of Ricci solitons and its different generalizations have evolved from the ground breaking work of Perelman [13] based on Hamilton's [12] Ricci flow. Modifying the definition of Ricci soliton Pigola et al [15] introduced the definition of Ricci almost soliton as follows:

A Riemannian manifold (M^n, g) is said to be a Ricci almost soliton if there exist a complete vector field ξ and a smooth soliton function $\lambda : M^n \to R$ satisfying

$$(\pounds_\xi g)(X, Y) + 2S(X, Y) = 2\lambda g(X, Y), \tag{2.1}$$

where, S indicates the Ricci tensor and $\pounds_\xi$ stands for Lie derivative operator along the vector field ξ, and X, Y are arbitrary differentiable vector fields on the manifold M. When $\lambda < 0$, the soliton is termed as expanding. For $\lambda = 0$, it is called steady and it is shrinking whenever $\lambda > 0$. For details regarding the existence of such solitons one may consult [15–20].

The metric of a Riemannian manifold is called conformal Ricci soliton if

$$(\pounds_\xi g)(X, Y) + 2S(X, Y) = \left(2\lambda - \left(p + \frac{2}{2n+1}\right)\right) g(X, Y), \tag{2.2}$$

where S and $\pounds_\xi$ are as stated above. Here p is a positive constant and λ is an arbitrary constant [3,6]. If p is a positive function and λ is an arbitrary function, then such a soliton will be called conformal almost Ricci soliton as a generalization of Ricci almost soliton introduced by Pigola [15]. Conformal Ricci almost soliton will be called expanding, steady or shrinking according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.

A non-zero vector field ξ on a Riemannian or pseudo Riemannian manifold is called a torqued vector field if it satisfies the following conditions

$$\nabla_X \xi = lX + m(X)\xi, \quad \text{and} \quad m(\xi) = 0. \tag{2.3}$$

The function l is called torqued function and the 1-form m is called torqued form of ξ. The existence of such vector fields on Riemannian manifolds has been ensured by Chen [4,5]. Chen proved that every non-zero vector field on a 1-dimensional Riemannian manifold is torqued vector field. He also characterized the following:

If a Riemannian manifold M admits a torqued vector field ξ such that ξ is always tangent to I, an open interval, then M is locally a twisted product $I \times_f F$. Conversely, for a twisted product $I \times_f F$, there exists a torqued vector field ξ such that ξ is always tangent to I. A torqued vector field ξ is said to be associated with a twisted product $I \times_f F$ if ξ is always tangent to I. Every torqued vector field ξ associated with a twisted product $I \times_f F$ is of the form $\xi = \mu f \frac{\partial}{\partial s}$, where s is an arc length parameter on I, μ is a non-zero function defined on F and f is the twisting function.

For a twisted product $I \times_f F$, the torqued vector field $\xi = \mu f \frac{\partial}{\partial s}$ is called the canonical torqued vector field if $\mu = 1$. We shall denote the canonical torqued vector field $f \frac{\partial}{\partial s}$ by ξ_c^f. If f is a function of only s, then ξ_c^f becomes a concircular vector field [4]. It is known that [4] the Levi-Civita connection of $I \times_f F$ satisfies

$$\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} = 0, \quad \nabla_{\frac{\partial}{\partial s}} V = \nabla_V \frac{\partial}{\partial s} = \frac{\partial (lnf)}{\partial s} V$$

for any vector field V of F. From [4], we also have

$$l = \rho \frac{\partial (lnf)}{\partial s} \quad m(V) = V(ln\rho), \tag{2.4}$$

where $\rho = \mu f$. If $\mu = 1$, $\rho = f$. A vector field ξ on a Riemannian manifold (M, g) is called an infinitesimal harmonic transformation if the local 1-parameter group of infinitesimal point transformations produced by the vector field ξ constructs a group of harmonic transformations [19]. A characteristic feature of such a vector field is given by the equation trace$(\pounds_\xi \nabla) = 0$. Stepanov [19] proved that a vector field ξ generates an infinitesimal harmonic transformations on a Riemannian manifold (M, g) if and only if

$$\Delta\xi = 2Q\xi, \tag{2.5}$$

where Q is Ricci operator and Δ denotes Laplacian operator. A vector field ξ satisfying WeitzenbRck formula is given by

$$\Delta\xi = \nabla^*\nabla\xi + Q\xi, \tag{2.6}$$

where ∇^* is the formal adjoint of ∇ and Q is the Ricci operator associated with the $(0, 2)$ Ricci tensor S. The rough Laplacian $\bar{\Delta}$ of the vector field ξ is given by

$$\bar{\Delta}\xi = -\text{trace}\nabla^2\xi,$$

where

$$(\nabla^2\xi)(X, Y) = \nabla_X\nabla_Y\xi - \nabla_{\nabla_X V}\xi.$$

If $\{e_i\}$ is any orthonormal frame field, then we obtain

$$\bar{\Delta}\xi = \sum_i(\nabla_{\nabla_{e_i}e_i} - \nabla_{e_i}\nabla_{e_i}). \tag{2.7}$$

We also know that $\bar{\Delta}\xi = \nabla^*\nabla\xi$ and hence

$$\Delta\xi = \bar{\Delta}\xi + Q\xi. \tag{2.8}$$

From (2.5) and (2.8)

$$\bar{\Delta}\xi = Q\xi. \tag{2.9}$$

3 Conformal Ricci almost soliton with torqued soliton vector fields

Let us consider a conformal Ricci almost soliton with the soliton vector field as a torqued vector field ξ.

$$(\pounds_\xi g)(X, Y) + 2S(X, Y) = \left(2\lambda - \left(p + \frac{2}{2n+1}\right)\right)g(X, Y). \tag{3.1}$$

By virtue of (2.3) the above equation yields

$$S(X, Y) = \left(\lambda - l - \frac{p}{2} - \frac{1}{2n+1}\right)g(X, Y) - \frac{1}{2}m(X)q(Y) - \frac{1}{2}m(Y)q(X), \tag{3.2}$$

where q is dual to ξ. The above equation states the following:

Proposition 3.1 *A conformal Ricci almost soliton with torqued soliton vector field is a quasi-Einstein manifold.*
Since the metric g is Riemannian and ξ is non-zero, $g(\xi, \xi)$ is non-zero. Without loss of generality, let us assume ξ is a unit vector field. Let us construct an orthonormal frame $\{e_1, e_2, ...e_{n-1}, \xi\}$. Putting $X = Y = \xi$ and using (2.3) we infer that

$$S(\xi, \xi) = \left(\lambda - l - \frac{p}{2} - \frac{1}{2n+1}\right).$$

Similarly for any $i = 1, 2, .., n-1$

$$S(e_i, e_i) = \left(\lambda - l - \frac{p}{2} - \frac{1}{2n+1}\right).$$

Hence the scalar curvature of the manifold is

$$r = \left(\lambda - l - \frac{p}{2} - \frac{1}{2n+1}\right)(2n+1).$$

From above

$$l = \lambda - r - (2n+1)\frac{p}{2} - 1. \tag{3.3}$$

So, we have

Proposition 3.2 *A conformal Ricci almost soliton with torqued soliton vector field is expanding, steady or shrinking according as*
$r < -l - (2n+1)\frac{p}{2} - 1$,
$r = -l - (2n+1)\frac{p}{2} - 1$, *or,*
$r > -l - (2n+1)\frac{p}{2} - 1$.

Again from (3.2), putting $X = Y$, we have after simplification

$$m(X)g(\xi, X) = \left(\lambda - l - \frac{p}{2} - \frac{1}{2n+1}\right)g(X, X) - S(X, X). \tag{3.4}$$

If X is orthogonal to ξ.

$$\lambda = l + \frac{p}{2} + \frac{1}{2n+1}. \tag{3.5}$$

From (3.3) and (3.5) we get

$$r = -np - \frac{2n}{2n+1} \tag{3.6}$$

Since p is positive, the above equation gives us the following:

Theorem 3.1 *A conformal Ricci almost soliton with torqued soliton vector field is a manifold of negative scalar curvature.*

By Proposition 3.2 and equation (3.9) we are able to state the following:

Corollary 3.1 *A conformal Ricci almost soliton with torqued soliton vector field is expanding, steady or shrinking according as $p < \frac{4n}{2n+1} - 2(l+1)$, $p = \frac{4n}{2n+1} - 2(l+1)$, or $p > \frac{4n}{2n+1} - 2(l+1)$.*

Now consider a Ricci almost soliton with the soliton vector field as a conformal torqued vector field ξ_c^f. Then the manifold is locally a twisted product $I \times_f F$. where f is the twisting function and F is a Riemannian manifold of dimension $n-1$. By virtue of (3.1)

$$(\pounds_{\xi_c^f} g)(X, Y) = 2lg(X, Y) + m(X)g(\xi_c^f, Y) + m(Y)g(\xi_c^f, X), \tag{3.7}$$

for X, Y tangent to M. By virtue of (3.1), from the above equation we infer that

$$S(X, Y) = \left(\lambda - l - \frac{p}{2} - \frac{1}{2n+1}\right)g(X, Y) - \frac{1}{2}m(X)g(\xi_c^f, Y) - \frac{1}{2}m(Y)g(\xi_c^f, X). \tag{3.8}$$

Because of ξ_c^f is canonical torqued vector field on $I \times_f F$, we note that

$$\xi_c^f = f\frac{\partial}{\partial s} \quad \text{and} \quad g = ds^2 + f^2 g_F. \tag{3.9}$$

Now in view of (2.7) it follows that

$$l = \frac{\partial f}{\partial s} \quad \text{and} \quad m(V) = V(\ln f). \tag{3.10}$$

Since $m(\xi_c^f) = 0$, *from (3.8), (3.9) and (3.10), one obtains*

$$S(\xi_c^f, \xi_c^f) = \left(\lambda - \frac{p}{2} - \frac{1}{2n+1} - f_s\right) f^2. \tag{3.11}$$

$$S(\xi_c^f, V) = -\frac{1}{4} V(f^2). \tag{3.12}$$

$$S(V, W) = \left(\lambda - \frac{p}{2} - \frac{1}{2n+1} - f_s\right) g(V, W) \tag{3.13}$$

for any vector fields V *and* W *of* F. *Since* $M = I \times_f F$, *from Fernandez [8] we have*

$$S(\xi_c^f, \xi_c^f) = (1-n) f f_{ss}. \tag{3.14}$$

$$S(\xi_c^f, V) = (2-n) V(f_s). \tag{3.15}$$

$$\begin{aligned} S(V, W) = {} & S^F(V, W) + (3-n)(VW(\ln F)(\nabla_V^F W)(\ln f)) \\ & + (n-1) V(\ln f) W(\ln f) \\ & - g(V, W)(\Delta(\ln f) + g(\nabla(\ln f), \nabla(\ln f))) \end{aligned} \tag{3.16}$$

for any vector field V, W *of* F. *Here* Δ *is the Laplacian operator. From (3.11) and (3.14)*

$$(1-n) f_{ss} = (\lambda - \frac{p}{2} - \frac{1}{2n+1} - f_s) f. \tag{3.17}$$

In view of (3.12) and (3.15)

$$4(n-2) V(f_s) = V(f^2) \tag{3.18}$$

for any vector fields V *of* F. *Solving the equation (3.16) as in [4] one obtains*

$$4(n-2) f_s - f^2 = k(s), \tag{3.19}$$

where k *is a function of* s. *Now solving (3.19) yields*

$$f_{ss} = \frac{2 f f_s + k'(s)}{4(n-2)}. \tag{3.20}$$

By virtue of (3.17) and (3.20)

$$(n-3) f_s = \frac{n-1}{2f} k'(s) + 2(n-2)\left(\lambda - \frac{p}{2} - \frac{1}{2n+1}\right). \tag{3.21}$$

By solving (3.19) for f_s *and using it in (3.21) we infer*

$$(n-3) f^3 \left[(n-3)k(s) + 8(n-2)^2 \left(\lambda - \frac{p}{2} - \frac{1}{2n+1}\right)\right] f = 2(n^2 - 3n + 2) k'(s). \tag{3.22}$$

From above we see that the valus of f *depends on* $k(s)$, λ *and* p. *Now for conformal Ricci almost soliton* λ *and* p *are not necessarily function of* s *only. So we conclude the following:*

Theorem 3.4 *A conformal Ricci almost soliton* $(I \times_f F, g, \xi_c^f)$ *with soliton vector field as canonical torqued vector field is not necessarily a warped product space.*

If instead of conformal Ricci almost soliton, we take conformal Ricci soliton, then λ *and* p *are constants. Then* f *is function of* s *only. Thus, we have*

Corollary 3.2 *A conformal Ricci soliton* $(I \times_f F, g, \xi_c^f)$ *with canonical torqued soliton vector field is a warped product space.*

4 Conformal Ricci almost soliton with soliton vector as infinitesimal harmonic transformation

Stepanov [20] proved that the soliton vector field of a Ricci soliton is an infinitesimal transformation. Ghosh [11] applied this result to study Ricci almost soliton. Here, we opt to analyze conformal Ricci almost soliton having soliton vector field as infinitesimal harmonic transformations.

Let us consider a conformal Ricci almost soliton with soliton vector field ξ as an infinitesimal harmonic transformation. Then from (2.2) we infer that

$$(\nabla_W \pounds_\xi g)(X, Y) + 2(\nabla_W S)(X, Y) = (2(W\lambda) - Wp)\, g(X, Y). \tag{4.1}$$

By the consequence of commutation formula of Lie derivative and covariant derivative [21], we have

$$(\nabla_W \pounds_\xi g)(X, Y) = g((\pounds_\xi \nabla)(W, X), Y) + g((\pounds_\xi \nabla)(W, Y), X). \tag{4.2}$$

Combining (4.1) and (4.2), we have

$$g((\pounds_\xi \nabla)(W, X), Y) + g((\pounds_\xi \nabla)(W, Y), X) + 2(\nabla_W S)(X, Y) = (2W\lambda - Wp)g(X, Y). \tag{4.3}$$

Using cyclic permutation W, X, Y in (4.3), we obtain after some routine calculation

$$\begin{aligned} g((\pounds_\xi \nabla)(W, X), Y) = {} & (\nabla_Y S)(W, X) - (\nabla_W S)(X, Y) \\ & - (\nabla_X S)(W, Y) + \left(W\lambda - \frac{1}{2}Wp\right) g(X, Y) \\ & + \left(X\lambda - \frac{1}{2}Xp\right) g(W, Y) \\ & - \left(Y\lambda - \frac{1}{2}Yp\right) g(W, X). \end{aligned} \tag{4.4}$$

Again from Kon [21], we know the formula

$$g\left((\pounds_\xi \nabla)(W, X), Y\right) = g(\nabla_W \nabla_X V - \nabla_{\nabla_W X} V - R(W, V)X, Y). \tag{4.5}$$

By virtue of the above two equations, we have the following

$$\begin{aligned} g(\nabla_W \nabla_X \xi - \nabla_{\nabla_W X} \xi - R(W, \xi)X, Y) = {} & (\nabla_Y S)(W, X) - (\nabla_W S)(X, Y) \\ & - (\nabla_X S)(W, Y) + \left(W\lambda - \frac{1}{2}Wp\right) g(X, Y) \\ & + \left(X\lambda - \frac{1}{2}Xp\right) g(W, Y) \\ & - \left(Y\lambda - \frac{1}{2}Yp\right) g(W, X). \end{aligned} \tag{4.6}$$

After contraction of W and X in (4.6) and using (2.4) one obtains

$$\bar{\Delta}\xi - Q\xi = (n - 2)D\left(\lambda - \frac{p}{2}\right). \tag{4.7}$$

Now in view of (2.9), we state the following:

Proposition 4.1 *If the soliton vector field of a conformal Ricci almost soliton is an infinitesimal harmonic transformation, then $\lambda - \frac{p}{2}$ is a constant.*

To prove the converse, let us follow [20]. Let us consider a coordinate chart $x^1, x^2, x^3, \ldots.., x^n$ on the manifold. In that coordinate chart, equation (2.2) takes the form

$$-2S_{ij} = \nabla_i \xi_j + \nabla_j \xi_i - 2\left(\lambda - \frac{p}{2} - \frac{1}{2n+1}\right) g_{ij}.$$

Here $\xi_i = g_{ij}\xi^j$ are the components of the soliton vector field ξ. S_{ij} are the components of the Ricci tennsor S and g_{ij} are the components of the metric tensor g. If $\lambda - \frac{p}{2}$ is constant, then by the same technique used by Stepanov [20], ξ is infinitesimal harmonic transformation. Thus, we obtain

Proposition 4.2 *If $\lambda - \frac{p}{2}$ is a constant, then the soliton vector field of a conformal Ricci almost soliton is an infinitesimal harmonic transformation.*

Combining, the above two propositions, we obtain the following

Theorem 4.1 *The soliton vector field of a conformal Ricci almost soliton is an infinitesimal harmonic transformation if and only if $\lambda - \frac{p}{2}$ is a constant.*

5 Triviality of conformal Ricci almost solitons satisfying an integral inequality

A Ricci almost soliton is called trivial if it is Einstein. Barros and Ribeiro [1] deduced an integral inequalities under which a Ricci almost soliton becomes trivial by generalizing a result of Peterson [14]. In [11], Ghosh deduced another integral inequalities for Ricci almost solitons to be trivial. Here we would like to find the form of the integral inequality under which conformal Ricci almost soliton is trivial.

From Yano [21], we know the following formula

$$\int_M \left(g(\Delta\xi, \xi) - \frac{1}{2}|d\xi^*|^2 - (\text{div}\xi)^2 \right) dM = 0, \tag{5.1}$$

where ξ^* is the 1-form dual to the soliton vector field ξ. Now, if the manifold admits a conformal Ricci almost soliton, proceeding in the same way as in the previous section, we deduce that

$$\bar{\Delta}\xi - Q\xi = (n-2)D\left(\lambda - \frac{p}{2}\right). \tag{5.2}$$

By virtue of (2.8) and the above equation, we infer

$$\Delta\xi - (n-2)D\left(\lambda - \frac{p}{2}\right) = 2Q\xi. \tag{5.3}$$

In view of (5.1) and (5.3), we have

$$\int_M \left(2S(\xi, \xi) + (n-2)g\left(D\left(\lambda - \frac{p}{2}\right), \xi\right)\right) dM = \int_M \left(\frac{1}{2}|d\xi^2| + (\text{div}\xi)\right)^2 dM. \tag{5.4}$$

In the previous equation , if we assume the left hand side is ≤ 0, it gives us

$$\int_M \left(\frac{1}{2}|d\xi^*| + \text{div}\xi \right)^2 dM = 0. \tag{5.5}$$

Consequently, $d\xi^* = 0$ and $\text{div}\xi = 0$. In other words, the one form ξ^* is harmonic. Hence integrating the Bochner formula one can obtain

$$\int_M \left(S(\xi.\xi) + |\nabla\xi^*|^2 \right) dM = \int_M \frac{1}{2}\Delta|\xi^*| dM = 0.$$

Again from (5.4), applying harmonicity of ξ^*, we have

$$\int_M \left(2S(\xi, \xi) + (n-2)g\left(D\left(\lambda - \frac{p}{2}\right), \xi\right)\right) dM = 0.$$

Since $g\left(D\left(\lambda - \frac{p}{2}\right), \xi\right) = \text{div}\left(\left(\lambda - \frac{p}{2}\right)\xi\right) = 0$, the above equation gives $\int_M S(\xi, \xi)dM = 0$. Then following Ghosh [11] we get S is identically zero. Thus the manifold is Ricci flat. Hence, we are in a position to state the following:

Theorem 5.1 *If a compact conformal Ricci almost soliton of dimension greater than one satisfies $\int_M \left(2S(\xi, \xi) + (n-2)g\left(D\left(\lambda - \frac{p}{2}\right), \xi\right)\right) dM \leq 0$, then the soliton is trivial.*

Since for conformal Ricci soliton λ and p are constants, as a consequence of the above theorem, we obtain the following:

Theorem 5.1 *If a compact conformal Ricci soliton of dimension greater than one satisfies $\int_M 2S(\xi, \xi) \leq 0$, then the soliton is trivial.*

Data availability The manuscript has no associated data.

Declarations

Competing interests There is no competing interest.

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